5. SEQUENCES AND SERIES

§5.1. Limits of Sequences

Let \( N = \{0, 1, 2, \ldots \} \) be the set of natural numbers and let \( \mathbf{R} \) be the set of real numbers. An infinite real sequence \( u_0, u_1, u_2, \ldots \) is a function from \( N \) to \( \mathbf{R} \), where we write \( u(n) \) as \( u_n \). We often denote the sequence as \((u_n)\). Note that this is quite different to a set. For a start there can be repetitions and the order of a a sequence is important, whereas for a set it is not. A limit of a sequence \((u_n)\) is a real number \( L \) such that:

for all \( \varepsilon > 0 \) there exists a natural number \( N \) such that for all \( n \geq N, |u_n - L| < \varepsilon. \)

If a limit exists we say that the sequence converges and if \( L \) is a limit we say that it converges to \( L \). If no limit exists we say that the sequence diverges. We can capture the gist of this definition by saying that “\( L \) is a limit of \((u_n)\) if we can make \( u_n \) as close as we like to \( L \) by making \( n \) sufficiently large”. But remember that the actual definition is the precise one given above.

A sequence \((u_n)\) is bounded if there exists \( K \) such that \( |u_n| \leq K \) for all \( n \).

Example 1: The sequence \( 1/2, 3/4, 7/8, 15/16, \ldots \) converges to 1.

The sequence \( 1, 2, 3, \ldots \) diverges and is unbounded.

The sequence \( 1, 0, 1, 0, \ldots \) is bounded, but diverges.

We said a limit, but if a limit exists it must be unique and so would be the limit.

Theorem 1: If \((u_n)\) converges to \( L_1 \) and also \( L_2 \) then \( L_1 = L_2 \).

Proof: Suppose \( L_1 \neq L_2 \) and let \( \varepsilon = |L_1 - L_2|/2 > 0 \).

Since \( u_n \) converges to \( L_1 \) there exists \( N_1 \) such that if \( n \geq N_1 \) then \( |u_n - L_1| < \varepsilon. \)

Since \( u_n \) converges to \( L_2 \) there exists \( N_2 \) such that if \( n \geq N_2 \) then \( |u_n - L_2| < \varepsilon. \)

Let \( N \) be the maximum of \( N_1 \) and \( N_2 \). Then if \( |u_N - L_1| < \varepsilon \) and \( |u_N - L_2| < \varepsilon. \)

Hence, by the Triangle Inequality, \(|L_1 - L_2| = |(L_1 - u_N) + (u_N - L_2)| \)

\[ \leq |L_1 - u_N| + |u_N - L_2| \]

\[ < 2\varepsilon \]

\[ < |L_1 - L_2|, \] a contradiction.

Theorem 2: Convergent sequences are bounded.

Proof: Suppose \((u_n)\) converges to \( L \).

Then there exists \( N \) such that for all \( n \geq N, |u_n - L| < 1. \) (Here we take \( \varepsilon = 1. \))

The for all \( n \geq N, |u_n| = |(u_n - L) + L| \)

\[ \leq |u_n - L| + |L| \]

\[ < 1 + |L|. \]

Now we have to deal with the terms before \( u_N \). Since there are only a finite number of these we can take the maximum of their absolute values and \( 1 + |L| \) and this will be an upper bound for \( (u_n) \).
**Theorem 3:** If \((a_n)\) converges to \(L\) and \((b_n)\) converges to \(M\) then:
1. \((a_n + b_n)\) converges to \(L + M\);
2. \((a_n - b_n)\) converges to \(L - M\);
3. \((a_n b_n)\) converges to \(LM\);
4. \(\left(\frac{a_n}{b_n}\right)\) converges to \(\frac{L}{M}\), provided \(M \neq 0\).

**Proof:** Let \(\varepsilon > 0\).
1. Then there exists \(N_1\) such that whenever \(n \geq N_1\), \(|a_n - L| < \varepsilon/2\) and there exists \(N_2\) such that whenever \(n \geq N_2\), \(|b_n - M| < \varepsilon/2\).

Let \(N\) be the maximum of \(N_1\) and \(N_2\). Then if \(n \geq N\):

\[
|a_n + b_n - (L + M)| = |(a_n - L) + (b_n - M)| \\
\leq |a_n - L| + |b_n - M| \\
< \varepsilon/2 + \varepsilon/2 \\
< \varepsilon.
\]

(2) This is proved similarly.
(3) Since \((b_n)\) converges, it is bounded. Suppose \(|b_n| \leq K\) for all \(n\), where \(K > 0\).

There exists \(N_1\) such that for all \(n \geq N_1\), \(|a_n - L| < \frac{\varepsilon}{2K}\) and there exists \(N_2\) such that for all \(n \geq N_2\), \(|b_n - L| < \frac{\varepsilon}{2(|L| + 1)}\).

Let \(N\) be the maximum of \(N_1\) and \(N_2\). Then for all \(n \geq N\),

\[
|a_n b_n - LM| = |(a_n b_n - Lb_n) + (Lb_n - LM)| \\
\leq |a_n b_n - Lb_n| + |Lb_n - LM| \\
\leq |a_n - L| |b_n| + |L| |b_n - M| \\
\leq |a_n - L| K + |L| |b_n - M| \\
< \varepsilon/2 + \varepsilon/2 \\
< \varepsilon.
\]

(4) Since \(M \neq 0\) there exists \(N_1\) such that for all \(n \geq N_1\), \(|b_n - M| < |M|/2\).

Then, for all \(n \geq N_1\), \(|M - b_n| + |b_n| < |M|/2 + |b_n|\) so \(|b_n| > |M|/2\).

Now there exists \(N_2\) such that for all \(n \geq N_2\), \(|b_n - M| < \frac{\varepsilon|M|^2}{2} = \frac{\varepsilon M^2}{2}\).

Let \(N\) be the maximum of \(N_1\) and \(N_2\). Then for all \(n \leq N\),

\[
\left|\frac{1}{b_n} - \frac{1}{M}\right| = \left|\frac{M - b_n}{b_n M}\right| \\
< \frac{\varepsilon(M^2/2)}{M^2/2} \\
< \varepsilon.
\]

If \((u_n)\) is a sequence then a **subsequence** is a sequence of the form \((u_{f(n)})\) where \(f(n)\) is an increasing sequence of \(n\). We can prove by induction that such an increasing function has the property that \(f(n) \geq n\) for all \(n \in \mathbb{N}\). The following theorem is then easily proved.

**Theorem 4:** If \((u_n)\) converges to \(L\) then so does every subsequence \((u_{f(n)})\).

**Theorem 5:** If \((u_n)\) is a sequence such that the subsequences \((u_{2n})\) and \((u_{2n+1})\) both converge to \(L\) then \((u_n)\) converges to \(L\).

**Proof:** Let \(\varepsilon > 0\). Then there exists \(N_1\) such that if \(n \geq N_1\) then \(|u_{2n} - L| < \varepsilon\).
Also there exists $N_2$ such that if $n \geq N_2$ then $|u_{2n+1} - L| < \varepsilon$.

Let $N$ be twice the maximum of $N_1$ and $N_2$ and suppose $n \geq N$.

If $n = 2k$ then $k \geq N_1$ and so $|u_n - L| = |u_{2k} - L| < \varepsilon$.

If $n = 2k + 1$ then $k \geq N_2$ and so $|u_n - L| = |u_{2k+1} - L| < \varepsilon$.

A sequence $(u_n)$ is increasing if $u_n < u_{n+1}$ for all $n$ and is non-decreasing if $u_n \leq u_{n+1}$ for all $n$. In a similar way we define decreasing and non-increasing sequences.

**Theorem 6:** A non-decreasing sequence that is bounded above converges.

**Proof:** Suppose $(u_n)$ is a non-decreasing sequence that is bounded above.

Then the set $\{u_n \mid n \in \mathbb{N}\}$ is non-empty and bounded above. By completeness it has a least upper bound $L$. We shall show that $(u_n)$ converges to $L$.

Let $\varepsilon > 0$. Since $L - \varepsilon < L$ there exists $N$ such that $u_N > L - \varepsilon$ and so $L - u_N < \varepsilon$.

If $n \geq N$, $|u_n - L| = L - u_n \leq L - u_N < \varepsilon$.

§ 5.2. Infinite Series

Suppose $(u_n)$ is an infinite sequence of real numbers. We define its sequence of partial sums by $s_0 = 0$ and $s_{n+1} = s_n + u_n$. Then $s_n$ is the sum of the first $n$ terms. In this context we denote the sequence by the notation $\sum_{n=0}^{\infty} u_n$ and call it a series. We say that the series converges to the limit $L$ if the sequence of partial sums $(s_n)$ converges to $L$. Otherwise we say that the series diverges. If the series converges we also denote its limit by $\sum_{n=0}^{\infty} u_n$ and call this the sum of the series.

Sometimes it is convenient to omit the term $u_0$, in which case we write $\sum_{n=1}^{\infty} u_n$. But often, when the sum goes from $n = 0$, we omit the upper and lower limits and write $\sum_{0}^{\infty} u_n$, both for the series itself as well as its sum.

**Example 2:** $\sum_{n=0}^{\infty} \frac{1}{2^n}$ represents the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$

This converges and its sum is $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1$.

This is because the $n$’th partial sum is $s_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$ which converges to 1 as $n \to \infty$.

**Example 3:** $\sum_{n=0}^{\infty} (-1)^n$ represents the series $1 - 1 + 1 - 1 + ...$

The sequence of partial sums is 0, 1, 0, 1, 0, ....

Since this sequence diverges the series diverges.

**Example 4:** $\sum_{n=0}^{\infty} 2^n$ represents the series $1 + 2 + 4 + ...$

Clearly the sequence of partial sums diverges and so the series diverges.

These last three are examples of the Geometric Series $1 + r + r^2 + ...$

If $|r| < 1$ this series converges and we are able to find the sum.
**Theorem 7:** If \(|r| \geq 1\) the series \(\sum r^n\) diverges.
If \(|r| < 1\) the series \(\sum r^n\) converges to \(\frac{1}{1 - r}\).

**Proof:** If \(|r| \geq 1\) the n’th term does not converge to zero and so the series diverges.
Suppose \(|r| < 1\).
Let \(s_n = 1 + r + r^2 + \ldots + r^{n-1}\).
Then \(rs_n = r + r^2 + r^3 + \ldots + r^n\).
Subtracting we get \((1 - r)s_n = 1 - r^n\) and so \(s_n = \frac{1 - r^n}{1 - r}\).
As \(n\) approaches infinity this converges to \(\frac{1}{1 - r}\).

The case \(r = -1\) is interesting. This has been called Grandi’s Series. In the early 18th century, when mathematicians were struggling to come up with a proper definition of limits there was a great controversy over this series \(1 - 1 + 1 - 1 + \ldots\). Some declared that it does not converge while others argued that it converges to \(\frac{1}{2}\) on the grounds that if you take an odd number of terms the partial sum is 1 and if you take an even number of terms the partial sum is 0. The average of these is \(\frac{1}{2}\). This goes to show that if a concept is not precisely defined there can be disagreement.

**Example 5:** \(\sum \left(\frac{1}{2}\right)^n = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}\).

**Theorem 8:** If the series \(\sum u_n\) converges then the sequence \((u_n)\) converges to 0.

**Proof:** Suppose \(\sum u_n\) converges to \(L\).
If \((s_n)\) is the sequence of partial sums then \(s_n\) converges to \(L\).
But then, so does \(s_{n-1}\).
Since \(u_n = s_n - s_{n-1}\), \(a_n\) converges to \(L - L = 0\).

The converse to this theorem is most definitely not true. This is slightly counter-intuitive because if the terms that we are adding at each step get smaller and smaller, approaching zero, it is reasonable to expect that the sum will converge. Reasonable, perhaps, but false nonetheless. This is one of the most common errors students make. Not only do students tend to turn one-way theorems into “if and only if’s” but the false converse is so plausible. The classic counter-example is the so-called Harmonic Series.

### §5.3. The Harmonic Series

The **Harmonic Series**, so named because of some connection with music, is:

\[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots\]

Clearly the n’th term approaches zero. And if you compute the partial sums, though they are increasing, they slow down to such an extent that one can really believe that they will be bounded above. And, being an increasing sequence that is bounded above, it would have to converge.

The sum of the first 10 terms is about 2.93 while the sum of the first 100 terms is about 5.19. The sum of the first 1000 terms is about 7.48. It is inconceivable that the sum could ever exceed a million. And yet it does, if we take an enormous number of terms.
Theorem 9: The Harmonic Series diverges.
Proof: The trick is to group terms together in ever increasing blocks. We can write the Harmonic Series as:
\[
1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \ldots + \left( \frac{1}{2^m+1} + \ldots + \frac{1}{2^{m+1}} \right) + \ldots
\]
After the first term each group of terms totals more than \(\frac{1}{2}\).
The typical group has \(2^m\) terms and is
\[
\left( \frac{1}{2^m+1} + \ldots + \frac{1}{2^{m+1}} \right) > \left( \frac{1}{2^m+1} + \ldots + \frac{1}{2^{m+1}} \right) = \frac{1}{2}.
\]
Clearly the sequence of partial sums is unbounded and hence does not converge.

Example 6: For how many terms will the partial sum exceed a million?
Solution: If we take 2 million groups we will exceed a million. So \(2^{2000000}\) terms will achieve a total more than a million. Of course this is not the number of terms for which the sum will first exceed a million. That would be quite hard to determine.

It is often said that computers have superseded mathematics. Ask a question like the above and we can just program a computer to do the calculation! The problem is that on a computer the Harmonic Series converges! What it converges to will depend on the computer, or at least the number of decimal places it can work to. Eventually each extra term will make no different to the total and the illusion of convergence will result. Never mindlessly believe what a calculator or a computer tells you.

Example 7: An insect crawls along a 1 metre long rubber band at the rate of 1 cm per minute. After every minute the band stretches uniformly by a metre. Will the insect ever reach the end? Note that this is a theoretical question so the rubber band is assumed to be stretch indefinitely and the insect is assumed to be immortal. The question is not whether the insect will reach the end before it dies or before the band snaps.
Solution: Crawling at 1 cm per minute, and with the band lengthening by a metre per minute, it would seem that the insect has no hope of reaching the end. But remember that in the stretching process the insect gets moved a little extra. But this extra movement becomes quite small. For example after 100 minutes this extra boost is only 1 per cent.

After \(n\) minutes, just before the \(n\)’th stretch, let \(a_n\) be the distance, in centimetres that the insect has moved from the start and let \(b_n\) be the length of the rubber band. Clearly \(b_n = 100n\).

For \(n > 1\), \(a_n = a_{n-1} \left( \frac{b_n}{b_{n-1}} \right) + 1\). The \(\left( \frac{b_n}{b_{n-1}} \right)\) factor is the stretching and the 1 is the extra centimetre the insect walks.

Hence \(a_n = a_{n-1} \left( \frac{n}{n - 1} \right) + 1\).

Let \(p_n = 100 \left( \frac{a_n}{b_n} \right)\). This is the percentage of the rubber band the insect has walked (just before the \(n\)’th stretch). We wish to determine if \(p_n\) will ever reach 100.

Now \(p_n = 100 \left( \frac{a_n}{100n} \right) = \frac{a_n}{n}\). So \(p_n = p_{n-1} + \frac{1}{n}\). So \((p_n)\) is the sequence of partials sums of the Harmonic Series. Since this sequence is not bounded above, eventually \(p_n\) will exceed 100 and so the insect will have travelled 100% of the length of the band. So it will reach the end. That’s if it doesn’t die of old age first. It would take about \(3 \times 10^{37}\) years by which time the rubber band will be about \(3 \times 10^{34}\) kilometres long!

This might seem a highly artificial problem, and so it is. But one version of it relates to the question of light travelling in an expanding universe.
Another famous interpretation of the divergence of the Harmonic Series is a problem involving stacking dominos. Suppose each domino measures 2cm long. The width and thickness are irrelevant. Place a domino on a flat surface and place a second one on top, so that it overhangs 1cm.  

Now place a third domino underneath so that it sticks out ½ cm to the left.

Continue stacking the dominos so the successive distances between the left hand edges are, in centimetres: 1, \(\frac{1}{2}\), \(\frac{1}{3}\), \(\frac{1}{4}\) etc.

If we continue adding dominos to the bottom the stack, in theory, it will not topple over in that the centre of gravity at each stage will not be beyond the base. But the measurements will need to be exact, and even so, a puff of wind would cause the stack to collapse. So this is a theoretical problem rather than a practical one.

The question is, how far will the stack hang over the base. The answer is “as far as you like”, because after \(n\) dominos have been put in place the overhang will be \(1 + \frac{1}{2} + \ldots + \frac{1}{n-1}\). Since this is unbounded as \(n\) approaches infinity we can, in theory, have such a stack extend from one side of the Pacific to the other. Mind you, the number of dominos would be far more than the number of atoms in the universe! As, I said, this is a theoretical question.

To substantiate the claim that the stack is theoretically stable we show that the centre of gravity of a stack of \(n\) dominos lies directly above the right hand end of the \((n+1)\)st. If we suppose this true for \(n\) dominos, as depicted above, the horizontal coordinate of the centre of gravity of \(n + 1\) dominos, relative to the right hand end of the \((n + 2)\)nd is \(\frac{1}{n+1} \cdot \left( -1 + \frac{1}{n} \right) = 0.\)

Relative to the right hand end of the \((n + 1)\)st domino the centre of gravity of the \((n + 1)\)st domino has x-coordinate \(-1\) and the centre of gravity of the top \(n\) dominos has x-coordinate 0. Taking the weighted sum of these the x-coordinate of the centre of gravity of the stack of \(n + 1\)
dominos is \( \frac{1}{n+1} (-1 + n.0) = - \frac{1}{n+1} \). This lies directly above the right hand end of the \((n+2)\text{nd}\) domino.

**n’th Term Test:** If \(a_n\) does not converge to 0 then \((a_n)\) diverges.

Beware. This is most definitely NOT a 2 way test. If the n’th term *does* converge to 0 we can say NOTHING. The series may converge, but it may still diverge, like the Harmonic Series.

**Example 8:**
1. \(1 + \sqrt{2} + \sqrt{3} + ... + \sqrt{n} + ...\) diverges because \((\sqrt{n})\) is unbounded and so does not converge.
2. \(1 - 1 + 1 - 1 + ... + (-1)^n + ...\) diverges because although \((-1)^n\) is bounded it does not converge.
3. \(\cos 1 + \cos \frac{1}{2} + \cos \frac{1}{3} + ... + \cos \frac{1}{n} + ...\) diverges because although the n’th term converges it converges to 1 and not 0.
4. \(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} + ...\) Here the n’th term converges to 0. Hence the n’th term test tells us NOTHING. We will show that it actually diverges.

§5.4. Series With Positive Terms

Given a series it is not always easy to determine whether or not it converges or diverges. In this section we present a number of tests. None of them gives a necessary and sufficient test so it sometimes happens that none of our tests is able to answer the question. So far we have just the n’th Term Test. This is a very weak test in so far as it only detects series that very clearly diverge. There are many series where the n’th term does converge to zero but which still diverge. Remember that all of these tests are one-way tests and can easily be inconclusive.

We begin with some tests for series of positive terms. Throughout this section the series \((a_n)\) and \((b_n)\) are assumed to be such series.

**Comparison Test for Convergence:**
If \(a_n \leq b_n\) for all \(n\) and \(\sum b_n\) converges then \(\sum a_n\) converges.

**Proof:** Let \((s_n)\) be the sequence of partial sums. The fact that all \(a_n\) are positive means that \((s_n)\) is an increasing sequence. For each \(n\), \(s_n \leq b_0 + b_1 + b_2 + ... + b_n < \Sigma b_n\) and so \((s_n)\) is bounded above. Hence \(\Sigma a_n\) converges.

[In this case it is obvious that \(\Sigma a_n \leq \Sigma b_n\).]

**Example 9:** \(\sin 1 + \sin \frac{1}{2} + \sin \frac{1}{4} + ... + \sin \frac{1}{2^n} + ...\) converges.

**Proof:** For all \(x > 0\), \(\sin x < x\). Since \(1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n} + ...\) converges to 2, \(\sin 1 + \sin \frac{1}{2} + \sin \frac{1}{4} + ... + \sin \frac{1}{2^n} + ...\) converges and its sum is less than or equal to 2.

**Comparison Test for Divergence:**
If \(a_n \geq b_n\) for all \(n\) and \(\sum b_n\) is unbounded then \(\sum a_n\) diverges.

**Proof:** This is proved similarly.
Example 10: \(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots\) diverges.

**Proof:** For all \(n\), \(\sqrt{n} \leq n\) so \(\frac{1}{\sqrt{n}} \geq \frac{1}{n}\). Since \(\sum \frac{1}{n}\) diverges \(\sum \frac{1}{\sqrt{n}}\) diverges.

**Ratio Test for Convergence:**

If there is some \(k\) such that \(\frac{a_{n+1}}{a_n} \leq k < 1\) for all \(n\) then \(\Sigma a_n\) converges.

**Proof:** Suppose that \(\frac{a_{n+1}}{a_n} \leq k < 1\) for all \(n\). Then by induction we can prove that \(a_n \leq a_0 k^n\) for all \(n\).

Since \(k < 1\), \(a_0 \Sigma k^n\) converges and hence, by the Comparison Test, so does \(\Sigma a_n\).

Example 11: \(\sum \frac{2^n}{n!}\) converges for all \(x\).

**Proof:** Here \(a_n = \frac{2^n}{n!}\) and so \(\frac{a_{n+1}}{a_n} = \frac{2}{n+1}\).

Now \(\frac{a_1}{a_0} = 2\) and \(\frac{a_2}{a_1} = 1\). But \(\frac{a_{n+1}}{a_n} \leq \frac{2}{3} < 1\) for all \(n \geq 2\).

This is sufficient for the Ratio Test because it shows that the series converges if we omit the first two terms. Clearly removing or adding a finite number of terms does not affect convergence. So the original series must converge.

This illustrates an important principle. Any test for convergence or divergence can be applied after removing any finite number of terms without affecting convergence or divergence.

**Ratio Test for Divergence:**

If there is some \(k\) such that \(\frac{a_{n+1}}{a_n} \geq k > 1\) for all \(n\) then \(\Sigma a_n\) diverges.

**Proof:** Suppose that \(\frac{a_{n+1}}{a_n} \geq k > 1\) for all \(n\). Then by induction we can prove that \(a_n \geq a_0 k^n\) for all \(n\).

Since \(k > 1\), \(a_0 \Sigma k^n\) diverges and hence, by the Comparison Test, so does \(\Sigma a_n\).

**Ratio Limit Test for Convergence:**

If \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1\) then \(\Sigma a_n\) converges.

**Proof:** Suppose that \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L < 1\). Then there exists \(N\) such that for all \(n \geq N\),

\[
\left| \frac{a_{n+1}}{a_n} - L \right| \leq \frac{1 - L}{2}.
\]

Hence, by the Triangle Inequality, \(\frac{a_{n+1}}{a_n} \leq \frac{1 - L}{2} + L \leq \frac{L + 1}{2} < 1\).

Hence, by the Ratio Test, \(\Sigma a_n\) converges.
**Ratio Limit Test for Divergence:**

If \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} > 1 \) then \( \Sigma a_n \) converges.

**Proof:** Suppose that \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L > 1 \). Then there exists \( N \) such that for all \( n \geq N \),

\[
\left| \frac{a_{n+1}}{a_n} - L \right| \leq \frac{L - 1}{2},
\]

Hence, \( \frac{a_{n+1}}{a_n} \geq \frac{L + 1}{2} > 1 \).

Hence, by the Ratio Test, \( \Sigma a_n \) diverges.

The Ratio Limit Tests are the easiest to use but they are less powerful than the Ratio Tests themselves. These in turn are simpler to use than the Comparison Tests, but again, they are a little less powerful. As a rule of thumb one should try to use the tests in the order:

Ratio Limit Test
Ratio Test
Comparison Test

However, the Ratio Tests are not powerful enough for the Harmonic Series.

If \( a_n = \frac{1}{n} \) then \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \). And \( \frac{a_{n+1}}{a_n} < 1 \) for all \( n \). These facts show nothing. We could prove the divergence of the Harmonic Series by the Comparison Test, but only by using another series where the Ratio Tests fail.

Luckily we have been able to show that the Harmonic Series diverges. It becomes a powerful tool to use in conjunction with the Comparison Test in proving the divergence of other series.

**n’th Root Test For Convergence:**

If there exists \( k \) such that \( a_n^{1/n} \leq k < 1 \) then \( (a_n) \) converges.

**Proof:** Suppose \( a_n^{1/n} \leq k < 1 \) for all \( n \). Then \( a_n \leq k^n \) for all \( n \). By the Comparison Test, \( \Sigma a_n \) converges.

**n’th Root Limit Test for Convergence:**

If \( \lim_{n \to \infty} a_n^{1/n} < 1 \) then \( (a_n) \) converges.

**Proof:** Similar to the Ratio Limit Test.

**§5.5. The p-Series**

The p-series is \( \sum \frac{1}{n^p} \). For \( p = 1 \) this is the Harmonic Series and, as we have seen, it diverges. But if \( p > 1 \), even if \( p = 1.0000001 \), it converges. The Harmonic Series is the boundary between convergence and divergence of this type of series.
Theorem 10: The p-series is \( \sum \frac{1}{n^p} \) converges for all \( p > 1 \) and diverges for all \( p \leq 1 \).

Proof:

Case 1: \( p < 1 \): \( 1 + \frac{1}{2^p} + \frac{1}{3^p} + ... + \frac{1}{n^p} < \frac{n}{n^p} = n^{1-p} \). Since this is unbounded, the series diverges.

Case 2: \( p = 1 \): This is the Harmonic Series which we have seen diverges.

Case 3: \( p > 1 \): Group the terms as follows:

\[
1 + \left( \frac{1}{2^p} + \frac{1}{3^p} \right) + \left( \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \right) + ... + \left( \frac{1}{(m-1)^p} + \frac{1}{m^p} \right) + ...
\]

\[
< 1 + \left( \frac{1}{2^p} + \frac{1}{2^p} \right) + \left( \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} \right) + ... + \left( \frac{1}{(m-1)^p} + \frac{1}{m^p} \right) + ...
\]

\[
< 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + ... + \frac{1}{2^{(m-1)(p-1)}} + ...
\]

\[
< 1 + r + r^2 + ... + r^{m-1} + ... \text{ where } r = \frac{1}{2^p} < 1 \text{ since } p > 1
\]

\[
< \frac{1}{1-r}.
\]

Hence the series is bounded above and so, being an increasing series, it converges.

§5.6. Alternating Series

If there are only finitely many negative terms we can ignore these and omit finitely many terms so that we are left with a series of positive terms. This will not affect the convergence or divergence. Also, we can clearly omit any zero terms, even if there are infinitely many of them.

Example 12: The series \( -\frac{1}{27} + 0 - \frac{1}{27} + 0 + 1 + 0 + \frac{1}{27} + 0 + \frac{1}{64} + 0 + ... + \frac{1}{(2n+1)^3} + 0 + ... \) converges. This is because we can omit the first three terms as well as all the zero terms to obtain

\[
1 + \frac{1}{27} + \frac{1}{64} + ... + \frac{1}{(2n+1)^3} + ...
\]

Since \( 3 > 1 \) the p-series \( \sum \frac{1}{n^3} \) converges.

Since \( \frac{1}{(2n+1)^3} < \frac{1}{n^3} \) \( \sum \frac{1}{(2n+1)^3} \) converges by the Comparison Test.

Hence the original series converges.

Consider a series \( \sum a_n \) of alternating positive and negative terms where, for example, \( a_n > 0 \) if \( n \) is even and \( a_n < 0 \) if \( n \) is odd. For convenience we shall consider an alternating series \( \sum (-1)^n a_n \) where each \( a_n \) is positive.

Alternating Signs Test: Suppose \( a_n > 0 \) for all \( n \) and \( a_n \geq a_{n+1} \) for all \( n \).

Then \( \sum (-1)^n a_n \) converges if and only if \( \lim_{n \to \infty} a_n = 0 \).

Proof: Suppose \( a_n > 0 \) for all \( n \) and \( a_n \geq a_{n+1} \) for all \( n \).

We have already shown that if \( \sum (-1)^n a_n \) converges then \( \lim_{n \to \infty} a_n = 0 \).
So suppose that \( \lim_{n \to 0} a_n = 0 \).

Let \( s_n = a_0 - a_1 + a_2 - \ldots + (-1)^n a_n \)

Then \( s_{2n} = (a_0 - a_1) + (a_2 - a_3) + (a_4 - a_5) + \ldots + a_{2n} \geq 0 \).

But \( s_{2n+2} - s_{2n} = -(a_{2n+1} - a_{2n}) \leq 0 \).

So \( (s_{2n}) \) is a non-increasing sequence bounded below and so converges. Let \( \lim_{n \to \infty} s_{2n} = L \).

Now \( s_{2n+1} = s_{2n} - a_{2n+1} \) and so

\[
\lim_{n \to \infty} s_{2n+1} = \lim_{n \to \infty} s_{2n} - \lim_{n \to \infty} a_{2n+1} = L - 0 = L.
\]

Hence \( (s_n) \) converges to \( L \).

**Example 13:** The series \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \) converges.

We conclude that this converges, by the Alternating Signs Test. We can even conclude that its limit is less than 1. But at this stage we have no way of determining what the limit is.

The assumption that \( (a_n) \) is a non-increasing sequence is vital. Consider the following example. Although \( a_n \) converges to zero, the odd numbered terms do so far more quickly than the even numbered terms. This results in the series diverging.

**Example 14:** Consider the series:

\[
1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{8} + \ldots + \frac{1}{n} - \frac{1}{2n-1} + \ldots
\]

This is an alternating series for which the \( n \)th term converges to zero.

But \( s_{2n+1} = 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{8} + \ldots + \frac{1}{n} - \frac{1}{2n-1} \)

\[
= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}\right) - \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{n-1}}\right).
\]

Now \( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{n-1}} < 2 \) and so \( s_{2n+1} > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} - 2 \).

Since \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} \) is unbounded, so is \( s_{2n+1} \) and hence the series diverges.

\[\text{§5.7. Absolute Convergence}\]

With a finite sum the order of the terms is irrelevant. The commutative law shows that we can rearrange the terms without affecting the sum. With an infinite sum a convergent series can be changed into a divergent one, and vice versa, by simply rearranging the terms.

**Example 15:** Consider the convergent series

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots
\]

If we take just the negative terms we get the series \( -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \ldots \).

This is minus half the Harmonic Series and so diverges.

The series of positive terms is \( 1 + \frac{1}{3} + \frac{1}{5} + \ldots \).

By the Comparison Test, comparing this with the divergent series \( \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots \) the series of positive terms is also divergent.
So we have one divergent series minus another divergent series. But the terms are interleaved in such a way that the series converges. Let’s rearrange the terms.

Begin by taking \(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\). The sum of these terms is bigger than 2.

Now include the first negative term: \(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{2}\). Then take enough positive terms so that the sum exceeds 3. We can do this because the partial sums of the positive terms are unbounded.

So far we will have:

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19}
\]

We now dip into the negative term bucket and include the next one to get

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{2} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} - \frac{1}{3}.
\]

Again we add enough positive terms to make the sum exceed 4. This will give:

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + ... + \frac{1}{19} - \frac{1}{3} + \frac{1}{20} + ... + \frac{1}{71}.
\]

We alternately take several terms from the positive ones with one term from the negative ones. After a short while we will find ourselves having to take millions of positive terms for each negative one. But both pots being infinite we will never run out.

At each stage, just prior to taking \(-\frac{1}{n}\), the partial sum will exceed n. Clearly the partial sums will be unbounded and so the series diverges. Yet this divergent series will contain exactly the same terms as the original convergent one. It’s just that they will be in a different order.

It’s because we have both positive and negative terms that this can happen. If all the terms are positive we can show that rearrangement will not affect convergence. In fact this property extends to certain series with mixed signs, the so-called **absolutely convergent series**.

**Example 16:** Consider the following table, with infinitely many rows and infinitely many columns.

<table>
<thead>
<tr>
<th>(0)</th>
<th>(-1)</th>
<th>(-\frac{1}{2})</th>
<th>(-\frac{1}{4})</th>
<th>(-\frac{1}{8})</th>
<th>(-\frac{1}{16})</th>
<th>(-\frac{1}{32})</th>
<th>(-\frac{1}{64})</th>
<th>......</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>(-\frac{1}{2})</td>
<td>(-\frac{1}{4})</td>
<td>(-\frac{1}{8})</td>
<td>(-\frac{1}{16})</td>
<td>(-\frac{1}{32})</td>
<td>(-\frac{1}{64})</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>(-\frac{1}{2})</td>
<td>(-\frac{1}{4})</td>
<td>(-\frac{1}{8})</td>
<td>(-\frac{1}{16})</td>
<td>......</td>
</tr>
<tr>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{2})</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>(-\frac{1}{2})</td>
<td>(-\frac{1}{4})</td>
<td>(-\frac{1}{8})</td>
<td>......</td>
</tr>
<tr>
<td>(\frac{1}{8})</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{2})</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>(-\frac{1}{2})</td>
<td>(-\frac{1}{4})</td>
<td>......</td>
</tr>
<tr>
<td>(\frac{1}{16})</td>
<td>(\frac{1}{8})</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{2})</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>(-\frac{1}{2})</td>
<td>......</td>
</tr>
<tr>
<td>(\frac{1}{32})</td>
<td>(\frac{1}{16})</td>
<td>(\frac{1}{8})</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{2})</td>
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<td>......</td>
<td>......</td>
</tr>
</tbody>
</table>

Now, as you know, you can get the grand total of a table of figures by adding the rows and then adding the row totals. Or you can add the columns and then add the column totals. It is a good idea to do both, as a check. That’s fine for finite tables but not so for infinite tables.
The row and column totals are shown in bold. But the sum of the row totals is \(-4\) while the sum of the column totals is \(+4\). This discrepancy comes about because the two ways of computing the grand total involve taking the terms in different orders.

A series \(\sum u_n\) is defined to be **absolutely convergent** if \(\sum |u_n|\) is convergent. As we will show, absolutely convergent series are convergent. But there are convergent series that are not absolutely convergent. These are called **conditionally convergent**. A conditionally convergent series is one that only convergent because the positive and negative terms more or less cancel each other because of the order in which they are in.

**Example 13:** The series \(\sum (-1)^n \frac{1}{n}\) is conditionally convergent.
This is because we get the divergent Harmonic Series when we take absolute values.

**Example 13:** The series \(\sum (-1)^n \frac{1}{n^2}\) is absolutely convergent because \(\sum \frac{1}{n^2}\) is convergent (it is a p-series with \(p > 1\)).

**Theorem 11:** An absolutely convergent series is convergent.  
**Proof:** Suppose \(\sum |a_n|\) converges.  
Define \(u_n = |a_n| - a_n\). Then \(0 \leq u_n \leq 2|a_n|\). (\(u_n = 0\) or \(2|a_n|\) defending whether \(a_n\) is positive or not.)  
Hence, by the Comparison Test, \(\sum u_n\) converges.  
Since \(a_n = |a_n| - u_n\) we conclude that \(\sum a_n\) converges.

**Example 14:** \(\sum \frac{\sin n}{n^2}\) converges.  
**Solution:** \(\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}\) so \(\sum \frac{\sin n}{n^2}\) is absolutely convergent.
A more difficult question is whether \( \sum \frac{\sin n}{n} \) converges. It can be shown that it is not absolutely convergent, but it is not an alternating series. It turns out that it is conditionally convergent with a sum of \( \frac{1}{2}(\pi - 1) \).

A rearrangement of the series \( \Sigma a_n \) is a series of the form \( \Sigma a_{f(n)} \) where \( f: \mathbb{N} \to \mathbb{N} \) is a 1-1 and onto map. The difference between absolutely and conditionally convergent series, so far as rearrangement is concerned, is highlighted by the following theorem that we state without proof.

**Theorem 12:**

1. If \( \Sigma b_n \) is a rearrangement of the absolutely convergent series \( \Sigma a_n \) then \( \Sigma b_n \) is absolutely convergent and \( \Sigma a_n = \Sigma b_n \).
2. If \( \Sigma a_n \) is conditionally convergent then there exists a rearrangement that is divergent.
3. If \( \Sigma a_n \) is conditionally convergent and \( K \) is any real number then there exists a rearrangement that converges to \( K \).

§5.8. Power Series

Many important functions can be expressed as a series that involves powers of \( x \). For example \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \) A series of the form \( \sum a_n x^n \) is called a **power series**.

Of course an important question is “for what values of \( x \) does the series converge?” We define the **region of convergence** of a power series \( \sum a_n x^n \) to be the set of all \( x \) for which it converges. For some power series, such as the one above, the region of convergence is the whole real line. At the other extreme for some power series the region of convergence is just \( \{0\} \). (Of course these power series are not particularly useful! But in all other cases the region of convergence is an interval, centred about the origin.

We will show that if a power series does not converge for all \( x \), there is a number \( R \), called the **radius of convergence**, such that the power series converges for all \( x \) with \( |x| < R \) and diverges if \( |x| > R \). This leaves the question of what happens when \( x = \pm R \). Sometimes the power diverges for both of these values and sometimes it converges at both values. Sometimes it converges at one of these points and diverges at the other. In the case where the region of convergence is the whole real line we say that the radius of convergence is \( \infty \). If the power series converges only for \( x = 0 \) the radius of convergence is 0.

The reason why \( R \) is called the **radius of convergence** is that power series exist in the complex plane. Here the region of convergence is typically the interior of a circle of radius \( R \), perhaps with some points on the circumference. For real power series we have a 1-dimensional analogue of a circle, that is, an interval.

**Theorem 13:** If the power series \( \Sigma a_n x^n \) converges when \( x = b \) then it converges absolutely whenever \( |x| < |b| \).

**Proof:** Suppose \( \Sigma a_n b^n \) converges. Then \( \lim_{n \to \infty} a_n b^n = 0 \).

Hence there is a positive integer \( N \) such that whenever \( n \geq N \), \( |a_n b^n| < 1 \).

Suppose \( |x| < |b| \) and let \( r = \frac{|x|}{|b|} < 1 \).

Then if \( n \geq N \), \( |a_n x^n| = |a_n b^n| r^n < r^n \).

By the Comparison Test with the Geometric Progression, \( \Sigma r^n \), \( \Sigma a_n x^n \) is absolutely convergent.
Corollary: If the power series $\sum a_n x^n$ diverges when $x = b$ then it diverges whenever $|x| > |b|$.

Proof: Suppose $\Sigma a_n b^n$ diverges.
Suppose $|x| > |b|$. If $\Sigma a_n x^n$ converges then $\Sigma a_n b^n$ converges by the theorem. Hence $\Sigma a_n b^n$ diverges.

Theorem 14: The set of points where the power series $\sum a_n x^n$ converges is one of the following:
- the whole real line $(-\infty, \infty)$,
- $\{0\}$
- $(-R, R), (-R, R], [-R, R), [-R, R]$ for some $R > 0$.

Proof: Let $S = \{x \mid \sum a_n x^n \text{ converges}\}$.
Clearly $0 \in S$ so $S$ is non-empty.
Suppose that $S \neq \mathbb{R}$ and suppose that $\sum a_n b^n$ diverges.
Suppose $x \geq |b| + 1$. Then by the corollary to Theorem 13 $\sum a_n x^n$ diverges and so $S$ is bounded above.
Let $R$ be the least upper bound of $S$.
Suppose $|x| < R$. Then there exists $y \in S$ such that $|x| < y < R$.
Since $\sum a_n y^n$ converges so does $\sum a_n x^n$ by Theorem 13.
Suppose $|x| > R$. Then $x \not\in S$ and so $\sum a_n x^n$ diverges.
It follows that $S$ is one of the intervals $(-R, R), (-R, R], [-R, R), [-R, R]$.

Theorem 15: Suppose $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = R$. Then $R$ is the radius of convergence of the power series $\sum a_n x^n$.

Proof: Suppose $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = R$ and let $|x| < R$.
Then $\lim_{n \to \infty} \frac{a_{n+1} x^{n+1}}{a_n x^n} = \lim_{n \to \infty} \frac{a_{n+1} x}{a_n} = x \cdot \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = x \cdot R$.
If $|x| < R$ then $\lim_{n \to \infty} \frac{a_{n+1} x^{n+1}}{a_n x^n} < 1$ and so $\sum a_n x^n$ converges.
If $|x| > R$ then $\lim_{n \to \infty} \frac{a_{n+1} x^{n+1}}{a_n x^n} > 1$ and so $\sum a_n x^n$ diverges.

Example 15: The region of convergence of $\sum \frac{x^n}{2^n}$ is $(-2, 2)$.

Solution: Here $a_n = \frac{1}{2^n}$ and $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \lim_{n \to \infty} \frac{2(n+1)}{n} = 2$ so the radius of convergence is 2.

When $x = -2$ the series become $\sum \frac{(-1)^n}{n}$ which, as we have seen, converges.

When $x = 2$ the series become the Harmonic Series $\sum \frac{1}{n}$ which, as we have seen, diverges.

Example 16: The region of convergence of the power series $\sum n! x^n$ is $\{0\}$.

Solution: Here $a_n = n!$ and so $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \lim_{n \to \infty} \frac{1}{n+1} = 0$.
So the radius of convergence is 0 and the region of convergence is $\{0\}$. 63
Theorem 16: Suppose \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0 \). Then the radius of convergence of the power series \( \sum a_n x^n \) is \( \infty \), that is, the series converges for all \( x \).

Proof: Similar to the proof of Theorem 15.

Notice that the ratio is inverted compared with Theorem 15. Although this is not a proof, one can compute \( \lim_{n \to \infty} \frac{a_n}{a_{n+1}} \) and if is \( \infty \) we can say that the radius of convergence is \( \infty \), that is, the region of convergence is the whole real line.

Example 17: The region of convergence of the power series \( \sum \frac{x^n}{n!} \) is the whole real line.

Solution: Here \( a_n = \frac{1}{n!} \) and so \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{n+1} = 0 \).

§5.9. Differentiability of Power Series

We can use a power series \( \sum a_n x^n \) to define a function of \( x \) \( f(x) \). The domain of this function will be the region of convergence. We write \( f(x) = \sum a_n x^n \). It is tempting to want to differentiate this function by differentiating the power series term by term and indeed it is possible within the circle of convergence.

Theorem 17: Suppose \( f(x) = \sum a_n x^n \).

If the radius of convergence is \( \infty \) then for all \( x \), \( f(x) \) is differentiable and \( f'(x) = \sum n a_n x^{n-1} \).

If the radius of convergence is \( R \) then if \( |x| < R \), \( f(x) \) is differentiable and \( f'(x) = \sum n a_n x^{n-1} \).

We omit the proof of this theorem.

Example 18: Define \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \). Then \( \frac{d(e^x)}{dx} = e^x \).

Differentiating term by term we get \( \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \).

§5.10. Taylor and Maclaurin Series

A function \( f(x) \) is said to be **analytic at the origin** if there exists a power series \( \sum a_n x^n \) whose radius of convergence is positive if \( f(x) = \sum a_n x^n \) within the interval of convergence. Such a power series is called the **Maclaurin series** of \( f(x) \).

Theorem 18: Suppose \( f(x) \) is analytic at the origin. Then \( f(x) \) is infinitely differentiable at the origin and the Maclaurin series is \( \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \).

Proof: Let \( f(x) = \sum a_n x^n \). Then \( f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1) \ldots (n-k+1) a_n x^{n-k} \).

When \( x = 0 \) the RHS becomes \( k! \ a_k \). Hence \( a_k = \frac{f^{(k)}}{k!} \).
A function is analytic at \( x = c \) if it can be represented by a power series in \( x - c \). In that case the Maclaurin series is called the **Taylor series**.

**Theorem 19:** Suppose \( f(x) \) is analytic at \( x = c \). Then \( f(x) \) is infinitely differentiable at \( x = c \) and the Taylor series is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n.
\]

If a function is analytic at \( x = c \) then it is infinitely differentiable at \( x = c \). But the converse does not always hold. There are highly unusual functions that are not. It is not easy to prove that the familiar elementary functions, such as the exponential and trigonometric functions are analytic at \( x = 0 \) and we do not attempt to show this. However, assuming that they are analytic it is easy to compute their Maclaurin series, or indeed their Taylor series at any point.

The following are the Maclaurin series for certain functions.

<table>
<thead>
<tr>
<th>FUNCTION</th>
<th>MACLAURIN SERIES</th>
<th>CONVERGES FOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^x )</td>
<td>( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ... )</td>
<td>all ( x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^n \frac{x^{2n}}{(2n)!} + ... )</td>
<td>all ( x )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ... )</td>
<td>all ( x )</td>
</tr>
<tr>
<td>( \frac{1}{1-x} )</td>
<td>( 1 + x + x^2 + ... + x^n + ... )</td>
<td>(</td>
</tr>
<tr>
<td>( \log(1-x) )</td>
<td>( x + \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^n}{n} + ... )</td>
<td>( -1 \leq x &lt; 1 )</td>
</tr>
</tbody>
</table>