The 3-cocycle condition
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This is a slightly extended version of my handwritten note [S]. The purpose is to show that the 3-cocycle condition in the cohomology of groups comes straight from the pasting geometry of higher-order category theory: the fact that a particular cube can be assembled at all, with appropriate source and target compatibility, leads to the equation automatically (or should I say, chaotically).

The chaotic category \( X_c \) on a set \( X \) has the elements of \( X \) as objects and \( X_c(x,y) = 1 \) for all \( x, y \in X \). This gives a functor \((-)_c: \text{Set} \rightarrow \text{Cat} \) with a left adjoint. Hence (by the general theory of monoidal functors [EK], or otherwise) we obtain a functor \((-)_c: \text{Cat} \rightarrow \text{2-Cat} \) with a left adjoint; for a category \( C \), the 2-category \( C_c \) has underlying category \( C \) and \( C_c(d,e) = C(d,e)_c \). Since \((-)_c: \text{Cat} \rightarrow \text{2-Cat} \) has a left adjoint, it preserves products and so takes groups in \( \text{Cat} \) into groups in \( \text{2-Cat} \).

Recall that groups in \( \text{Cat} \) all arise from crossed modules (see [BS] for the history). Suppose \( \partial: N \rightarrow E \) is a group homomorphism (in \( \text{Set} \)) and \( \cdot: E \times N \rightarrow N \) is an action satisfying the crossed module properties

\[
\partial(e \cdot n) = e \partial(n) e^{-1}, \quad \partial(n) \cdot m = n m n^{-1}.
\]

The corresponding group \( C \) in \( \text{Cat} \) is described as follows:

- objects of \( C \) are elements \( e \) of \( E \);
- morphisms \( n: e \rightarrow e' \) have \( n \in N \) with \( e = \partial(n) e' \);
- composition is multiplication in \( N \);
- multiplication \( \otimes : C \times C \rightarrow C \) is defined by

\[
\left( d \xrightarrow{m} d', e \xrightarrow{n} e' \right) \mapsto \left( d e \xrightarrow{m(d' n)} d' e' \right).
\]

Notice that two morphisms \( n, q: e \rightarrow e' \) in \( C \) must have \( \partial(n) e' = e = \partial(q) e' \) so that \( \partial(n) = \partial(q) \). This means that we can regard 2-cells in the monoidal 2-category \( C_c \) as diagrams

\[
\begin{array}{ccc}
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\end{array}
\end{array}
\]

where \( n = a \cdot q \) with \( a \) in the kernel \( A \) of \( \partial: N \rightarrow E \). Notice that \( a \cdot n = n \cdot a \) for \( a \in A \) and \( n \in N \) (since \( a \cdot n^{-1} = \partial(a) \cdot n = 1 \cdot n = n \)). Vertical and horizontal composition in \( C_c \) are given by multiplication in \( A \).

Let \( G \) denote the cokernel of \( \partial: N \rightarrow E \) so that there is an exact sequence

\[
1 \rightarrow A \rightarrow N \xrightarrow{\partial} E \xrightarrow{\pi} G \rightarrow 1
\]

of groups (in \( \text{Set} \)). Since \( \pi \) is surjective, we can choose \( \sigma(x) \in E \) with \( \pi(\sigma(x)) = x \) for all...
Since $\sigma(x)\sigma(y)$ and $\sigma(x \cdot y)$ are in the same fibre of $\pi$, we can choose $\tau(x,y) \in N$ with $\sigma(x \cdot y) = \partial(\tau(x,y)) \sigma(x)\sigma(y)$.

This gives morphisms

$$\tau(x,y) : \sigma(x \cdot y) \longrightarrow \sigma(x)\sigma(y)$$

in $C$ for all $x, y \in G$.

For all $x, y, z \in G$, we can define $\lambda(x,y,z) \in A$ to be the unique 2-cell in $C_c$ fitting in the square below.

$$
\begin{array}{c}
\sigma(x \cdot y \cdot z) \\
\tau(x \cdot y \cdot z)
\end{array}
\begin{array}{c}
\sigma(x \cdot y) \sigma(z) \\
\sigma(x) \sigma(y) \sigma(z)
\end{array}
\begin{array}{c}
\tau(x, y \cdot z) \\
\tau(x, y \cdot z)
\end{array}
\begin{array}{c}
\lambda(x, y, z) \\
\lambda(x, y, z)
\end{array}
\begin{array}{c}
\sigma(x) \sigma(y) \sigma(z) \\
\sigma(x) \sigma(y) \sigma(z)
\end{array}
$$

Consequently, by mere source and target requirements (because of local chaos), there is a commutative cube

$$
\begin{array}{c}
\sigma(u \cdot x \cdot y) \sigma(z) \\
\tau(u \cdot x, y \cdot z)
\end{array}
\begin{array}{c}
\sigma(u \cdot x) \sigma(y) \sigma(z) \\
\sigma(u) \sigma(x) \sigma(y) \sigma(z)
\end{array}
\begin{array}{c}
\tau(u, x \cdot y \cdot z) \\
\tau(u, x \cdot y \cdot z)
\end{array}
\begin{array}{c}
\lambda(u, x, y \cdot z) \\
\lambda(u, x, y \cdot z)
\end{array}
\begin{array}{c}
\sigma(u) \sigma(x) \sigma(y) \sigma(z) \\
\sigma(u) \sigma(x) \sigma(y) \sigma(z)
\end{array}
$$

This gives the usual equation

$$\lambda(u, x, y \cdot z) \lambda(u \cdot x, y \cdot z) (u \cdot \lambda(x, y, z)) \lambda(u, x \cdot y) \lambda(u, x, y) = (u \cdot \lambda(x, y, z)) \lambda(u, x \cdot y) \lambda(u, x, y)$$

for a 3-cocycle $\lambda : G^3 \longrightarrow A$ on $G$ with coefficients in $A$. 

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References


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