Lawvere Theories

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Abstract

In his 1963 PhD dissertation, F. William Lawvere presented a categorical formulation of universal algebra. To explain this we begin by providing the standard definitions and theorems for adjoints and Kan extension. We examine the standard approach to describing algebraic structure and its corresponding formulation in category theory. An example of this correspondence is demonstrated by considering the theory for monoids and we explore the special case of the "empty" theory. Using properties of the left Kan extension we show that algebraic functors have left adjoints and list some familar examples. Finally, proofs are given that the category of models for a Lawvere theory is both complete and cocomplete and that the forgetful functor has a left adjoint which is the free algebra construction.

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Chapter 1

Introduction

When describing algebraic structure, traditional mathematical (eg. Bourbaki) practice is to define a collection of operations on a set and then specify a list of axioms that they must obey. This has proved to be a very successful approach, a great deal of rigor is achieved and many interesting results have come about by defining new and exotic structures in this fashion. As mathematics has developed, there have been many efforts to make these definitions more abstract and to specify how to describe a theory for each algebraic structure. This is the domain of universal algebra.

After defining the concept of group, mathematicians proceeded to develop "group theory". A certain amount of this is routine and applies analogously to other structures such as commutative rings – matters such as products, images, quotients, first isomorphism theorem and so on. To deal with this, we seek a "theory of groups" which crystallizes in a higher structure the essence of these commonalities. This theory will then have models which deliver back the groups as sets with structure.

As category theory has grown in popularity in the past century, one of the early persuasive examples of its power is found in dealing with this problem of algebraic theory. In his 1963 dissertation, F. William Lawvere provided a very accurate description of how algebraic theories should be represented in terms of categories and outlined how complete this representation is. This work by Lawvere was so accurate that, even unpublished, the ideas were very quickly understood and extended by authors such as Benabou, Mac Lane and Pareigis and the results made appearances in text books only a few years later. It has also been applied to computer science in recent years.

We will provide an outline of some of the concepts behind Lawvere's dissertation, though it is presented in a form much different from the original. First some standard theory on adjunction and Kan extension needs developing.

A Lawvere theory exists for every variety of algebras and captures the theory as a category $\mathbb{T}$ with finite coproducts. We may consider models of a Lawvere theory in any category with finite products, though the case of most interest is that of models in **Set**, called $\mathbb{T}$-algebras. The Lawvere theory for monoids is constructed and indeed the models are proved to be monoids. The special case of the "empty" theory is outlined, giving an equivalence between the models of the theory in a category $\mathcal{X}$ and the category itself.

By proving a result about left Kan extensions, one shows that all algebraic functors have a left adjoint and a few familar constructions are provided as examples. The category of $\mathbb{T}$-algebras is complete and cocomplete implying that there are always universal constructions such as product, coproduct and equalizer of $\mathbb{T}$-algebras and their morphisms. The universal algebra matters mentioned earlier all depend on the availability of these constructions. The forgetful functor that sends each $\mathbb{T}$-algebra to its underlying set has a left adjoint which is the free $\mathbb{T}$-algebra construction. This is pretty amazing because even the example of free group on a set requires a delicate construction and a lot of work to prove its universal property.

These results demonstrate the power of Lawvere theories and the extent to which category theory is capable of describing existence results in algebra with extreme depth and accuracy.

Chapter 2

Preliminaries

We begin by presenting some definitions and theorems that are required for our work regarding Lawvere theories. The standard definitions of category, functor and natural transformation are omitted along with concepts such as duality, universals and limits. The standard theory treated here is that of adjunction and Kan extension. The majority of definitions in this chapter are contained in standard texts on category theory such as [1] or [2]. Most proofs have been omitted, except where they do not appear in standard texts.

2.1 Adjunction

Definition Let $\mathcal{A}$ and $\mathcal{B}$ be categories. An *adjunction* from $\mathcal{A}$ to $\mathcal{B}$ is a triple $\langle S, T, \phi \rangle : \mathcal{B} \to \mathcal{A}$, where S and T are functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "T"'] & \mathcal{B} \arrow[l, shift right, "S"'] \end{tikzcd}$$

while ϕ is a function which assigns to each pair of objects $b \in \mathcal{B}$, $a \in \mathcal{A}$, a bijection of sets

$$\phi = \phi_{b,a} : \mathcal{A}(Sb, a) \cong \mathcal{B}(b, Ta)$$

which is natural in b and a.

Here the left hand side $\mathcal{A}(Sb, a)$ is the functor

$$\mathcal{B}^{\mathrm{op}} \times \mathcal{A} \xrightarrow{S^{\mathrm{op}} \times I_{\mathcal{A}}} \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \xrightarrow{\text{hom}} \mathbf{Set}$$

which sends each pair of objects $\langle b, a \rangle$ to the homset $\mathcal{A}(Sb, a)$, and the right hand side is a similar such functor $\mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Set}$. The naturality of the bijection ϕ means just that it is natural in each place.

When we have an adjunction $\langle S, T, \phi \rangle : \mathcal{B} \to \mathcal{A}$, we then say S is a *left adjoint* for T and T is a *right adjoint* for S. For notation, we use

$$S \dashv T \qquad \text{or} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, bend right, "T"'] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend right, "S"'] \end{tikzcd}$$

to mean "S is a left adjoint for T" and "T is a right adjoint for S".

In order to present enough theory to support later work, but not wanting to dwell too long on the more standard theory, four theorems will now be presented without proof. They may be found with proof in [1].

Theorem 2.1.1 *An adjunction $\langle S, T, \phi\rangle : \mathcal{B} \to \mathcal{A}$ determines the following.*

i) A natural transformation $\eta : I_{\mathcal{B}} \to TS$ such that for each object b the arrow η_b is universal to T from b, while the right adjunct [1] *of each $f : Sb \to a$ is*

$$\phi f = Tf \circ \eta_b : b \to Ta. \tag{2.1}$$

ii) A natural transformation $\epsilon : ST \to I_{\mathcal{A}}$ such that for each object b the arrow ϵ_a is universal from a to S, while the left adjunct of each $g : b \to Ta$ is

$$\phi^{-1} g = \epsilon_a \circ Sg : Sb \to a. \tag{2.2}$$

Moreover, each of the following composites are identities (of T and S respectively),

$$T \xrightarrow{\eta T} TST \xrightarrow{T\epsilon} T\ , \qquad S \xrightarrow{S\eta} STS \xrightarrow{\epsilon S} S\ , \tag{2.3}$$

We call η the *unit* and ϵ the *counit* of the adjunction. In fact, the given adjunction is already determined by various portions of all these data, in the following sense.

Theorem 2.1.2 *Suppose we have functors $S : \mathcal{B} \to \mathcal{A}$ and $T : \mathcal{A} \to \mathcal{B}$, and natural transformations $\eta : I_{\mathcal{B}} \to TS$ and $\epsilon : ST \to I_{\mathcal{A}}$ such that both composites in (2.3) are identity transformations. Then there is an adjunction $\langle S, T, \phi\rangle : \mathcal{B} \to \mathcal{A}$ with ϕ defined by (2.1) and ϕ^{-1} by (2.2).*

In light of this result, we often denote the adjunction $\langle S, T, \phi\rangle : \mathcal{B} \to \mathcal{A}$ by $\langle S, T; \eta, \epsilon\rangle : \mathcal{B} \to \mathcal{A}$.

Definition An *adjoint equivalence* of categories is an adjunction $\langle S, T; \eta, \epsilon\rangle : \mathcal{B} \to \mathcal{A}$ in which both the unit and counit are natural isomorphisms: $I_{\mathcal{B}} \cong TS$, $ST \cong I_{\mathcal{A}}$.

As equivalence and adjunction play a major role in later results, the following theorem is of particular importance.

Theorem 2.1.3 *The following properties of a functor $S : \mathcal{B} \to \mathcal{A}$ are equivalent:*

i) S is part of an equivalence of categories,

ii) S is part of an adjoint equivalence $\langle S, T; \eta, \epsilon\rangle : \mathcal{B} \to \mathcal{A}$,

iii) S is essentially surjective, full and faithful.

Theorem 2.1.4 *Suppose we have two adjunctions*

$$\langle S\,, T\,; \eta\,, \epsilon\rangle : \mathcal{B} \to \mathcal{A} \quad \textit{and} \quad \langle \hat{S}\,, \hat{T}\,; \hat{\eta}\,, \hat{\epsilon}\rangle : \mathcal{C} \to \mathcal{B},$$

they compose to give an adjunction specified by

$$\langle \hat{S}S\,, T\hat{T}\,; T\hat{\eta}S \circ \eta\,, \hat{\epsilon} \circ \hat{S}\epsilon\hat{T}\rangle : \mathcal{C} \to \mathcal{A}.$$

[1] If $f : Sb \to a$ then we call $\phi f : b \to Ta$ the *right adjunct* of f.

Consider now an adjunction $\langle S, T, \phi\rangle : \mathcal{B} \to \mathcal{A}$ and the associated natural isomorphism $\phi_{a,b} : \mathcal{A}(Sb, a) \cong \mathcal{B}(b, Ta)$. The functor S is called a "left" adjoint because it appears on the left in the hom-functor $\mathcal{A}(S-, -)$. The right adjoint is named in a similar fashion. However, if we introduce a third category $\mathcal{X}$ and a pair of functors $F : \mathcal{A} \to \mathcal{X}$ and $G : \mathcal{B} \to \mathcal{X}$, then we can obtain a bijection with S on the right (the "wrong" side). This fact is properly expressed and proved in the following theorem.

Theorem 2.1.5 (Adjoints on the wrong side) *Suppose that we have an adjunction $\langle S, T; \eta, \epsilon\rangle$ and two functors F and G,*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, bend right, "T"'] \arrow[dr, "F"'] \arrow[rr, phantom, "\perp"] & & \mathcal{B} \arrow[ll, bend right, "S"'] \arrow[dl, "G"] \\
& \mathcal{X} &
\end{tikzcd}$$

then there is a bijection

$$\phi : \mathrm{Nat}_a(GTa, Fa) \to \mathrm{Nat}_b(Gb, FSb).$$

Proof Let $\phi : \mathrm{Nat}_b(Gb, FSb) \to \mathrm{Nat}_a(GTa, Fa)$ be defined by sending $\alpha : G \to FS$ to the composite

$$GT \xrightarrow{\alpha T} FST \xrightarrow{F\epsilon} F$$

so that $\phi(\alpha) = F\epsilon \circ \alpha T$.

Let $\theta : \mathrm{Nat}_a(GTa, Fa) \to \mathrm{Nat}_b(Gb, FSb)$ be defined by sending $\beta : GT \to S$ to the composite

$$G \xrightarrow{G\eta} GTS \xrightarrow{\beta S} FS$$

so that $\theta(\beta) = \beta S \circ G\eta$.

It is easy to show that θ is inverse to ϕ. We can already see that $\theta(\phi(\alpha)) = (F\epsilon \circ \alpha T)S \circ G\eta = F\epsilon S \circ \alpha TS \circ G\eta$ and if we consider the commuting diagram below,

$$\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[d, "G\eta"'] & FS \arrow[d, "FS\eta"'] \arrow[dr, "I_{FS}"] & \\
GTS \arrow[r, "\alpha TS"'] & FSTS \arrow[r, "F\epsilon S"'] & FS
\end{tikzcd}$$

we can see also that $F\epsilon S \circ \alpha TS \circ G\eta = I_{FS} \circ \alpha = \alpha$ and hence $\theta(\phi(\alpha)) = \alpha$. The square on the left commutes because of the naturality of α, and the commuting triangle on the right is obtained from one of the unit-counit identities. We can show in a similar fashion that $\phi(\theta(\beta)) = \beta$ and so ϕ is a bijection as required. ∎

This result has been proved with much greater generality by Kelly in [5].

2.2 Kan Extension

Given a functor $J : \mathcal{A} \to \mathcal{B}$ and a category $\mathcal{X}$, consider the functor category $[\mathcal{B}, \mathcal{X}]$ with objects the functors $G : \mathcal{B} \to \mathcal{X}$ and arrows the natural transformations $\sigma : G \to G'$. We define the functor $\mathrm{res}_J : [\mathcal{B}, \mathcal{X}] \to [\mathcal{A}, \mathcal{X}]$ to be "pre-compose with J", more specifically, res_J is defined by the assignments

$$\langle \sigma : G \to G' \rangle \longmapsto \langle \sigma J : GJ \to G'J \rangle .$$

The problem of Kan extension is to find left and right adjoints to res_J. Here we will consider the problem only for left adjoints.

Definition Given functors $J : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{A} \to \mathcal{X}$, a *left Kan extension of F along J* is a pair L, $\eta : F \to LJ$ consisting of a functor $L \in [\mathcal{B}, \mathcal{X}]$ and a natural transformation η which is universal as an arrow from F to $\mathrm{res}_J : [\mathcal{B}, \mathcal{X}] \to [\mathcal{A}, \mathcal{X}]$. The diagram is shown below.

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "J"] \arrow[dr, "F"'] & \arrow[d, phantom, "\xrightarrow{\eta}"] & \mathcal{B} \arrow[dl, "L"] \\
& \mathcal{X} &
\end{tikzcd}$$

As always, this universality determines the functor L uniquely up to isomorphism. In detail, this universality means that for each pair S and $\alpha : F \to SJ$ there is a unique natural transformation $\sigma : L \to S$ such that $\alpha = \sigma J \circ \eta$.

The assignment $\sigma \mapsto \sigma J \circ \eta$ is a bijection

$$\mathrm{Nat}(L, S) \cong \mathrm{Nat}(F, \mathrm{res}_J S)$$

natural in S; again, this natural bijection determines L from J and F. It is called a "left" Kan extension because L appears on the left in the bijection. It follows that if every functor $F \in [\mathcal{A}, \mathcal{X}]$ has a left Kan extension $\langle L, \eta_F : F \to LJ \rangle$, then the assignment $F \mapsto L$ is the object function of a left adjoint to res_J and η is the counit of this adjunction. When this is the case we denote L by $\mathrm{Lan}_J F$ and have $\mathrm{Lan}_J \dashv \mathrm{res}_J$.

Theorem 2.2.1 *Given $J : \mathcal{A} \to \mathcal{B}$, let $F : \mathcal{A} \to \mathcal{X}$ be a functor such that the composite $(J \downarrow b) \to \mathcal{A} \to \mathcal{X}$ has for each $b \in \mathcal{B}$ a colimit*

$$Lb = \mathrm{Colim}((J \downarrow b) \xrightarrow{Q} \mathcal{A} \xrightarrow{F} \mathcal{X}) = \mathrm{Colim}_f\, Fa, \qquad f \in (J \downarrow b)$$

in $\mathcal{X}$, with limiting cocone λ.

Each $g : b \to b'$ induces a unique arrow

$$Lg : \mathrm{Colim}\, FQ \to \mathrm{Colim}\, FQ'$$

commuting with the limiting cocones. These formulas define a functor $L : \mathcal{B} \to \mathcal{X}$, and for each $a \in \mathcal{A}$, the components $\lambda_{I_{Ja}} = \eta_a$ of the limiting cocones define a natural transformation $\eta : F \to LJ$, and L, η is a left Kan extension of F along J.

Corollary 2.2.2 *If $\mathcal{A}$ is small, $\mathcal{B}$ is locally small and $\mathcal{X}$ is cocomplete, any functor $F : \mathcal{A} \to \mathcal{X}$ has a left Kan extension along any $J : \mathcal{A} \to \mathcal{B}$, and* res_J *has a left adjoint.*

Corollary 2.2.3 *The pair L, $\eta : F \to LJ$ is a pointwise Kan extension of F along J if and only if, for all $x \in \mathcal{X}$ and $b \in \mathcal{B}$,*

$$\mathcal{X}(Lb, x) \longrightarrow \mathrm{Nat}_a(\mathcal{B}(Ja, b), \mathcal{X}(Fa, x)),$$

sending $g : Lb \to x$ to the transformation with the component

$$\mathcal{B}(Ja, b) \xrightarrow{\quad L \quad} \mathcal{X}(LJa, Lb) \xrightarrow{\mathcal{X}(\eta_a, g)} \mathcal{X}(Fa, x)$$

at $a \in \mathcal{A}$, is a bijection.

Chapter 3

Algebraic Theories

F. William Lawvere's 1963 dissertation titled *Functorial Semantics of Algebraic Theories* (reprinted as [4]) introduced some very significant ideas that demonstrated how theories for general algebra could be condensed into certain kinds of categories. These categories contain within them all the information required to describe a given algebraic structure independently of the context in which they are usually studied. Lawvere puts it himself:

> Algebras (and other structures, models, etc.) are actually functors to a background category from a category which abstractly concentrates the essence of a certain general concept of algebra, and indeed homomorphisms are nothing but natural transformations between such functors. Categories of algebras are very special, and explicit axiomatic characterizations of them can be found, thus providing a general guide to the special features of construction in algebra.

In this chapter we define what we mean by Lawvere theory and model of a Lawvere theory and explain the above quotation.

3.1 Lawvere theories

When one considers how many standard algebraic structures are described, one sees that many are defined by a collection of operations on a set together with some axioms that the operations must obey. A standard example is that of group structure. We describe a group as a set X together with three operations:

$$\begin{aligned} &\mu : X^2 \longrightarrow X^1 && \mu(x,y) = xy \\ &\sigma : X^1 \longrightarrow X^1 && \sigma(x) = x^{-1} \\ &\eta : X^0 \longrightarrow X^1 && \eta(\ ) = 1 \end{aligned}$$

a binary operation called "multiplication", a unary operation called "inverse" and a nullary operation called "identity". The axioms that they must obey are

$$x(yz) = (xy)z\,, \quad x1 = x = 1x \quad \text{and} \quad xx^{-1} = 1 = x^{-1}x\,.$$

There are many other examples that are described in a similar manner, such as rings, R-modules and R-algebras (R is a commutative ring). These considerations led to a subject called *universal algebra* and the concept, within that subject, of *variety*.

Consider a set Ω, whose elements are thought of as *abstract operations*, which is graded by a function $\mathrm{ari} : \Omega \to \mathbb{N}$, called arity. Let $F(\Omega)$ denote the category whose objects are sets X equipped with a function

$$a_\omega : X^n \to X$$

for each $\omega \in \Omega$ with $\mathrm{ari}(\omega) = n$; we call a_ω an operation of type ω. The morphisms of $F(\Omega)$ are functions which preserve the operations of all the types.

A *variety* is any category $\mathcal{A}$ which is a full subcategory of some $F(\Omega)$ closed under products, subobjects and homomorphic images. By the Garrett Birkhoff theorem, the objects of $\mathcal{A}$ are obtained from those of $F(\Omega)$ by imposing equational axioms.

A *derived operation* in this setting is a function $a : X^n \to X^m$ all of whose projections

$$X^n \xrightarrow{a} X^m \xrightarrow{\pi_i} X,$$

$i = 1, \ldots, m$, is of the form a_ω for some $\omega \in \Omega$.

Though varieties cover many common algebraic structures, there are some that cannot be described in this fashion. Examples include fields, partially ordered sets and categories.

Categories are well suited to describing varieties. Each (derived) operation $\rho : X^n \to X^m$ may be considered as a map $\hat{\rho} : c^n \to c^m$ in a category $\mathcal{C}$ with finite products and each equational axiom may be regarded as a commuting diagram in $\mathcal{C}$. It is with these considerations in mind that we give the following definition.

Definition Let $\mathcal{C}$ be a category with finite coproducts, we call $\mathcal{C}$ a *Lawvere theory* when every object $c \in \mathcal{C}$ is a finite copower of one object $c_0 \in \mathcal{C}$. We call c_0 the *generating object* and say that $\mathcal{C}$ is *one-sorted* [1].

The choice to use coproduct here instead of product (by duality) does not have any great significance and actually differs from the usage of other authors. We define it this way only because it makes our examples easier to describe. For example, the skeletal category $\mathbb{S}$ of finite sets is a Lawvere theory with generating object $c_0 = \langle 1 \rangle$ (since $\langle n \rangle = \langle 1 \rangle .n$); this is the subject of section 3.4.

Now with this definition in mind, consider a Lawvere theory $\mathbb{T}$ with generating object 1 (generally not terminal) and three maps

$$1 \xrightarrow{\mu} 1+1\ , \qquad 1 \xrightarrow{\sigma} 1\ , \qquad \text{and} \qquad 1 \xrightarrow{\eta} 0$$

such that the following diagrams commute.

$$\begin{tikzcd}
1+1+1 & 1+1 \arrow[l, "1+\mu"'] \\
1+1 \arrow[u, "\mu+1"] & 1 \arrow[l, "\mu"] \arrow[u, "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
1 & 1+1 \arrow[l, "1+\eta"'] \arrow[r, "\eta+1"] & 1 \\
& 1 \arrow[ul, "I"] \arrow[u, "\mu"'] \arrow[ur, "I"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1+1 \arrow[dl, "1+\sigma"'] & \\
1 & 0 \arrow[l, "!"'] & 1 \arrow[ul, "\mu"'] \arrow[l, "\eta"'] \arrow[dl, "\mu"] \\
& 1+1 \arrow[ul, "\sigma+1"] &
\end{tikzcd}$$

[1] In general, if $\mathcal{C}$ had n generating objects then we would say that it is *n-sorted*.

It would seem here that the object 1 behaves as a group in $\mathbb{T}^{\text{op}}$ and that maybe this category contains all the structure needed to define groups in all kinds of contexts. However, how is this category linked to the category of groups **Grp**? How does one describe group homomorphisms using this category? Is this category exactly the one we are looking for? This leads us neatly to the next section.

3.2 Models

While we are able to abstractly present the operations and axioms required to specify a group, we have not yet specified how to link this to concrete examples. We use the word model to describe any mathematical object that may reasonably be said to satisfy the requirements of the theory, a more exact definition is provided shortly. Intuitively, if we understand what structure the theory is specifying, a model is just a set (or a function, or a space etc.) equipped with the specified structure.

Definition Let $\mathcal{X}$ be a category with finite products and let $\mathbb{T}$ be a Lawvere theory. A finite product preserving functor $F : \mathbb{T}^{\text{op}} \to \mathcal{X}$ is called a model of $\mathbb{T}$ in $\mathcal{X}$. The full subcategory of $[\mathbb{T}^{\text{op}}, \mathcal{X}]$ containing these is called the category of models of $\mathbb{T}$ in $\mathcal{X}$ and is denoted by $\text{Prod}(\mathbb{T}^{\text{op}}, \mathcal{X})$.

When $\mathcal{X} = \textbf{Set}$ we will call $\text{Prod}(\mathbb{T}^{\text{op}}, \textbf{Set})$ the category of $\mathbb{T}$-algebras, or $\mathbb{T}$- **Alg**.

The consequences of being a functor are very helpful in this context, especially where they preserve products. For each commuting diagram in $\mathbb{T}$, there is a corresponding commuting diagram in $\mathcal{X}$ and if $F1 = x$, then $F2 = x^2$, $F3 = x^3$ and so on. As an example, if there were a map $\sigma : 1 \to 2$ in $\mathbb{T}$ satisfying

$$\begin{tikzcd} 3 & 2 \arrow[l, "1+\sigma"'] \\ 2 \arrow[u, "\sigma+1"] & 1 \arrow[l, "\sigma"] \arrow[u, "\sigma"'] \end{tikzcd}$$

and $F1 = x$ then x satisfies

$$\begin{tikzcd} x^3 \arrow[r, "1\times F\sigma"] \arrow[d, "F\sigma\times 1"'] & x^2 \arrow[d, "F\sigma"] \\ x^2 \arrow[r, "F\sigma"'] & x \end{tikzcd}$$

in $\mathcal{X}$.

It should be noted that $x = F1$ alone has no structure, it is just an object in $\mathcal{X}$ (usually a set). In an opposing sense, the Lawvere theory $\mathbb{T}$ has clear structure, but it is too far removed from the category of sets to be of use. It is precisely the product preserving structure that allow us the connect the two together.

This approach also allows us to describe the (homo-)morphisms of the algebra. Suppose there are two functors F and G in $\text{Prod}(\mathbb{T}^{\text{op}}, \mathcal{X})$ and a natural transformation $\alpha : F \to G$. Then by considering the following diagrams

$$\begin{array}{c} n \\ {\scriptstyle \iota_i}\uparrow \\ 1 \end{array} \qquad \begin{tikzcd} F1^n \arrow[r, "\alpha_n"] \arrow[d, "\pi_i"'] & G1^n \arrow[d, "\pi_i"] \\ F1 \arrow[r, "\alpha_1"'] & G1 \end{tikzcd}$$

one can see that α_n is completely determined by α_1, in fact $\alpha_n = \alpha_1^n$. When you combine this with the maps specified in $\mathbb{T}$, one sees that α_1 preserves all the operations, just as a morphism of algebras should. For example, for a binary operation σ we have

$$\begin{tikzcd} F2 \arrow[r, "\alpha_1 \times \alpha_1"] \arrow[d, "F\sigma"'] & G2 \arrow[d, "F\sigma"] \\ F1 \arrow[r, "\alpha_1"'] & G1. \end{tikzcd}$$

We will see all of these properties in the following section.

3.3 An example: monoids

We begin by reminding the reader of the structure of a monoid.

Definition A *monoid* is a triple $\langle X, *, 1\rangle$ where X is a set, $*$ is an associative binary operation and 1 is an element of X that acts as an identity under the binary operation. So for all $x, y, z \in X$ we have the equational axioms

$$x * (y * z) = (x * y) * z \quad \text{and} \quad x * 1 = x = 1 * x \quad .$$

Some familiar monoids are $\langle \mathbb{N}, +, 0\rangle$ and $\langle \mathbb{Z}, \times, 1\rangle$. A *monoid morphism* from $\langle X, *, 1\rangle$ to $\langle Y, \circ, 0\rangle$ is a function $f : X \to Y$ which preserves the binary operation and sends the identity in X to that in Y. That is,

$$f(a * b) = f(a) \circ f(b) \quad \text{and} \quad f(1) = 0\,.$$

Monoid morphisms compose as functions and so we have a category **Mon** whose objects are monoids and whose arrows are monoid morphisms.

We can construct a monoid from any set X. Let X^* be the set of "words" in elements of X, that is, the set of tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of elements in X. Let it have a binary operation $\oplus$ which is concatenation of words; that is,

$$(x_1, \ldots, x_n) \oplus (y_1, \ldots, y_m) = (x_1, \ldots, x_n, y_1, \ldots, y_m)\,.$$

Concatenation is an associative operation and the empty word $o = ()$ is an identity. Hence $\langle X^*, \oplus, o\rangle$ is a monoid, we call it the free monoid on X and denote it by X^*.

The formulation given here is the traditional approach; where monoids are regarded as sets with the given structure. However, monoids may be generalised to mathematical objects other than sets in the following manner.

Definition Let $\mathcal{C}$ be a category with finite products and terminal object t. Then a *monoid* in $\mathcal{C}$ is a triple $\langle c, \mu : c \times c \to c, \eta : t \to c \rangle$ such that the following diagrams commute.

$$\begin{tikzcd}
c \times c \times c \arrow[r, "\mu \times I_c"] \arrow[d, "I_c \times \mu"'] & c \times c \arrow[d, "\mu"] \\
c \times c \arrow[r, "\mu"'] & c
\end{tikzcd}
\qquad
\begin{tikzcd}
t \times c \arrow[r, "\eta \times I_c"] \arrow[dr] & c \times c \arrow[d, "\mu"'] & c \times t \arrow[l, "I_c \times \eta"'] \arrow[dl] \\
& c &
\end{tikzcd}$$

A morphism $f : \langle c, \mu, \eta \rangle \to \langle c', \mu', \eta' \rangle$ of monoids is an arrow $f : c \to c'$ such that both diagrams below

$$\begin{tikzcd}
c \times c \arrow[r, "\mu"] \arrow[d, "f \times f"'] & c \arrow[d, "f"] \\
c \times c \arrow[r, "\mu'"'] & c
\end{tikzcd}
\qquad
\begin{tikzcd}
t \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & c \arrow[d, "f"] \\
& c
\end{tikzcd}$$

commute. Together with these morphisms, the monoids in $\mathcal{C}$ constitute a category $\mathbf{Mon}(\mathcal{C})$. [2]

We may intuitively understand the maps μ and η in $\mathcal{C}$ as a binary operation and a constant, then the first set of commuting diagrams describes the associativity of the binary operation and the identity nature of the constant. The second set of commuting diagrams then describes how monoid morphisms preserve structure. Hence, if $\mathcal{C} = \mathbf{Set}$ then this definition reduces to that given initially and $\mathbf{Mon}(\mathbf{Set}) = \mathbf{Mon}$.

We claim that there is a Lawvere theory, call it $\mathbb{M}$, such that $\mathrm{Prod}(\mathbb{M}^{\mathrm{op}}, \mathcal{X}) \simeq \mathbf{Mon}(\mathcal{X})$. This theory concentrates all the structure of a monoid and any models of this theory will have monoid structure.

Let $\mathbb{M}$ have as objects the finite sets $\langle n \rangle$ (including the empty set), that is, $\mathbb{M}$ has the same objects as $\mathbb{S}$. We will abuse standard notation somewhat and refer to the objects in $\mathbb{M}$ by their cardinality, that is, m is the set with m elements and $0 = \varnothing$. Where there might be any confusion, we will return to the original notation, $\langle n \rangle = \{1, 2, \ldots, n\}$. The arrows in $\mathbb{M}$ are defined by $\mathbb{M}(n, m) = \mathbf{Mon}(n^*, m^*)$, that is, the arrows from n to m are all the monoid morphisms from n^* to m^*. This category will be called *the Lawvere theory for monoids*, we will see shortly why.

It should be noted here that because monoid morphisms preserve the multiplication in the monoid, each morphism $f : n^* \to m^*$ on a free monoid is completely determined by its values on the one-element words. It must also map the empty word to the identity. Hence there is a bijection

$$\mathbf{Mon}(n^*, m^*) \cong \mathbf{Set}(\langle n \rangle, m^*)$$

natural in m^*.

Proposition 3.3.1 *$\mathbb{M}$ is a Lawvere theory.*

[2] This definition is less general than the standard definition for monoid in a monoidal category but it fits our needs here.

Proof For any n and m in $\mathbb{M}$ we have the following sequence of isomorphisms,

$$\begin{aligned}
\mathbb{M}(m+n,p) &= \mathbf{Mon}((m+n)^*,p^*)\\
&\cong \mathbf{Set}(\langle m+n\rangle, p^*)\\
&\cong \mathbf{Set}(\langle m\rangle + \langle n\rangle, p^*)\\
&\cong \mathbf{Set}(\langle n\rangle, p^*)\times \mathbf{Set}(\langle m\rangle, p^*)\\
&\cong \mathbf{Mon}(m^*,p^*)\times \mathbf{Mon}(n^*,p^*)\\
&\cong \mathbb{M}(m,p)\times \mathbb{M}(n,p)
\end{aligned}$$

natural in p. Then by noting that

$$\mathbb{M}(0,p) = \mathbf{Mon}(0^*,p^*) \cong \mathbf{Set}(\varnothing,p^*)\cong \langle 1\rangle$$

we see that $\mathbb{M}$ has finite coproducts. Each object n is therefore a coproduct of 1 with itself n times.

■

Shortly, we will show that the category of models of $\mathbb{M}$ in a category $\mathcal{X}$ is equivalent to the category of monoids in $\mathcal{X}$. First however, it is useful to note that the generating object of $\mathbb{M}$, call it 1, is a monoid in $\mathbb{M}^{\mathrm{op}}$. Consider the maps

$$\begin{array}{rcl}
\mu : 1 & \longrightarrow & 2\\
o & \longmapsto & o\\
(1) & \longmapsto & (1,2)
\end{array}
\qquad
\begin{array}{rcl}
\eta : 1 & \longrightarrow & 0\\
o & \longmapsto & o\\
(1) & \longmapsto & o
\end{array}$$

in $\mathbb{M}$, where o denotes the empty word. They satisfy the following pair of commuting diagrams,

$$\begin{tikzcd}
3 & 2 \arrow[l, "\mu+1"'] \\
2 \arrow[u, "1+\mu"] & 1 \arrow[l, "\mu"] \arrow[u, "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
1 & 2 \arrow[l, "1+\eta"'] \arrow[r, "\eta+1"] & 1 \\
& 1 \arrow[ul, "1"] \arrow[u, "\mu"'] \arrow[ur, "1"'] &
\end{tikzcd}$$

in $\mathbb{M}$. Hence, in $\mathbb{M}^{\mathrm{op}}$, 1 is a monoid.

Proposition 3.3.2 *Suppose a category $\mathcal{X}$ has finite products, then*

$$\mathrm{Prod}(\mathbb{M}^{\mathrm{op}},\mathcal{X})\simeq \mathbf{Mon}(\mathcal{X}).$$

Proof Let $E : \mathrm{Prod}(\mathbb{M}^{\mathrm{op}},\mathcal{X})\to \mathbf{Mon}(\mathcal{X})$ be be the functor "evaluate at 1" sending $F : \mathbb{M}^{\mathrm{op}}\to\mathcal{X}$ to $\langle F1, F\mu, F\eta\rangle$. The functor E is defined on $\alpha : F\to G$ in $\mathrm{Prod}(\mathbb{M}^{\mathrm{op}},\mathcal{X})$ by $E\alpha = \alpha_1 : F1\to G1$. To see that E is well defined, first observe that $F\mu$ and $F\eta$ make following diagrams commute,

$$\begin{tikzcd}
3 \arrow[r, "F\mu\times 1"] \arrow[d, "1\times F\mu"'] & 2 \arrow[d, "\mu"] \\
2 \arrow[r, "\mu"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "1\times F\eta"] \arrow[dr, "1"'] & 2 \arrow[d, "F\mu"] & 1 \arrow[l, "F\eta\times 1"'] \arrow[dl, "1"] \\
& 1 &
\end{tikzcd}$$

so the triple $\langle F1, F\mu, F\eta\rangle$ is in fact a monoid in $\mathbf{Mon}(\mathcal{X})$. Also,

$$E(\alpha\beta) = (\alpha\beta)_1 = \alpha_1\beta_1 = E(\alpha)E(\beta)$$

and it is easy to show that

$$\begin{tikzcd}
F2 \arrow[r, "\alpha_1 \times \alpha_1"] \arrow[d, "F\mu"'] & G2 \arrow[d, "F\mu"] \\
F1 \arrow[r, "\alpha_1"'] & G1
\end{tikzcd}$$

commutes and hence $E(\alpha) = \alpha_1$ is a monoid morphism.

For any monoid $\langle X, \mu, \eta\rangle$ in $\mathbf{Mon}(\mathcal{X})$, we define $\hat{F} = F_{\langle X,\mu,\eta\rangle} : \mathbb{M}^{\mathrm{op}} \to \mathcal{X}$ on objects by $\hat{F}n = X^n$ and on arrows by sending $\xi : \langle n\rangle \to \langle m\rangle$ to the unique arrow making the diagram,

$$\begin{tikzcd}
X^n \arrow[dr, "\pi_i"'] & & X^m \arrow[ll, dotted, "\hat{F}\xi"'] \arrow[dl, "\pi_{\xi(i)}"] \\
& X &
\end{tikzcd}$$

commute for $i = 1, \ldots, n$. Then define $\hat{F}$ on $a : \langle n\rangle \to m^*$ by sending it to the unique arrow in the the diagram on the left where $a(i) : \langle k_i\rangle \to \langle m\rangle$ is the function describing the word $a(i)$ of length k_i.

$$\begin{tikzcd}
X^m \arrow[r, dotted, "\hat{F}a"] \arrow[d, "\hat{F}a(i)"'] & X^n \arrow[d, "\pi_i"] \\
X^{k_i} \arrow[r, "\mu_{k_i}"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X^{n+1} \arrow[rr, "\mu_{n+1}"] \arrow[dr, "\mu \times 1_{X^{n-1}}"'] & & X \\
& X^n \arrow[ur, "\mu_n"'] &
\end{tikzcd}$$

We define μ_{k_1} using the diagram on the right.

Using this definition for $\hat{F}$, it is easy to show that $\hat{F}(\mu) = \mu$, $\hat{F}(\eta) = \eta$ and $\hat{F}$ preserves finite products. It is not so easy to see that $\hat{F}$ is well defined as a functor, but this this too is true. Hence, $E(\hat{F}) = \left\langle \hat{F}1, \hat{F}\mu, \hat{F}\eta \right\rangle = \langle X, \mu, \eta\rangle$ and so E is essentially surjective.

Suppose we have $\alpha : F \to G$ in $\mathrm{Prod}(\mathbb{M}^{\mathrm{op}}, \mathcal{X})$ and consider the following diagrams,

$$\begin{tikzcd}
n \\
1 \arrow[u, "\iota_i"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Fn \arrow[r, "\alpha_n"] \arrow[d, "F\iota_i"'] & Gn \arrow[d, "G\iota_i"] \\
F1 \arrow[r, "\alpha_1"'] & G1
\end{tikzcd}$$

which commute for $i = 1, 2, \ldots, n$ because α is natural. Since F and G preserve finite products, $F\iota_i = \pi_i$ and $G\iota_i = \pi_i$. Then the diagram above shows that α_n is completely determined by α_1 (it is the unique arrow satisfying $\pi_i \circ \alpha_n = \alpha_1 \circ \pi_i$ for $i = 1, 2, \ldots, n$). Thus, the whole of α is completely determined by α_1. Let $\alpha, \beta : F \to G$ be two natural transformations and suppose $E\alpha = E\beta$, then $\alpha_1 = \beta_1$ and so $\alpha = \beta$. Hence E is faithful.

Showing that E is full is more difficult, but in the same way, given a monoid morphism $f : E(F) \to E(G)$ we can construct a natural transformation $\phi : F \to G$ by letting $\phi_1 = f$ and defining ϕ_n to be the unique arrow making the diagram below commute for $i = 1, 2, \ldots, n$.

$$\begin{tikzcd}
Fn \arrow[r, dotted, "\phi_n"] \arrow[d, "\pi_i"'] & Gn \arrow[d, "\pi_i"] \\
F1 \arrow[r, "f"'] & G1
\end{tikzcd}$$

Showing that this actually defines a natural transformation is difficult, but the construction here suffices. Now since $E(\phi) = \phi_1 = f$, for all $f \in \hom(E(F), E(G))$ there exists $\phi \in \mathrm{Nat}(F, G)$ such that $E(\phi) = f$. Hence E is full.

■

3.4 The initial theory

As before, let $\mathbb{S}$ be the full subcategory of **Set** with finite sets as objects and for each finite cardinal n there is precisely one set of that cardinality. In particular, $\varnothing$ is in $\mathbb{S}$. An alternative definition could be: $\mathbb{S}$ is the full subcategory of **Set** with objects the sets $\langle n \rangle$ for $n = 0, 1, 2, \ldots$. These two categories are clearly isomorphic. Note that the empty set $\varnothing$ is an initial object in $\mathbb{S}$ (empty coproduct) and each $\langle n \rangle \in \mathbb{S}$ is an n-fold disjoint union (coproduct) of $\langle 1 \rangle$ with itself. We have proved the following proposition.

Proposition 3.4.1 $\mathbb{S}$ *is a Lawvere theory.*

Suppose now that we have a covariant functor $F : \mathbb{S}^{\mathrm{op}} \to \mathcal{A}$ which is a bijection on objects and which preserves finite products. Then $F1$ is an object in $\mathcal{A}$ and every other object in $\mathcal{A}$ is of the form $Fn = F(1.n) = F1^n$. So $\mathcal{A}^{\mathrm{op}}$ is a Lawvere theory.

This is the approach taken in [3] and [4]. More explicitly, let $\mathbb{S}$ be the full subcategory of **Set** described above. A covariant functor $A : \mathbb{S}^{\mathrm{op}} \to \mathcal{A}$ which is a bijection on objects and preserves finite products is referred to as an *algebraic theory.* The notation is then softened slightly and algebraic theories are referred to by naming the target category (in this case $\mathcal{A}$). Our definition of Lawvere theory differs from this definition in only one way: we choose to name the category with finite coproducts instead of that with finite products.

Having defined the category $\mathbb{S}$, observed it as a Lawvere theory and seeing its significance in both [3] and [4], it seems relevant to ask "what are the models of $\mathbb{S}$?". The answer is that $\mathbb{S}$ is the "empty" Lawvere theory, that is, the Lawvere theory imposing no structure. This is properly expressed in the following proposition.

Proposition 3.4.2 *Suppose a category $\mathcal{X}$ has finite products, then*

$$\mathrm{Prod}(\mathbb{S}^{\mathrm{op}}, \mathcal{X}) \simeq \mathcal{X}.$$

Proof Let $E : \mathrm{Prod}(\mathbb{S}^{\mathrm{op}}, \mathcal{X}) \to \mathcal{X}$ be the functor "evaluate at 1". More specifically, if $F : \mathbb{S}^{\mathrm{op}} \to \mathcal{X}$ then $E(F) = F1$ and if $\alpha : F \to G$ then $E(\alpha) = \alpha_1 :$

$F1 \to G1$. To see that E is well defined, note that $E(I_F) = I_{F(1)} = I_{E(F)}$ and if $M \xrightarrow{\beta} N \xrightarrow{\alpha} P$ then $E(\alpha\beta) = (\alpha\beta)_1 = \alpha_1\beta_1 = E(\alpha)E(\beta)$. We aim to show that E is essentially surjective and fully faithful.

For each $x \in \mathcal{X}$ we have the functor $M_x : \mathbb{S}^{\mathrm{op}} \to \mathcal{X}$ which is defined on objects by $M_x n = x^n$ and on arrows by sending each arrow $f : n \to m$ in $\mathbb{S}$ to the unique arrow making

$$\begin{tikzcd} x^n \arrow[dr, "\pi_i"'] & & x^m \arrow[ll, dotted, "M_x f"'] \arrow[dl, "\pi_{f(i)}"] \\ & x & \end{tikzcd}$$

commute for $i = 1, \ldots, n$. It is easy to see that that $M_x I_n = I_{x^n} = I_{M_x n}$. Then the diagrams,

$$\begin{tikzcd} n \arrow[r, "f"'] \arrow[rr, bend left, "g \circ f"] & m \arrow[r, "g"'] & p \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x^n \arrow[dr, "\pi_i"'] & x^m \arrow[l, "M_x f"] \arrow[d, "\pi_{f(i)}"] & x^p \arrow[l, "M_x g"] \arrow[dl, "\pi_{g(f(i))}"] \arrow[ll, bend right, "M_x(g \circ f)"'] \\ & x & \end{tikzcd}$$

which commutes for $i = 1, 2, \ldots, n$ demonstrate that $M_x(g \circ f) = M_x f \circ M_x g$. Hence M_x is well defined. It is easily verifiable that M_x also preserves products (as a covariant functor). Then for all $x \in \mathcal{X}$ there exists a functor M_x in $\mathrm{Prod}(\mathbb{S}^{\mathrm{op}}, \mathcal{X})$ such that $E(M_x) = M_x 1 = x$, so E is essentially surjective.

Suppose we have $\alpha : M \to N$ in $\mathrm{Prod}(\mathbb{S}^{\mathrm{op}}, \mathcal{X})$ and consider the following diagrams,

$$\begin{tikzcd} n \\ 1 \arrow[u, "\iota_i"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Mn \arrow[r, "\alpha_n"] \arrow[d, "M\iota_i"'] & Nn \arrow[d, "N\iota_i"] \\ M1 \arrow[r, "\alpha_1"'] & N1 \end{tikzcd}$$

which commutes for $i = 1, 2, \ldots, n$ because α is natural. Since M and N preserve finite products, $M\iota_i = \pi_i$ and $N\iota_i = \pi_i$. Then the diagram above shows that α_n is completely determined by α_1 (it is the unique arrow satisfying $\pi_i \circ \alpha_n = \alpha_1 \circ \pi_i$ for $i = 1, 2, \ldots, n$). Thus, the whole of α is completely determined by α_1. Let $\alpha, \beta : M \to N$ be two natural transformations and suppose $E\alpha = E\beta$, then $\alpha_1 = \beta_1$ and so $\alpha = \beta$. Hence E is faithful.

In the same way, given an arrow $f : E(M) \to E(N)$ we can construct a natural transformation $\phi : M \to N$ by letting $\phi_1 = f$ and defining ϕ_n to be the unique arrow making the diagram below commute for $i = 1, 2, \ldots, n$.

$$\begin{tikzcd} Mn \arrow[r, dotted, "\phi_n"] \arrow[d, "\pi_i"'] & Nn \arrow[d, "\pi_i"] \\ M1 \arrow[r, "f"'] & N1 \end{tikzcd}$$

It is straight forward to verify that ϕ is a well defined natural transformation. Now since $E(\phi) = \phi_1 = f$, for all $f \in \hom(E(M), E(N))$ there exists $\phi \in$

$\mathrm{Nat}(M, N)$ such that $E(\phi) = f$. Hence E is full.

■

Note how similar this proof is to that of Proposition 3.3.2. This proposition expresses exactly what we mean when we say that $\mathbb{S}$ is the empty Lawvere theory. This is also a result that will come in very handy later on.

3.5 Varieties

The example given illustrates a relationship between Lawvere theories and algebraic structure that is true for every variety of algebras. For every variety of algebras there is a Lawvere theory $\mathbb{T}$ such that

$$\mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set}) \simeq \mathbb{T}\text{-}\mathbf{Alg}.$$

An excellent list of examples can be found in [3] pp.145–148. A few of the more familiar ones are listed here.

1. **Group** (abelian)
   Operations:

$$\begin{aligned} \times : X^2 &\longrightarrow X^1 \\ (\ )^{-1} : X^1 &\longrightarrow X^1 \\ 1 : X^0 &\longrightarrow X^1 \end{aligned}$$

   Axioms:

$$x \times (y \times z) = (x \times y) \times z$$

$$x \times x^{-1} = 1 = x^{-1} \times x$$

$$1 \times x = x = x \times 1$$

$$(\, x \times y = y \times x \,)$$

2. **Ring** (commutative)
   Operations:

$$\begin{aligned} + : X^2 &\longrightarrow X^1 \\ -(\ ) : X^1 &\longrightarrow X^1 \\ 0 : X^0 &\longrightarrow X^1 \\ \times : X^2 &\longrightarrow X^1 \\ 1 : X^0 &\longrightarrow X^1 \end{aligned}$$

   Axioms:

$$x + (y + z) = (x + y) + z$$

$$x + -x = 0 = -x + x$$

$$0 + x = x = x + 0$$

$$x + y = y + x$$

$$x \times (y \times z) = (x \times y) \times z$$

$$1 \times x = x = x \times 1$$

$$(x \times y = y \times x)$$

$$x \times (y + z) = x \times y + x \times z$$
$$(x + y) \times z = x \times z + y \times z$$

3. **Lie Ring**
   Operations:

$$+ : X^2 \longrightarrow X^1$$
$$-(\ ) : X^1 \longrightarrow X^1$$
$$0 : X^0 \longrightarrow X^1$$
$$[\ ,\ ] : X^2 \longrightarrow X^1$$
$$1 : X^0 \longrightarrow X^1$$

   Axioms:

$$x + (y + z) = (x + y) + z$$
$$x + -x = 0 = -x + x$$
$$0 + x = x = x + 0$$
$$x + y = y + x$$

$$[x, y] + [y, x] = 0$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

$$[x + y, z] = [x, z] + [y, z]$$
$$[x, y + z] = [x, y] + [x, z]$$

4. **R-module** (R a commutative ring)
   Operations:

$$+ : X^2 \longrightarrow X^1$$
$$-(\ ) : X^1 \longrightarrow X^1$$
$$0 : X^0 \longrightarrow X^1$$

   and for each $r \in R$, an operation

$$r : X^1 \longrightarrow X^1$$

   Axioms:

$$x + (y + z) = (x + y) + z$$
$$x + -x = 0 = -x + x$$
$$0 + x = x = x + 0$$
$$x + y = y + x$$

$$(r + r')x = rx + r'x$$
$$r(x + x') = rx + rx'$$
$$(r \times r')x = r(r'x)$$
$$1x = x$$

5. **R-algebra** (R a commutative ring)
   Operations:

$$
\begin{aligned}
+ : X^2 &\longrightarrow X^1 \\
-(\ ) : X^1 &\longrightarrow X^1 \\
0 : X^0 &\longrightarrow X^1 \\
\times : X^2 &\longrightarrow X^1 \\
1 : X^0 &\longrightarrow X^1
\end{aligned}
$$

   and for each $r \in R$, an operation

$$r : X^1 \longrightarrow X^1$$

   Axioms:

$$x + (y + z) = (x + y) + z$$
$$x + -x = 0 = -x + x$$
$$0 + x = x = x + 0$$
$$x + y = y + x$$

$$x \times (y \times z) = (x \times y) \times z$$
$$1 \times x = x = x \times 1$$

$$x \times (y + z) = x \times y + x \times z$$
$$(x + y) \times z = x \times z + y \times z$$

$$(r + r')x = rx + r'x$$
$$r(x + y) = rx + ry$$
$$(r \times r')x = r(r'x)$$
$$1x = x$$
$$r(x \times y) = (rx) \times y = x \times (ry)$$

6. **Jonsson-Tarski** [7]
   Operations:

$$
\begin{aligned}
b : X^2 &\longrightarrow X^1 \\
l : X^1 &\longrightarrow X^1 \\
r : X^1 &\longrightarrow X^1
\end{aligned}
$$

   Axioms:

$$b(l(x), r(x)) = x$$
$$l(b(x, y)) = x$$
$$r(b(x, y)) = y$$

7. M-**Set** (M a monoid)
   Operations: for each $m \in M$, an operation

$$m : X^1 \longrightarrow X^1$$

   Axioms:

$$1x = x$$

$$(mm')x = m(m'x)$$

The final two examples are of particular interest because both categories of models are also toposes and are essentially the only two toposes that can be described by means of a variety. In particular, models of the Jonsson-Tarski theory are sets with a bijection $X \cong X \times X$ and hence the only finite models in the category of models are 0 and 1.

Chapter 4

Existence and properties of adjoints

There are many universal constructions in algebra that, while discovered using techniques of the original subjects, were later observed to be left (or right) adjoint to a functor an elementary functor between categories of algebras. We will prove that if we consider only algebraic functors and we work with models in **Set** then the left adjoint always exists. A list of familiar examples of this result will follow.

4.1 $\mathrm{Lan}_J F$ preserves finite products

Let $J : \mathbb{R} \to \mathbb{T}$ be a coproduct preserving functor between Lawvere theories. Then there is a standard functor $\mathrm{res}_J^* : \mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathcal{X}) \to \mathrm{Prod}(\mathbb{R}^{\mathrm{op}}, \mathcal{X})$ which is "pre-compose with J^{op}". Such functors as are called *algebraic functors*.

Suppose we have an algebraic functor $K : \mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathcal{X}) \to \mathrm{Prod}(\mathbb{R}^{\mathrm{op}}, \mathcal{X})$, then the problem of finding a left adjoint to K is similar to that of left Kan extension. The difference however is that Kan extensions, as they are defined, do not necessarily preserve products. Hence, we cannot immediately solve the problem. However, we do have the following result.

Theorem 4.1.1 (**$\mathrm{Lan}_J F$ preserves finite products**) *Suppose a functor $F : \mathcal{A} \to \mathcal{X}$ has a left Kan extension along $J : \mathcal{A} \to \mathcal{B}$, that $\mathcal{A}$ and $\mathcal{B}$ have finite products and $\mathcal{X}$ is cartesian closed.*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "J"] \arrow[dr, "F"'] & & \mathcal{B} \arrow[dl, "\mathrm{Lan}_J F"] \\ & \mathcal{X} & \end{tikzcd}$$

If F preserves finite products then $\mathrm{Lan}_J F$ preserves finite products.

Proof Let $K = \mathrm{Lan}_J F$. Then we have the sequence of isomorphisms below, each of which is natural in b, c and x:

$$
\begin{aligned}
&\mathcal{X}(Kb \times Kc, x) \\
&\cong \mathcal{X}(Kb, x^{Kc}) && (\mathcal{X} \text{ is cartesian closed}) \\
&\cong \mathrm{Nat}_a(\mathcal{B}(Ja, b), \mathcal{X}(Fa, x^{Kc})) && (\text{Corollary 2.2.3}) \\
&\cong \mathrm{Nat}_a(\mathcal{B}(Ja, b), \mathcal{X}(Kc, x^{Fa})) && (\mathcal{X} \text{ is cartesian closed}) \\
&\cong \mathrm{Nat}_a(\mathcal{B}(Ja, b), \mathrm{Nat}_{a'}(\mathcal{B}(Ja', c), \mathcal{X}(Fa', x^{Fa}))) && (\text{Corollary 2.2.3}) \\
&\cong \mathrm{Nat}_{a,a'}(\mathcal{B}(Ja, b) \times \mathcal{B}(Ja', c), \mathcal{X}(Fa', x^{Fa})) && (\mathbf{Set} \text{ is cartesian closed}) \\
&\cong \mathrm{Nat}_{a,a'}(\mathcal{B}(Ja, b) \times \mathcal{B}(Ja', c), \mathcal{X}(Fa \times Fa', x)) && (\mathcal{X} \text{ is cartesian closed}) \\
&\cong \mathrm{Nat}_{a,a'}(\mathcal{B}(Ja, b) \times \mathcal{B}(Ja', c), \mathcal{X}(F(a \times a'), x)) && (F \text{ preserves products}) \\
&\cong \mathrm{Nat}_{a''}(\mathcal{B}(Ja'', b) \times \mathcal{B}(Ja'', b), \mathcal{X}(Fa'', x)) && (\text{Theorem 2.1.5}) \\
&\cong \mathrm{Nat}_{a''}(\mathcal{B}(Ja'', b \times c), \mathcal{X}(Fa'', x)) && (\mathcal{B} \text{ has finite products}) \\
&\cong \mathcal{X}(K(b \times c), x) && (\text{Corollary 2.2.3})
\end{aligned}
$$

The sixth isomorphism is achieved by noting that a natural transformation is natural in two variables if and only if it is natural in each variable separately. The ninth isomorphism is obtained by applying Theorem 2.1.5 to the functors in the diagram below.

$$
\begin{tikzcd}
\mathcal{A}^{\mathrm{op}} \arrow[rr, bend right, "\Delta"'] \arrow[dr, "\mathcal{X}(F-{,}x)"'] & & (\mathcal{A} \times \mathcal{A})^{\mathrm{op}} \arrow[ll, bend right, "-\times\sim"'] \arrow[ll, phantom, "\perp"] \arrow[dl, "\mathcal{B}(J-{,}b)\times\mathcal{B}(J\sim{,}c)"] \\
& \mathbf{Set} &
\end{tikzcd}
$$

By considering the terminal object 1 in $\mathcal{B}$, we get the following sequence of natural isomorphisms, each of which is natural in x.

$$
\begin{aligned}
&\mathcal{X}(K1, x) \\
&\cong \mathrm{Nat}_a(\mathcal{B}(Ja, 1), \mathcal{X}(Fa, x)) && (\text{Corollary 2.2.3}) \\
&\cong \mathrm{Nat}_a(1, \mathcal{X}(Fa, x)) && (1 \text{ is terminal}) \\
&\cong \mathrm{Nat}_a(\mathcal{A}(a, 1), \mathcal{X}(Fa, x)) && (1 \text{ is terminal}) \\
&\cong \mathcal{X}(F1, x) && (\text{Yoneda's Lemma}) \\
&\cong \mathcal{X}(1, x) && (F \text{ preserves products})
\end{aligned}
$$

Then since each resultant isomorphism above is natural in x, by Yoneda's lemma, $Kb \times Kc \cong K(b \times c)$ and $K1 = 1$. Hence $K \cong \mathrm{Lan}_J F$ preserves finite products.

■

4.2 Algebraic functors have left adjoints

Though this result is helpful, it does not yet complete the picture. We cannot be sure that the Kan extension actually exists, let alone whether it can be restricted to act on the category of models as we require. However, as we choose to consider only models in **Set**, we have the following result.

Theorem 4.2.1 *Suppose that* $\mathbb{T}$ *and* $\mathbb{R}$ *have finite coproducts and* $J : \mathbb{T}^{\mathrm{op}} \to \mathbb{R}^{\mathrm{op}}$ *is finite product preserving. Then the functor "compose with* J*",*

$$\mathrm{res}_J^* : \mathrm{Prod}(\mathbb{R}^{\mathrm{op}}, \mathbf{Set}) \to \mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set})$$

has a left adjoint.

Proof Begin by considering the following diagram,

$$\begin{tikzcd}
{[\mathbb{R}^{\mathrm{op}}, \mathbf{Set}]} \arrow[r, bend right, "\mathrm{res}_J"'] \arrow[r, phantom, "\perp"] & {[\mathbb{T}^{\mathrm{op}}, \mathbf{Set}]} \arrow[l, bend right, "\mathrm{Lan}_J"'] \\
\mathrm{Prod}(\mathbb{R}^{\mathrm{op}}, \mathbf{Set}) \arrow[u] \arrow[r, bend right, "\mathrm{res}_J^*"'] \arrow[r, phantom, "\perp"] & \mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set}) \arrow[u] \arrow[l, bend right, "\mathrm{Lan}_J^*"']
\end{tikzcd}$$

where the vertical arrows are inclusions.

Since **Set** is cocomplete and $\mathcal{A}$ is small, res_J has a left adjoint given by the left Kan extension (Corollary 2.2.2), call it Lan_J. Since J is product preserving, res_J sends product preserving functors to product preserving functors and so can be restricted to act just on $\mathrm{Prod}(\mathbb{R}^{\mathrm{op}}, \mathbf{Set})$, this restriction is precisely the functor res_J^*. Now by Theorem 4.1.1, Lan_J sends product preserving functors to product preserving functors and so it too can be restricted to act just on $\mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set})$, call this restriction Lan_J^*. Now because the inclusions are full and faithful and Lan_J is left adjoint to res_J, Lan_J^* is left adjoint to res_J^*. Thus, res_J^* has a left adjoint.

■

4.3 An example: groups to monoids

Consider the forgetful functor

$$U : \mathbf{Grp} \to \mathbf{Mon}$$

which sends each group in **Grp** to its underlying monoid in **Mon** by ignoring inverses. Note also that each homomorphism in **Grp** is mapped into **Mon** without change, for it is simply a function on sets.

Let $\mathbb{G}$ and $\mathbb{M}$ denote the Lawvere theories for groups and monoids respectively. Then the generating object 1 in $\mathbb{G}^{\mathrm{op}}$ has group structure and therefore monoid structure (just ignore the inverse maps), hence it belongs to $\mathbf{Mon}(\mathbb{G}^{\mathrm{op}})$. Then remembering the equivalence,

$$\phi : \mathbf{Mon}(\mathbb{G}^{\mathrm{op}}) \simeq \mathrm{Prod}(\mathbb{M}^{\mathrm{op}}, \mathbb{G}^{\mathrm{op}})\,,$$

let $J = \phi(1) : \mathbb{M}^{\mathrm{op}} \to \mathbb{G}^{\mathrm{op}}$ be the functor in $\mathrm{Prod}(\mathbb{M}^{\mathrm{op}}, \mathbb{G}^{\mathrm{op}})$ corresponding to 1 in $\mathbf{Mon}(\mathbb{G}^{\mathrm{op}})$. The resulting algebraic functor $\mathrm{res}_J^* : \mathbf{Grp} \to \mathbf{Mon}$ is prescisely the forgetful functor U.

It follows immediately from Theorem 4.2.1 that since U is an algebraic functor, it has a left adjoint specified by the left Kan extension. The adjoint Lan_J^* is the construction "free group on a monoid" and can be described explicitly by making use of the colimit formula for left Kan extension given in Theorem 2.2.1. This example U also has a right adjoint taking each monoid to the group of invertible elements inside it. This is unusual because algebraic functors do not, in general, have right adjoints.

An excellent list of further examples can be found in [3] pp148–149. We list just four here.

1. The forgetful functor from abelian groups to groups is algebraic. The left adjoint is most commonly called "factor commutator group".

2. The forgetful functor from commutative R-algebras to R-modules is algebraic. The left adjoint is most commonly called "symmetric algebra".

3. The forgetful functor from anti-commutative R-algebras to R-modules is algebraic. The left adjoint is most commonly called "exterior algebra".

4. The functor from R-algebras to R-Lie-algebras defined by letting $[a\,,b] = ab - ba$ is algebraic. The left adjoint is most commonly called "universal enveloping algebra".

Chapter 5

Applications to universal algebra

The result given in the previous chapter is an important one. Using Theorem 4.2.1 alone we can prove quite significant results that can be applied to any category of algebras definable by a Lawvere theory $\mathbb{T}$. First, the category of models for a Lawvere theory is complete (and cocomplete). Second, there is a left adjoint to the forgetful functor on $\mathbb{T}$-algebras which is the free $\mathbb{T}$-algebra construction.

5.1 Families

Before exploring these results, we need to define a standard categorical construction called the category of finite families. For any category $\mathcal{A}$, there is the category of finite families of $\mathcal{A}$, call it $\mathrm{Fam}_f\,\mathcal{A}$. It is in fact the coproduct completion of $\mathcal{A}$.

Definition Let $\mathcal{A}$ be a category, then construct $\mathrm{Fam}_f\,\mathcal{A}$ as follows. The objects in $\mathrm{Fam}_f\,\mathcal{A}$ are finite families of objects in $\mathcal{A}$, that is, tuples $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ where $a_i \in \mathcal{A}$ for $i = 1, 2, \ldots, n$. An arrow from $\mathbf{a} = (a_1, \ldots, a_n)$ to $\mathbf{b} = (b_1, \ldots, b_m)$ in $\mathrm{Fam}_f\,\mathcal{A}$ is a pair $(\xi, \mathbf{f})$ where $\xi : \langle n\rangle \to \langle m\rangle$, $\mathbf{f} = (f_1, \ldots, f_n)$ and $f_i : a_i \to b_{\xi(i)}$ for $i = 1, 2, \ldots, n$.

It should be noted that Fam_f has all finite coproducts; for the empty family $()$ is always in $\mathrm{Fam}_f\,\mathcal{A}$ and is initial and each family $\mathbf{a} = (a_1, \ldots, a_n)$ is a coproduct of the 1-families (a_i) and we have injections

$$(a_i) \xrightarrow{(i^*, I_{a_i})} (a_1, \ldots, a_n) \qquad i = 1, 2, \ldots, n$$

where $i^* : \langle 1\rangle \to \langle n\rangle$ is the map $1 \mapsto i$.

There is also the canonical functor

$$\mathcal{A} \xrightarrow{N} \mathrm{Fam}_f\,\mathcal{A}$$

which sends each $a \in \mathcal{A}$ to the one object family (a) and each arrow $f : a \to b$ to the pair $(I_{\langle 1\rangle}, (f))$.

If $\mathcal{A}$ has all finite coproducts then N has a left inverse, call it L

$$\mathrm{Fam}_f\,\mathcal{A} \xrightarrow{L} \mathcal{A}$$

which sends each family $(a_1, \ldots, a_n)$ to the coproduct $a_1 + \ldots + a_n$ in $\mathcal{A}$ and each arrow $(\xi, \mathbf{f}) : \mathbf{a} \to \mathbf{b}$ to the unique map in the following diagram.

$$\begin{tikzcd} a_i \arrow[r, "\iota_i"] \arrow[d, "f_i"'] & a_1 + \ldots + a_n \arrow[d, dotted, "L(\xi,\mathbf{f})"] \\ b_{\xi(i)} \arrow[r, "\iota_{\xi(i)}"'] & b_1 + \ldots + b_m \end{tikzcd}$$

It is easily observed that $L \circ N = I_{\mathcal{A}}$. A more interesting fact however is that L preserves finite coproducts. Observe that when L acts on the injections $\iota_i : (a_i) \to (a_1, \ldots, a_n)$ we have the following diagram,

$$\begin{tikzcd} a_i \arrow[r, "\iota_1"] \arrow[d, "I_{a_i}"'] & a_i \arrow[d, "L\iota_i"] \\ a_i \arrow[r, "\iota_i"'] & a_1 + \ldots + a_n \end{tikzcd}$$

which commutes, hence $L\iota_i = \iota_i$. Finally, L also maps the empty family () to the empty coproduct and so L preserves finite coproducts.

Proposition 5.1.1 *Suppose a category $\mathcal{X}$ has finite products, then there is an equivalence*

$$\mathrm{Prod}((\mathrm{Fam}_f\,\mathcal{A})^{\mathrm{op}}, \mathcal{X}) \simeq [\mathcal{A}^{\mathrm{op}}, \mathcal{X}].$$

Proof Let $\phi : \mathrm{Prod}(\mathrm{Fam}_f\,\mathcal{A}^{\mathrm{op}}, \mathcal{X}) \to [\mathcal{A}^{\mathrm{op}}, \mathcal{X}]$ be defined on objects by sending the functor $K \in \mathrm{Prod}((\mathrm{Fam}_f\,\mathcal{A})^{\mathrm{op}}, \mathcal{X})$ to $K \circ N$. That is, if $\phi(K) = L$ then $La = K(a)$ and $Lf = K(I_{\langle 1 \rangle}, (f))$ as shown.

$$\begin{tikzcd} a \arrow[d, "f"] & K(a) \arrow[d, "K(I_{\langle 1 \rangle},(f))"] \\ b & K(b) \end{tikzcd}$$

If $\alpha : K \to M$ in $\mathrm{Prod}(\mathrm{Fam}_f\,\mathcal{A}^{\mathrm{op}}, \mathcal{X})$, then define ϕ on natural transformations by $\phi(\alpha)_a = \alpha_{(a)}$. That is, if $\phi(\alpha) = \beta$ then $\beta_a = \alpha_{(a)}$ as shown.

$$K(a) \xrightarrow{\alpha_{(a)}} M(a)$$

We aim to show that ϕ is essentially surjective and fully faithful.

Suppose we have a functor $G \in [\mathcal{A}, \mathcal{X}]$, then there exists a functor F in $\mathrm{Prod}((\mathrm{Fam}_f\,\mathcal{A})^{\mathrm{op}}, \mathcal{X})$ that sends $(a_1, \ldots, a_n)$ to $Ga_1 \times \ldots \times Ga_n$ and sends $(\xi, \mathbf{f}) : \mathbf{a} \to \mathbf{b}$ to the unique arrow making the diagram on the right

$$\begin{tikzcd} (a_1, \ldots, a_n) \arrow[d, "(\xi,\mathbf{f})"] & Ga_i & Ga_1 \times \ldots \times Ga_n \arrow[l, "\pi_i"'] \\ (b_1, \ldots, b_m) & Gb_{\xi(i)} \arrow[u, "Gf_i"] & Gb_1 \times \ldots \times Gb_m \arrow[l, "\pi_{\xi(i)}"'] \arrow[u, dotted, "F(\xi,\mathbf{f})"'] \end{tikzcd}$$

commute for $i = 1, 2, \ldots, n$. By observing how F behaves on composites, identities and injections in $\mathrm{Fam}_f\,\mathcal{A}^{\mathrm{op}}$ one can easily show that F is well defined and preserves finite products. Then since $\phi(F) = F \circ N$, we get $\phi(F)a = Ga$ and the following commuting diagram,

$$\begin{tikzcd} a \arrow[d, "f"] & Ga & Ga \arrow[l, "\pi_1"'] \\ b & Gb \arrow[u, "Gf"] & Gb \arrow[l, "\pi_1"] \arrow[u, dotted, "F(I,(f))"'] \end{tikzcd}$$

so $\phi(F)f = F(I,(f)) = Gf$. Thus, $\phi(F) = G$ and ϕ is essentially surjective.

Suppose that $\alpha, \beta : M \to N$ in $\mathrm{Prod}((\mathrm{Fam}_f\,\mathcal{A})^{\mathrm{op}}, \mathcal{X})$, then by naturality of α we get the diagrams,

$$\begin{tikzcd} (a_1, \ldots, a_n) & M(a_1) \times \ldots \times M(a_n) \arrow[r, "\alpha_{\mathbf{a}}"] \arrow[d, "\pi_i"'] & N(a_1) \times \ldots \times N(a_n) \arrow[d, "\pi_i"'] \\ (a_i) \arrow[u, "\iota_i"] & M(a_i) \arrow[r, "\alpha_{(a_i)}"'] & N(a_i) \end{tikzcd}$$

which commute for $i = 1, \ldots, n$. The injections are mapped to projections because M and N preserve finite products (as covariant functors). The diagram above demonstrates that the component of α at some $(a_1, \ldots, a_n)$ is completely determined by the components $\alpha_{(a_i)}$ for $i = 1, \ldots, n$. Suppose that $\phi(\alpha) = \phi(\beta)$, then $\alpha_{(a)} = \beta_{(a)}$ for all $a \in \mathcal{A}$ and hence $\alpha_{\mathbf{a}} = \beta_{\mathbf{a}}$ for all $\mathbf{a} \in \mathrm{Fam}_f\,\mathcal{A}$. Therefore $E(\alpha) = E(\beta)$ implies $\alpha = \beta$ and ϕ is faithful.

In the same way as above, suppose that $M, N \in \mathrm{Prod}(\mathrm{Fam}_f\,\mathcal{A}^{\mathrm{op}}, \mathcal{X})$ and we have $\gamma : \phi(M) \to \phi(N)$, then we can define a natural transformation $\delta : M \to N$ by letting $\delta_{(a)} = \gamma_a$ for all $a \in \mathcal{A}$ and defining $\delta_{\mathbf{a}}$ to be the unique arrow making

$$\begin{tikzcd} M(a_1) \times \ldots \times M(a_n) \arrow[r, "\delta_{\mathbf{a}}"] \arrow[d, "\pi_i"'] & N(a_1) \times \ldots \times N(a_n) \arrow[d, "\pi_i"'] \\ M(a_i) \arrow[r, "\gamma_{a_i}"'] & N(a_i) \end{tikzcd}$$

commute for $i = 1, \ldots, n$. It can be verified that δ is a well defined natural transformation. Then $\phi(\delta) = \gamma$ and ϕ is full.

■

This proof followed very similar lines to that in Proposition 3.4.2, in fact it is a more general result. If $\mathcal{A}$ is the category with one object and one arrow, then $\mathrm{Fam}_f\,\mathcal{A} \cong \mathbb{S}$ and $[\mathcal{A}^{\mathrm{op}}, \mathcal{X}] \cong \mathcal{X}$, so $\mathrm{Prod}(\mathbb{S}^{\mathrm{op}}, \mathcal{X}) \simeq \mathcal{X}$.

5.2 The category of $\mathbb{T}$-algebras is complete

It is clear that $\mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set})$ is a full subcategory of $[\mathbb{T}^{\mathrm{op}}, \mathbf{Set}]$ and hence there is a canonical inclusion functor

$$\mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set}) \xrightarrow{Q} [\mathbb{T}^{\mathrm{op}}, \mathbf{Set}]\ .$$

Theorem 5.2.1 *For any Lawvere theory $\mathbb{T}$, the inclusion*

$$\mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set}) \xrightarrow{Q} [\mathbb{T}^{\mathrm{op}}, \mathbf{Set}]$$

has a left adjoint.

Proof The proof is obtained by contemplating the following diagram.

$$\begin{tikzcd}
\mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set}) \arrow[r, shift right, "\mathrm{res}_L"'] \arrow[r, phantom, "\perp"] \arrow[rr, bend right, "Q"'] & \mathrm{Prod}((\mathrm{Fam}_f\,\mathbb{T})^{\mathrm{op}}, \mathbf{Set}) \arrow[l, shift right, "\mathrm{Lan}_L"'] \arrow[r, shift right, "\mathrm{res}_N"'] \arrow[r, phantom, "\perp"] & {[\mathbb{T}^{\mathrm{op}}, \mathbf{Set}]} \arrow[l, shift right, "S"'] \arrow[ll, bend right]
\end{tikzcd}$$

The adjunction on the left is given by application of Theorem 4.2.1 to the coproduct preserving functor L. The adjunction on the right is the adjoint equivalence given by Proposition 5.1.1, call the left adjoint S. Since adjunctions can be composed, it remains only to show that the composite $\mathrm{res}_N \circ \mathrm{res}_L$ is the inclusion functor. This is straight forward because $L \circ N = I$ implies that $\mathrm{res}_N \circ \mathrm{res}_L = \mathrm{res}_I = Q$. Thus the inclusion Q has a left adjoint.

■

The following result comes as a corollary.

Corollary 5.2.2 *Suppose $\mathbb{T}$ is a Lawvere theory, then*

$$\mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set})$$

is complete and cocomplete.

Proof In Theorem 5.2.1 we proved that the inclusion

$$Q : \mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set}) \to [\mathbb{T}^{\mathrm{op}}, \mathbf{Set}]$$

has a left adjoint, and hence $\mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set})$ is a full reflective subcategory of $[\mathbb{T}^{\mathrm{op}}, \mathbf{Set}]$. We know also that **Set** is complete (and cocomplete), therefore $[\mathbb{T}^{\mathrm{op}}, \mathbf{Set}]$ is complete (and cocomplete). Finally, every full reflective subcategory of a complete (and cocomplete) category is also complete (and cocomplete); see [1] exercise 3, p120 for the cocomplete case [1].

■

This is a significant result. The notions of product, coproduct, equalizer and coequalizer are all examples of limits and colimits, and hence exist in every category of $\mathbb{T}$-algebras. For example, in **Ab**, product is the direct product $G \times H$, coproduct is direct sum $G \oplus H$ (isomorphic to direct product in this case), the equalizer object is the subgroup $\{x \in G : \ f(x) = g(x)\}$ and the coequalizer object is the factor group $G/\operatorname{im}(f - g)$. If the category of $\mathbb{T}$-algebras has a zero object (that is, both initial and terminal) then the category also has kernels and cokernels.

[1] The complete case is also well known; see [8] Corollary 8.

5.3 The free $\mathbb{T}$-algebra construction

Suppose $\mathbb{T}$ is a Lawvere theory with generating object 1. Then consider the functor U with domain $\mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set})$ which is "evaluate at 1" (in some previous cases we called this functor E). The functor U sends $F : \mathbb{T}^{\mathrm{op}} \to \mathbf{Set}$ to $F1$ and $\alpha : F \to G$ to $\alpha_1 : F1 \to G1$. We say that U is forgetful because it takes each model (with structure from $\mathbb{T}$) to a set (with no structure). We choose to name this functor with the letter "U" because it takes each model to it's "underlying" set. We have enough preliminary results to prove the following theorem.

Theorem 5.3.1 *For any Lawvere theory $\mathbb{T}$, the forgetful functor*

$$U : \mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set}) \longrightarrow \mathbf{Set}$$

has a left adjoint.

Proof Let 1 denote the generating object of $\mathbb{T}$. Now consider the product preserving functor $J : \mathbb{S}^{\mathrm{op}} \to \mathbb{T}^{\mathrm{op}}$ defined on objects by $Jn = 1^n$ and on arrows by sending $f : n \to m$ to the unique arrow making,

$$\begin{tikzcd} 1^n \arrow[dr, "\pi_i"'] & & 1^m \arrow[ll, dotted, "Jf"'] \arrow[dl, "\pi_{f(i)}"] \\ & 1 & \end{tikzcd}$$

commute for $i = 1, 2, \ldots, n$. This is precisely the functor M_1 in the proof of Proposition 3.4.2, where it was also shown to be well defined and product preserving. The rest of the proof then comes from contemplating the following diagram.

$$\begin{tikzcd} \mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set}) \arrow[r, bend right, "\mathrm{res}_J"'] \arrow[r, phantom, "\perp"] & \mathrm{Prod}(\mathbb{S}^{\mathrm{op}}, \mathbf{Set}) \arrow[l, bend right, "\mathrm{Lan}_J"'] \arrow[r, bend right, "E"'] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, bend right] \end{tikzcd}$$

Using Theorem 4.1.1 with the product preserving functor J we obtain the adjunction shown on the left. Then by Proposition 3.4.2 we get the adjoint equivalence shown on the right. The functor E is "evaluate at $\langle 1 \rangle$". These two adjunctions compose, so the composite $E \circ \mathrm{res}_J$ has a left adjoint.

Now if $F \in \mathrm{Prod}(\mathbb{T}^{\mathrm{op}}, \mathbf{Set})$ then

$$(E \circ \mathrm{res}_J)F = E(F \circ J) = (F \circ J)1 = F(J\,\langle 1 \rangle) = F1$$

and if $\alpha : F \to G$ then

$$((E \circ \mathrm{res}_J)\alpha) = E(\alpha J) = \alpha_{J\langle 1 \rangle} = \alpha_1.$$

Thus, $E \circ \mathrm{res}_J$ is precisely the forgetful functor U and it has a left adjoint.

■

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