The 3-cocycle condition

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3 October 2001

This is a slightly extended version of my handwritten note [S]. The purpose is to show that the 3-cocycle condition in the cohomology of groups comes straight from the pasting geometry of higher-order category theory: the fact that a particular cube can be assembled at all, with appropriate source and target compatibility, leads to the equation automatically (or should I say, chaotically).

The *chaotic category* X_c *on a set* X has the elements of X as objects and $X_c(x, y) = 1$ for all $x, y \in X$. This gives a functor $(-)_c : \mathbf{Set} \longrightarrow \mathbf{Cat}$ with a left adjoint. Hence (by the general theory of monoidal functors [EK], or otherwise) we obtain a functor $(-)_c : \mathbf{Cat} \longrightarrow 2\text{-}\mathbf{Cat}$ with a left adjoint; for a category $\mathcal{C}$, the 2-category $\mathcal{C}_c$ has underlying category $\mathcal{C}$ and $\mathcal{C}_c(d, e) = \mathcal{C}(d, e)_c$. Since $(-)_c : \mathbf{Cat} \longrightarrow 2\text{-}\mathbf{Cat}$ has a left adjoint, it preserves products and so takes groups in $\mathbf{Cat}$ into groups in $2\text{-}\mathbf{Cat}$.

Recall that groups in $\mathbf{Cat}$ all arise from *crossed modules* (see [BS] for the history). Suppose $\partial : N \longrightarrow E$ is a group homomorphism (in $\mathbf{Set}$) and $\bullet : E \times N \longrightarrow N$ is an action satisfying the crossed module properties

$$\partial(e \bullet n) = e\,\partial(n)\,e^{-1}, \qquad \partial(n) \bullet m = n\,m\,n^{-1}.$$

The corresponding group $\mathcal{C}$ in $\mathbf{Cat}$ is described as follows:

objects of $\mathcal{C}$ are elements e of E ;
morphisms $n : e \longrightarrow e'$ have $n \in N$ with $e = \partial(n)\,e'$;
composition is multiplication in N ;
multiplication $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is defined by

$$\left(d \xrightarrow{m} d',\; e \xrightarrow{n} e'\right) \longmapsto \left(de \xrightarrow{m(d' \bullet n)} d'e'\right).$$

Notice that two morphisms $n, q : e \longrightarrow e'$ in $\mathcal{C}$ must have $\partial(n)\,e' = e = \partial(q)\,e'$ so that $\partial(n) = \partial(q)$. This means that we can regard 2-cells in the monoidal 2-category $\mathcal{C}_c$ as diagrams

$$\begin{tikzcd} e \arrow[r, bend left, "n"] \arrow[r, bend right, "q"'] \arrow[r, phantom, "\downarrow a"] & e' \end{tikzcd}$$

where $n = a\,q$ with a in *the kernel* A *of* $\partial : N \longrightarrow E$. Notice that $a\,n = n\,a$ for $a \in A$ and $n \in N$ (since $a\,n\,a^{-1} = \partial(a) \bullet n = 1 \bullet n = n$). Vertical and horizontal composition in $\mathcal{C}_c$ are given by multiplication in A.

Let G denote *the cokernel of* $\partial : N \longrightarrow E$ so that there is an exact sequence

$$1 \longrightarrow A \longrightarrow N \xrightarrow{\partial} E \xrightarrow{\pi} G \longrightarrow 1$$

of groups (in $\mathbf{Set}$). Since π is surjective, we can choose $\sigma(x) \in E$ with $\pi(\sigma(x)) = x$ for all

$x \in G$. Since $\sigma(x)\sigma(y)$ and $\sigma(x\,y)$ are in the same fibre of π, we can choose $\tau(x,y) \in N$ with

$$\sigma(x\,y) = \partial(\tau(x,y))\,\sigma(x)\sigma(y).$$

This gives morphisms

$$\tau(x,y) : \sigma(x\,y) \longrightarrow \sigma(x)\sigma(y)$$

in $\mathcal{C}$ for all $x, y \in G$.

For all $x, y, z \in G$, we can define $\lambda(x,y,z) \in A$ to be the unique 2-cell in $\mathcal{C}_c$ fitting in the square below.

$$\begin{tikzcd}
\sigma(x\,y\,z) \arrow[r, "\tau(x\,y,\, z)"] \arrow[d, "\tau(x,\, y\,z)"'] \arrow[dr, phantom, "\Downarrow \lambda(x,\, y,\, z)"] & \sigma(x\,y)\,\sigma(z) \arrow[d, "\tau(x,\, y) \otimes \sigma(z)"] \\
\sigma(x)\,\sigma(y\,z) \arrow[r, "\sigma(x) \otimes \tau(y,\, z)"'] & \sigma(x)\,\sigma(y)\,\sigma(z)
\end{tikzcd}$$

Consequently, by mere source and target requirements (because of local chaos), there is a commutative cube

$$\begin{tikzcd}
 & \sigma(u\,x\,y)\,\sigma(z) \arrow[rr, "\tau(ux,y)\otimes\sigma(z)"] \arrow[d, phantom, "\Downarrow \lambda(ux,y,z)"] & & \sigma(u\,x)\,\sigma(y)\,\sigma(z) \arrow[dr, "\tau(u,x)\otimes(\sigma(y)\sigma(z))"] & \\
\sigma(u\,x\,y\,z) \arrow[ur, "\tau(uxy,z)"] \arrow[rr, "\tau(ux,yz)"] \arrow[dr, "\tau(u,xyz)"'] & {} \arrow[d, phantom, "\Downarrow \lambda(u,x,yz)"] & \sigma(u\,x)\,\sigma(y\,z) \arrow[ur, "\sigma(ux)\otimes\tau(y,z)"'] \arrow[dr, "\tau(u,x)\otimes\sigma(yz)"] & & \sigma(u)\,\sigma(x)\,\sigma(y)\,\sigma(z) \\
 & \sigma(u)\,\sigma(x\,y\,z) \arrow[rr, "\sigma(u)\otimes\tau(x,yz)"'] & & \sigma(u)\,\sigma(x)\,\sigma(y\,z) \arrow[ur, "(\sigma(u)\sigma(x))\otimes\tau(y,z)"'] &
\end{tikzcd}$$

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$$\begin{tikzcd}
 & \sigma(u\,x\,y)\,\sigma(z) \arrow[rr, "\tau(ux,y)\otimes\sigma(z)"] \arrow[dr, "\tau(u,xy)\otimes\sigma(z)"] & & \sigma(u\,x)\,\sigma(y)\,\sigma(z) \arrow[dr, "\tau(u,x)\otimes(\sigma(y)\sigma(z))"] & \\
\sigma(u\,x\,y\,z) \arrow[ur, "\tau(uxy,z)"] \arrow[dr, "\tau(u,xyz)"'] \arrow[r, phantom, "\Downarrow \lambda(u,xy,z)"] & {} & \sigma(u\,)\,\sigma(x\,y)\,\sigma(z) \arrow[rr, "(\sigma(u)\otimes\tau(x,y))\otimes\sigma(z)"] & {} \arrow[u, phantom, "\Downarrow \lambda(u,x,y)\otimes\sigma(z)"] \arrow[d, phantom, "\Downarrow \sigma(u)\otimes\lambda(x,y,z)"] & \sigma(u)\,\sigma(x)\,\sigma(y)\,\sigma(z) \\
 & \sigma(u)\,\sigma(x\,y\,z) \arrow[rr, "\sigma(u)\otimes\tau(x,yz)"'] \arrow[ur, "\sigma(u)\otimes\tau(xy,z)"] & & \sigma(u)\,\sigma(x)\,\sigma(y\,z) \arrow[ur, "(\sigma(u)\sigma(x))\otimes\tau(y,z)"'] &
\end{tikzcd}$$

This gives the usual equation

$$\lambda(u,x,yz)\,\lambda(ux,y,z) = (u \bullet \lambda(x,y,z))\,\lambda(u,xy,z)\,\lambda(u,x,y)$$

for a 3-cocycle $\lambda : G^3 \longrightarrow A$ on G with coefficients in A.

References

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[EK] S. Eilenberg and G.M. Kelly, Closed categories, *Proceedings of the Conference on Categorical Algebra at La Jolla* (Springer, 1966) 421-562.

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