Gray's tensor product of 2-categories

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There is a particularly simple locally-ordered 2-category $\mathbb{Q}(T)$ obtained from each totally ordered set T. After describing this 2-category we shall produce a presentation which shows that, for T of cardinality N, it is Gray's <u>free</u> <u>N-dimensional cube with commutative 3-dimensional faces</u> [ ].

Gray [ ] proved that the 2-category with this presentation is locally ordered without describing an explicit model such as our $\mathbb{Q}(T)$. He made use of Artin's braid group [ ]. Our approach uses the normal form for elements of the <u>braid monoid with zero</u> which we learned from Eilenberg [ ]. (Eilenberg–Street [ ] have shown that this normal form can be obtained from a "rewrite system".)

Let T be any totally ordered set. For any list $a = a_1 a_2 \dots a_r$ of elements of T, let

$$\sigma(a) = \{(a_i, a_j) \mid i < j,\ a_i < a_j\}.$$

The relation between σ and concatenation of lists is

$$\sigma(ab) = \sigma(a) + \{(a_i, b_j) \mid a_i < b_j\} + \sigma(b). \qquad (1)$$

The objects of the 2-category $\mathcal{Q}(T)$ are finite subsets x of T. For $y \subseteq x$, the ordered set $\mathcal{Q}(T)(x,y)$ consists of listings a of all the elements of $x-y$ together with the order:

$$a \le a' \quad \text{when} \quad \sigma(a) \subseteq \sigma(a').$$

The other homs of $\mathcal{Q}(T)$ are empty. Composition

$$\mathcal{Q}(T)(x,y) \times \mathcal{Q}(T)(y,z) \longrightarrow \mathcal{Q}(T)(x,z)$$

is concatenation of lists, which is order preserving by equation (1).

Let $\mathcal{G}(T)$ denote the graph whose vertices are finite subsets x of T and whose edges $u : x \to y$ are elements u of x such that $x = y + \{u\}$. We can identify the free category on $\mathcal{G}(T)$ with the underlying category of $\mathcal{Q}(T)$.

A graph together with some 2-cells between coterminal paths in the graph was called a <u>computad</u> by Street [ ].

Enrich $\mathcal{G}(T)$ with the structure of computad by giving it a 2-cell

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\langle u,v\rangle}{\Longrightarrow}"] & x-\{u\} \arrow[d, "v"] \\ x-\{v\} \arrow[r, "u"'] & x-\{u,v\} \end{tikzcd} \qquad u<v \qquad (2)$$

for each $u, v \in x$ with $u < v$. In this situation,

$\sigma(vu) = \emptyset \subseteq \{(u,v)\} = \sigma(uv)$; so we have a morphism of computads $\mathcal{G}(T) \longrightarrow \mathbb{Q}(T)$ which extends to a unique 2-functor

$$h : \mathcal{G}(T)^* \longrightarrow \mathbb{Q}(T) \tag{3}$$

from the free 2-category on $\mathcal{G}(T)$.

The following two diagrams denote two particular 2-cells in $\mathcal{G}(T)^*$ for all $u < v < w$ in x:

$$\begin{tikzcd}
 & & x \arrow[dll, "w"'] \arrow[drr, "u"] & & \\
x-\{w\} \arrow[drr, "u"] \arrow[dd, "v"'] \arrow[rrrr, phantom, "\overset{\langle u,w\rangle}{\Longrightarrow}"] & & & & x-\{u\} \arrow[dll, "w"'] \arrow[dd, "v"] \\
 & & x-\{u,w\} \arrow[dd, "v"] \arrow[ll, phantom, "\overset{\langle u,v\rangle}{\Longrightarrow}"] \arrow[rr, phantom, "\overset{\langle v,w\rangle}{\Longrightarrow}"] & & \\
x-\{v,w\} \arrow[drr, "u"'] & & & & x-\{u,v\} \arrow[dll, "w"] \\
 & & x-\{u,v,w\} & &
\end{tikzcd} \tag{4}$$

$$\begin{tikzcd}
 & & x \arrow[dll, "w"'] \arrow[drr, "u"] \arrow[dd, "v"] & & \\
x-\{w\} \arrow[dd, "v"'] \arrow[rr, phantom, "\overset{\langle v,w\rangle}{\Longrightarrow}"] & & & & x-\{u\} \arrow[dd, "v"] \arrow[ll, phantom, "\overset{\langle u,v\rangle}{\Longrightarrow}"] \\
 & & x-\{v\} \arrow[dll, "w"'] \arrow[drr, "u"] & & \\
x-\{v,w\} \arrow[drr, "u"'] \arrow[rrrr, phantom, "\overset{\langle u,w\rangle}{\Longrightarrow}"] & & & & x-\{u,v\} \arrow[dll, "w"] \\
 & & x-\{u,v,w\} & &
\end{tikzcd} \tag{5}$$

The 2-functor h takes (4), (5) to the same 2-cell in $\mathbb{Q}(T)$ since $\mathbb{Q}(T)$ is locally ordered and the 2-cells (4), (5) have the same source path wvu and same target path uvw.

<u>Theorem 1.</u> <u>The 2-functor</u> (3) <u>provides a presentation of $\mathbb{Q}(T)$ as the quotient of the free 2-category $\mathcal{G}(T)^*$ modulo the relations</u> (4) = (5).

<u>Proof.</u> Consider a non-identity 2-cell $\begin{tikzcd} x \arrow[r, bend left, "a"] \arrow[r, bend right, "a'"'] \arrow[r, phantom, "\Uparrow"] & y \end{tikzcd}$ in $\mathbb{Q}(T)$. Then there exists a consecutive pair (v, u) in the list a which occurs in reverse order in the list a' (or else we would have $a = a'$). If $v < u$ then $(v, u) \in \sigma(a) \subseteq \sigma(a')$, a contradiction. So $u < v$ and $(u, v) \in \sigma(a') - \sigma(a)$. This gives a decomposition of the 2-cell under consideration as

$$\begin{tikzcd} & & \cdot \arrow[dr, "u"] & & \\ x \arrow[r, "c"] \arrow[rrrr, bend right=40, "a'"'] \arrow[rrrr, phantom, bend right=20, "\Uparrow"] & \cdot \arrow[ur, "v"] \arrow[dr, "u"'] \arrow[rr, phantom, "\Downarrow h\langle u,v\rangle"] & & \cdot \arrow[r, "d"] & y, \\ & & \cdot \arrow[ur, "v"'] & & \end{tikzcd}$$

where $a = c\,v\,u\,d$. By induction on the cardinality of $\sigma(a') - \sigma(a)$, the image of h generates $\mathbb{Q}(T)$.

It remains to show that the relations (4) = (5) imposed on $\mathcal{G}(T)^*$ force local order. This is where we need the braid monoid with zero.

<u>A monoid with zero</u> is a monoid M with an element 0 satisfying $0\mu = \mu 0 = 0$ for all $\mu \in M$. A monoid is a set with an associative binary operation having an identity element 1.

Let $\mathbb{B}_r$ denote the monoid with zero presented by generators $2, 3, \ldots, r$ subject to the relations

$$ji = ij \quad \text{for } i < j-1 \tag{6}$$

$$(i+1)i(i+1) = i(i+1)i \tag{7}$$

$$ii = 0. \tag{8}$$

Relations (6), (7) are those for <u>Artin's braid group</u> [ ]. So $\mathbb{B}_r$ is called <u>the braid monoid with 0</u>. Recall [ ] that the monoid presented by $2, \ldots, r$ subject to (6), (7) and

$$ii = 1, \tag{8$'$}$$

is <u>the symmetric group</u> $\mathbb{S}_r$ <u>on</u> r <u>symbols</u>.

For $i \geq j$, consider the particular words

$$[i, j] = i(i-1)(i-2)\ldots(j+1).$$

They satisfy the equations

$$[i, i] = 1, \qquad [i, j][j, k] = [i, k].$$

Eilenberg's result is that <u>each element</u> ξ <u>of</u> $\mathbb{B}_r$ <u>is either</u> 0 <u>or uniquely of the form</u>

$$\xi = [2, i_2][3, i_3]\ldots[r, i_r] \tag{9}$$

<u>where</u> $1 \leq i_n \leq n$. It follows that $\mathbb{B}_r$ has cardinality $r! + 1$. Notice also that, since the words (9) all represent different permutations (where i represents the transposition of the $(i-1)$-st symbol

Better as action in category of pointed sets.

with the i-th), and since there are $r!$ of them, every element of $\mathfrak{S}_r$ is uniquely of the form (9).

There is a partial right action of $\mathbb{B}_r$ on the set of listings a of $x-y$ where $y \subseteq x$ and the cardinality of $x-y$ is r. Define $a * i$ precisely when $a_{i-1} > a_i$, and, in that case, to be the list obtained from a by interchanging a_{i-1} and a_i. Extend this associatively to all words in $2, 3, \ldots, r$; so we have

$$(a * \xi) * \eta = a * (\xi \eta) \quad , \quad a * 1 = a ,$$

and we declare $a * 0$ undefined. To see that $a * \xi = a * \xi'$ when ξ, ξ' represent the same element of $\mathbb{B}_r$, notice that, for each of the relations (6), (7), (8), a list a can be acted on by the left side if and only if it can be acted on by the right side, and the results are equal when they can be.

The category $\mathcal{G}(T)^*(x, y)$ is the quotient of the free category on the graph with edges

$$\begin{tikzcd} & & \cdot \arrow[dr, "u"] & & \\ x \arrow[r, "c"] & \cdot \arrow[ur, "v"] \arrow[dr, "u"'] & \Downarrow\langle u,v\rangle & \cdot \arrow[r, "d"] & y \\ & & \cdot \arrow[ur, "v"'] & & \end{tikzcd} \quad \text{for } u < v, \quad (10)$$

modulo the relations which identify the two paths from $cvudv'u'e$ to $cuvdu'v'e$ obtainable from

$$\begin{tikzcd}
 & & \bullet \arrow[dr, "u"] & & & \bullet \arrow[dr, "u'"] & & \\
x \arrow[r, "c"] & \bullet \arrow[ur, "v"] \arrow[dr, "u"'] \arrow[rr, phantom, "\Downarrow"] & & \bullet \arrow[r, "d"] & \bullet \arrow[ur, "v'"] \arrow[dr, "u'"'] \arrow[rr, phantom, "\Downarrow"] & & \bullet \arrow[r, "e"] & y \\
 & & \bullet \arrow[ur, "v"'] & & & \bullet \arrow[ur, "v'"'] & &
\end{tikzcd} \tag{11}$$

where $u<v$ and $u'<v'$. We can rewrite (10) as

$$\begin{tikzcd}
x \arrow[r, bend left, "a"] \arrow[r, bend right, "a'"'] \arrow[r, phantom, "\Downarrow i"] & y
\end{tikzcd}$$

where i is the length of the list cvu, where $a = cvu$ and where $a * i = a'$. In this notation, the relations (11) precisely become the relations (6).

Furthermore, the relations (4) = (5) become the relations (7). So, in the quotient of $\mathcal{G}(T)^*$ by the relations (4) = (5), a 2-cell

$$\begin{tikzcd}
x \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow \xi"] & y
\end{tikzcd}$$

is precisely a non-zero element of B_r such that $a * \xi = b$. But we can assume that ξ has the normal form (9). By looking at b we see that i_r is completely determined. By induction on r, it follows that ξ is uniquely determined by a and b if it exists. This gives the desired local orderedness. $\square$

Let 2-Cat denote the category whose objects are small 2-categories and whose arrows are 2-functors. Let $\mathcal{C}u$ denote the full subcategory of 2-Cat consisting of the objects $\mathbb{Q}(T)$ for T a finite totally ordered set.

Theorem 2. The inclusion of $\mathcal{C}u$ in 2-Cat is dense.

Proof. In fact we prove more: we only need those $\mathbb{Q}(T)$ with the cardinality of T less than 4. For simplicity we write (A, B) for the set of 2-functors from A to B.

We must show that, for each family of four functions

$$\theta_n : (\mathbb{Q}(T), A) \longrightarrow (\mathbb{Q}(T), B)$$

where T runs over four totally ordered sets of cardinality $0, 1, 2, 3$ $0 \leq n \leq 3$ such that the square

$$\begin{tikzcd}
(\mathbb{Q}(T), A) \arrow[r, "\theta_n"] \arrow[d, "(f,1)"'] & (\mathbb{Q}(T), B) \arrow[d, "(f,1)"] \\
(\mathbb{Q}(T'), A) \arrow[r, "\theta_{n'}"'] & (\mathbb{Q}(T'), B)
\end{tikzcd} \tag{12}$$

commutes for all 2-functors $f : \mathbb{Q}(T') \longrightarrow \mathbb{Q}(T)$, there is a unique 2-functor $t : A \longrightarrow B$ such that $\theta = (1, t)$. So take such a θ.

For T of cardinality 0 and 1, we can identify $(\mathbb{Q}(T), A)$ with the set of objects and arrows of A, respectively. So we define t on objects and arrows to be given by θ_0 and θ_1. By applying (12) with $f : \mathbb{Q}(0) \longrightarrow \mathbb{Q}(1)$

we see that, for $\alpha : a \to a'$ an arrow of A, we have $t(\alpha) : t(a) \to t(a')$ in B.

Elements s of $(Q(2), A)$ can be identified with diagrams

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\overset{p}{\Rightarrow}"] & a' \arrow[d, "\sigma'"] \\ b \arrow[r, "\beta"'] & b' \end{tikzcd} \tag{13}$$

in A. By applying (12) with $f : Q(1) \to Q(2)$, we see that Θ_3 takes (13) to a diagram of the form

$$\begin{tikzcd} t(a) \arrow[r, "t(\alpha)"] \arrow[d, "t(\sigma)"'] \arrow[dr, phantom, "\overset{\bar{s}}{\Rightarrow}"] & t(a') \arrow[d, "t(\sigma')"] \\ t(b) \arrow[r, "t(\beta)"'] & t(b') . \end{tikzcd}$$

By taking $f : Q(2) \to Q(2)$ to be the diagram

$$\begin{tikzcd} \{u,v\} \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\leqslant"] & \{u,v\} \arrow[d, "uv"] \\ \{u,v\} \arrow[r, "vu"'] & \emptyset \end{tikzcd}$$

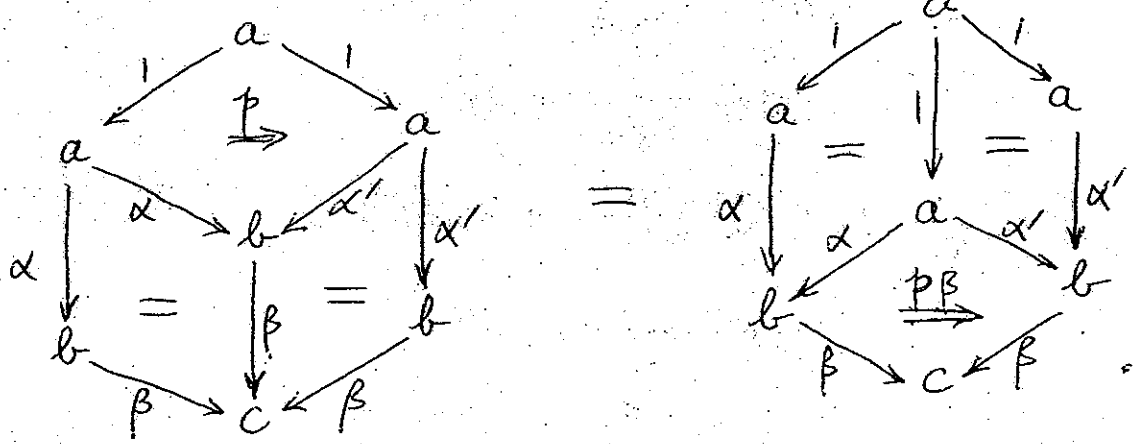
in (12) and evaluating at s, we see that $\bar{s}$ depends only on the 2-cell $p : \sigma\beta \Rightarrow \alpha\sigma'$ so that it can be written $t(p)$, and that t preserves composition of arrows.

Elements h of $(Q(3), A)$ can be identified with commutative cubes in A (see (4), (5) with $x = \{u, v, w\}$). By applying (12) with $f : \mathbb{Q}(2) \to \mathbb{Q}(3)$, we see that the proper faces of $\Theta_3(h)$ are just obtained by applying t to the proper faces of h. So in fact Θ_3 is determined already by $\Theta_0, \Theta_1, \Theta_2$; however, it is needed to prove that t preserves compositions involving 2-cells.

Given a diagram in A of the form

$$\begin{tikzcd} a \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\alpha'"'] \arrow[r, phantom, "\Downarrow p"] & b \arrow[r, "\beta"] & c \end{tikzcd},$$

we obtain an element of $(\mathbb{Q}(3), A)$ depicted by



Since Θ_3 takes this to a commutative 3-cube in B, it follows that $t(p\beta) = t(p)\,t(\beta)$. A symmetric argument applies to show that t preserves composition of a 2-cell p with a 1-cell on the left side.

Given a diagram in A of the form

$$\begin{tikzcd} a \arrow[r, bend left=60, "\alpha"] \arrow[r, "\beta" description] \arrow[r, bend right=60, "\gamma"'] \arrow[r, phantom, shift left=3, "\Downarrow p"] \arrow[r, phantom, shift right=3, "\Downarrow q"] & b \end{tikzcd},$$

we obtain an element of $(Q(\mathcal{B}), A)$ of the form

$$
\begin{tikzcd}[column sep=small]
 & a \arrow[dl, "1"'] \arrow[dr, "1"] \arrow[d, phantom, "="] & \\
a \arrow[dr, "1"] \arrow[dd, "1"'] & {} & a \arrow[dl, "1"'] \arrow[dd, "1"] \\
 & a \arrow[dd, "\beta"] & \\
a \arrow[r, phantom, "\overset{p}{\Rightarrow}"] \arrow[dr, "\alpha"'] & {} \arrow[r, phantom, "\overset{q}{\Rightarrow}"] & a \arrow[dl, "\gamma"] \\
 & b &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=small]
 & a \arrow[dl, "1"'] \arrow[dr, "1"] \arrow[dd, "1"] & \\
a \arrow[dd, "1"'] \arrow[r, phantom, "="] & {} \arrow[r, phantom, "="] & a \arrow[dd, "1"] \\
 & a \arrow[dl, "1"'] \arrow[dr, "1"] & \\
a \arrow[dr, "\alpha"'] \arrow[rr, phantom, "\overset{p\circ q}{\Rightarrow}"] & & a \arrow[dl, "\gamma"] \\
 & b &
\end{tikzcd}
$$

Applying Θ_3 to this, we obtain $t(p\circ q) = t(p)\circ t(q)$. So Θ uniquely determines a 2-functor t, as required. □

Recall (Gray[,]) that a <u>quasi-functor</u> $f : A \times B \longrightarrow C$ <u>of two variables</u> where A, B, C are 2-categories assigns to 2-cells $a \underset{\gamma}{\overset{\alpha}{\Downarrow p}} a'$, $b \underset{\delta}{\overset{\beta}{\Downarrow q}} b'$ in A, B the data displayed in the diagram

$$
\begin{tikzcd}[column sep=large, row sep=large]
f(a,b) \arrow[r, bend left=50, "{f(\gamma,b)}"] \arrow[r, bend right=50, "{f(\alpha,b)}"'] \arrow[r, phantom, "{\Uparrow f(p,b)}"] \arrow[d, bend right=50, "{f(a,\beta)}"'] \arrow[d, bend left=50, "{f(a,\delta)}"] \arrow[d, phantom, "{\overset{f(a,q)}{\Rightarrow}}"] \arrow[dr, phantom, "{\underset{f_{\alpha,\delta}}{\Rightarrow}}"] & f(a',b) \arrow[d, "{f(a',\delta)}"] \\
f(a,b') \arrow[r, "{f(\alpha,b')}"'] & f(a',b')
\end{tikzcd}
=
\begin{tikzcd}[column sep=large, row sep=large]
f(a,b) \arrow[r, "{f(\gamma,b)}"] \arrow[d, "{f(a,\beta)}"'] \arrow[dr, phantom, "{\overset{f_{\gamma,\beta}}{\Rightarrow}}"] & f(a',b) \arrow[d, bend right=50, "{f(a',\beta)}"'] \arrow[d, bend left=50, "{f(a',\delta)}"] \arrow[d, phantom, "{\overset{f(a',q)}{\Rightarrow}}"] \\
f(a,b') \arrow[r, bend left=50, "{f(\gamma,b')}"] \arrow[r, bend right=50, "{f(\alpha,b')}"'] \arrow[r, phantom, "{\Uparrow f(p,b')}"] & f(a',b')
\end{tikzcd}
\qquad (14)
$$

in C in such a way that equality (14) holds, each $f(-,b) : A \longrightarrow C$, $f(a,-) : B \longrightarrow C$ is a 2-functor, the 2-cell $f_{\alpha\alpha',\beta}$ is obtained by pasting the square $f_{\alpha,\beta}$ next to $f_{\alpha',\beta}$, the 2-cell $f_{\alpha,\beta\beta'}$ is obtained by pasting the square for $f_{\alpha,\beta}$ on top of the square for $f_{\alpha,\beta'}$, and $f_{\alpha,\beta}$ is an identity 2-cell if either α or β is an identity arrow.

If $s: A' \to A$, $t: B' \to B$, $r: C \to C'$ are 2-functors, it is possible to compose in the obvious way with such an $f: A \times B \to C$ to obtain a quasi-functor of two variables

$$(s \times t) f r : A' \times B' \longrightarrow C'.$$

The <u>ordered sum</u> $S \dotplus T$ of ordered sets S, T is the disjoint union of the underlying sets of S and T ordered so that it contains S, T as sub-ordered sets and so that $u < v$ for all $u \in S, v \in T$.

Define the canonical quasi-functor of two variables

$$k : Q(S) \times Q(T) \longrightarrow Q(S \dotplus T) \tag{15}$$

by setting, for $a: x \to x'$, $b: y \to y'$ in $Q(S), Q(T)$:

$$\begin{tikzcd} k(x,y) \arrow[r, "k(a,y)"] \arrow[d, "k(x,b)"'] \arrow[dr, phantom, "\overset{k_{a,b}}{\Rightarrow}"] & k(x',y) \arrow[d, "k(x',b)"] \\ k(x,y') \arrow[r, "k(a,y')"'] & k(x',y') \end{tikzcd} \quad = \quad \begin{tikzcd} x+y \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\leqslant"] & x'+y \arrow[d, "b"] \\ x+y' \arrow[r, "a"'] & x'+y'. \end{tikzcd}$$

<u>Theorem 3. Composition with (15) establishes a bijection between 2-functors</u> $g : Q_2(S \underset{0}{+} T) \longrightarrow A$ <u>and quasi-functors of two variables</u> $f : Q(S) \times Q(T) \longrightarrow A$.

<u>Proof</u>. We begin by examining the computad $\mathcal{G}(S \underset{0}{+} T)$. Each object z can be written uniquely as a disjoint union $z = x + y$ where $x \subseteq S$, $y \subseteq T$. The edges are either of the form $u : x + y \longrightarrow x' + y$ with $u : x \to x'$ in $\mathcal{G}(S)$ or of the form $v : x + y \to x + y'$ where $v : y \to y'$ in $\mathcal{G}(T)$. The 2-cells are of three kinds (refer to (2)) : $\langle u, u' \rangle$ with $u < u'$ in S, $\langle v, v' \rangle$ with $v < v'$ in T, and $\langle u, v \rangle$ with $u \in S$, $v \in T$.

Given f, in order to define g, we make use of Theorem 1 to see that the equations

$$g(x+y) = f(x,y), \quad g(u) = f(u,y), \quad g(v) = f(x,v),$$
$$g(h\langle u,u' \rangle) = f(\langle u,u' \rangle, y), \quad g(h\langle v,v' \rangle) = f(x, \langle v,v'$$
$$\text{and} \quad g(h\langle u,v \rangle) = f_{u,v}$$

determine the unique g whose composite with k is f. $\square$

Using this Theorem 3, we see that 2-functor

$$s : Q(S) \longrightarrow Q(S'), \qquad t : Q(T) \longrightarrow Q(T')$$

induce a 2-functor $s \otimes t$ uniquely determined by commutativity of the square

$$\begin{tikzcd} \mathbb{Q}(S)\times\mathbb{Q}(T) \arrow[r, "k"] \arrow[d, "s\times t"'] & \mathbb{Q}(S \underset{\circ}{+} T) \arrow[d, "s\otimes t"] \\ \mathbb{Q}(S')\times\mathbb{Q}(T') \arrow[r, "k"'] & \mathbb{Q}(S' \underset{\circ}{+} T') . \end{tikzcd} \tag{16}$$

This defines a functor $\otimes : \mathcal{Cu}\times\mathcal{Cu} \longrightarrow \mathcal{Cu}$ given by $\mathbb{Q}(S)\otimes\mathbb{Q}(T) = \mathbb{Q}(S \underset{\circ}{+} T)$ and $(s,t) \longmapsto s\otimes t$.

Let Δ denote the category of finite totally ordered sets (including the empty one) with order-preserving functions. We have a functor

$$\mathbb{Q} : \Delta \longrightarrow \mathcal{Cu} \tag{17}$$

taking T to $\mathbb{Q}(T)$. For an order-preserving $\rho : S \longrightarrow T$, the 2-functor $\mathbb{Q}(\rho) : \mathbb{Q}(S) \longrightarrow \mathbb{Q}(T)$ takes x to $\rho(x)$, takes $u : x \longrightarrow x'$ to $\rho(u) : \rho(x) \longrightarrow \rho(x')$ when $\rho(u) \notin \rho(x')$, and takes $\langle u, v\rangle$ as in (2) to $\langle \rho(u), \rho(v)\rangle$ when $u : x \to x'$, $v : x' \to x''$ with $\rho(u) \notin \rho(x')$, $\rho(v) \notin \rho(x'')$.

Ordered sum provides Δ with the structure of monoidal category. The canonical associativity in Δ transfers to $\mathcal{C}u$ via (17) to yield an isomorphism

$$a : (\mathbb{Q}(R)\otimes\mathbb{Q}(S))\otimes\mathbb{Q}(T) \cong \mathbb{Q}(R)\otimes(\mathbb{Q}(S)\otimes\mathbb{Q}(T)).$$

Naturality of a in $\mathbb{Q}(R), \mathbb{Q}(S), \mathbb{Q}(T)$ as objects of $\mathcal{C}u$ is obvious from (16). All the coherence conditions (including Mac Lane's pentagon) satisfied by the associativity in Δ transfer to a via (17).

Thus we have defined a monoidal structure on the category $\mathcal{C}u$ in such a way that (17) becomes a monoidal functor.

The monoidal category Δ does not admit a symmetry in the usual sense of Eilenberg-Kelly [ ]; yet there is a canonical isomorphism

$$c \; : \; (S \underset{0}{+} T)^{op} \cong T^{op} \underset{0}{+} S^{op}$$

(where S^{op} denotes S with the reverse order) satisfying various coherence conditions including commutativity of the triangle

$$\begin{tikzcd} S \underset{0}{+} T \arrow[r, "c_{S,T}^{\ op}"] \arrow[dr, "1"'] & (T^{op} \underset{0}{+} S^{op})^{op} \arrow[d, "c_{T^{op},S^{op}}"] \\ & S \underset{0}{+} T \end{tikzcd} \qquad (18)$$

.

There is an equally coherent isomorphism

$$c : (\mathbb{Q}(S) \otimes \mathbb{Q}(T))^{co} \cong \mathbb{Q}(T)^{co} \otimes \mathbb{Q}(S)^{co} \qquad (19)$$

on Cu obtained by applying (17), using (16), and using

$$\mathbb{Q}(S^{op}) = \mathbb{Q}(S)^{co}, \qquad (20)$$

where A^{co} denotes the 2-category obtained from A by reversing 2-cells.

We are now in the familiar situation of a cocomplete category 2-Cat with a full dense subcategory Cu bearing a monoidal structure. It is possible (see Day [ ]) by means of left Kan extension to extend the monoidal structure on Cu to one on 2-Cat. Define the <u>tensor product</u> $\otimes$ on 2-Cat to be the 2-functor in the diagram

$$\begin{tikzcd}
\mathrm{Cu} \times \mathrm{Cu} \arrow[r, hook] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\underset{\cong}{\Rightarrow}"] & 2\text{-}\mathrm{Cat} \times 2\text{-}\mathrm{Cat} \arrow[d, "\otimes"] \\
\mathrm{Cu} \arrow[r, hook] & 2\text{-}\mathrm{Cat}
\end{tikzcd}$$

obtained as the left Kan extension of the composite of the left and bottom sides along the top side. The formula for the tensor product can be written (Day-Kelly [ ], Mac Lane [ ]):

$$A \otimes B = \int^{Q(S),\,Q(T)} \big((Q(S), A) \times (Q(T), B) \big) \cdot Q(S \mathbin{\overset{+}{\circ}} T). \tag{21}$$

This means that there is a natural bijection between 2-functors $A \otimes B \to C$ and families of functions

$$(Q(S), A) \times (Q(T), B) \longrightarrow (Q(S \mathbin{\overset{+}{\circ}} T), C) \tag{22}$$

which are natural in $Q(S), Q(T)$ as objects of Cu.

This monoidal structure on 2-Cat is not genuinely symmetric, but we do have a canonical isomorphism

$$c : (A \otimes B)^{co} \cong B^{co} \otimes A^{co} \tag{23}$$

induced by, and as coherent as, (19). We now turn to the question of closedness.

Let $s, t : A \to C$ be 2-functors. <u>A lax natural transformation</u> $\theta : s \to t$ (also called "quasi-natural" [ ]) is given by the data displayed in the square

$$\begin{tikzcd} s(a) \arrow[r, "s(\alpha)"] \arrow[d, "\theta_a"'] \arrow[dr, phantom, "\overset{\theta_\alpha}{\Longrightarrow}"] & s(a') \arrow[d, "\theta_{a'}"] \\ t(a) \arrow[r, "t(\alpha)"'] & t(a') \end{tikzcd} \tag{24}$$

satisfying the obvious axioms [ ], [ ].

Let $[\![A, C]\!]$ denote the 2-category whose objects are 2-functors $s : A \to C$, whose arrows are lax natural transformations $\theta : s \to t$, and whose 2-cells are modifications [ ], [ ], [ ].

There is an evaluation quasi-functor of 2-variable

$$e : A \times [\![A, C]\!] \longrightarrow C \tag{25}$$

given by $e(a, s) = s(a)$, $e(a, \theta) = \theta_a$, $e(\alpha, s) = s(\alpha)$ and $e_{\alpha,\theta} = \theta_\alpha$ (see (24)). The following result is in Gray [ ]; the proof is just a matter of interpretation and bookkeeping.

Theorem 4. The function taking f to the composite of $A \times f$ and e is a bijection between 2-functors $f : B \longrightarrow [\![A, C]\!]$ and quasi-functors of 2-variables $q : A \times B \longrightarrow C$. $\square$

It is also clear that quasi-functors of two variables $g : A \times B \longrightarrow C$ are in bijection with quasi-functors of two variables $\bar{g} : B^{co} \times A^{co} \longrightarrow C^{co}$ via $g(a, b) = \bar{g}(b, a)$.

Theorem 5. There are bijections

$$(B, [\![A, C]\!]) \cong (A \otimes B, C) \cong (A, [\![B^{co}, C^{co}]\!]^{co}) \tag{26}$$

natural in $A, B, C \in$ 2-Cat.

Proof. Theorem 2 amounts to the formula

$$A \cong \int^{Q(S)} (Q(S), A) \cdot Q(S). \tag{27}$$

Together Theorems 3 and 4 yield

$$(\mathbb{Q}(T), [\![\mathbb{Q}(S), C]\!]) \cong (\mathbb{Q}(S) \otimes \mathbb{Q}(T), C). \tag{28}$$

Theorem 4 and the remark following it yield

$$(B, [\![A, C]\!]) \cong (A, [\![B^{co}, C^{co}]\!])^{co}. \tag{29}$$

The calculation

$$(B, [\![\mathbb{Q}(S), C]\!]) \overset{(27)}{\cong} \left(\int^{\mathbb{Q}(T)} (\mathbb{Q}(T), B) \cdot \mathbb{Q}(T), [\![\mathbb{Q}(S), C]\!] \right)$$

$$\cong \int_{\mathbb{Q}(T)} (\mathbb{Q}(T), [\![\mathbb{Q}(S), C]\!])^{(\mathbb{Q}(T), B)} \quad \text{(by definition of coend)}$$

$$\overset{(28)}{\cong} \int_{\mathbb{Q}(T)} (\mathbb{Q}(S) \otimes \mathbb{Q}(T), C)^{(\mathbb{Q}(T), B)}$$

$$\cong \left(\int^{\mathbb{Q}(T)} (\mathbb{Q}(T), B) \cdot (\mathbb{Q}(S) \otimes \mathbb{Q}(T)), C \right) \quad \text{(by definition of coend)}$$

$$\cong (\mathbb{Q}(S) \otimes B, C) \quad \text{(by (21) and Yoneda)}.$$

Thus we have (26) for $A = \mathbb{Q}(S)$.

A similar calculation using (29) gives the result. $\square$

Thus we have described a biclosed monoidal structure on the category 2-Cat. There is another one which we shall now describe.

Let $\{A, C\}$ denote the sub-2-category of $[\![A, C]\!]$ consisting of the same objects, the lax natura transformations $\theta : s \to t$ for which each θ_α (see (24)) is invertible (these θ are called <u>pseudo-natural transformations</u>), and all modifications. There is an isomorphism of 2-categories

$$\{A, C\} \cong \{A^{co}, C^{co}\}^{co} \tag{30}$$

obtained by replacing θ_α by θ_α^{-1}.

The tensor product $A \boxtimes B$ for this smaller internal hom will now be constructed. The non-identity 2-cell of the "square" $\mathbb{Q}(\mathbb{2})$ can be regarded as a modification

$$\begin{tikzcd} \mathbb{1} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow \Rrightarrow \Downarrow"] & \mathbb{Q}(\mathbb{2}). \end{tikzcd} \tag{31}$$

Arrows $\alpha : a \to a'$, $\beta : b \to b'$ in A, B can be regarded as 2-functors $\bar{\alpha} : \mathbb{Q}(\mathbb{1}) \to A$, $\bar{\beta} : \mathbb{Q}(\mathbb{1}) \to B$, so, we obtain a 2-functor

$$\mathbb{Q}(\mathbb{2}) \cong \mathbb{Q}(\mathbb{1}) \otimes \mathbb{Q}(\mathbb{1}) \xrightarrow{\bar{\alpha} \otimes \bar{\beta}} A \otimes B. \tag{32}$$

Combining (31), (32), we obtain a modification

$$\begin{tikzcd} \operatorname{arr} A \times \operatorname{arr} B \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow \Rrightarrow \Downarrow"] & A \otimes B, \end{tikzcd} \tag{33}$$

where $\operatorname{arr} A$ denotes the set of arrows of A. Let

$$A \otimes B \longrightarrow A \boxtimes B \tag{34}$$

be the universal 2-functor which inverts the modification (33).

Theorem 6. *The first bijection of Theorem 5 induces a bijection*

$$(B, \{A, C\}) \cong (A \boxtimes B, C)$$

natural in $A, B, C \in 2\text{-Cat}$.

Proof. Write $[A, C]$ for the sub-2-category of $[\![A, C]\!]$ where we restrict to arrows which are 2-natural transformations (each θ_α an identity). It is easy to see that

$$[B, \{A, C\}] \longrightarrow [B, [\![A, C]\!]]$$

is the universal 2-functor which inverts the modification

$$\begin{tikzcd} {[B, [\![A, C]\!]]} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow \Rrightarrow \Downarrow"] & {[\mathrm{arr}A \times \mathrm{arr}B, C]} \end{tikzcd}.$$

This gives a 2-natural isomorphism of 2-categories

$$[B, \{A, C\}] \cong [A \boxtimes B, C];$$

the object bijection gives the result. $\square$

Theorem 7. *The tensor product $\boxtimes$ (together with associativity isomorphism induced from $\otimes$ and symmetry induced from (23) along (34)) enriches 2-Cat with a symmetric monoidal closed structure.*

Proof. It is easy to see that the associativity isomorphism for $\otimes$ induces one for $\boxtimes$; coherence follows since (34) is epic. Combining (30), (18), (23), we obtain all that is needed for a symmetry except for commutativity of the triangle (omitting associativities)

$$\begin{tikzcd}
A \boxtimes B \boxtimes C \arrow[rr, "c"] \arrow[dr, "1 \boxtimes c"'] & & C \boxtimes A \boxtimes B \\
& A \boxtimes C \boxtimes B \arrow[ur, "c \boxtimes 1"'] &
\end{tikzcd}$$

which could not even be drawn in the $\otimes$ case with $(\;)^{co}$ involved. Now that it can be drawn, it is easy. [