VI. Distances and Luminosities
Expansion and Redshift

\[ dv = Hdr = \frac{\dot{a}}{a} dr \quad ; \quad \Delta \lambda \equiv \lambda_0 - \lambda_e \quad ; \quad \frac{\Delta \lambda}{\lambda_e} = \frac{\Delta v}{c} \quad ; \quad dt = \frac{dr}{c} \]

\[ \frac{\Delta \lambda}{\lambda_e} = \frac{\dot{a}}{a} \frac{dr}{c} = \frac{\dot{a}}{a} dt = \frac{da}{a} \quad \Rightarrow \quad \ln \lambda = \ln a + \ldots \quad \lambda \propto a \]

Crude derivation – now a somewhat more rigorous one
Expansion and Redshift II

\[ ds^2 = -c^2 dt^2 + a^2(t)[dr^2 + S_k(r)^2 d\Omega^2] \]; \( ds = 0 \); \( c^2 dt^2 = a^2(t) dr^2 \)

(light follows a geodesic)

\[ c \frac{dt}{a(t)} = dr \]

LHS only function(t), RHS independent of t

Suppose distant galaxy emits light \( \lambda_e \) at \( t_e \), wavecrest observed at \( t_0 \):

\[ c \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^r dr = r \]

Next wave crest emitted at \( t_e + \lambda_e/c \), observed at \( t_0 + \lambda_0/c \).

For second wavecrest,

\[ c \int_{t_e + \lambda_e/c}^{t_0 + \lambda_0/c} \frac{dt}{a(t)} = \int_0^r dr = r \]
Expansion and Redshift III

The integral of $\frac{dt}{a(t)}$ between the time of emission and the time of observation is the same for every wavecrest.

Subtract the integral from each side, to get

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e + \lambda e / c}^{t_0 + \lambda_0 / c} \frac{dt}{a(t)}$$
Expansion and Redshift IV

\[ \int_{t_e}^{t_e + \lambda_e / c} \frac{dt}{a(t)} = \int_{t_0}^{t_0 + \lambda_0 / c} \frac{dt}{a(t)} \]

- The integral of \( dt/a(t) \) between emission of wavecrests = the integral of \( dt/a(t) \) between observation of wavecrests

- Between emission/observation of successive wavecrests, Universe doesn’t expand significantly \( \Rightarrow a(t) \sim \text{constant} \)

\[ \frac{1}{a(t_e)} \int_{t_e}^{t_e + \lambda_e / c} dt = \frac{1}{a(t_0)} \int_{t_0}^{t_0 + \lambda_0 / c} dt ; \quad \frac{\lambda_e}{a(t_e)} = \frac{\lambda_0}{a(t_0)} \]

\[ z = \frac{(\lambda_0 - \lambda_e)}{\lambda_e} \Rightarrow 1 + z = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)} \quad \text{using } a(t_0) = 1 \]
We are observing a galaxy at $z=2$ as it was when the Universe had a scale factor $a(t_e) = 1/3$.

- Redshift for a distant object only depends on the relative scale factors at the time of emission and the time of observation – what has happened in-between (e.g., abrupt expansion vs. gradual expansion, monotonic vs. oscillatory).
The Observable Universe

- How far could light have travelled during the Universe’s lifetime?

\[ ds^2 = -c^2 dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \]

\[ \int_0^{r_0} \frac{dr}{\sqrt{1 - kr^2}} = \int_0^{t_0} \frac{c \, dt}{a(t)} \]

- Assume a matter-dominated Universe with \( k = \Lambda = 0 \)

\[ a(t_0) = \left( \frac{t}{t_0} \right)^{2/3} \]

\[ \int_0^{r_0} dr = ct_0^{2/3} \int_0^{t_0} \frac{dt}{t^{2/3}} \Rightarrow r_0 = 3ct_0 \]
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- Finite result, even though \(1/a(t) \rightarrow \infty\) as \(t \rightarrow 0\); we can only see part of the Universe
- The light has travelled a distance \(> c \times (\text{the age of the Universe})\)...(?)...but \(r_0\) is the distance in the present-day Universe
Comoving Distance

- **Comoving distance** $\chi$: does not change with the expansion of the Universe (vs. proper distance $d_p = a(t) \chi$)

- Again, follow a geodesic $\Rightarrow ds = 0 \Rightarrow c \, dt = a \, d\chi$

$$\frac{c \, da}{\dot{a}} = a \, d\chi \ ; \ \chi = \int_0^{\frac{1}{1+z}} \frac{c \, da}{a \, \dot{a}} = \int_0^z \frac{c \, a}{H} \, dz = \int_0^z \frac{c}{H} \, dz$$

- So the proper distance was $a \, \chi = \chi / (1 + z)$ at $t_e$ and $\chi$ at $t_0$
Luminosity Distance

- Based on the amount of light received from a distant object
- Not the actual distance to the object, because inverse square law doesn’t apply
  - the geometry of the Universe isn’t necessarily flat
  - the Universe is expanding
- Luminosity \( L = \frac{\text{energy}}{\text{unit solid angle} \cdot \text{second}} = \frac{\text{total power}}{4\pi \text{ steradians}} \)
- Radiation flux density \( S \) received by us = energy received / unit area / second
• \((d_{\text{lum}})^2 = L/S\) : imagine a sphere with comoving radius \(r_0\)
• physical radius is \(a_0r_0\), so total surface area is \(4\pi a_0^2r_0^2\)
  (geometry of Universe contained within \(r_0\))
• if Universe were static, \(S = L / a_0^2r_0^2\) ...but it’s not!
• The expansion of the Universe affects photons in two ways:
  – individual photons lose energy: \(E = \frac{hc}{\lambda} \Rightarrow E = \frac{E_{\text{emit}}}{(1 + z)}\)
  – photons arrive less frequently: \(1/(1 + z)\)
• Received flux \(S = L / (a_0^2r_0^2(1 + z)^2)\) \(\Rightarrow\)
• Luminosity distance \(d_{\text{lum}} = a_0r_0(1+z)\) (flux = \(L/(4\pi d_{\text{lum}}^2)\))
• (Note: for observations, must also take into account the fact that the emitted spectrum of an object shifts toward longer \(\lambda\))
Angular Diameter Distance

• How large does an object appear? In Euclidean geometry, an object perpendicular to the line of sight with intrinsic physical size $l$ has an angular diameter distance of

$$d_{diam} \equiv \frac{l}{\sin \theta} \approx \frac{l}{\theta}$$

in the small angle approximation. If we imagine ourselves at the origin and the object at comoving coordinate $r_0$, use the metric at the time the light was emitted, $t_e$, and align our object in the $\theta$ direction, we get a physical size $l$ measured using $d\theta$ (entirely in the $\theta$ direction) of:

$$l = ds = r_0 a(t_e) d\theta ; d\theta = \frac{l}{r_0 a(t_e)} = \frac{l(1 + z)}{a_0 r_0}$$
Angular Diameter Distance II

\[ l = d_s = r_0 \, a(t_e) \, d\theta ; \quad d\theta = \frac{l}{r_0 \, a(t_e)} = \frac{l(1+z)}{a_0 \, r_0} \]

\[ d_{diam} = \frac{a_0 r_0}{1+z} \left[ = \frac{d_{lum}}{(1+z)^2} \right] \]

- Like \(d_{lum}\), angular diameter distance \(\sim\) physical distance for nearby objects, but for distant objects \(d_{diam} \rightarrow 0\) as \(z \rightarrow \infty\)
- When objects are far enough away, they can appear larger (!)
- Objects of a given size appear smallest @ \(z \sim 1\) (depending on cosmology)
- However, for observational purposes the luminosity distance and angular diameter distance effects combine \(\rightarrow\) “surface brightness dimming” makes it difficult to observe resolved objects at high \(z\)