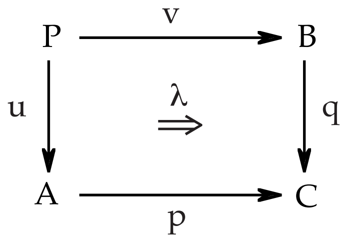
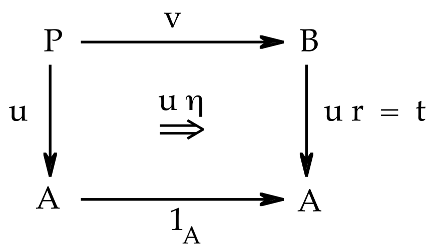
A 2-categorical version of a result mentioned by Lack on Wednesday 21 January 1998

We work in a 2-category which admits the following comma square.

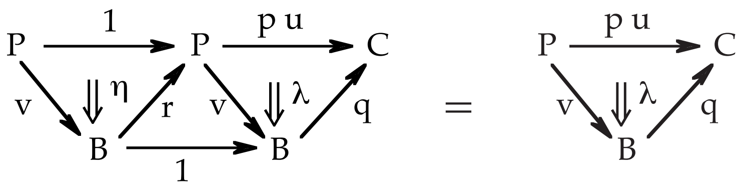


Proposition *If* v *has a right adjoint* r *with identity counit and with unit* $\eta : 1_P \Rightarrow r\ v$ *then the following is a comma square.*



Proof One adjunction triangle for $v \dashv r$ gives $v\,\eta = 1$ and hence the equality $(*)$ below.

$(*)$

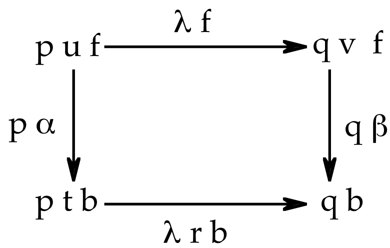


Another consequence of $v \dashv r$ is that, for all arrows $f : X \longrightarrow P$, $b : X \longrightarrow B$, there is a bijection between 2-cells $\rho : f \Rightarrow r\,b$ and 2-cells $\beta : v\,f \Rightarrow b$ determined by the equations

$$\beta = v\,\rho, \qquad \rho = r\,\beta \circ \eta\,f\,.$$

By the 2-cell property of the comma object P, the 2-cells $\rho : f \Rightarrow r\,b$ are in bijection with pairs of 2-cells $\alpha : u\,f \Rightarrow t\,b$, $\beta : v\,f \Rightarrow b$ such that the square $(**)$ below commutes.

$(**)$



The bijection is determined by the equations $\alpha = u\,\rho$, $\beta = v\,\rho$. It follows that, for each 2-cell $\beta : v\,f \Rightarrow b$, there exists a unique 2-cell $\alpha : u\,f \Rightarrow t\,b$ such that $(**)$ commutes.

Now we prove the 2-cell $u\,\eta$ has the claimed comma property. Take any 2-cell $\alpha : a \Rightarrow$

t b. By the comma property of λ, there exists a unique arrow $f : X \longrightarrow P$ such that

$$\begin{array}{ccccc} X & \xrightarrow{f} & P & \xrightarrow{v} & B \\ & & {\scriptstyle u}\downarrow & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle q} \\ & & A & \xrightarrow[p]{} & C \end{array} \quad = \quad \begin{array}{ccccc} X & \xrightarrow{b} & B & & \\ {\scriptstyle a}\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle t} & \overset{\lambda r}{\Rightarrow} & \searrow{\scriptstyle q} \\ A & \xrightarrow[1_A]{} & A & \xrightarrow[p]{} & C \end{array}$$

So α satisfies $(**)$ with $\beta = 1$. By $(*)$, the 2-cell $u\,\eta\,f$ also satisfies $(*)$ as the α with $\beta = 1$. So $\alpha = u\,\eta\,f$, as required. If $u\,\eta\,f = u\,\eta\,g$ for some g, the equality $(*)$ gives $\lambda\,f = \lambda\,g$, so, by the comma property of λ, we obtain $f = g$. So f is unique with $\alpha = u\,\eta\,f$.

The further 2-cell property required of $u\,\eta$ is immediate. **Q.E.D.**