Mackey functors

The talk has three parts:

I. a categorical view of the definitions;

II. structure of the category of Mackey functors;

III. application to classifying spaces of Lie groups.

Parts I and II are joint work with Elango Panchadcharam

Part III is work of Jackowski–McClure:

Homotopy decomposition of classifying spaces via elementary abelian subgroups, *Topology* **31** (1992) 113–132

Part I The definitions

Recall that left Kan extension Lan_i along a functor $i : H \longrightarrow G$ is the functor between functor categories which is left adjoint to restriction Res_i along i:

$$\begin{tikzcd} {[G, \mathcal{M}]} \arrow[r, shift right, "\mathrm{Res}_i"'] \arrow[r, phantom, "\perp"] & {[H, \mathcal{M}]} \arrow[l, shift right, "\mathrm{Lan}_i"'] \end{tikzcd}$$

$$(G \xrightarrow{V} \mathcal{M}) \longmapsto (H \xrightarrow{i} G \xrightarrow{V} \mathcal{M})$$

When $\mathcal{M}$ is cocomplete and i satisfies smallness requirements, the natural transformation λ in the comma category square below induces a natural isomorphism

$$\mathrm{Res}_j \circ \mathrm{Lan}_i \cong \mathrm{Lan}_q \circ \mathrm{Res}_p .$$

$$\begin{tikzcd} i \downarrow j \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & K \arrow[d, "j"] \arrow[ddl, bend left=60, "\mathrm{Lan}_q(Wp)"] \\ H \arrow[r, "i"'] \arrow[d, "W"'] \arrow[dr, phantom, "\Rightarrow"] & G \arrow[dl, "\mathrm{Lan}_i W"] \arrow[d, phantom, "\cong"] \\ \mathcal{M} & {} \end{tikzcd}$$

If a category D is a disjoint union

$$D = \sum_{\alpha \in \Lambda} D_\alpha$$

of subcategories $m_\alpha : D_\alpha \longrightarrow D$ then, for all functors $r : D \longrightarrow K$, there is a canonical isomorphism

$$\mathrm{Lan}_r \cong \sum_{\alpha \in \Lambda} \mathrm{Lan}_{r \circ m_\alpha} \circ \mathrm{Res}_{m_\alpha} .$$

$$\begin{tikzcd} D_\alpha \arrow[r, "m_\alpha"] & D \arrow[r, "r"] \arrow[dr, "T"'] & K \arrow[d, "\mathrm{Lan}_r T \cong \sum_\alpha \mathrm{Lan}_{rm_\alpha}(Tm_\alpha)"] \\ & & \mathcal{M} \end{tikzcd}$$

$$\Longrightarrow$$

A groupoid is a category D with every morphism invertible. Each object $d \in D$ determines a group $D(d) = D(d, d)$ under composition. Let Λ be a representative set of objects in D for all isomorphism classes. Then we have an equivalence of categories

$$D \simeq \sum_{d \in \Lambda} D(d).$$

Suppose H and K are subgroups of a group G. We apply the above to the inclusion functors $i : H \to G$, $j : K \to G$, and to $D = i \downarrow j$. The groupoid D has as objects the elements g (morphisms!) of the group G while the morphisms $(h, k) : g \to g'$ are elements of $H \times K$ such that $kg = g'h$. Isomorphism classes of objects of D are in bijection with double cosets

$$KgH = \{kgh \mid k \in K, h \in H\}.$$

Let $[K\backslash G/H] \subseteq G$ represent all double cosets KgH; so we have an equivalence of categories

$$\sum_{g \in [K\backslash G/H]} (i\downarrow j)(g) \simeq i \downarrow j, \quad \text{and}$$

$$\mathrm{Res}_j \circ \mathrm{Lan}_i \cong \sum_{g \in [K\backslash G/H]} \mathrm{Lan}_{q_g} \circ \mathrm{Res}_{p_g}$$

where $p_g : (i\downarrow j)(g) \to H$ and $q_g : (i\downarrow j)(g) \to K$ are restrictions of p and q. The last two restrictions induce isomorphisms

$$(i\downarrow j)(g) \cong H \cap K^g$$

$$(i\downarrow j)(g) \cong {}^gH \cap K$$

where $K^g = g^{-1}Kg$ and ${}^gH = gHg^{-1}$.

Define γ_g by commutativity of

$$
\begin{tikzcd}
{}^{g}H \cap K \arrow[rr, "\gamma_g"] \arrow[dr, "\cong"] \arrow[dd, "\leq"'] & & H \cap K^{g} \arrow[dl, "\cong"'] \arrow[dd, "\leq"] \\
& (i \downarrow j)(g) \arrow[dl, "q_g"] \arrow[dr, "p_g"'] & \\
K & & H
\end{tikzcd}
$$

In representation theory, for $\mathcal{M} = \mathrm{Mod}_k$ (k commutative ring), note that Lan_i and Res_i are called Ind_H^G and Res_H^G.

Mackey Decomposition Theorem

$$\mathrm{Res}_K^G \circ \mathrm{Ind}_H^G \cong \sum_{g \in [K \backslash G / H]} \mathrm{Ind}^K_{{}^gH \cap K} \circ \mathrm{Res}_{\gamma_g} \circ \mathrm{Res}^H_{H \cap K^g}.$$

This inspires the technical axiom 4 in the first definition of Mackey functor which is as follows:

Green's definition of Mackey functor (over k) for a group G

- to each subgroup $H \leqslant G$, a k-module $M(H)$,
- for $K \leq H \leq G$, module morphisms
$$t_K^H : M(K) \longrightarrow M(H) \quad \text{and} \quad r_K^H : M(H) \longrightarrow M(K),$$
- for $H \leqslant G$ and $g \in G$, a module isomorphism
$$c_{g,H} : M(H) \longrightarrow M({}^gH),$$

satisfying the following four axioms:

1. $t_K^H t_L^K = t_L^H$ and $r_L^K r_K^H = r_L^H$ $(L \leq K \leqslant H)$,
2. $c_{g',{}^gH}\, c_{g,H} = c_{g'g,H}$ and $c_{h,H} = 1_{M(H)}$ $(H \leqslant G,\, g, g' \in G,\, h \in H)$,
3. $c_{g,H}\, t_K^H = t_{{}^gK}^{{}^gH}\, c_{g,K}$ and $c_{g,K}\, r_K^H = r_{{}^gK}^{{}^gH}\, c_{g,H}$ $(K \leq H \leqslant G \ni g)$,
4. $r_K^L t_H^L = \sum_{g \in [K\backslash L/H]} t_{{}^gH \cap K}^K\, c_{g, H\cap K^g}\, r_{H \cap K^g}^H$ $(H \leqslant L, K \leqslant L)$.

t_K^H is called <u>transfer</u>, <u>trace</u> or <u>induction</u>,

r_K^H is called <u>restriction</u>, and

$c_{g,H}$ is called a <u>conjugation map</u>.

Suppose now that $\mathcal{M}$ is a cocomplete monoidal category whose tensor preserves colimits in each variable. Suppose $i : H \to G$ is the inclusion of a subgroup $H \leq G$ and that the categories $[H, \mathcal{M}]$ and $[G, \mathcal{M}]$ are equipped with the pointwise tensor products. For $V \in [G, \mathcal{M}]$ and $W \in [H, \mathcal{M}]$, there is an isomorphism

$$V \otimes \mathrm{Lan}_i(W) \cong \mathrm{Lan}_i(\mathrm{Res}_i(V) \otimes W).$$

In the case $\mathcal{M} = \mathrm{Mod}_k$, we obtain

Frobenius Reciprocity

$$V \otimes \mathrm{Ind}_H^G(W) \cong \mathrm{Ind}_H^G(\mathrm{Res}_H^G(V) \otimes W).$$

A <u>Green functor</u> A (<u>over</u> k) <u>for</u> G is a Mackey functor A equipped with a k-algebra structure on each k-module $A(H)$ $(H \leq G)$ satisfying:

5. t_K^H, r_K^H, $c_{g,K}$ are k-algebra morphisms
6. $a \cdot t_K^H(b) = t_K^H(r_K^H(a) \cdot b)$ and
   $t_K^H(b) \cdot a = t_K^H(b \cdot r_K^H(a))$

$$(K \leq H,\ a \in A(H),\ b \in A(K)).$$

Axioms 4 and 6 are obvious abstractions of Mackey Decomposition & Frobenius Reciprocity. Yet, moreover, we obtain an example of a Green functor (over $\mathbb{Z}$) by defining

$$A(H) = K_0 \, \mathrm{Rep}_k(H).$$

Green [JPAA 1 (1971)] looked at the category $\mathcal{S}(G)$ defined as follows:

- objects are subgroups H of G,
- morphisms $g : H \to K$ are elements $g \in G$ satisfying $H^g \leq K$,
- composition is reverse product in G.

$\mathcal{S}(G)$ is equivalent to the category $\mathcal{C}_G$ of connected G-sets (transitive and non-empty). Each $g : H \to K$ determines a G-set morphism

$$G/H \to G/K \quad , \quad xH \mapsto xgK.$$

Green re-expressed:

A Mackey functor M for G defines two functors

$$M^* : \mathcal{S}(G)^{op} \longrightarrow \mathrm{Mod}_k \, , \quad M_* : \mathcal{S}(G) \longrightarrow \mathrm{Mod}_k$$

with $M^*(H) = M_*(H) = M(H)$ and, for $g : H \to K$,

$$\begin{tikzcd} M(K) \arrow[r, "r^K_{H^g}"] \arrow[d, "c_{g,K}"'] \arrow[dr, "M^*(g)"] & M(H^g) \arrow[d, "c_{g,H^g}"] \\ M({}^gK) \arrow[r, "r^{{}^gK}_H"'] & M(H) \end{tikzcd} \qquad \begin{tikzcd} M(H) \arrow[r, "t^{{}^gK}_H"] \arrow[d, "c_{H,g^{-1}}"'] \arrow[dr, "M_*(g)"] & M({}^gK) \arrow[d, "c_{g^{-1},{}^gK}"] \\ M(H^g) \arrow[r] & M(K) \, . \end{tikzcd}$$

Dress [SLNM $\underline{342}$ (1973)] looked at the case of a finite group G and extended M^* and M_* to functors on the category G-set of finite G-sets. Note that G-set is the free completion of $\mathcal{C}_G$ under finite coproducts:

$$G\text{-set} \simeq \mathrm{Fam}_{fin}(\mathcal{C}_G).$$

In this form, the axioms 1 to 4 neatly reduce to two.

A <u>Mackey functor</u> M (over k) for G consists of a pair of functors

$$M^* : G\text{-set}^{op} \longrightarrow \mathrm{Mod}_k \ , \quad M_* : G\text{-set} \longrightarrow \mathrm{Mod}_k$$

which agree $M(X) := M^*(X) = M_*(X)$ on objects X of G-set and satisfy:

1. $M^*(\beta)M_*(\alpha) = M_*(\delta)M^*(\gamma)$ for every pullback

$$\begin{tikzcd} P \arrow[r, "\delta"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd} ,$$

2. $$M(X) \underset{M_*(i)}{\overset{M^*(i)}{\rightleftarrows}} M(X+Y) \underset{M_*(j)}{\overset{M^*(j)}{\rightleftarrows}} M(Y)$$

is a direct sum situation in Mod_k for every coproduct $X \xrightarrow{i} X+Y \xleftarrow{j} Y$ in G-set.

Harald Lindner [<u>Manuscripta Math.</u> 18 (1976)] realized M^* and M_* could be combined into a single functor defined on spans in G-set.

A category $\mathcal{E}$ is <u>lextensive</u> when it has finite coproducts, finite limits, and

$$\mathcal{E}/X \times \mathcal{E}/Y \xrightarrow{\sim} \mathcal{E}/X+Y$$

$$(U \to X,\ V \to Y) \longmapsto (U+V \longrightarrow X+Y) .$$

Definition of the autonomous monoidal, commutative monoid enriched category $\mathrm{Spn}(\mathcal{E})$:

the objects are those of $\mathcal{E}$;

the morphisms $[S] : X \to Y$ are isomorphism classes of "spans" $X \xleftarrow{u} S \xrightarrow{v} Y$;

composition is by pullback

$$\begin{tikzcd}[column sep=small] & & S \times_Y T \arrow[dl] \arrow[dr] & & \\ & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd};$$

using lextensivity, coproduct in $\mathcal{E}$ becomes direct sum in $\mathrm{Spn}(\mathcal{E})$

$$\mathrm{Spn}(\mathcal{E})(X, Y+Z) \cong \mathrm{Spn}(\mathcal{E})(X,Y) \times \mathrm{Spn}(\mathcal{E})(X,Z)$$
$$\mathrm{Spn}(\mathcal{E})(X+Y, Z) \cong \mathrm{Spn}(\mathcal{E})(X,Z) \times \mathrm{Spn}(\mathcal{E})(Y,Z);$$

cartesian product in $\mathcal{E}$ gives the tensor product in $\mathrm{Spn}(\mathcal{E})$

$$\left(\begin{tikzcd}[column sep=small] & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}\right) \times \left(\begin{tikzcd}[column sep=small] & S' \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}\right) = \begin{tikzcd}[column sep=small] & S \times S' \arrow[dl] \arrow[dr] & \\ X \times X' & & Y \times Y' \end{tikzcd};$$

each object is its own dual

$$\mathrm{Spn}(\mathcal{E})(X \times Y, Z) \cong \mathrm{Spn}(\mathcal{E})(Y, X \times Z).$$

A <u>Mackey functor on</u> $\mathcal{E}$ is an additive functor

$$M : \mathrm{Spn}(\mathcal{E}) \longrightarrow \mathrm{Mod}_k .$$

($\mathcal{E} = G$-set gives back Dress' definition)

Part II The category of Mackey functors

$$\mathrm{Mky}_k(\mathscr{E}) = [\mathrm{Spn}(\mathscr{E}), \mathrm{Mod}_k]_{add}$$

is the category of Mackey functors on $\mathscr{E}$.
It is closed monoidal by Day convolution:

$$(M * N)(Z) = \int^Y M(Z \times Y) \otimes_k NY$$

$$\mathrm{Hom}(M,N)(Z) = \mathrm{Mky}_k(\mathscr{E})(M(Z\times -), N)$$

The unit J is the <u>Burnside functor</u>:

JX is the free k-module on the commutative monoid (under coproduct) of isomorphism classes of $\mathscr{E}/X$.

<u>Green functors</u> A are monoids in $\mathrm{Mky}_k(\mathscr{E})$:

$$AX \otimes_k AY \xrightarrow{\mu} A(X \times Y), \quad k \xrightarrow{\eta} A1$$

<u>Fact</u> A monoid Z in the monoidal lax centre $\mathscr{Z}_\ell \mathscr{C}$ of a monoidal category $\mathscr{C}$ becomes a monoidal functor $Z \otimes - : \mathscr{C} \to \mathscr{C}$.

<u>Remark</u> $\mathscr{Z}_\ell(G\text{-set}) = \mathscr{Z}(G\text{-set}) = G\text{-set}/G_{conj}$

The Dress construction:

$Z\times - : \mathscr{E} \longrightarrow \mathscr{E}$ preserves pullbacks and finite coproducts and so induces an additive functor

$$Z\times - : \mathrm{Spn}(\mathscr{E}) \longrightarrow \mathrm{Spn}(\mathscr{E}).$$

Each Mackey functor M on $\mathscr{E}$ defines a Mackey functor

$$M_Z := M\circ(Z\times -) : \mathrm{Spn}(\mathscr{E}) \longrightarrow \mathrm{Mod}_k .$$

Consequently, if Z is a monoid in $\mathcal{Z}_\ell\,\mathscr{E}$ (where $\mathscr{E}$ has the cartesian monoidal structure) then each Green functor A on $\mathscr{E}$ defines a Green functor

$$A_Z = \left(\mathrm{Spn}(\mathscr{E}) \xrightarrow{Z\times -} \mathrm{Spn}(\mathscr{E}) \xrightarrow{\ A\ } \mathrm{Mod}_k\right).$$

A monoidal category $\mathscr{C}$ is <u>$*$-autonomous</u> (Barr) when there exists an equivalence of categories $S : \mathscr{C}^{op} \longrightarrow \mathscr{C}$ and a natural isomorphism

$$\mathscr{C}(B, SA) \cong \mathscr{C}(I, S(A\otimes B)).$$

<u>Case</u>: $\mathscr{E} = G$-set, k field

A Mackey functor M is <u>finite</u> when each MX is finite dimensional. Write fMky_k for the full subcategory of such.

<u>Theorem</u> [PS] <u>fMky_k is a monoidal full subcategory of</u> Mky_k <u>which is $*$-autonomous with</u> $S(M)X = (MX)^*$.

Let fRep_k denote category of finite dimensional k-linear representations of G. We have the functor

$$\tilde{k}_* : \mathrm{fRep}_k \longrightarrow \mathrm{fMky}_k$$
$$R \longmapsto G\text{-Set}(-, R)$$

where $(u, S, v) : X \to\!\!\!\!\!+ \; Y$ goes to

$$G\text{-Set}(X, R) \longrightarrow G\text{-Set}(Y, R)$$
$$\tau \longmapsto \tau_S$$
$$\tau_S(y) = \sum_{v(s)=y} \tau(u(s))$$

<u>Proposition</u> $\tilde{k}_*$ <u>is fully faithful with a strong monoidal left adjoint.</u>

Modules for Green functors

We study Morita equivalence of Green functors in terms of adjoint two-sided (bi)modules rather than "Morita contexts".

Put $\mathcal{W} = Mky_k(\mathcal{E})$ as a symmetric cocomplete closed monoidal category.

Let $\mathcal{M} = Mod(\mathcal{W})$ be the monoidal bicategory whose objects are Green functors (monoids in $\mathcal{W}$), whose morphisms $M : A \nrightarrow B$ are modules

$$AU \otimes_k MV \otimes_k BW \longrightarrow M(U \times V \times W), \text{ and}$$

whose composition $N \circ M = M \ast_B N$ is obtained as the expected coequalizer.

We define Green functors A and B to be <u>Morita equivalent</u> when they are equivalent in $\mathcal{M}$.

Each Green functor A has a $\mathcal{W}$-category $\hat{A}$ of right A-modules where $\hat{A}(M, N) \in \mathcal{W}$ is defined by the expected equalizer.

The <u>Cauchy completion</u> $Q A$ of A is the full sub-$\mathcal{W}$-category of $\hat{A}$ consisting of the modules $M : J \nrightarrow A$ with right adjoints in $\mathcal{M}$.

<u>Theorem</u> [PS] $M \in QA$ <u>iff</u> $\exists Z_1, \ldots, Z_k \in \mathrm{Spn}(\mathcal{E})$ <u>such that</u> M <u>is a retract of</u>

$$\bigoplus_{i=1}^{k} A_{Z_i} .$$

<u>Theorem</u> <u>Green functors</u> A <u>and</u> B <u>are Morita equivalent iff</u> $QA \simeq QB$ <u>as</u> $\mathcal{W}$-<u>categories.</u> <u>This implies an equivalence of ordinary categories</u>

$$\mathrm{Mod}\,A \simeq \mathrm{Mod}\,B$$

<u>of right modules.</u>

($\mathrm{Mod}\,A = \mathcal{M}(J, A)$ is the underlying category of $\hat{A}$)

Part III Application?

Reference: Jackowski – McClure 1992

Here G is a compact (connected) Lie group.

BG is the classifying space of G
(homotopy classes of continuous functions $X \to BG$ are in bijection with isomorphism classes of G-fibre bundles over X).

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "pb"] & EG \arrow[d, "p"] \\ X \arrow[r, "f"'] & BG \end{tikzcd}$$ universal G-fibre bundle

For some classical examples of G there are nice explicit models of BG.

Calculating the cohomology groups of BG is a major research problem.

A. Borel (1967) Let NT denote the normalizer of a maximal torus T of G. If the prime p does not divide the order of the Weyl group $W_G = NT/T$ then $BNT \to BG$ induces an isomorphism on mod p cohomology.

<u>Dwyer–Miller–Wilkerson</u> (1987) $G = SO(3)$, $p = 2 \mid \# W_G$

<u>The square</u>

$$\begin{tikzcd} BD_8 \arrow[r] \arrow[d] & BO_{24} \arrow[d] \\ BO(2) \arrow[r] & BSO(3) \end{tikzcd}$$

is seen as a homotopy pushout by mod 2 cohomology.

<u>Jackowski–McClure</u> (1992) (any prime p)

Consider the category $\mathcal{A}_p(G)$:

objects are nontrivial elementary abelian p-subgroups of G (they prove there are only finitely many conjugacy classes of these subgroups);

morphisms are restrictions of inner automorphisms of G.

Let $C(E)$ be the centralizer of $E \in \mathcal{A}_p(G)$ in G.

$BC(E)$ is homotopy equivalent to a functor in $E \in \mathcal{A}_p(G)$. The result is:

$$\operatorname{hocolim}_{E \in \mathcal{A}_p(G)^{op}} BC(E) \longrightarrow BG$$

induces an isomorphism on mod p cohomology.

(This implies the Borel and Dwyer-M-W results.)

The proof of [J-McC] uses Mackey and Green functors.

<u>Step 1</u> There is a standard spectral sequence associated with the cohomology of a homotopy colimit. As usual, to use the spectral sequence, we need it to collapse. This involves proving:

(a) $\lim\limits_{E \in \mathcal{A}_p(G)^{op}}{}^{i} H^j(BC(E)) = 0$ for all j and all $i > 0$.
(acyclicity)

(b) $H^*(BG) \cong \lim\limits_{E \in \mathcal{A}_p(G)^{op}} H^*(BC(E))$
induced by restriction.

Here $\lim^i$ is the i-<u>th derived functor</u> of

$$\lim : [\mathcal{A}_p(G)^{op}, Ab] \longrightarrow Ab.$$

Consider a functor $M : \mathcal{B}^{op} \longrightarrow Ab$. Then

$$\lim M = [\mathcal{B}^{op}, Ab](\Delta \mathbb{Z}, M)$$

so the i-th derived functor of $\lim$ is

$$H^i(\mathcal{B}; M) := \mathrm{Ext}^i_{[\mathcal{B}^{op}, Ab]}(\Delta \mathbb{Z}, M).$$

Call M <u>acyclic</u> when $H^i(\mathcal{B}; M) = 0$ for $i > 0$.

Step 2

A functor $M : \mathcal{B}^{op} \longrightarrow Ab$ is <u>proto-Mackey</u> when its extension $M_+ : \mathrm{Fam}_{fin}(\mathcal{B})^{op} \longrightarrow Ab$ extends to a Mackey functor $M : \mathrm{SpnFam}_{fin}(\mathcal{B}) \rightarrow Ab$.

<u>Observation</u> $H^i(\mathcal{B}; M) \cong H^i(\mathrm{Fam}_{fin}(\mathcal{B}); M_+)$

[J-McC] prove that $\mathrm{Fam}_{fin}\mathcal{A}_p(G)$ <u>is extensive enough for Mackey theory</u> to be useful and that $M(E) = H^j(BC(E); \mathbb{Z}/p)$ <u>is a proto-Mackey functor on</u> $\mathcal{A}_p(G)$.

It remains to prove, for this M,

(a) M is acyclic

(b) $H^j(BG; \mathbb{Z}/p) \cong \lim M$

The proof of (b) imitates the Dress induction theorem [<u>SLNM</u> 342 (1973) p. 199] in Mackey functor theory.

Step 3

The proof of (a) follows from

(a)′ every proto-Mackey functor $\mathcal{A}_p(G)^{op} \to \mathbb{Z}/p\text{-mod}$ is acyclic.

The inspiration for the proof of (a)′ comes from two simple lemmas:

Lemma 1 Every functor $M : \mathcal{C}^{op} \to Ab$ is acyclic if $\mathcal{C}$ has a terminal object.

Proof $\lim M = M1$ is exact in M and so the derived functors of $\lim$ vanish. □

Lemma 2 If multiplication by $\#\Gamma$ as a function $M \to M$ is invertible, where Γ is a finite group and M is a Γ-module, then

$$H^i(\Gamma; M) = 0 \quad \text{for all } i > 0.$$

Proof

$$\begin{tikzcd} {[\Gamma^{op}, Ab]} \arrow[r, bend left, "U"] \arrow[r, phantom, "\perp"] & Ab \arrow[l, bend left, "Set(\Gamma, -)"] \end{tikzcd}$$

$$\Delta\mathbb{Z} \in [\Gamma^{op}, Ab], \qquad N \in Ab$$

$$N \cong Ab(\mathbb{Z}, N) \cong [\Gamma^{op}, Ab](\Delta\mathbb{Z}, Set(\Gamma, N))$$

$$\Longrightarrow \quad 0 \cong H^i(\Gamma; Set(\Gamma, N)) \qquad i > 0$$

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend right, "\text{mult. by } \#\Gamma"'] & Set(\Gamma, M) \arrow[r] & M \end{tikzcd}$$

M is a retract of $Set(\Gamma, M)$. □