III. Newtonian Cosmology
• Construct an imaginary volume $V$ containing mass $M$, use Gauss’ Law:
\[ \oint G \frac{M}{r^2} \hat{r} \cdot dA = < g > A \]
\[ = 4\pi GM ; \quad g = 0, \quad 4\pi GM = 0 (!) \]
• $V$ is arbitrary $\Rightarrow$ $M = 0$ everywhere in the Universe...?
• Assume problem with Newtonian gravity:
  – velocity of a galaxy $v$ at a distance of $r$ from us is only affected by the gravitational pull of matter within $r$
  – Birkhoff’s rule: from GR, OK if $r$ not too large

\[ v = H_0 r \]
Using Birkhoff’s rule, we can estimate the mass within radius $r$:

$$M = \rho_0 \frac{4}{3} \pi r^3$$

As time increases, $r$ also increases, but $M$ affecting galaxy at $r$ remains constant.

Equation of motion for galaxy:

$$E = \frac{1}{2} mv^2 - \frac{GMm}{r} = \text{constant}$$

\[v = H_0 r\]
Critical Density

- Set $E = 0$, and replace velocity at $t_0$ with $v = H_0 r$:

$$\frac{1}{2} mv^2 = \frac{G M m}{r} ; \quad \frac{1}{2} (H_0 r)^2 = \frac{G \rho \frac{4}{3} \pi r^3}{r}$$

$$\rho = \frac{3 H_0^2}{8 \pi G} \equiv \rho_{\text{crit}}$$

- $\rho_{\text{crit}}$ is the density at which the Universe is marginally bound:
  - $\rho < \rho_{\text{crit}} \Rightarrow E > 0$, Universe is unbound and will expand forever
  - $\rho > \rho_{\text{crit}} \Rightarrow E < 0$, Universe is bound and will eventually contract
Friedmann Equation

- Material in sphere has total mass $M = 4\pi \rho r^3/3$, with force
  \[ F = \frac{GMm}{r^2} = \frac{4\pi G\rho rm}{3} \]
- Particle has gravitational potential energy
  \[ V = -\frac{GMm}{r} = -\frac{4\pi G\rho r^2m}{3} \]
- Kinetic energy
  \[ T = \frac{1}{2} m\dot{r}^2 \]
- Energy conservation
  \[ U = T + V \]
Friedmann Equation II

- Substitute:
  \[ U = \frac{1}{2} m \dot{r}^2 - \frac{4\pi}{3} G \rho r^2 m \]

- Universe is homogeneous, can apply to any two particles\( \Rightarrow \) comoving coordinates, \( x \), with scale factor \( a(t) \):

  \[
  \bar{r} = a(t) \bar{x} \\
  \dot{x} = 0; \quad U = \frac{1}{2} m \ddot{a} x^2 - \frac{4\pi}{3} G \rho a^2 x^2 m
  \]

- Multiply each side by \( 2/ma^2x^2 \):

  \[
  \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{a^2} \quad ; \quad k c^2 = -\frac{2U}{mx^2}
  \]
Fluid Equation

- **First Law of Thermodynamics**

\[ dQ = dE + PdV; \quad TdS = dE + PdV \]

\[ E = \frac{4\pi}{3} a^3 \rho c^2 \]

\[ \frac{dE}{dt} = 4\pi a^2 \rho c^2 \frac{da}{dt} + \frac{4\pi}{3} a^3 \frac{d\rho}{dt} c^2 \]

\[ \frac{dV}{dt} = 4\pi a^2 \frac{da}{dt}; \quad dS = 0 \]

(assuming a reversible expansion)

\[ \dot{\rho} + 3 \frac{\dot{a}}{a} \left( \rho + \frac{P}{c^2} \right) = 0 \]
Fluid Equation II

- Fluid Equation:
  \[ \dot{\rho} + 3 \frac{\dot{a}}{a} \left( \rho + \frac{P}{c^2} \right) = 0 \]

- First term in brackets: dilution in density because volume ↑
- Second term in brackets: loss of energy because pressure has done work as Universe’s volume ↑ (energy ➔ gravitational potential energy)
- Only know \( \rho \) if we know \( P \), need \( P = P(\rho) \) – need *Equation of State*
• We have the Friedmann Equation and the Fluid Equation:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8 \pi G}{3} \rho - \frac{kc^2}{a^2} \quad \quad \dot{\rho} + 3 \frac{\dot{a}}{a} \left( \rho + \frac{P}{c^2} \right) = 0
\]

• Differentiate Friedmann Eq. w/rt time, substitute in for \( \rho \cdot \) from Fluid Eq. and cancel factor \( 2a \cdot /a \) in each term:

\[
2 \frac{\dot{a}}{a} = \frac{8 \pi G}{3} \dot{\rho} + 2 \frac{kc^2 \dot{a}}{a^3} \quad \quad \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 = -4 \pi G \left( \rho + \frac{P}{c^2} \right) + \frac{k c^2}{a^2}
\]

• Using Friedmann Eq. again, we get the acceleration equation

\[
\frac{\dddot{a}}{a} = -\frac{4 \pi G}{3} \left( \rho + \frac{3P}{c^2} \right)
\]
The acceleration equation:

\[
\frac{\ddot{a}}{a} = - \frac{4\pi G}{3} \left( \rho + \frac{3P}{c^2} \right)
\]

- If material has any pressure, this increases the gravitational force and thus decelerates the expansion even more!
- Note that in an isotropic Universe, there are no pressure gradients, hence no forces associated with pressure.