1. FINITE-DIMENSIONAL VECTOR SPACES

§1.1. Fields

By now you’ll have acquired a fair knowledge of matrices. These are a concrete embodiment of something rather more abstract. Sometimes it is easier to use matrices, but at other times the abstract approach allows us more freedom.

Recall that a field is a mathematical system having two operations + and \( \times \), where we write the composition of two elements \( a, b \) as \( a + b \) and \( ab \) respectively. There are 11 field axioms that must be satisfied. For addition we have the closure law, the associative law, the commutative law and the existence of an identity, 0, and inverses, \(-a\). We have the corresponding five axioms for multiplication, but note that there are subtle changes when it comes to the existence of identity, 1, and inverses \( a^{-1} \). We insist that 0 and 1 are distinct and we only insist on multiplicative inverses for all non-zero elements. Finally there’s the distributive law to hold both structures together. A field underlies every vector space and the elements of that field are called scalars.

Example 1: The following are examples of fields:

- \( \mathbb{Q} \), the set of rational numbers.
- \( \mathbb{R} \), the set of real numbers.
- \( \mathbb{C} \), the set of complex numbers.
- \( \mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \} \).

The associative, commutative and distributive laws hold throughout the system of complex numbers, so the only axioms that need to be checked are the closure, identity and inverse laws. For \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{C} \) these are obvious. Let us check them for \( \mathbb{Q}[\sqrt{2}] \).

\[
\begin{align*}
(a + b\sqrt{2}) + (c + d\sqrt{2}) &= (a + c) + (b + d)\sqrt{2}, \text{ so } \mathbb{Q}[\sqrt{2}] \text{ is closed under addition.} \\
(a + b\sqrt{2})(c + d\sqrt{2}) &= (ac + 2bd) + (ad + bc)\sqrt{2}, \text{ so } \mathbb{Q}[\sqrt{2}] \text{ is closed under addition.} \\
\mathbb{Q}[\sqrt{2}] \text{ clearly contain both 0 and 1 so the identity laws hold.}
\end{align*}
\]

\[
-(a + b\sqrt{2}) = (-a) + (-b)\sqrt{2} \text{ and } \frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} + \left(\frac{-b}{a^2 - 2b^2}\right)\sqrt{2} \text{ so } \mathbb{Q}[\sqrt{2}] \text{ is closed under both additive and multiplicative inverses.}
\]

There are certain properties of field that are consequences of these axioms. For example there is no axiom that says that \( 0 \times 0 = 0 \). However if we consider that \( 0 \times (0 + 0) = 0 \times 0 \) we reach this conclusion (with the help of the additive identity axiom and the distributive law).

The cancellation law that we constantly use in basic algebra is not one of the 11 axioms but is a consequence of them. If \( ab = 0 \) then \( a = 0 \) or \( b = 0 \). For if \( a \neq 0 \) then \( a^{-1} \) exists and \( a^{-1}(ab) = a^{-1}0 = 0 \) and so \( b = 0 \).

**Theorem 1:** If \( p \) is prime then \( \mathbb{Z}_p = \{0, 1, 2, \ldots, p - 1\} \) is a field under addition and multiplication modulo \( p \) (where we add and multiply normally but then take the remainder on dividing by \( p \)).

**Proof:** All the field axioms are obvious except for inverses under multiplication.

Suppose \( x \) is a non-zero element of \( \mathbb{Z}_p \). Regarding \( x \) as an integer this means that \( x \) is not divisible by \( p \). In other words, \( x \) is coprime to \( p \). This means that \( ax + bp = 1 \) for some integers \( a, b \). Now, interpreting this modulo \( p \) this becomes \( ax = 1 \), so \( x \) has an inverse under multiplication.
Example 2: In \( \mathbb{Z}_{17} \) the following table gives the inverses under addition and multiplication for the non-zero elements.

| x   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| −x  | 16  | 15  | 14  | 13  | 12  | 11  | 10  | 9   | 8   | 7   | 6   | 5   | 4   | 3   | 2   | 1   |
| x^{-1} | 1   | 9   | 6   | 13  | 7   | 3   | 5   | 15  | 2   | 12  | 14  | 10  | 4   | 11  | 8   | 16  |

Integers that are 1 plus a multiple of 17 include 18, 35, 52, 69, 86, 103 and 120.
So, for example, \( 10 \times 12 = 1 \mod 17 \) and so 10 and 17 are inverses of one another.
Then, since \( 7 = -10 \), \( 7^{-1} = -10^{-1} = -12 = 5 \).

If \( p \) is not prime, however, \( \mathbb{Z}_p \) is not a field because, if \( p = ab \) for some integers \( a, b \) where \( 1 < a, b < p \), then modulo \( p \) we would have \( ab = 0 \) while \( a \neq 0 \) and \( b \neq 0 \).
So the only systems of integers modulo \( p \) that give fields are those where \( p \) is prime. But these are not the only finite fields. For every prime power \( p^n \) there exists a field with \( p^n \) elements (and for no other sizes). But the field of order \( p^n \) (there is only one) is not \( \mathbb{Z}_p^n \), unless \( n = 1 \).

Example 3: The following are the addition and multiplication tables for the field with 4 elements:

\[
\begin{array}{c|ccc}
  + & 0 & 1 & 2 & 3 \\
  \hline
  0 & 0 & 1 & 2 & 3 \\
  1 & 1 & 0 & 3 & 2 \\
  2 & 2 & 3 & 0 & 1 \\
  3 & 3 & 2 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
  \times & 0 & 1 & 2 & 3 \\
  \hline
  0 & 0 & 0 & 0 & 0 \\
  1 & 0 & 1 & 2 & 3 \\
  2 & 0 & 2 & 3 & 1 \\
  3 & 0 & 3 & 1 & 2 \\
\end{array}
\]

§1.2. Vector Spaces

A **vector space** \( V \), over a field \( F \), is a set, together with two operations, addition \((u + v)\) and scalar multiplication \((\lambda v)\) such that the following properties (called the vector space axioms) hold.

1. (Closure under addition) For all \( u, v \in V \), \( u + v \in V \);
2. (Closure under scalar multiplication) For all \( \lambda \in F \) and all \( v \in V \), \( \lambda v \in V \);
3. (Associative law under addition) For all \( u, v, w \in V \), \( u + (v + w) = (u + v) + w \);
4. (Commutative law under addition) For all \( u, v \in V \), \( u + v = v + u \);
5. (Identity under addition) There exists \( 0 \in V \) such that for all \( v \in V \), \( v + 0 = v \);
6. (Inverses under addition) For all \( v \in V \) there exists \( -v \in V \) such that \( v + (-v) = 0 \);
7. (Distributive) For all \( \lambda \in F \) and all \( u, v \in V \), \( \lambda (u + v) = \lambda u + \lambda v \);
8. (Distributive) For all \( \lambda, \mu \in F \) and all \( v \in V \), \( (\lambda + \mu) v = \lambda v + \mu v \);
9. (Associative under scalar multiplication) For all \( \lambda, \mu \in F \) and all \( v \in V \), \( (\lambda \mu) v = \lambda (\mu v) \);
10. (Identity under scalar multiplication) For all \( v \in V \), \( 1 v = v \).

You may wonder why we haven’t printed the vectors \( u, v, w \) in bold type. That is a practice when you first learn about vectors so that you can see clearly which are vectors and which are scalars. So we would write \( \lambda v \) to emphasise that \( \lambda \) is a scalar and \( v \) is a vector. However there is nothing in the set of axioms for a vector space that says that vectors and scalars are different things. There are examples where some \( F \) is a subset of \( V \) and where there are elements that are both vectors and scalars.

Example 4: \( \mathbb{C} \) is a vector space over \( \mathbb{R} \). Here the real numbers are scalars, and the complex numbers are the vectors. But because every real number is also a complex number, they are both scalars and vectors.
This situation occurs whenever you have a subfield of a larger field. If you go through the 10 axioms, in the situation where F is a subfield of V, and you will find that the 10 vector space axioms are direct consequences of the 11 field axioms.

In the area of mathematics that studies fields (Galois Theory) the theory of vector spaces plays an important role.

**Example 5:** For any field \( F \), \( F^n \) is the set of all n-tuples \((x_1, x_2, \ldots, x_n)\) where each \( x_i \in F \) and where addition and scalar multiplication are defined in the usual way. It is easily seen to be a vector space over \( F \).

**Example 6:** Consider 3-dimensional Euclidean space. This is a vector space over \( \mathbb{R} \) as follows. The vectors are the directed line segments from the origin to a point. The scalars are the real numbers. Addition is defined by completing a parallelogram.

![Vector Addition](image)

Multiplying a non-zero vector \( v \) by a positive scalar \( \lambda \) produces a vector with the same direction as \( v \) but with \( \lambda \) times the length. Multiplying by a negative scalar magnifies the length and reverses the direction. And, of course \( 0v = 0 = \lambda 0 \) for all vectors \( v \) and scalars \( \lambda \).

Clearly this is essentially the same as \( \mathbb{R}^3 \). We will make the concept of “essentially the same” precise in a later chapter.

**Example 7:** For any field \( F \), \( F^\infty \) is the set of all infinite sequences \((x_1, x_2, \ldots)\) with each \( x_i \in F \) and with addition and multiplication defined in the obvious way.

**Example 8:** For any field \( F \), \( F[x] \) is the set of all polynomials \( a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) where each \( x_i \in F \) and \( n \) is any non-negative integer. Addition and scalar multiplication are defined in the obvious way. \( F[x] \) is clearly a vector space over \( F \). Note that we could write the polynomial as an infinite sequence \((a_0, a_1, \ldots)\), but \( F[x] \) differs from \( F^\infty \) in that here all the components from some point on are zero.

**Example 9:** For any field \( F \), \( M_n(F) \) is the set of all \( n \times n \) matrices with the usual addition and scalar multiplication. This is clearly a vector space over \( F \).

Up to this point the vector spaces have had recognisable components. In the next example this is not the case.

**Example 10:** \( \text{Diff}(\mathbb{R}) \) is the set of all differentiable functions from the reals to the reals. The sum of two differentiable functions \( f(x) \) and \( g(x) \) is the differentiable function \( f(x) + g(x) \) and the scalar multiple of the differentiable function \( f(x) \) by the scalar \( \lambda \) is the differentiable function \( \lambda f(x) \). Thus the closure laws hold. The remaining axioms are just as obvious.

The function \( f(x) = x^2 \) belongs to \( \text{Diff}(\mathbb{R}) \). But what are its components?

Spaces of functions are very important in the deeper study of analysis.
§1.3. Elementary Properties of Vector Spaces

There are a number of additional properties that vector spaces satisfy. However these are not included in the list of axioms because they are consequences of them.

**Theorem 2:** If \( V \) is a vector space over the field \( F \) then for all \( u, v \in V \) and all \( \lambda, \mu \in F \):

1. \( 0v = 0 \);
2. \((-1)v = -v\);
3. \( 0 = 0 \);
4. \( \lambda v = 0 \) implies that \( \lambda = 0 \) or \( v = 0 \).
5. \( \lambda u = \lambda v \) and \( \lambda \neq 0 \) implies that \( u = v \).
6. \( \lambda v = \mu v \) and \( v \neq 0 \) implies that \( \lambda = \mu \).

**Proof:**

1. \( 0v = (0 + 0)v = 0v + 0v \), so the result follows by subtraction.
2. \( 0 = 0v = (-1 + 1)v = (-1)v + v \).
3. \( \lambda 0 = \lambda (0 + 0) = \lambda 0 + \lambda 0 \), so the result follows by subtraction.
4. If \( \lambda \neq 0 \) then \( \lambda^{-1} \) exists. Hence if \( \lambda v = 0 \) then \( v = 1v = (\lambda^{-1}\lambda)v = \lambda^{-1}(\lambda v) = \lambda^{-1}0 = 0 \) by (4).
5. If \( \lambda u = \lambda v \) then \( \lambda(u - v) = 0 \), so if \( \lambda \neq 0 \) then \( u - v \).
6. If \( \lambda v = \mu v \) then \( (\lambda - \mu)v = 0 \), so if \( v \neq 0 \) then \( \lambda - \mu = 0 \).

§1.4. Subspaces

A **subspace** of a vector space is a non-empty subset that is closed under addition and scalar multiplication. The fact that \( -v = (-1)v \) guarantees that a subspace is also closed under inverse. Note too that if \( V \) is any vector space \( \{0\} \) is a subspace of \( V \), as is \( V \) itself.

If \( U \) is a subspace of \( V \) we write \( U \leq V \). Every vector space is a subspace of itself, but if we want to emphasise that \( U \) is not the same as \( V \) we can write \( U < V \) and say that \( U \) is a proper subspace of \( V \).

**Examples 11:**

1. \( \{(x, y, z) \in \mathbb{R}^3 \mid z = x + y\} \) is a subspace of \( \mathbb{R}^3 \).
2. \( \{(x, y, z) \in \mathbb{R}^3 \mid 3x + 2y + 5z = 0 \text{ and } 7x - y + 2z = 0\} \leq \mathbb{R}^3 \).
3. A plane that passes through the origin is a subspace of 3-dimensional space.
4. The set of differentiable functions from \( \mathbb{R} \) to \( \mathbb{R} \) is a subspace of the set of continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \), which in turn is a subspace of the set of all functions from \( \mathbb{R} \) to \( \mathbb{R} \).
5. The set of convergent sequences is a subspace of \( \mathbb{R}^\infty \).
6. The set of diagonal \( n \times n \) matrices is a subspace of the set of all \( n \times n \) matrices.
7. \( \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C} \).
8. In Example 3, \( \mathbb{Z}_2 \) is a subspace of the field with 4 elements.
9. For any vector space \( V \) \( \{0\} \leq V \).

There are 10 axioms for a vector space, but axioms (3), (4), (7), (8), (9) and (10) will automatically be inherited by any subset. If the subset is non-empty we can dispense with (5) and (6) as well.

**Theorem 3:** If a non-empty subset is closed under addition and scalar multiplication then it is a subspace.

**Proof:** Closure under inverses follows from closure under scalar multiplication and the fact that: \( -v = (-1)v \). Closure under zero follows from the fact that \( 0v = 0 \), provided the subset is non-empty.
But note that the empty set is (vacuously) closed under addition, inverses and scalar multiplication, but it is not a subspace.

If \( U \) and \( V \) are subspaces of \( W \) there are two other subspaces that can be formed from them (though in special cases these may coincide with \( U \) or \( V \)). The intersection \( U \cap V \) is a subspace, as is the sum, \( U + V \), which is defined to be \( \{ u + v \mid u \in U, v \in V \} \).

**Theorem 4:** If \( U, V \) are subspaces of \( W \) (a vector space over \( F \)) then so are:

1. \( U \cap V \)
2. \( U + V \).

**Proof:**

(1) Since \( 0 = 0 + 0 \) \( \in U \cap V \), it is non-empty.

Let \( u, v \in U \cap V \) and \( \lambda \in F \). Then \( u, v \in U \) and \( \lambda v \), for all scalars \( \lambda \), belong to \( U \). Similarly they both belong to \( V \) and so belong to \( U \cap V \).

(2) Since \( 0 = 0 + 0 \in U + W \), it is non-empty.

Let \( w_1 = u_1 + v_1 \) and \( w_2 = u_2 + v_2 \) belong to \( U + V \), with \( u_1, u_2 \in U \) and \( v_1, v_2 \in V \).

Then \( w_1 + w_2 = (u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2) \in U + V \).

Let \( \lambda \in F \) and \( w = u + v \) where \( u \in U \) and \( v \in V \). Then \( \lambda w = \lambda(u + v) = \lambda u + \lambda v \in U + V \).

**Example 12:** If \( U, V \) are distinct lines through the origin (in 3-dimensional space) \( U \cap V \) is \( \{0\} \) and \( U + V \) is the plane that passes through both lines. If \( U, V \) are distinct planes through the origin then \( U \cap V \) is the line where the planes intersect and \( U + V \) is the 3-dimensional space.

### §1.5. Finite-Dimensional Vector Spaces and Their Bases

A **linear combination** of \( v_1, v_2, \ldots, v_n \in V \) is any vector of the form \( \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n \).

The **span** of \( v_1, v_2, \ldots, v_n \) is the subspace \( \langle v_1, v_2, \ldots, v_n \rangle \), which is the set of all linear combinations of them. The space spanned by the empty set is defined to be \( \{0\} \). So the span of a finite set is the smallest subspace that contains it. If this is the space \( V \) we say that \( v_1, v_2, \ldots, v_n \) span \( V \).

**Examples 13:**

1. \( \langle v \rangle \) is the set of all scalar multiples of \( v \);
2. \( (1, 1, 0), (1, 0, 0) \) and \( (0, 1, 1) \) span \( \mathbb{R}^3 \) since 

\[
\begin{align*}
(x, y, z) &= (y - z)(1, 1, 0) + (x + z - y)(1, 0, 0) + z(0, 1, 1) \\
\end{align*}
\]

3. \( (1, 1, 0), (1, 2, 1) \) and \( (0, 1, 1) \) span the plane \( \{ (x, y, z) \mid x + z = y \} \), not the whole of \( \mathbb{R}^3 \).

Clearly \( \langle v_1, v_2, \ldots, v_n \rangle = \langle v_1 \rangle + \langle v_2 \rangle + \ldots + \langle v_n \rangle \).

**Theorem 5:** If \( u \in \langle v_1, v_2, \ldots, v_n \rangle \) then \( \langle u, v_1, v_2, \ldots, v_n \rangle = \langle v_1, v_2, \ldots, v_n \rangle \).

**Proof:** Suppose \( u \in \langle v_1, v_2, \ldots, v_n \rangle \) and let \( u = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n \).

Clearly \( \langle v_1, v_2, \ldots, v_n \rangle \leq \langle u, v_1, v_2, \ldots, v_n \rangle \).

Since \( \lambda u + \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n = (\lambda x_1 + \lambda_1) v_1 + (\lambda x_2 + \lambda_2) v_2 + \ldots + (\lambda x_n + \lambda_n) v_n \), the inequality holds in reverse.

The vectors \( v_1, v_2, \ldots, v_n \) are defined to be **linearly independent** if 

\[
\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n = 0 \implies \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0.
\]

If they are not linearly independent they are said to be **linearly dependent**.

**Example 14:** In 3-dimensional space two vectors are linearly dependent if they are in the same line. Three vectors are linearly dependent if they are in the same plane.
Example 15: Are \((1, 1, 1, 4), (1, 4, 7, 13), (3, 1, 8, 6), (4, 2, 6, 10)\) linearly independent?

Solution: Suppose 
\[\lambda_1(1, 1, 1, 4) + \lambda_2(1, 4, 7, 13) + \lambda_3(3, 1, 8, 6) + \lambda_4(4, 2, 6, 10) = (0, 0, 0, 0)\]

We solve the resulting system of equations by reducing the coefficient matrix to echelon form.

\[
\begin{pmatrix}
1 & 1 & 3 & 4 \\
1 & 4 & 7 & 6 \\
1 & 8 & 6 & 10
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 3 & 4 \\
0 & 3 & 2 & 2 \\
0 & 6 & 2 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 3 & 4 \\
0 & 3 & 2 & 2 \\
0 & 0 & 9 & 6
\end{pmatrix}
\]

so there is a non-zero solution and hence the vectors are linearly dependent. Solving the system of equations we obtain the non-zero solution 
\[\lambda_4 = 9, \lambda_3 = -6, \lambda_2 = 2, \lambda_1 = -20.\]

So \(2(1, 4, 7, 13) + 9(4, 2, 6, 10) = 6(3, 1, 8, 6) + 20(1, 1, 1, 4).\)

A basis for a finite-dimensional vector space \(V\) is a linearly independent set that spans \(V\).

Example 16: The set \({(1, 0, 0, \ldots, 0, 0), (0, 1, 0, \ldots, 0, 0), \ldots, (0, 0, 0, \ldots, 0, 1)\}\) is a basis for \(F^n\), where \(F\) is any field. This is called the standard basis. We often denote it by \({e_1, e_2, \ldots, e_n}\).

A minimal spanning set for \(V\) is a set of vectors that spans \(V\) and has smallest size of any spanning set.

Theorem 6: If \(V\) has a spanning set of size \(m\) and a linearly independent set of size \(n\) then \(m \geq n\).

Proof: Let \(A\) be a spanning set for \(V\) and let \(B\) be a linearly independent subset of \(V\). Suppose \(\#A = m\) and \(\#B = n\). Let \(b \in B\). Since \(A\) spans \(V\), \(b\) is a linear combination of the vectors in \(A\). Now some of the vectors in \(A\) might also be in \(B\). But since \(B\) is linearly independent the coefficient of some element \(a \in A - B\) must be non-zero. Then \(a\) can be expressed as a linear combination of \(A - \{a\} + \{b\}\) and so this set spans \(V\). In other words, we can replace \(a\) by \(b\) and the resulting set still spans \(V\). Continuing in this way we can transfer all the vectors from \(B\) into \(A\), displacing an equal number of vectors. Hence \(m \geq n\).

Corollary: All bases for a finite-dimensional vector space have the same number of elements.

Proof: If two bases have \(m\) and \(n\) elements respectively then \(m \leq n\) and \(n \leq m\).

The unique number of vectors in a basis of a finite-dimensional vector space \(V\) is called the dimension of \(V\) and is denoted by \(\dim(V)\). A set of vectors in \(V\) whose size exceeds \(\dim(V)\) is always linearly dependent. A set of vectors whose size is less than \(\dim(V)\) cannot span \(V\).

Examples 17:

1. \(\dim F^n = n\) because the vectors \((1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots (0, 0, 0, \ldots, 0, 1)\) form a basis (called the standard basis).

2. The dimension of three dimensional Euclidean space is 3, of course!

3. \(\dim M_n(F) = n^2\). The standard basis consists of the matrices \(E_{ij}\) which have a “1” in the \(i\)-\(j\) position and 0’s elsewhere.

4. \(\dim \mathbb{C}\) (as a vector space over \(\mathbb{R}\)) is 2, with \(\{1, i\}\) as an obvious basis.

5. The space \(\text{Diff}(\mathbb{R})\) is infinite dimensional.

6. The dimension of the zero subspace is 0.
§1.6. Sums and Direct Sums

Recall that the sum of two subspaces $U, V$ is $U + V = \{ u + v \mid u \in U, v \in V \}$ and their intersection is $U \cap V = \{ v \mid v \in U \text{ and } v \in V \}$. The sum $U + V$ is called a direct sum whenever $U \cap V = 0$. (Here we are writing the zero subspace as 0, instead of $\{0\}$). If $W$ is the direct sum of $U$ and $V$ we write $W = U \oplus V$.

**Example 18:** In $\mathbb{R}^3$, if $U, V$ are two distinct planes through the origin then $U + V = \mathbb{R}^3$. The sum is not direct, however, because $U \cap V$ will be a line through the origin. On the other hand if $U$ is a plane through the origin and $V$ is a line through the origin that doesn’t lie in the plane, then $U + V = \mathbb{R}^3$ as before, but this time $U \cap V = 0$. We can therefore write $\mathbb{R}^3 = U \oplus V$.

**Theorem 7:** If $W = U \oplus V$ then every element of $W$ can be expressed uniquely as $u + v$ for $u \in U$ and $v \in V$.

**Proof:** The only part that isn’t immediately obvious is the directness of the sum.

Suppose $u_1 + v_1 = u_2 + v_2$ where $u_1, u_2 \in U$ and $v_1, v_2 \in V$.

Then $u_1 - u_2 = v_2 - v_1 \in U \cap V = 0$. Hence $u_1 = u_2$ and $v_1 = v_2$.

The converse also holds. If every element of $W$ can be expressed uniquely as $u + v$ for $u \in U$ and $v \in V$ then $W = U \oplus V$.

The dimension of $U \cap V$ can be expressed in terms of the dimensions of $U, V$ and $U + V$.

**Theorem 8:** If $U, V$ are subspaces of $W$ then

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V).$$

**Proof:** Let $m = \dim(U)$, $n = \dim(V)$ and $r = \dim(U \cap V)$.

Take a basis $w_1, w_2, \ldots, w_r$ for $U \cap V$.

Extend this to a basis $w_1, w_2, \ldots, w_r, u_1, u_2, \ldots, u_{m-r}$ for $U$.

Now extend the basis for $U \cap V$ to a basis $w_1, w_2, \ldots, w_r, v_1, v_2, \ldots, v_{n-r}$ for $V$.

We shall show that $v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_{m-r}, v_1, v_2, \ldots, v_{n-r}$ is a basis for $U + V$.

**They span $V$:**

Let $u + v \in U + V$ where $u \in U$ and $v \in V$.

Then $u = \alpha_1 w_1 + \ldots + \alpha_r w_r + \beta_1 u_1 + \ldots + \beta_{m-r} u_{m-r}$ for some $\alpha_i$’s and $\beta_j$’s in $F$.

and $v = \gamma_1 w_1 + \ldots + \gamma_r w_r + \delta_1 v_1 + \ldots + \delta_{n-r} v_{n-r}$ for some $\gamma_i$’s and $\delta_i$’s in $F$.

Hence $u + v = [\alpha_1 w_1 + \ldots + \alpha_r w_r + \beta_1 u_1 + \ldots + \beta_{m-r} u_{m-r}] + [\gamma_1 w_1 + \ldots + \gamma_r w_r + \delta_1 v_1 + \ldots + \delta_{n-r} v_{n-r}]$

This is a linear combination of $\alpha_1 w_1 + \ldots + \alpha_r w_r + \beta_1 u_1 + \ldots + \beta_{m-r} u_{m-r} + \delta_1 v_1 + \ldots + \delta_{n-r} v_{n-r}$, i.e., the sum of linear combinations.

**They are linearly independent:**

Suppose $\alpha_1 w_1 + \ldots + \alpha_r w_r + \beta_1 u_1 + \ldots + \beta_{m-r} u_{m-r} + \delta_1 v_1 + \ldots + \delta_{n-r} v_{n-r} = 0$ \hspace{1cm} (*)

Then $\delta_1 v_1 + \ldots + \delta_{n-r} v_{n-r} = -\alpha_1 w_1 - \ldots - \alpha_r w_r - \beta_1 u_1 - \ldots - \beta_{m-r} u_{m-r}$ \hspace{1cm} $\in U \cap V = 0$.

Hence $\delta_1 v_1 + \ldots + \delta_{n-r} v_{n-r} = 0$ and $\alpha_1 w_1 + \ldots + \alpha_r w_r + \beta_1 u_1 + \ldots + \beta_{m-r} u_{m-r} = 0$.

Since $v_1 \ldots v_{n-r}$ are linearly independent (they are a basis for $V$) it follows that

$\delta_1 = \ldots = \delta_{n-r} = 0$.

Since $w_1, \ldots, w_r, u_1 \ldots u_{m-r}$ are linearly independent (they are a basis for $V$) it follows that

$\alpha_1 = \ldots = \alpha_r = \beta_1, \ldots, \beta_{m-r} = 0$.

Hence $\dim(U + V) = r + (m - r) + (n - r) = m + n - r$.

We can generalise a sum to a sum of any finite number of terms. The sum of $U_1, \ldots, U_k$ is $U_1 + \ldots + U_k$, the set of all vectors of the form $u_1 + \ldots + u_k$ where $u_r \in U_r$ for each $r$.

The above sum is a direct sum if, whenever $u_1 + \ldots + u_k = 0$, with each $u_r \in U_r$, implies that each $u_r = 0$. 

11
EXERCISES FOR CHAPTER 1

Exercise 1: Prove that the set of all real symmetric matrices is a vector space over \( \mathbb{R} \).

Exercise 2: Prove that for functions of a real variable \( a(x), b(x), c(x) \) the solutions to the differential equation \( a(x) \frac{d^2y}{dx^2} + b(x) \frac{dy}{dx} + c(x)y = 0 \) form a subspace of the space of all differentiable functions of \( x \).

Exercise 3: Prove that the set of bounded sequences of real numbers is a vector space over \( \mathbb{R} \).

Exercise 4: Show that the set \( S \) of all real sequences \( \{a_n\} \) where \( \lim_{n \to \infty} a_n = \pm \infty \) is NOT a vector space.

Exercise 5: If \( U = \{(x, y, z) \mid 3x - 2y + 5z = 0\} \) and \( V = \{(x, y, z) \mid x - y + 3z = 0\} \) find \( U \cap V \) and \( U + V \).

Exercise 6: Show that \( (1, 2, 3), (4, 5, 6), (7, 8, 9) \) are linearly dependent.

Exercise 7: If \( u = (5, -2, 3) \) and \( v = (1, 4, -2) \) show that \( (13, -14, 13) \in \langle u, v \rangle \).

Exercise 8: Show that \( \{(5, 4, 2), (1, 2, 3), (0, 2, 1)\} \) are linearly independent.

Exercise 9: Is \( \{(1, 5, 7), (2, -1, 3), (6, 2, 8), (0, 5, 1)\} \) linearly dependent or independent.

Exercise 10: Find the dimension of the space of all \( 3 \times 3 \) symmetric matrices.

Exercise 11: Suppose \( U, V \) are subspaces of \( \mathbb{R}^7 \) with dimensions 4 and 5 respectively. Suppose too that \( U + V = \mathbb{R}^7 \). Find \( \dim(U \cap V) \).

SOLUTIONS FOR CHAPTER 1

Exercise 1: Closure under +: Suppose \( A, B \) are real symmetric matrices. Then \( A^T = A \) and \( B^T = B \). Hence \( (A + B)^T = A^T + B^T = A + B \), so \( A + B \) is symmetric. 
Closure under scalar \( \lambda \): For any scalar \( k \), \( (kA)^T = kA^T = kA \) so the set is closed under scalar multiplication.

Hence the set is a subspace.

Exercise 2: Closure under +: Suppose \( f(x), g(x) \) are solutions to the differential equation.

Then \( a(x) \frac{d^2f(x)}{dx^2} + b(x) \frac{df(x)}{dx} + c(x)f(x) = 0 \) and \( a(x) \frac{d^2g(x)}{dx^2} + b(x) \frac{dg(x)}{dx} + c(x)g(x) = 0 \).

Hence \( a(x) \frac{d^2(f(x) + g(x))}{dx^2} + b(x) \frac{d(f(x) + g(x))}{dx} + c(x)(f(x) + g(x)) \)

\[ = a(x) \left( \frac{d^2f(x)}{dx^2} + \frac{d^2g(x)}{dx^2} \right) + b(x) \left( \frac{df(x)}{dx} + \frac{dg(x)}{dx} \right) + c(x)f(x) + c(x)g(x) \]

\[ = a(x) \frac{d^2f(x)}{dx^2} + b(x) \frac{df(x)}{dx} + c(x)f(x) + a(x) \frac{d^2g(x)}{dx^2} + b(x) \frac{dg(x)}{dx} + c(x)g(x) \]

\[ = 0. \]
Closure under scalar multiplication: For any real number \( k \)
\[
a(x)\frac{d^2(kf(x))}{dx^2} + b(x)\frac{d(kf(x))}{dx} + c(x)kf(x) = ka(x)\left(\frac{d^2f(x)}{dx^2} + b(x)\frac{df(x)}{dx} + c(x)f(x)\right) = 0
\]

Exercise 3: Closure under +: Suppose \((a_n)\) and \((b_n)\) are bounded sequences. Then, there exist \( K, L \)
such that for all \( n |a_n| \leq K \) and \(|b_n| \leq L \).
Now \(|a_n + b_n| \leq |a_n| + |b_n| \leq K + L \) for all \( n \).
Hence the sequence \((a_n + b_n) = (a_n + b_n)\) is bounded.
Closure under scalar multiplication: For any real number \( k, |k||a_n| \leq |k|K \).
\( \therefore k(a_n) = (ka_n) \) is bounded. Hence the set is a subspace.

Exercise 4: It isn’t closed under addition. If \( a_n = n \) and \( b_n = -n \) then both \((a_n)\) and \((b_n)\) belong to \( S \).
But \((a_n) + (b_n)\) does not.

Exercise 5: \( U \cap V = \{(x, y, z) \mid 3x - 2y + 5z = 0 \text{ and } x - y + 3z = 0\} \).
We solve the homogeneous system \[
\begin{pmatrix} 1 & -1 & 3 \\ 3 & -2 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -4 \end{pmatrix}.
\]
Let \( z = k \). Then \( y = 4k \) and \( x = k \).
So \( U \cap V = \{k(1, 4, 1) \mid k \in \mathbb{R}\} \).
Now \( \dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V) \)
\( = 2 + 2 - 1 = 3 \) so \( U + V = \mathbb{R}^3 \).

Exercise 6: \((1, 2, 3) + (7, 8, 9) = 2(4, 5, 6) \).

Exercise 7: Suppose \((13, -14, 13) = k(5, -2, 3) + h(1, 4, -2) \).
We attempt to solve the system
\[
\begin{align*}
5k + h &= 13 \\
-2k + 4h &= -14 \\
3k - 2h &= 13
\end{align*}
\]
\[
\begin{pmatrix} 5 & 1 & 13 \\ -2 & 4 & -14 \\ 3 & -2 & 13 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 9 & -15 \\ -2 & 4 & -14 \\ 3 & -2 & 13 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 9 & -15 \\ 0 & 22 & -44 \\ 0 & -29 & 58 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 9 & -15 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.
\]
So \( h = -2, k = -15 - 9(-2) = 3 \).
So \((13, -14, 13) = 3(5, -2, 3) - 2(1, 4, -2) \in \langle u, v \rangle \).

Exercise 8: Suppose \( a(5, 4, 2) + b(1, 2, 3) + c(0, 2, 1) = (0, 0, 0) \).
We solve the homogeneous system
\[
\begin{align*}
5a + b &= 0 \\
4a + 2b + 2c &= 0 \\
2a + 3b + c &= 0
\end{align*}
\]
\[
\begin{pmatrix} 5 & 1 & 0 \\ 4 & 2 & 2 \\ 2 & 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -2 \\ 4 & 2 & 2 \\ 2 & 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -2 \\ 0 & 6 & 10 \\ 0 & 5 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}.
\]
\( \therefore a = b = c = 0 \) and so the vectors are linearly independent.
Alternatively we can evaluate the determinant\[
\begin{pmatrix}
5 & 1 & 0 \\
4 & 2 & 2 \\
2 & 3 & 1
\end{pmatrix}
\]
= 5(2 - 6) - (4 - 4) = -20.
Since this is non-zero the vectors are linearly independent.

**WARNING:** This second method only works for \( n \) vectors in an \( n \)-dimensional vector space.

**Exercise 9:** Linearly dependent. Whenever you have more vectors than the dimension of the vector space from which they come, they must be linearly dependent.

**Exercise 10:** A \( 3 \times 3 \) symmetric matrix has the form \[
\begin{pmatrix}
a & d & e \\
d & b & f \\
e & f & c
\end{pmatrix}
\]. Clearly \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]
is a basis so the dimension is 6.

**Exercise 11:** \( \dim(U \cap V) = \dim(U) + \dim(V) - \dim(U + V) = 4 + 5 - 7 = 2 \).