6. EIGENVALUES AND EIGENVECTORS

§6.1. The Characteristic Polynomial

If \( A \) is a square matrix and \( \mathbf{v} \) is a non-zero vector such that \( A\mathbf{v} = \lambda \mathbf{v} \) we say that \( \mathbf{v} \) is an eigenvector of \( A \) and \( \lambda \) is the corresponding eigenvalue.

Example 1: Let \( A = \begin{pmatrix} 2 & -2 \\ 1 & 5 \end{pmatrix} \) and \( \mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \). Then \( A\mathbf{v} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -1 \end{pmatrix} \).

So 3 is an eigenvalue of \( A \) and \( \begin{pmatrix} 2 \\ -1 \end{pmatrix} \) is a corresponding eigenvector.

The name “eigen” comes from German and means “characteristic”. The eigenvalues are important characteristics of the matrix in so far as they contain valuable information about the matrix. Eigenvalues are sometimes called “characteristic roots” or “latent roots”. The word “latent” means “hidden”. The latent roots, or eigenvalues, are buried somewhat and need a certain amount of work for them to be revealed. The traditional symbol for eigenvalues is the Greek letter lambda (\( \lambda \)) (corresponding to the English letter lower case L), probably because of the word “latent”.

The characteristic polynomial of a square matrix \( A \) is \(|\lambda I - A|\). This is a polynomial in \( \lambda \). We denote the characteristic polynomial of \( A \) by \( \chi_A(\lambda) \) or, if the matrix \( A \) is understood, just \( \chi(\lambda) \).

The word “characteristic” in English is derived from the Greek “\( \chiαρακτηριστικ\)” meaning the same thing. In turn this comes from the Greek word “\( \chiαρακτηρ\)” meaning a stamping tool, a tool for stamping a design or symbol on an object. This is why it is traditional to use the Greek letter \( \chi \).

Example 2: If \( A = \begin{pmatrix} 2 & -2 \\ 1 & 5 \end{pmatrix} \) find \( \chi_A(\lambda) \).

Solution: \[
\begin{vmatrix} \lambda - 2 & -2 \\ -1 & \lambda - 5 \end{vmatrix} = (\lambda - 2)(\lambda - 5) - (-2) = \lambda^2 - 7\lambda + 10.
\]

Theorem 1: The eigenvalues of a square matrix \( A \) are the zeros of its characteristic polynomial.

Proof: \( \lambda \) is an eigenvalue of \( A \) if and only if \( A\mathbf{v} = \lambda \mathbf{v} \) has a non-zero solution for \( \mathbf{v} \). This holds if and only if \(|\lambda I - A| = 0 \) which means that \( \lambda \) is a zero of the characteristic polynomial.

Example 3: Find the characteristic polynomial of the matrix \( A = \begin{pmatrix} 2 & -2 \\ 1 & 5 \end{pmatrix} \) and hence find its eigenvalues and eigenvectors.

Solution: \[
\chi_A(\lambda) = \begin{vmatrix} \lambda - 2 & 2 \\ -1 & \lambda - 5 \end{vmatrix}
\]
\[
\begin{align*}
= (\lambda - 2)(\lambda - 5) + 2 \\
= \lambda^2 - 7\lambda + 12 \\
= (\lambda - 3)(\lambda - 4).
\end{align*}
\]
Hence the eigenvalues are \( \lambda = 3, 4 \).

For each eigenvalue we must find the non-zero solutions to the equation

\[
\begin{pmatrix}
\lambda - 2 & 2 \\
-1 & \lambda - 5
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

However it is more convenient to use the equivalent equation \((A - \lambda I)v = 0\), because we can simply take the original matrix and subtract off the eigenvalue from the diagonal components.

For \( \lambda = 3 \) we consider

\[
\begin{pmatrix}
2 - 3 & -2 \\
1 & 5 - 3
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]
that is

\[
\begin{pmatrix}
-1 & -2 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
Now this has infinitely many solutions \( x = 2k, y = -k \), for any \( k \), but we simply quote any non-zero solution, such as \( \begin{pmatrix} 2 \\ -1 \end{pmatrix} \).

For \( \lambda = 4 \) we solve

\[
\begin{pmatrix}
-2 & -2 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]
giving the eigenvector \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).

A magic square is an \( n \times n \) matrix where each row, each column and each diagonal has the same total. It is generally arranged for the components to be the integers from 1 to \( n^2 \), each once but this is not obligatory. For example

\[
\begin{pmatrix}
2 & 7 & 6 \\
9 & 5 & 1 \\
4 & 3 & 8
\end{pmatrix}
\]
is a 3 \times 3 magic square. A very famous 4 \times 4 magic square appears in a woodcut by Albrecht Dürer [1471 – 1528]. It is:

\[
\begin{pmatrix}
16 & 3 & 2 & 13 \\
5 & 10 & 11 & 8 \\
9 & 6 & 7 & 12 \\
4 & 15 & 14 & 1
\end{pmatrix}
\]

For a magic square, with row/column/diagonal total \( T \), we can easily see that \( T \) is an eigenvalue since

\[
\begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix}
\]
is an eigenvector corresponding to \( T \).

For the Dürer square \( T = 34 \). The other eigenvalues are 0 and \( \pm 8 \).

Finding the characteristic polynomial and solving it is the normal way to find eigenvalues. And once we find the eigenvalues we can find the corresponding eigenvectors. But, in the case of magic squares, we can find an eigenvector first and then find its eigenvalue. Often we can guess all the eigenvectors and so get all the eigenvalues, and then the characteristic polynomial – the exact reverse of the usual procedure. Consider the following example.
Example 4: Find the eigenvalues and eigenvectors of \( A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix} \).

Solution: It would take a lot of work to follow the traditional path of computing the characteristic polynomial first. It turns out to be \( \lambda^4 - 12\lambda^3 + 20\lambda^2 + 16\lambda - 160 \) and so there would remain the difficult job of finding its zeros.

But note that the sum of every row is the same. This means that \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) is an eigenvector, for \[
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 10 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}
\] is just another way of saying that every row adds to 10. So in this case we’ve found the eigenvector first, and then the eigenvalue. What about the other eigenvalues?

Note that each row is the same as the one above but rotated one place to the left, with the component that falls off the left-hand end going down to the right-hand end. This pattern can be encapsulated by the equation
\[
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ k \\ k^2 \\ k^3 \end{pmatrix} = (1 + 2k + 3k^2 + 4k^3) \begin{pmatrix} 1 \\ k \\ k^2 \\ k^3 \end{pmatrix}
\]
if \( k^4 = 1 \).

So for every 4’th root of 1, \( \begin{pmatrix} 1 \\ k \\ k^2 \\ k^3 \end{pmatrix} \) is an eigenvector and \( 1 + 2k + 3k^2 + 4k^3 \) is the corresponding eigenvalue. Putting \( k = 1, i, -1 \) and \( -i \) we conclude that the eigenvalues of \( A \) are 10, \(-2 \pm i \) and \(-2 \).

So \( \chi_A(\lambda) = (\lambda - 10)(\lambda + 2)(\lambda + 2 + 2i)(\lambda + 2 - 2i) = \lambda^4 - 12\lambda^3 + 20\lambda^2 + 16\lambda - 160 \).

Theorem 2: Let \( A \) be an \( n \times n \) matrix with eigenvalues \( \lambda_1, \lambda_2, ..., \lambda_n \). (Here multiple zeros are counted according to their multiplicities.)
(i) \( \text{tr}(A) = \lambda_1 + \lambda_2 + ... + \lambda_n \) (trace is the sum of its eigenvalues);
(ii) \( |A| = \lambda_1 \lambda_2 ... \lambda_n \) (determinant is the product of the eigenvalues).

Proof: (i) Let \( A = (a_{ij}) \) be a square matrix and let its characteristic polynomial be \( \lambda^n + c_{n-1}\lambda^{n-1} + ... + c_1\lambda + c_0 \).
Then the sum of the eigenvalues is \(-c_{n-1} \). But the only term of degree \( n - 1 \) in \( |A - \lambda I| \) comes from \( (\lambda - a_{11})(\lambda - a_{22}) ... (\lambda - a_{nn}) \) and this has coefficient \( -(a_{11} + a_{22} + ... + a_{nn}) = -\text{tr}(A) \).
So \( c_{n-1} = -\text{tr}(A) \) and hence the sum of the eigenvalues is \( \text{tr}(A) \).
(ii) The product of the eigenvalues is \((-1)^n c_0 \). Now the constant term is the value of the characteristic polynomial when \( \lambda = 0 = |A| = (-1)^n |A| \). (Remember that taking out the factor of \(-1 \) from each row changes the sign of the determinant.) Hence \( |A| = \lambda_1 \lambda_2 ... \lambda_n \).
Theorem 3: The eigenvalues of an upper-triangular matrix, or a lower-triangular matrix, are the diagonal components.

\[
\begin{vmatrix}
\lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\
0 & \lambda - a_{22} & \cdots & -a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda - a_{nn}
\end{vmatrix} = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}).
\]

§6.2. Similar Matrices

Two \( n \times n \) matrices \( A, B \) are said to be similar if \( B = S^{-1}AS \) for some invertible matrix \( S \). In ordinary algebra, the algebra of a field, the equation \( B = S^{-1}AS \) would give us \( B = A \) by cancelling the \( S^{-1} \) and the \( S \). But while such “remote” cancelling is permissible in a field, it is not permitted for a non-commutative system such as the system of \( n \times n \) matrices. We can only cancel if the element and its inverse are adjacent. In a field it doesn’t matter if they’re not because we can rearrange terms until they are. For matrices we just can’t do that because matrices in general don’t commute.

Theorem 4: Similarity is an equivalence relation.

Proof: Reflexive: Suppose \( A \) is an \( n \times n \) matrix. Then \( A = I^{-1}AI \) so \( A \) is similar to \( A \).

Symmetric: Suppose \( A \) is similar to \( B \). Then \( A = S^{-1}BS \) for some invertible \( S \).

Hence \( B = S^{-1}BS = (S^{-1})^{-1}A(S^{-1}) \) and so \( B \) is similar to \( A \).

Transitive: Suppose \( A \) is similar to \( B \) and \( B \) is similar to \( C \).

Then \( A = S^{-1}BS \) and \( B = T^{-1}CT \) for some invertible \( S, T \).

\[ \therefore A = S^{-1}(T^{-1}CT)S = (TS)^{-1}C(TS) \] so \( A \) is similar to \( C \).

Similar matrices are indeed similar in the non-technical sense in that they have many common properties.

Theorem 5: Similar matrices have the same characteristic polynomial.

Proof: Suppose that \( B = S^{-1}AS \) for some invertible matrix \( S \). The characteristic polynomial for \( B \) is

\[ |\lambda I - B| = |\lambda I - S^{-1}AB| = |\lambda S^{-1}IS - S^{-1}AS| = |S^{-1}(\lambda I - A)S| = |S|^{-1}|\lambda I - A|.|S| = |\lambda I - A|. \]

Corollary: Similar matrices have the same determinant and the same trace.

§6.3. Properties of Eigenvalues

If \( p(x) \) is a polynomial and \( A \) is an \( n \times n \) matrix (with the coefficients of \( p(x) \) and the components of \( A \) coming from the same field) we define \( p(A) \) to be the \( n \times n \) matrix obtained by substituting \( A \) for \( x \) and replacing the constant term, \( a_0 \), by \( a_0I \).

Example 5: If \( p(x) = 3x^2 + 2x - 5 \) and \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \) then

\[
p(A) = 3 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^2 + 2 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[ = 3 \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} + 2 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 5 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}
\]

\[ = \begin{pmatrix} 18 & 24 \\ 36 & 54 \end{pmatrix}.\]
**Theorem 6:** If \( p(x) \) is a polynomial the eigenvectors of \( p(A) \) are the same as for \( A \). If the eigenvalues of \( A \) are \( \lambda_1, \lambda_2, \ldots, \lambda_n \), the eigenvalues of \( p(A) \) are \( p(\lambda_1), p(\lambda_2), \ldots, p(\lambda_n) \).

**Proof:** Suppose \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \) and suppose that \( Av = \lambda v \).
Then \( p(A)v = (a_n A^n + a_{n-1} A^{n-1} + \ldots + a_1 A + a_0)v = (a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0)v. 

**Theorem 7:** If \( A \) is invertible the eigenvectors of \( A^{-1} \) are the same as for \( A \). If the eigenvalues of \( A \) are \( \lambda_1, \lambda_2, \ldots, \lambda_n \), the eigenvalues of \( A^{-1} \) are \( \lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1} \).

**Proof:** Suppose \( Av = \lambda v \). Then multiplying both sides on the left by \( A^{-1} \) we get \( v = \lambda A^{-1}v \) so \( A^{-1}v = \lambda^{-1}v \).

**Theorem 8:** The eigenvalues of \( A^T \) are the same as those for \( A \). (The eigenvectors will generally be different.)

**Proof:** \( |\lambda I - A^T| = |(\lambda I - A)^T| = |\lambda I - A| \).

**Theorem 9:** If \( p(A) = 0 \) then every eigenvalue of \( A \) is a zero of \( p(x) \).

**Proof:** Suppose \( Av = \lambda v \). Then \( A^r = \lambda^r v \) for all \( r \) and so \( p(\lambda)v = p(A)v = 0 \). Since \( v \neq 0 \), \( p(\lambda) = 0 \).

**Example 6:**
(i) A nilpotent matrix is one for which \( A^m = 0 \) for some \( m \).
The eigenvalues of a nilpotent matrix satisfy \( \lambda^m = 0 \) and so are all zero.
(ii) An idempotent matrix is one where \( A^2 = A \).
The eigenvalues of an idempotent matrix satisfy \( \lambda^2 - \lambda = 0 \) and so are all 0’s and 1’s.

**§6.4. Cayley Hamilton Theorem**

The Cayley Hamilton Theorem states that a square matrix satisfies its characteristic polynomial. That is, \( \chi_A(A) = 0 \).

**Example 7:** Suppose \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \),
Then \( \chi_A(\lambda) = \lambda^2 - 5\lambda - 2 \).
\[
A^2 - 5A - 2I = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 5 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
= \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} - 5 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]
Since \( \chi_A(\lambda) = |\lambda I - A| \) it’s tempting to attempt to prove this by substituting \( \lambda = A \). The problem is that in doing this we would have the determinant of a matrix where the components are a mixture of scalars and matrices.

For example, if \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) then \( |\lambda I - A| = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} \). If we substitute \( A \) for \( \lambda \) we get \( \begin{vmatrix} A-a & -b \\ -c & A-d \end{vmatrix} \) which doesn’t make sense. We could replace the scalars \( a, b, c, d \) by \( aI, bI, cI, dI \) respectively, but we would get \( \begin{vmatrix} A-al & -bI \\ -cI & A-dI \end{vmatrix} \) and there is no way this is the determinant of the zero matrix the way \( |A - I| \) would be. The following proof is due to Yishao Zhou.
Theorem 10 (CAYLEY-HAMILTON & YISHAO ZHOU):
If A is an \( n \times n \) matrix then \( \chi_A(A) = 0 \). That is, every square matrix satisfies its characteristic polynomial.

Proof: Let \( \chi_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0 \).
Let \( B(\lambda) = \text{adj}(\lambda I - A) \) and write the components of \( B(\lambda) \) as \( b_{ij}(\lambda) \). Since these are cofactors of the \( n \times n \) matrix \( \text{adj}(\lambda I - A) \), they will each be a polynomial of degree at most \( n - 1 \).
Let \( b_{ij}(\lambda) = b_{ij}^{(r-1)}\lambda^{n-1} + \ldots + b_{ij}^{(r)}(\lambda) + b_{ij}^{(0)}(\lambda) \). Here the \( (r) \) are superscripts, not powers.
Define \( B(\lambda) = (b_{ij}^{(k)}) \). So the components of \( B(\lambda) \) are the coefficients of \( \lambda^k \) in the components of \( B(\lambda) \).
Hence \( \text{adj}(\lambda I - A) = B(\lambda) = \lambda^{n-1}B^{(n-1)} + \ldots + \lambda B^{(1)} + B^{(0)} \).

Remember that \( M\text{adj}(M) = |M|I \) for all square matrices \( M \) (this is what gives the cofactor formula for \( M^{-1} \) when \( |M| \neq 0 \)). So \( (\lambda I - A)\text{adj}(\lambda I - A) = |\lambda I - A|I \).
Hence \( (\lambda I - A)[\lambda^{n-1}B^{(n-1)} + \ldots + \lambda B^{(1)} + B^{(0)}] = |\lambda I - A|I \)
\[ = \chi_A(\lambda)I \]
\[ = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0I. \]
Equating coefficients of the various powers of \( \lambda \), we get
\[ B^{(n-1)} = I, \quad \text{whence } A^{n}B^{(n-1)} = A^n; \]
\[ B^{(n-2)} - AB^{(n-1)} = a_{n-1}I, \quad \text{whence } A^{n-1}B^{(n-2)} - A^2B^{(n-1)} = a_{n-1}A^{n-1}; \]
\[ B^{(n-3)} - AB^{(n-2)} = a_{n-2}I, \quad \text{whence } A^{n-2}B^{(n-3)} - A^3B^{(n-2)} = a_{n-2}A^{n-2}; \]
\[ \vdots \]
\[ B^{(0)} - AB^{(1)} = a_1I, \quad \text{whence } AB^{(0)} - A^2B^{(1)} = a_1A; \]
and finally
\[ -AB^{(0)} = a_0I. \]
Adding these equations we get \( 0 = A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \ldots + a_1A + a_0I. \)

You may find all this confusing. What a pity we can’t just put \( \lambda = A \) in \( |\lambda I - A| \). To help you, here’s an example that you can follow in parallel with the proof.

Example 8: Let \( A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \).
Then \( \chi(\lambda) = \begin{vmatrix} \lambda - 1 & -2 & 3 \\ -4 & \lambda - 5 & 6 \\ -7 & 8 & \lambda - 9 \end{vmatrix} \)
\[ = (\lambda - 1)[(\lambda - 5)(\lambda - 9) - 48] + 2[-4\lambda + 36 - 42] - 3[32 + 7\lambda - 35] \]
\[ = (\lambda - 1)[\lambda^2 - 14\lambda - 3] + 2[-4\lambda - 6] - 3[7\lambda - 3] \]
\[ = [\lambda^3 - 14\lambda^2 - 3\lambda - \lambda^2 + 14\lambda + 3] - [8\lambda + 12] + [-21\lambda + 9] \]
\[ = \lambda^3 - 15\lambda^2 - 18\lambda \]
\[ \lambda I - A = \begin{pmatrix} \lambda - 1 & -2 & -3 \\ -4 & \lambda - 5 & -6 \\ -7 & 8 & \lambda - 9 \end{pmatrix}. \]
\[ B(\lambda) = \text{adj}(\lambda I - A) = \begin{pmatrix} (\lambda - 5)(\lambda - 9) - 48 & 4(\lambda - 9) + 42 & 32 + 7(\lambda - 5) \\ 2(\lambda - 9) + 24 & (\lambda - 1)(\lambda - 9) - 21 & 8(\lambda - 1) + 14 \\ 12 + 3(\lambda - 5) & 6(\lambda - 1) + 12 & (\lambda - 1)(\lambda - 5) - 8 \end{pmatrix}^T \]
\[
\begin{pmatrix}
\lambda^2 - 14\lambda - 3 & 2\lambda + 6 & 3\lambda - 3 \\
4\lambda + 6 & \lambda^2 - 10\lambda - 12 & 6\lambda + 6 \\
7\lambda - 3 & 8\lambda + 6 & \lambda^2 - 6\lambda - 3
\end{pmatrix}
\]
so

\[
B^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};
\]

\[
B^{(1)} = \begin{pmatrix} -14 & 2 & 3 \\ 4 & -10 & 6 \\ 7 & 8 & -6 \end{pmatrix};
\]

\[
B^{(0)} = \begin{pmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{pmatrix}.
\]

\[
A^3B^{(2)} = A^3;
\]

\[
B^{(1)} - AB^{(2)} = \begin{pmatrix} -14 & 2 & 3 \\ 4 & -10 & 6 \\ 7 & 8 & -6 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{pmatrix} = -15I.
\]

So \(A^2(B^{(1)} - AB^{(2)}) = -15A^2\).

\[
B^{(0)} - AB^{(1)} = \begin{pmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -14 & 2 & 3 \\ 4 & -10 & 6 \\ 7 & 8 & -6 \end{pmatrix} = \begin{pmatrix} -18 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -18 \end{pmatrix} = -18I
\]

So \(A(B^{(0)} - AB^{(1)}) = -18I\).

\[
-AB^{(0)} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

So \(A^3 - 15A^2 - 18A = A^3B^{(2)} + [A^2B^{(1)} - A^3B^{(2)}] + [AB^{(0)} - A^2B^{(1)}] - AB^{(0)} = 0\).
§6.5. Calculating the Characteristic Polynomial

Finding the characteristic polynomial is difficult, and error prone, when doing it by hand. And, although there are algorithms for evaluating a determinant they are difficult to implement on a computer if the determinant involves a variable. What we need is an algorithm that calculates the coefficients of the characteristic polynomial and so only works with numbers. For matrices bigger than $2 \times 2$ the following algorithm is preferable.

If $k \geq 1$, the $k$-th trace, $\text{tr}_k(A)$, of a square matrix $A$ is the sum of all the $k \times k$ sub-determinants that can be obtained from $A$ by deleting corresponding rows and columns (so that the diagonal of each sub-determinant coincides with the diagonal of $A$). We define $\text{tr}_0(A) = 1$.

Example 9: If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$, $\text{tr}_0(A) = 1$, $\text{tr}_1(A) = 1 + 6 + 11 + 16 = 34$.

\[
\begin{align*}
\text{tr}_2(A) &= \begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 9 & 11 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 13 & 16 \end{vmatrix} + \begin{vmatrix} 6 & 7 \\ 10 & 11 \end{vmatrix} + \begin{vmatrix} 6 & 8 \\ 14 & 16 \end{vmatrix} + \begin{vmatrix} 11 & 12 \\ 15 & 16 \end{vmatrix} \\
&= 4 + 8 + 12 + 4 + 8 + 4 = 36.
\end{align*}
\]

\[
\begin{align*}
\text{tr}_3(A) &= \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 6 & 8 \\ 13 & 14 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 9 & 11 \\ 13 & 15 \end{vmatrix} + \begin{vmatrix} 6 & 7 \\ 10 & 11 \\ 14 & 15 \end{vmatrix} \\
&= 123 + 124 + 134 + 234 = 595.
\end{align*}
\]

\[
\begin{align*}
\text{tr}_4(A) &= \begin{vmatrix} 1 & 3 & 4 \\ 8 & 8 & 8 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 12 & 12 \\ 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 12 & 12 \\ 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 8 & 8 \\ 1 \end{vmatrix} \\
&= 32 + 96 + 48 + 32 = 188.
\end{align*}
\]

Finally $\text{tr}_4(A) = |A| = 0$. 

Lemma (COOPER): Suppose $A$ is an $n \times n$ matrix and $D = D(\lambda_1, \ldots, \lambda_n) = \begin{vmatrix} \lambda_1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda_2-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda_n-a_{nn} \end{vmatrix}$ is a polynomial in $n$ commuting variables $\lambda_1, \ldots, \lambda_n$. For every subset $S = \{s_1, \ldots, s_k\}$ of $\{1, 2, \ldots, n\}$ with $s_1 < s_2 < \cdots < s_k$ the coefficient of $\lambda_{s_1}\cdots\lambda_{s_k}$ in $D$ is $(-1)^{n-k}$ times the sub-determinant got by deleting all rows and columns except those corresponding to the elements of $S$.

Proof: We prove this by induction on $k$. If $k = 0$, $S = \emptyset$ and the required coefficient is the constant term of $D$ which is $|A| = (-1)^n|A|$. Suppose $k \geq 1$. Expanding $D$ along row $s_1$ the only term in $\lambda_{s_1}$ arises from $(\lambda_{s_1} - a_{s_1 s_1})$ times the determinant $D'$ got from $D$ by deleting row $s_1$ and column $s_1$. This is the coefficient of $\lambda_{s_2}\cdots\lambda_{s_k}$ in $D'$ which, by induction, is $(-1)^{(n-1)-(k-1)}$ times the sub-determinant $D''$ got from $D'$ by deleting the rows and columns corresponding to $s_2, \ldots, s_k$. But this is $(-1)^{n-k}$ times the sub-determinant got from $D$ by deleting the rows and columns corresponding to the elements of $S$.

Theorem 11 (COOPER): The characteristic polynomial of the $n \times n$ matrix $A$ is $\chi_A(\lambda) = \lambda^n - \text{tr}(A)\lambda^{n-1} + \text{tr}^2(A)\lambda^{n-2} + \ldots + (-1)^{k} \text{tr}^k(A)\lambda^{n-k} + \ldots + (-1)^n|A|$.

Proof: With $D(\lambda_1, \ldots, \lambda_n)$ defined above $\chi_A(A) = D(\lambda, \ldots, \lambda)$.

Example 10: Find the characteristic polynomial of $A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 5 \\ 6 & 2 & 7 \end{pmatrix}$.

Solution: $\text{tr}(A) = 1 + 1 + 7 = 9$, $\text{tr}^2(A) = \begin{vmatrix} 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 5 \end{vmatrix} = -25$,

$\text{tr}^3(A) = 3 + 1 + 5 = 15$ and so the characteristic polynomial is $\lambda^3 - 9\lambda^2 - 25\lambda - 15$.

Example 11: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

$\text{tr}(A) = 15$,

$\text{tr}^2(A) = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3 - 12 - 3 = -18.$

$|A| = \begin{vmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 3 \cdot 3 = 0$.

So $\chi_A(\lambda) = \lambda^3 - 15\lambda^2 - 18\lambda$.

Compare the amount of work in computing $\chi_A(A)$ to that in Example 8.
§6.6. Matrices and Graphs

A **directed graph** is a set of vertices, $G$, together with **directed edges** from some vertices to others. A directed edge is simply an ordered pair of vertices. Hence we can think of a directed graph as being a set $G$ together with a subset of $G \times G$.

**Example 12:**

The above picture represents a graph with 5 vertices and 8 edges. (The double-headed arrow represents two edges.)

As a set of ordered pairs the graph is $\{(1, 4), (2, 3), (3, 4), (4, 2), (4, 5), (5, 1), (5, 2), (5, 5)\}$.

A **path** in an ordered graph is a sequence of edges of the form

$$(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n).$$

The length of a path is the number of edges, in this case, $n$. A path of length 1 is simply an edge.

**Example 13:** In the above example there is a path $(2, 3), (3, 4), (4, 5)$ from 2 to 5. We can write this as $2 \rightarrow 3 \rightarrow 4 \rightarrow 5$. There is a path of length 1 from 5 to 5 (represented by the circular arrow), a path of length 2 $(5 \rightarrow 1 \rightarrow 4 \rightarrow 5)$ and a path of length 3 $(5 \rightarrow 1 \rightarrow 4 \rightarrow 5)$. There is even a path of length 5, namely $5 \rightarrow 4 \rightarrow 5 \rightarrow 1 \rightarrow 4 \rightarrow 5$. There are at least four paths of length 6 from 5 to 5:

$$5 \rightarrow 5 \rightarrow 5 \rightarrow 5 \rightarrow 5 \rightarrow 5$$

$$5 \rightarrow 4 \rightarrow 5 \rightarrow 4 \rightarrow 5 \rightarrow 5$$

$$5 \rightarrow 1 \rightarrow 4 \rightarrow 5 \rightarrow 1 \rightarrow 4 \rightarrow 5$$

$$5 \rightarrow 4 \rightarrow 5 \rightarrow 1 \rightarrow 4 \rightarrow 5$$

and many more. How can we count the number of paths of length $n$ from vertex $i$ to vertex $j$?

The **adjacency matrix** of a directed graph on vertices 1, 2, ..., $n$ is the matrix $A = (a_{ij})$ where $a_{ij} = 1$ if there’s an edge from $i$ to $j$ in the graph, and 0 otherwise.

**Example 14:** The matrix of the directed graph in example 12 is

$$A = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1
\end{pmatrix}$$

**Theorem 12:** If $A$ is the matrix of a directed graph, the number of paths of length $N$ from vertex $i$ to vertex $j$ is the $i-j$ component of $A^N$.

**Proof:** It is true for $N = 1$. Suppose that it is true for $N$.

A path of length $N + 1$ from $i$ to $j$ is a path from $i$ to some vertex $k$ followed by an edge from $k$ to $j$. By induction the number of paths from $i$ to $k$ is $a_{ik}^{(N)}$, the $i-k$ component of $A^N$. Since the number of paths from $k$ to $j$ is $a_{kj}$ the total number of paths of length $N + 1$ from $i$ to $j$ is

$$\sum_k a_{ik}^{(N)} a_{kj}$$

which is the $i-j$ component of $A^N A = A^{N+1}$. Hence it is true for all $N$.  

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Example 15: The number of paths in the directed graph in example 12, of length 4 between various vertices are given by the components of $A^4$, where $A$ is the adjacency matrix of that graph, as given in example 14.

\[
A^2 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

\[
A^3 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 2 & 1 & 0 \\
1 & 2 & 1 & 1 & 2 \\
1 & 1 & 1 & 0 & 1
\end{pmatrix}
\]

\[
A^4 = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 \\
1 & 2 & 1 & 0 & 2 \\
1 & 1 & 1 & 0 & 1 \\
1 & 2 & 1 & 2 & 2
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 & 2 & 1 \\
1 & 1 & 1 & 0 & 1 \\
2 & 2 & 0 & 2 & 2 \\
2 & 4 & 2 & 2 & 4
\end{pmatrix}
\]

So there are 4 paths of length 4 from vertex 5 to vertex 2. These are the 2 paths of length 3 from vertex 5 to vertex 4, followed by an edge from vertex 4 to vertex 5 plus the 2 edges of length 3 from vertex 4 to vertex 5 followed by the edge from vertex 5 to vertex 5. This gives us

5→5→3→4→2,
5→5→1→4→2,
5→1→4→5→2,
5→5→5→5→2.

**EXERCISES FOR CHAPTER 6**

Exercise 1: If $A = \begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix}$ find $\chi_A(\lambda)$.

Exercise 2: If $A = \begin{pmatrix} 5 & -1 & 2 \\ 3 & 2 & -1 \\ 0 & 5 & 2 \end{pmatrix}$ find $\chi_A(\lambda)$.

Exercise 3: If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 6 & 0 & 1 & 0 \end{pmatrix}$ find $\chi_A(\lambda)$. 
Exercise 4: Find the eigenvalues and eigenvectors of \( A = \begin{pmatrix} 1 & 1 & -2 \\ -3 & 0 & 3 \\ 2 & -1 & -1 \end{pmatrix} \).

Exercise 5: Find the eigenvalues and eigenvectors of \( A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \).

Exercise 6: Find the eigenvalues and eigenvectors of \( A = \begin{pmatrix} 9 & 14 & 21 \\ -7 & -12 & -21 \\ 2 & 4 & 8 \end{pmatrix} \).

Exercise 7: Find the eigenvalues of the \( n \times n \) matrix \( A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \).

Exercise 8: If \( v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \) find the eigenvalues of the \( n \times n \) matrix \( A = vv^T \).

Exercise 9: Prove that it’s impossible for a magic square to have all its eigenvalues real and positive.

Example 10: Find the eigenvalues of \( A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \).

SOLUTIONS FOR CHAPTER 6

Exercise 1: \( \text{tr}(A) = 10, |A| = 11 \). So \( \chi_A(\lambda) = \lambda^2 - 10\lambda + 11 \).

Exercise 2: \( \text{tr}(A) = 9 \), 
\( \text{tr}^2(A) = 13 + 10 + 9 = 32 \). 
\( |A| = 5(9) + 6 + 2(15) = 45 + 30 + 6 = 81 \). So \( \chi_A(\lambda) = \lambda^3 - 9\lambda^2 + 32\lambda + 81 \).

Exercise 3: \( \text{tr}(A) = 1 \), 
\( \text{tr}^2(A) = 1 + 2 + 3 + 4 + 0 + 5 + 0 + 0 + 0 + 3 = -2 + 0 - 24 - 5 + 0 - 3 = -34 \).
\[ \text{tr}_3(A) = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 3 \\ 1 & 0 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 5 & 0 \\ 6 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -2 + 0 -3(-17) - 0 = 49 \]

\[ |A| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 6 & 0 & 0 & 1 \end{vmatrix} = -60 + 2 = -58. \]

So \( \chi_A(\lambda) = \lambda^4 - \lambda^3 - 34\lambda^2 - 49\lambda - 58. \)

**Exercise 4:**

\( \text{tr}(A) = 0. \)

\( \text{tr}_2(A) = 3 + 3 + 3 = 9. \)

\( |A| = 3(-3) - 3(-3) = 0. \)

\( \therefore \chi_A(\lambda) = \lambda^3 + 9\lambda = \lambda(\lambda^2 + 9). \)

Hence the eigenvalues are \( \lambda = 0. \pm 3i. \)

\( \lambda = 0: \quad A = \begin{pmatrix} 1 & 1 & -2 \\ -3 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} . \) So \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) is an eigenvector.

\( \lambda = 3i: \quad A - 3I = \begin{pmatrix} 1-3i & 1 & -2 \\ -3 & -3i & 3 \\ 2 & 1 & 1-3i \end{pmatrix} \rightarrow \begin{pmatrix} 1-3i & 1 & -2 \\ -1 & -1-3i & 2-3i \\ 2 & -1 & -1-3i \end{pmatrix} \]

\[ \Rightarrow \begin{pmatrix} 1 & 1+3i & -2+3i \\ -3-6i & 3-9i \\ 1 & 1+3i & -2+3i \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1+3i & -2+3i \\ 0 & 1 & 1+i \\ 0 & 0 & 0 \end{pmatrix} \]

So \( \begin{pmatrix} -i \\ 1+i \\ -1 \end{pmatrix} \) is an eigenvector.
\(\lambda = -3i\): Similarly \(\begin{pmatrix} i \\ -i \\ -1 \end{pmatrix}\) is an eigenvector.

**Exercise 5:** \(\text{tr}(A) = 3, \text{tr}_2(A) = 0, |A| = 0\).

\[\therefore \chi_A(\lambda) = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda - 3).\]

Hence the eigenvalues are \(\lambda = 0\) (twice) and \(\lambda = 3\).

\(\lambda = 0\): \(A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) so \(\begin{pmatrix} -1 \\ -1 \end{pmatrix}\) and \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) are independent eigenvectors.

\(\lambda = 3\): \(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\) is an eigenvector.

**Exercise 6:** \(\text{tr}(A) = 5\),

\[\text{tr}_2(A) = (-108 + 98) + (72 - 42) + (-96 + 84) = -10 + 30 - 12 = 8\]

\[|A| = 9(-96 + 84) - 14(-56 + 42) + 21(-28 + 24)\]

\[= 9(-12) - 14(-14) + 21(-4) = -108 + 196 - 84 = 4.\]

\[\therefore \chi_A(\lambda) = \lambda^3 - 5\lambda^2 + 8\lambda - 4\] 

\[= (\lambda - 1)(\lambda - 2)^2.\]

So the eigenvalues are \(\lambda = 1\) and \(\lambda = 2\) (twice).

\(\lambda = 1\): \(A - I = \begin{pmatrix} 8 & 14 & 21 \\ -7 & -13 & -21 \\ 2 & 4 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ -7 & -13 & -21 \\ 2 & 4 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 7 \\ -7 \\ 2 \end{pmatrix}\) is an eigenvector.

\(\lambda = 2\): \(A - 2I = \begin{pmatrix} 7 & 14 & 21 \\ -7 & -14 & -21 \\ 2 & 4 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 7 & 14 & 21 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 7 & 14 & 21 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}\)
\[
R_1 \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, R_1 \leftrightarrow R_3
\]

Hence \(\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\) and \(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\) are independent eigenvectors.

**Exercise 7:** \(\text{tr}(A) = n\).

For \(r > 1\), \(\text{tr}(A) = 0\) because the \(r \times r\) sub determinants are all zero.

Hence \(\chi_A(\lambda) = \lambda^n - n \lambda^{n-1} = \lambda^{n-1}(\lambda - n)\).

Hence the eigenvalues are \(\lambda = n\) and \(\lambda = 0\) \((n - 1\) times).  

**Exercise 8:** The \(i-j\) component of \(A\) is \(a_ia_j\). Hence \(\text{tr}(A) = \sum a_i^2\). Let this be \(N\).

For \(r > 1\), \(\text{tr}(A) = 0\) because the \(r \times r\) sub determinants are all zero.

Hence \(\chi_A(\lambda) = \lambda^n - N \lambda^{n-1} = \lambda^{n-1}(\lambda - N)\).

Hence the eigenvalues are \(\lambda = \sum a_i^2\) and \(\lambda = 0\) \((n - 1\) times).

**Exercise 9:** Let \(T\) be the row/column/diagonal total. Since the row totals are all equal to \(T\) then

\[
\begin{bmatrix} 1 \\ 1 \\ \ldots \\ 1 \end{bmatrix}
\]

is an eigenvector for the eigenvalue \(T\). Since the trace is equal to \(T\) the sum of the remaining eigenvalues must be zero. Therefore they cannot all be positive.

**Example 10:** Find the eigenvalues of \(A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}\).

**Solution:**

\(\text{tr}(A) = 34\).

\[
\text{tr}_2(A) = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 11 \\ 9 & 13 & 16 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 13 & 16 \end{vmatrix} + \begin{vmatrix} 6 & 7 \\ 10 & 11 \end{vmatrix} + \begin{vmatrix} 6 & 8 \\ 14 & 16 \end{vmatrix} + \begin{vmatrix} 11 & 12 \\ 15 & 16 \end{vmatrix} = (-4) + (-16) + (-36) + (-4) + (-16) + (-4) = -80.
\]

\[
\text{tr}_3(A) = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 8 \\ 9 & 11 & 12 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 13 & 16 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 13 & 15 \end{vmatrix} + \begin{vmatrix} 6 & 7 \\ 14 & 15 \end{vmatrix} = 0.
\]

Rather than work out \(|A|\) in the standard way we can be a little smart. Note that \(C_1 + C_4 = C_2 + C_3\).

This means that \(\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\) is an eigenvector corresponding to the eigenvalue of 0. Hence \(|A| = 0\).

Hence \(\chi_A(\lambda) = \lambda^4 - 34 \lambda^3 - 80 \lambda^2 = \lambda^2(\lambda^2 - 34 \lambda - 80)\).

The eigenvalues are 0 \((\text{twice})\) plus the two zeros of the quadratic \(\lambda^2 - 34 \lambda - 80\), that is, 0, 0, 34 \(\pm \sqrt{41}\).