

Folklore proposition with string proof

In any bicategory, if  $\varepsilon : f u \Rightarrow 1$  is a counit for an adjunction  $f \dashv u$  and there exists an invertible 2-cell  $\theta : f u \Rightarrow 1$  then  $\varepsilon$  is invertible.

Proof.  $\varepsilon : f u \Rightarrow 1$  and the unit is  $u \dashv f$  so that

$$(i) \quad u \dashv = |^u \quad \text{and} \quad (ii) \quad f | = f \dashv$$

$\theta : \begin{array}{c} \cup \\ \ominus \end{array}$  and inverse  $\begin{array}{c} \oplus \\ \cap \end{array}$  so that

$$(iii) \quad \begin{array}{c} \oplus \\ \cap \end{array} = \emptyset \quad \text{and} \quad (iv) \quad \begin{array}{c} \cup \\ \ominus \end{array} = \begin{array}{c} f \\ | \\ | \\ u \end{array}$$

Put  $\tau : \begin{array}{c} \oplus \\ \cap \end{array}$  and calculate

$$\begin{array}{c} \oplus \\ \cap \end{array} \stackrel{(i)}{=} \begin{array}{c} \oplus \\ \cap \end{array} \stackrel{(iii)}{=} \emptyset \quad \text{so that} \quad \varepsilon \tau = 1;$$

and

$$\begin{array}{c} \oplus \\ \cap \end{array} = \begin{array}{c} \oplus \\ \cap \end{array} \stackrel{(iv)}{=} \begin{array}{c} \oplus \\ \cap \end{array} \stackrel{(ii)}{=} \begin{array}{c} \oplus \\ \cap \end{array}$$

so that  $\tau \varepsilon \theta^{-1} = \theta^{-1}$ . That is,  $\tau = \varepsilon^{-1}$ .  $\square$

Rider: (i) & (iii)  $\Rightarrow \varepsilon$  split epi

(ii) & (iv)  $\Rightarrow \varepsilon$  split mono