IV. The Geometry of the Universe
Curvature and Geometry

- **Flat Geometry**
  - $\Sigma$ of angles of a triangle = 180°
  - Circumference of a circle = $2\pi r$
  - $k=0$, infinite (Flat Universe)

- **Spherical Geometry**
  - $\Sigma$ of angles of a triangle > 180°
  - Circumference of a circle < $2\pi r$
  - $k>0$, finite (Closed Universe)

- **Hyperbolic Geometry**
  - $\Sigma$ of angles of a triangle < 180°
  - Circumference of a circle > $2\pi r$
  - $k < 0$, infinite (Open Universe)
Describing Curvature

- **Start in 2-D flat** (Euclidean) space: $ds^2 = dx^2 + dy^2$ (Cartesian), $ds^2 = dr^2 + r^2 d\Theta^2$ (polar)
- **2-D positively curved** space: $ds^2 = dr^2 + R^2 \sin^2 (r/R) d\Theta^2$
- **2-D negatively curved** space: $ds^2 = dr^2 + R^2 \sinh^2 (r/R) d\Theta^2$
- If homogeneous and isotropic, then curvature must be uniform, and can be characterised by two quantities:
  - curvature constant, $\kappa$: 0 = flat, +1 = positive, -1 = negative
  - radius of curvature, $R$ (dimensions of length)
- **3-D flat**: $ds^2 = dr^2 + r^2 [d\Theta^2 + \sin^2 \Theta d\Phi^2]$
- **3-D positively curved**: $ds^2 = dr^2 + R^2 \sin^2 (r/R) [d\Theta^2 + \sin^2 \Theta d\Phi^2]$
- **3-D negatively curved**: $ds^2 = dr^2 + R^2 \sinh^2 (r/R) [d\Theta^2 + \sin^2 \Theta d\Phi^2]$
3-D Curvature

Another form: \( ds^2 = dr^2 + S_\kappa(r)^2d\Omega^2 \), where

\[ d\Omega^2 = d\Theta^2 + \sin^2\Theta d\Phi^2 \]

and

\[ S_\kappa(r) = \begin{cases} 
\sin(r/R) & \text{for } \kappa = +1 \\
r & \text{for } \kappa = 0 \\
\sinh(r/R) & \text{for } \kappa = -1 
\end{cases} \]

Note that in the limit \( r < R \), \( S_\kappa \approx r \) \( \Rightarrow \) the Universe can look flat on small scales, even if it isn’t...
V. General Relativistic Cosmology
The Robertson-Walker Metric

- Start with Minkowski metric: $ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2$ (flat)
- Robertson/Walker: what if space-time homogeneous, isotropic, allowed to expand or contract as function of time?

$$ds^2 = -c^2 dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

or

$$ds^2 = -c^2 dt^2 + a^2(t) \left[ dr^2 + S_\kappa(r)^2 d\Omega^2 \right]$$

$d\Omega^2 = d\Theta^2 + \sin^2 \Theta d\Phi^2$; \quad $S_\kappa(r) = \begin{cases} r & \text{for } \kappa = 0 \\ R \sinh(r/R) & \text{for } \kappa = -1 \end{cases}$

$\kappa$, $R$: curvature constant/radius
The Einstein Equations

\[ G_{\mu
u} \equiv R_{\mu
u} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \]

- G: Einstein tensor, R: Ricci tensor / scalar, T: energy-momentum tensor
- Up to 10 Einstein equations -- coupled non-linear partial differential equations – but symmetries can simplify things
- For perfect fluids, two independent Einstein equations:

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k c^2}{a^2} = \frac{8\pi G}{3} \rho \quad ; \quad 2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k c^2}{a^2} = -8\pi G \frac{P}{c^2}
\]
After some maths, get same acceleration and fluid equations as with the pseudo-Newtonian method.