This is a slightly extended version of my handwritten note [S] which makes no claim to originality. The main result was obtained by Giraud [G] and later by Conduché [C]. The problem addressed is that of characterizing the powerful (or "exponentiable") morphisms in the category \textbf{Cat} of (small) categories: that is, those functors \( p : E \to B \) for which the functor \( p^* : \text{Cat}/B \to \text{Cat}/E \), given by pulling back along \( p \), has a right adjoint. The reason for the name is that \( p \) is powerful if and only if raising to the power \( p \) exists in the full slice category \( \text{Cat}/B \) (that is, the cartesian internal hom \((A,u)^{(E,p)}\) exists for all objects \((A,u)\) of \( \text{Cat}/B \)).

We write \( \text{Mod} \) for the bicategory whose objects are (small) categories and for which the hom category \( \text{Mod}(A,B) \) is the functor category \([B^{\text{op}} \times A, \text{Set}]\). The morphisms of \( \text{Mod} \) are called \textit{modules} while the 2-cells are called \textit{module morphisms}. Composition of modules is given by the usual coend formula. We identify \( \text{Cat} \) as a sub-2-category of the bicategory \( \text{Mod} \) by thinking of a functor \( f : A \to B \) as the module defined by taking \( f(b,a) \) to be \( B(b,f(a)) \).

For any functor \( p : E \to B \), the \textit{fibre} over an object \( b \) of \( B \) is the subcategory \( E_b \) of \( E \) given by the pullback

\[
\begin{array}{ccc}
E_b & \to & E \\
\downarrow & & \downarrow p \\
1 & \to & B \\
b & \downarrow & \\
\end{array}
\]

Each \( \beta : b \to b' \) determines a module \( m_E(\beta) : E_{b'} \to E_b \) defined by

\[
m_E(\beta)(e,e') = \{ \xi : e \to e' \mid p(\xi) = \beta \}
\]

for objects \( e \) of \( E_b \) and \( e' \) of \( E_{b'} \). Notice immediately that \( m_E(1_b) \) is the identity module of \( E_b \) (that is, the hom-functor \( E_b(-, -) : E_b^{\text{op}} \times E_b \to \text{Set} \)), and yet, for each composable pair of morphisms \( \beta : b \to b' \) and \( \beta' : b' \to b'' \) in \( B \), we only have a module morphism

\[
\mu_{\beta,\beta'} : m_E(\beta) \otimes m_E(\beta') \to m_E(\beta''),
\]

which is induced by the composition functions \( E(e',e'') \times E(e,e') \to E(e,e'') \). In fact, we have defined a normal lax functor\(^1\)

\[
m_E : B^{\text{op}} \to \text{Mod}.
\]

Write \( \text{Cat}/B \) for the usual slice category of objects \( p : E \to B \) of \( \text{Cat} \) over \( B \) in which the morphisms \( f : (E,p) \to (F,q) \) are commutative triangles over \( B \); however, we enrich \( \text{Cat}/B \) to become a 2-category by accepting those 2-cells \( \theta : f \Rightarrow g : (E,p) \to (F,q) \) satisfying \( q \theta = p \). Write \( \text{Bicat}(B^{\text{op}}, \text{Mod}) \) for the bicategory of lax functors \( B^{\text{op}} \to \text{Mod} \), lax

\(^1\)Lax functors are Bénabou’s "morphisms of bicategories" while here "normal" means strictly identity preserving.
Proposition (Bénabou [B]) The slice 2-category \( \text{Cat}/B \) is equivalent to the sub-2-category of the bicategory \( \text{Bicat}(B^{\text{op}}, \text{Mod}) \) whose objects are the normal lax functors, whose morphisms are the lax transformations with components at objects \( b \) of \( B \) being actual functors, and whose 2-cells are all the modifications.

Proof (sketch) The value at the object \( (E, p) \) of a 2-functor \( P : \text{Cat}/B \to \text{Bicat}(B^{\text{op}}, \text{Mod}) \) is defined to be the normal lax functor \( m_E \). For a morphism \( f : (E, p) \to (F, q) \) over \( B \) we define a lax transformation \( P(f) : m_E \Rightarrow m_F \) by defining the component \( Pf_b : E_b \to F_b \) to be the functor induced by \( f \) (meaning that \( P(f)_b(e) = f(e) \)), and by defining the component

\[
\begin{array}{ccc}
m_E(\beta) & \xrightarrow{P(f)_b} & m_F(\beta) \\
E_b & \downarrow \quad \quad & \downarrow \\
F_b & \quad \quad \quad \quad & \\
\end{array}
\]

at \( \beta : b \to b' \) to be the function \( F_b(x, f(e)) \times m_E(\beta)(e, e') \to m_F(\beta)(x, f(e')) \) taking the equivalence class of the pair \( (\chi, \xi) \in F_b(x, f(e)) \times m_E(\beta)(e, e') \) to \( f(\xi) \chi \in m_F(\beta)(x, f(e')) \). It is easy to see that a 2-cell \( \theta : f \Rightarrow g : (E, p) \to (F, q) \) induces a modification \( P(\theta) : P(f) \Rightarrow P(g) \) in an obvious way and that what we have is a 2-functor \( P \) landing in the specified sub-2-category of \( \text{Bicat}(B^{\text{op}}, \text{Mod}) \). Every lax transformation \( \lambda : m_E \Rightarrow m_F \) having each \( \lambda_b \) a functor is of the form \( P(f) \) for a unique \( f : (E, p) \to (F, q) \). Similarly, each modification \( P(f) \Rightarrow P(g) \) is of the form \( P(\theta) \) for a unique \( \theta \).

The inverse equivalence for \( P \) is a generalization of the so-called "Grothendieck construction" of a fibration from a category-valued pseudo-functor (which itself is a generalization of the classical category of elements of a presheaf). Given a normal lax functor \( N : B^{\text{op}} \to \text{Mod} \), we obtain a category \( E = \text{coll} N \) as the lax colimit (or "collage") of \( N \) and a functor \( p : E \to B \) induced by the lax cocone

\[
\begin{array}{ccc}
N(b') & \xrightarrow{1} & B \\
\downarrow N(\beta) & \quad & \\
N(b) & \xrightarrow{1} & b \\
\end{array}
\]

Explicitly, the objects of \( E \) are pairs \( (b, x) \) where \( b \) is an object of \( B \) and \( x \) is an object of \( Nb \); a morphism \( (\beta, \chi) : (b, x) \to (b', x') \) consists of a morphism \( \beta : b \to b' \) in \( B \) and an element \( \chi \in N(\beta)(x, x') \); and composition uses composition in \( B \) and the composition constraints for \( N \). Of course, \( p(b, x) = b \) and \( p(\beta, \chi) = \beta \). Clearly there is a canonical
isomorphism $P(E, p) \cong N$ of lax functors. \textbf{q.e.d.}

For any functor $p : E \to B$ and any morphism $\beta : b \to b'$ in $B$, we can also form the pullback

$$
\begin{array}{ccc}
E_{\beta} & \to & E \\
\downarrow & & \downarrow p \\
2 & \to & B
\end{array}
$$

Notice that $E_{\beta}$ contains $E_b$ and $E_{b'}$ as disjoint full subcategories, and $E_{\beta}(e, e') = m_E(\beta)(e, e')$ and $E_{\beta}(e', e) = \emptyset$ for $e \in E_b$ and $e' \in E_{b'}$. This means that

$$
E_b \xrightarrow{c} E_{\beta} \xleftarrow{\gamma_{\beta}} E_{b'}
$$

is a codiscrete cofibration from $E_{b'}$ to $E_b$ and we have the collage (or lax colimit)

$$
\begin{array}{ccc}
E_{b'} & \xrightarrow{m_E(\beta)} & E_b \\
\downarrow & & \downarrow
\end{array}
$$

in $\text{Mod}$. 

Now we come to our main business: that of investigating what it means for the functor $p^* : \text{Cat}/B \to \text{Cat}/E$ given by pulling back along $p$, to have a right adjoint. Since the domain functor $\text{Cat}/E \to \text{Cat}$ is comonadic (in fact the counit with the right adjoint is a split monomorphism), the functor $p^*$ has a right adjoint if and only if the functor

$$
- \times E : \text{Cat}/B \to \text{Cat}
$$

has a right adjoint [D]. Such an adjoint is determined by its value $h : Z \to B$ on each object $X \in \text{Cat}$; such an $h$ is called a right lifting of $X$ through $- \times E$ and participates in a bijection

$$(\text{Cat}/B)((A, u), (Z, h)) \cong \text{Cat}(A \times E, X)$$

which is natural in $(A, u)$. As is so often the case with right adjoints, this allows us to discover what the category $Z$ must be. Take $A = 1$ and $u = b : 1 \to B$ to find that an object of $Z$ over $b$ amounts to a functor $s : E_b \to X$. So the objects of $Z$ are pairs $(b, s)$ where $b \in B$ and $s$ is such a functor. Now take $A = 2$ and $u = \beta : 2 \to B$ to find that a morphism of $Z$ over $\beta$ amounts to a functor $E_{\beta} \to X$. 

$$
\begin{array}{ccc}
E_{b'} & \xrightarrow{m_E(\beta)} & E_b \\
\downarrow & & \downarrow
\end{array}
$$
By the collage property, this is the same as a diagram

\[
\begin{array}{ccc}
E_{b'} & \xrightarrow{m_E(\beta)} & E_b \\
\downarrow{\sigma'} & \searrow{\sigma} & \downarrow{\sigma} \\
X & = & X
\end{array}
\]

in \textbf{Mod}. So a \textit{morphism} \((\beta, \sigma) : (b, s) \rightarrow (b', s')\) in \(Z\) amounts to a morphism \(\beta : b \rightarrow b'\) in \(B\) together with a \(\sigma\) as in the above triangle.

The problem comes when we try to define \textit{composition} in \(Z\). The appropriate diagram

\[
\begin{array}{ccc}
E_{b''} & \xrightarrow{m_E(\beta'\beta)} & E_b \\
\downarrow{\sigma''} & \searrow{\sigma} & \downarrow{\sigma} \\
X & = & X
\end{array}
\]

is not well formed for pasting. However, if each \(\mu_{\beta,\beta'}\) is invertible then \(Z\) becomes a category and \(h : Z \rightarrow B\), where \(h(b, s) = b\) and \(h(\beta, \sigma) = \beta\), is a right lifting of \(X\) through the functor \(\times_{\beta} B\).

To say each \(\mu_{\beta,\beta'}\) is invertible is to say \(m_E : B^{\text{op}} \rightarrow \textbf{Mod}\) is a pseudofunctor (or "homomorphism" in Bénabou's terminology). Yet what does it mean combinatorially for each \(\mu_{\beta,\beta'}\) to be invertible? Take a composable pair of morphisms \(\beta : b \rightarrow b'\) and \(\beta' : b' \rightarrow b''\) in \(B\) and take \(e \in E_b\) and \(e'' \in E_{b''}\). Consider the category \(M_E(\beta, \beta')(e, e'')\) whose objects are composable pairs of morphisms \(\xi : e \rightarrow e'\) and \(\xi' : e' \rightarrow e''\) in \(E\) such that \(p\xi = \beta\) and \(p\xi' = \beta'\), and whose morphisms are commutative diagrams

\[
\begin{array}{ccc}
e & \xrightarrow{\xi} & e' \\
\downarrow{\eta} & \searrow{\chi} & \downarrow{\eta'} \\
e_1 & \xrightarrow{\xi'} & e''
\end{array}
\]

in which \(\chi : e' \rightarrow e_1\) is in the fibre \(E_{b'}\) over \(b'\). Then \((m_E(\beta') \otimes m_E(\beta))(e, e'')\) is the set of path components of the category \(M_E(\beta, \beta')(e, e'')\), and \(\mu_{\beta,\beta'}\) has component at \((e, e'')\) induced by

\[
M_E(\beta, \beta')(e, e'') \rightarrow m_E(\beta'\beta)(e, e''), \quad (\xi, \xi') \mapsto \xi' \xi.
\]

With these preliminaries, the following precise statement is easily verified.
Theorem (Giraud [G], Conduché [C]) For a functor \( p : E \rightarrow B \), the following conditions are equivalent:

(i) the functor \( p^* : \text{Cat}/B \rightarrow \text{Cat}/E \) has a right adjoint;

(ii) the normal lax functor \( m_E : \text{B}^{\text{op}} \rightarrow \text{Mod} \) is a pseudofunctor;

(iii) for all \( \beta : pe \rightarrow b' \) and \( \beta' : b' \rightarrow pe'' \) in \( B \), and \( \zeta : e \rightarrow e'' \) in \( E \) over \( \beta \beta' \), there exist \( \xi : e \rightarrow e' \) and \( \xi' : e' \rightarrow e'' \) over \( \beta \) and \( \beta' \), respectively, with composite \( \zeta \), and any two such pairs \((\xi, \xi')\) are connected by a path in the category \( M_E(\beta, \beta')(e, e'') \).

References


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