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Weak omega-categories

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This paper proposes to define a weak higher-dimensional category to be a simplicial set satisfying properties. The definition is a refinement of that suggested at the end of [St3] which required extra structure on the simplicial set.

The paper [St3] constructed the simplicial nerve of a (strict) ω-category. The principal aim of the paper was to prove[1] that the construction was right adjoint to the functor which assigned, to the n-simplex, the free n-category thereon (called the n-th *oriental*). However, I also provided emperical evidence for a precise conjecture characterizing nerves of ω-categories. Roberts [Rs] had distinguished certain simplicial sets with extra structure that he called *complicial sets.* Verity and I have been using the adjective *stratified* for a simplicial set with certain elements distinguished, and called *thin*[2], such that no elements of dimension 0 are thin and all degenerate elements are thin. For nerves of strict ω-categories, the thin elements of the nerve are the "commutative simplexes". Without actually having defined the nerve, Roberts achievement was to recognize which unique horn filler conditions might possibly characterize a stratified simplicial set as isomorphic to the nerve of an ω-category: Roberts' conjecture was that his category of complicial sets was equivalent to the category of ω-categories.

In [St3] I made explicit a larger class of unique horn filler conditions and conjectured that the nerve I had constructed provided the equivalence between the category of ω-categories and a category of stratified simplicial sets satisfying these stronger conditions. I still expected that Roberts' conditions would be enough.

Right at the very end of [St3] I recorded my recognition that it should be possible to define the nerve of what were coined there *weak ω-categories.* However, these simplicial sets should be stratified by taking the thin elements to be those simplexes which "commute up to weak equivalence". My main idea was that the same horn filler conditions should still be satisfied but without uniqueness[3].

In [St4] I proved that the nerve of a strict ω-category as constructed in [St3] satisfied my more general unique horn filler conditions, and hence, also those of Roberts.

[1] I must admit a small correction is needed to one lemma in this proof; see [St5].

[2] Roberts originally used the term "neutral" but later suggested "hollow" to me instead, and I used that in [St3]. I am happy to use the terminology "thin" that M.K. Dakin [D] used for the nerves of (strict) ω-groupoids (although his School calls them "∞-groupoids").

[3] Peter Freyd used the terminology "weak limit" for a cone which satisfies the existence property of a limit cone but not necessarily the uniqueness property. He also used "prelimit" for uniqueness without existence. Unique horn filler conditions are limit-like in a sense that can be made precise.

With these solid theorems as base and some reasonable conjectures, I assigned projects in this subject to my graduate student Michael Zaks. We outlined a strategy for proving the Roberts conjecture. At the CT 90 Conference in Como, Italy, Wesley Phoa introduced me to Martin Hyland's student Dominic Verity who said he was interested in proving the conjecture. During the CT 91 Conference at McGill University, Wesley brought me a handwritten manuscript from Dominic which proved the full faithfulness of a purported equivalence between ω-categories and complicial sets; he had independently come up with some of our strategies and pushed them further than Zaks and I had. At that same conference, Bob Gordon, John Power and I planned the tricategories paper [GPS] to publish an explicit definition of weak 3-category and to prove a coherence theorem.

The article [St6], completed in November 1992, is an attempt to make some of the ideas of higher category theory more accessible. In particular, I define the nerve $N(A)$ of a bicategory A to be the simplicial set whose elements of dimension n are normal[4] lax functors from the ordinal $[n] = \{0, 1, \ldots, n\}$ to A (although we need to reverse the 2-cells in A to meet the "odds to evens" convention of [St3]). Recently Jack Duskin (see [Dus1] and [Dus2]) has documented the results of his detailed examination of the nerves of bicategories. His nerve is defined by taking coskeletons to obtain the elements of dimension 4 and higher which side-steps a basic (for me) coherence question of why any simplex with commutative 3-faces in a bicategory is commutative. This is fine for producing a characterization of nerves of bicategories; however, if we could obtain higher coskeletalness from the weak ω-category axioms on a simplicial set, it would provide one strong test of those axioms.

After completing a Cambridge University PhD thesis, which involved higher categories but not the Roberts conjecture, Dominic Verity took up an appointment at Macquarie University. He proved the Roberts conjecture in July 1993, exposing his work in seminars; however, the written version [Vy] has still not appeared in full. At the end of 1993 we were joined at Macquarie by Todd Trimble whose interest was aroused by the connection between my orientals and the Stasheff associahedra. Todd made the connection with operads and tried to apply them to obtain a definition of weak ω-category.

In November 1995, John Baez and James Dolan sent me an email explaining their definition of weak n-category which was motivated to some extent by my suggestion at the end of [St3]. They moved away from simplexes to other pasting diagrams called *opetopes* defined using operads. In February 1996, Michael Batanin moved to Macquarie and before the end of the year had his own definition of weak ω-category based on the construction of the free strict ω-category on a globular set (or ω-graph) and a higher-dimensional notion of operad. Other definitions of weak n-category have also appeared.

Up to this point, for several reasons, I have not tried to promote my attempted definition of weak ω-category as in [St3]. First of all, it is certainly not quite correct as it stands in that paper. Secondly, I always believed that in the weak case the thin elements should be

[4] Normal here meaning preserving identity morphisms strictly.

determined by the simplicial set itself, unlike the strict case. Finally, I was convinced that the attempt to restrict the types of diagrams in the nerve to be simplexes was rather constraining, and that the Baez-Dolan definition was a step forward from that idea.

However, now I am not convinced by the last reason. Simplicial sets are lovely objects about which algebraic topologists know a lot. If something is described as a simplicial set, it is ready to be absorbed into topology. Or, in other words, no matter which definition of weak ω-category eventually becomes dominant, it will be valuable to know its simplicial nerve. So, prompted by the appearance of [Lr], which takes the definition of [St3] seriously and corrects inaccuracies in my account, I am ready to describe my more recent thinking on correcting my attempted definition.

As usual, write $\mathbf{\Delta}$ for the topologists' simplicial category whose objects are the non-empty ordinals $[n] = \{0, 1, \ldots, n\}$. A simplicial set is a functor $X : \mathbf{\Delta}^{op} \longrightarrow \mathbf{Set}$; we put $X_n = X[n]$ and write $x\xi$ for $X(\xi)(x) \in X_n$ where $x \in X_m$ and $\xi : [n] \longrightarrow [m]$ in $\mathbf{\Delta}$. We write $\partial_k : [n-1] \longrightarrow [n]$ for the monomorphism in $\mathbf{\Delta}$ whose image $\mathrm{im}\partial_k$ does not contain $k \in [n]$ and write $\sigma_k : [n] \longrightarrow [n-1]$ for the epimorphism in $\mathbf{\Delta}$ which identifies only k and $k+1$. We call $x\partial_k$ the k-*face* of an element x of X; these are the *codimension 1 faces* of x. The category of simplicial sets is denoted by $\mathbf{Ss}$; it is the category $[\mathbf{\Delta}^{op}, \mathbf{Set}]$ of presheaves on $\mathbf{\Delta}$; the morphisms are natural transformations, called *simplicial maps*.

A *stratification* t of a simplicial set X is a choice of subset t_nX of X_n for all $n > 0$ such that t_nX contains all the degenerate[5] elements of X_n. A stratification is called m-*trivial* when $t_nX = X_n$ for all $n > m$. The category of stratified simplicial sets is denoted by $\mathbf{Sss}$; the morphisms are simplicial maps which preserve thinness, called *stratified simplicial maps*.

Proposition *The category* $\mathbf{Sss}$ *is a quasi-topos in the sense of* [Pn] *(that is, each slice category is cartesian closed and there is a regular subobject classifier).*

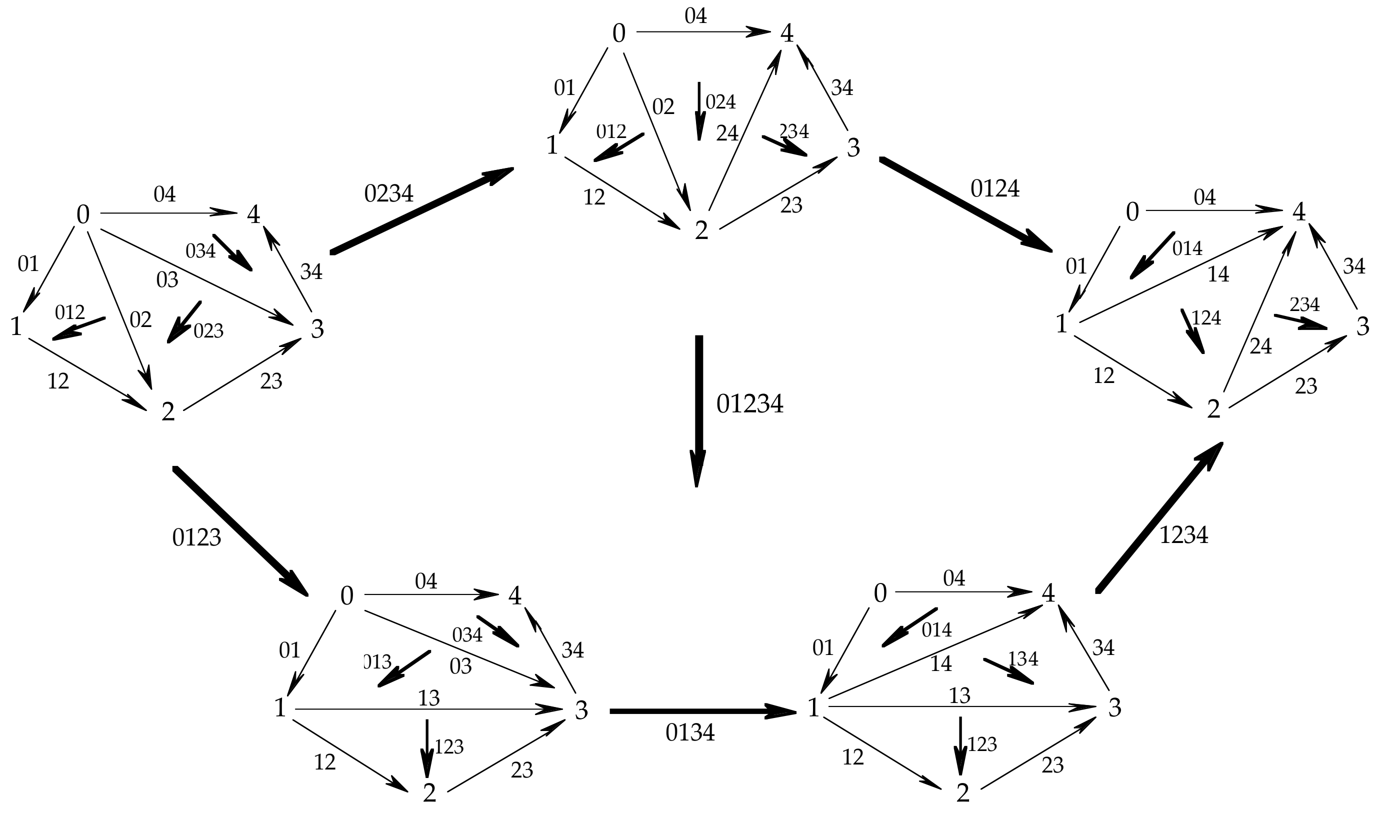
Proof Let $\mathbf{s\Delta}$ be the category obtained from $\mathbf{\Delta}$ by adjoining extra objects $[n]_s$ for $n > 0$, and morphisms $\tau : [n] \longrightarrow [n]_s$, $\sigma'_k : [n]_s \longrightarrow [n-1]$ such that $\sigma_k = \sigma'_k \tau$; there is a full inclusion $\iota : \mathbf{\Delta} \longrightarrow \mathbf{s\Delta}$ taking $[n]$ to $[n]$. Then $\mathbf{Sss}$ is the full subcategory of the presheaf category $[\mathbf{s\Delta}^{op}, \mathbf{Set}]$ on $\mathbf{s\Delta}$ consisting of those functors $X : \mathbf{s\Delta}^{op} \longrightarrow \mathbf{Set}$ which take each $\tau : [n] \longrightarrow [n]_s$ to a subset inclusion $X(\tau) : sX_n \hookrightarrow X_n$. The subcategory inclusion $\mathbf{Sss} \longrightarrow [\mathbf{s\Delta}^{op}, \mathbf{Set}]$ has a left adjoint L which takes each $Z : \mathbf{s\Delta}^{op} \longrightarrow \mathbf{Set}$ to the simplicial set $Z \circ \iota$ stratified by taking as thin elements those in the images of the functions $Z(\tau)$. It is easy to see that L preserves pullbacks of pairs of morphisms into objects in the subcategory $\mathbf{Sss}$. The result now follows using results of [St1; Section 7]. **q.e.d.**

Recall that the n-*simplex* $\mathbf{\Delta}[n]$ is the simplicial set defined by the representable functor $\mathbf{\Delta}(-, [n]) : \mathbf{\Delta}^{op} \longrightarrow \mathbf{Set}$. The k-*horn* $\Lambda^k[n]$ in $\mathbf{\Delta}[n]$ is the simplicial subset of $\mathbf{\Delta}[n]$ consisting

[5] Degenerate elements are those of the form $x\varepsilon$ for a non-identity epimorphism ε in $\mathbf{\Delta}$.

of those $\xi : [m] \longrightarrow [n]$ for which there exists $i \in [n]$, $i \neq k$, with i not in the image of ξ; that is, ξ factors through some ∂_i with $i \neq k$. A k-*horn of dimension* n in a simplicial set X is a simplicial map $h : \Lambda^k[n] \longrightarrow X$; sometimes the k-horn is identified with the list $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ of elements of X_{n-1} obtained as the values of the simplicial map h at the elements $\partial_0, \partial_1, \ldots, \partial_{k-1}, \partial_{k+1}, \ldots, \partial_n$ of $\Lambda^k[n]_{n-1}$. A k-horn of dimension n is called *inner* when $0 < k < n$ and *outer* when $k = 0$ or $k = n$. A *filler* for a k-horn in X is an element of X_n whose corresponding simplicial map $\mathbf{\Delta}[n] \longrightarrow X$ (under the Yoneda Lemma) restricts to the given horn $\Lambda^k[n] \longrightarrow X$.

For $0 \leq k \leq n$, we define the *Roberts stratification* r^k of $\mathbf{\Delta}[n]$ by taking the non-degenerate elements of $r^k_m\mathbf{\Delta}[n]$ to be the monomorphisms $\mu : [m] \longrightarrow [n]$ whose image contains $\{k-1, k, k+1\} \cap [n]$. This induces a stratification r^k of the k-horn $\Lambda^k[n]$ whose thin elements are those elements of $\Lambda^k[n]$ that are thin in the Roberts stratification of $\mathbf{\Delta}[n]$. A k-horn in a stratified simplicial set X is called *complicial* when it is a stratified simplicial map $\Lambda^k[n] \longrightarrow X$ where $\Lambda^k[n]$ has the Roberts stratification.



The fourth oriental of [St3]

In order to make some sense of the Roberts stratification, consider the fourth oriental pictured above. The arrows are all labelled by subsets of the ordinal [4]. The way we picture an element $x \in X_4$ of a stratified simplicial set X is to imagine, at the arrow labelled by the subset S of [4], the label $x\mu$ where μ is the monomorphism into [4] whose image is the complement of S; in particular, 01234 is replaced by x itself. Similarly, if we have a complicial 1-horn $x_0, x_2, x_3, x_4 \in X_3$ of dimension 4, we put these elements respectively at positions $1234, 0134, 0124, 0123$ in the fourth oriental as if we had $x_i = x\,\partial_i$, and then we label the lower-dimensional faces similarly. The positions S corresponding to the $x\mu$ with μ in the Roberts stratification are exactly those S which contain 0, 1 and 2. Mark those positions on the fourth oriental and regard the corresponding $x\mu$ as an identity arrow. In particular, position 01234 is regarded as an identity and the whole diagram can be regarded as an equation for determining x_1 given the complicial horn. If we were in an n-category, we could indeed solve the equation for x_1.

A *complicial set* (in the sense of Roberts) is a stratified simplicial set X satisfying the following conditions:

- (o) every thin element of dimension 1 is degenerate;
- (i)! every complicial horn has a unique thin filler;
- (ii) if a thin filler of a complicial horn has all but one of its codimension 1 faces known to be thin then the remaining codimension 1 face is thin.

Let **Cs** denote the full subcategory of **Sss** consisting of the complicial sets. It is proved in [St4] that the nerve of an ω-category, stratified by the commutative simplexes as thin, is a complicial set. The author also proved in [St2] that the full subcategory of **Cs** consisting of the 2-trivial complicial sets is equivalent to the category 2-**Cat** of 2-categories and 2-functors. Moreover, Roberts and I proved a variety of properties of complicial sets including some other characterizations (see [Rs] and [St2]); in particular, it is worth mentioning here the easy fact that inner horns suffice in condition (i)!.

Theorem (Verity [Vy]) **Cs** *is equivalent to the category* ω-**Cat** *of* ω-*categories and* ω-*functors.*

Let us define a *weak complicial set* to be a stratified simplicial set satisfying the two conditions:

- (i) every complicial horn has a thin filler;
- (ii) if a thin filler of a complicial horn has all but one of its codimension 1 faces known to be thin then the remaining codimension 1 face is thin.

Notice that a 0-trivial weak complicial set is precisely a *Kan complex:* that is, a simplicial set in which each horn has a filler.

For each m-trivial stratified simplicial set X, we shall now construct an (m+1)-trivial stratified simplicial set sX whose underlying simplicial set is the same as for X. The thin elements of dimension $n \leq m$ in sX are the same as those of X. An element x of dimension $m+1$ in sX is thin when there exists an m-trivial stratified simplicial subset of X which is a weak complicial set and contains the element x.

This allows us to build up what we call the *equivalence* stratification of any simplicial set

X. Begin with X and its 0-trivial stratification and iterate the construction s. An element x of dimension n in X is thin for the equivalence stratification when it is thin in $s^n X$.

Definition A *weak* ω-*category* is a simplicial set which is weak complicial when equipped with the equivalence stratification.

Consider the nerve $N(A)$ of a category A. Let G denote the subcategory of A consisting of all the objects yet only the invertible morphisms. The nerve $N(G)$ of G is the maximum simplicial subset of $N(A)$ which is a Kan complex. From this we see that the 1-dimensional thin elements are the invertible ones. The stratified simplicial set $sN(A)$ is a weak complicial set, so that the equivalence stratification of $N(A)$ is 1-trivial. Therefore, $N(A)$ is indeed a 1-trivial weak ω-category.

Consider the nerve $N(A)$ of a bicategory A. Let G denote the subbicategory of A consisting of all the objects, all the equivalence morphisms, and all the invertible 2-cells. Let H denote the subbicategory with the same objects and morphisms as A but only the invertible 2-cells. The nerve of G is the maximum simplicial subset of $N(A)$ which is a Kan complex. The nerve of H is the maximum 1-trivial stratified simplicial subset of $sN(A)$ which is a weak complicial set. Furthermore, $s^2N(A)$ is a weak complicial set. Therefore, the nerve $N(A)$ of A is a 2-trivial weak ω-category.

There are many questions. The first is whether, in any weak ω-category, every *admissible* horn (see [St3], [St4] and [Lr]) has a thin filler. If not, perhaps my admissible horns (including the outer ones) should be used in the definition of weak ω-category in place of Roberts' complicial horns. I used the complicial horns here because there are fewer of them and so, in principle, make the conditions easier to verify.

Another question is how we obtain from these ideas a definition of *weak* n-*category* for finite n. Certainly a weak n-category should be an n-trivial weak ω-category, however, some further uniqueness restriction needs to be imposed. I agree with Leinster's suggestion in [Lr] to ask for uniqueness of the filler in condition (i) for horns of dimension greater than n. Yet this may not be enough, as Duskin has pointed out. Presumably the requirement that the simplicial set should be an n-*dimensional Postnikov complex* (in the sense of [Dus1]) would suffice, yet it is a pity for this not to come as a consequence of horn-filler conditions.

The category **Cs** of complicial sets is cartesian closed using Verity's theorem since ω-**Cat** is cartesian closed (as proved in [St3] for example). Also, the category of Kan complexes is cartesian closed. I am tempted to conjecture that the category ω-**wCat** of weak ω-categories is also cartesian closed but have not proved it. Quite possibly the quasi-topos **Sss** supports a relevant Quillen model homotopy structure.

As a starting point for comparison with the more globular notions of weak ω-category (see [Lr]), we point out the important functor Cell : **Ss** $\longrightarrow$ **Glob**, where **Glob** denotes the category of globular sets (or ω-graphs). This construction appears in [Rs], [St2] and [St3; page 330]. For any simplicial set X, we put

$$\mathrm{Cell}_n(X) = \{ x \in X_n \mid x\,\partial_i = x\,\partial_i\,\partial_j\,\sigma_j \ \text{ for } \ j+1 < i \}$$

with the source and target functions s and t : $\mathrm{Cell}_n(X) \longrightarrow \mathrm{Cell}_{n-1}(X)$ induced by ∂_0 and ∂_1. This is clearly functorial in $X \in$ **Ss**. In particular, this means that each of our weak ω-categories X has an underlying globular set Cell(X).

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