Frobenius algebras and monoidal categories

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The Plan

Step 1 Recall the ordinary notion of Frobenius algebra over a field k .

Step 2 Lift the concept from linear algebra to a general monoidal category and justify this with examples and theorems.

Step 3 Lift the concept up a dimension so that monoidal categories themselves can be examples.

Frobenius algebras

An algebra A over a field k is called *Frobenius* when it is finite dimensional and equipped with a linear function $\varepsilon : A \longrightarrow k$ such that:

$$\varepsilon(ab) = 0 \text{ for all } a \in A \text{ implies } b = 0.$$

Example

$A = M_n(k) =$ the algebra of $n \times n$ matrices over k

$$\varepsilon(a) = \text{the trace } \mathrm{Tr}(a) \text{ of } a.$$

More generally, for any Frobenius algebra A, we can enrich the algebra $M_n(A)$ with the Frobenius structure $M_n(A) \xrightarrow{\mathrm{Tr}} A \xrightarrow{\varepsilon} k$. It follows, using Wedderburn Theory, that every finite-dimensional semisimple algebra admits a Frobenius structure.

Example

X an n-dimensional oriented compact manifold

$H^m(X)$ = de Rham cohomology of X of degree m

= closed differentiable m-forms on X modulo exact forms.

$$H^*(X) = \bigoplus_{m=0}^{n} H^m(X)$$

is a real algebra under wedge product

integration $\int_X : H^*(X) \longrightarrow \mathbf{R}$ over X provides a Frobenius structure.

Monoidal categories

A category $\mathcal{V}$ is *monoidal* when it is equipped with a functor $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ (called the *tensor product*), an object I of $\mathcal{V}$ (called the *tensor unit*), and three natural families of isomorphisms

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C), \quad I \otimes A \cong A \cong A \otimes I$$

in $\mathcal{V}$ (called *associativity* and *unital constraints*), such that the pentagon, involving the five ways of bracketing four objects, commutes, and the associativity constraint with $B = I$ is compatible with the unit constraints.

Example $\mathcal{V} = \mathrm{Vect}_k$ = the category of k-linear spaces with usual tensor product

Example $\mathcal{V} = \mathrm{Vect}_k^G = \mathrm{Rep}_k G$ = the category of k-linear representations of the group G with usual tensor product

Braided and symmetric monoidal categories

Call $\mathcal{V}$ *braided* when it is equipped with a natural family of isomorphisms

$$c_{A,B} : A \otimes B \cong B \otimes A$$

(called the *braiding*) satisfying two conditions (one expressing $c_{A\otimes B,C}$ in terms of associativity constraints, $1_A \otimes c_{B,C}$ and $c_{C,A} \otimes 1_B$, and a similar one for $c_{A,B\otimes C}$).

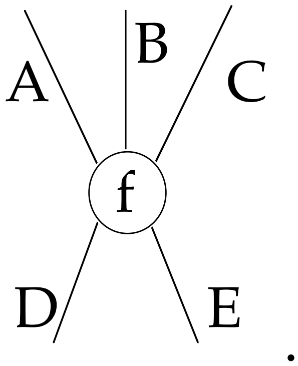
A braiding is a *symmetry* when $c_{B,A} \circ c_{A,B} = 1_{A\otimes B}$.

Example

Vect_k is symmetric as is the more general $\mathrm{Rep}_k G$

String diagrams

Morphisms $f : A \otimes B \otimes C \longrightarrow D \otimes E$ in a monoidal category $\mathcal{V}$ can be represented by diagrams in the Euclidean plane:



The strings are labelled by objects and the nodes are labelled by morphisms.
Composition of morphisms is performed vertically while tensoring is horizontal, creating more complicated plane graphs.

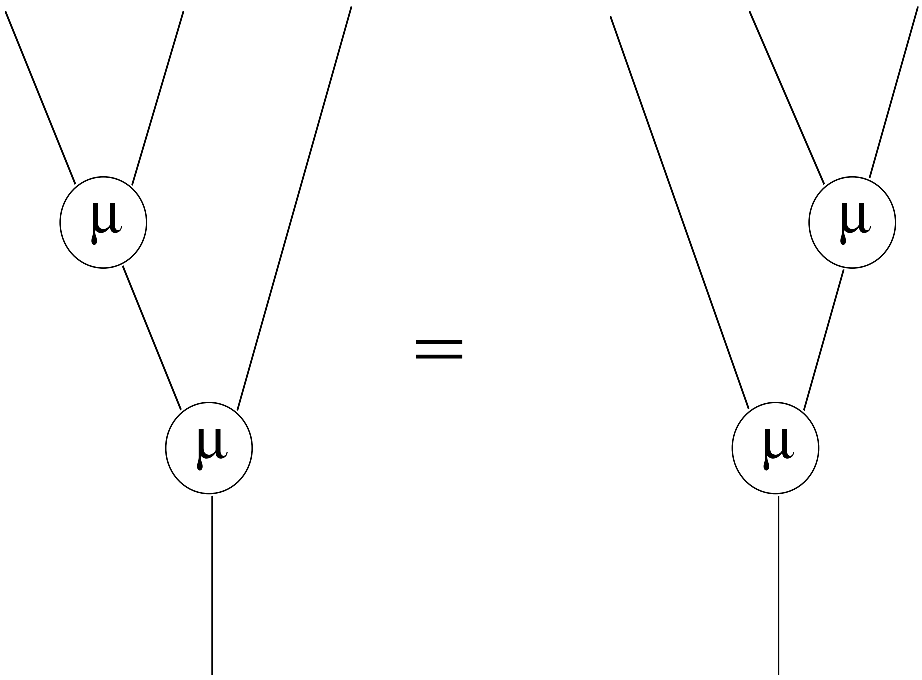
This geometric calculus in the plane faithfully represents calculations in monoidal categories.

We shall see how this works as we continue.

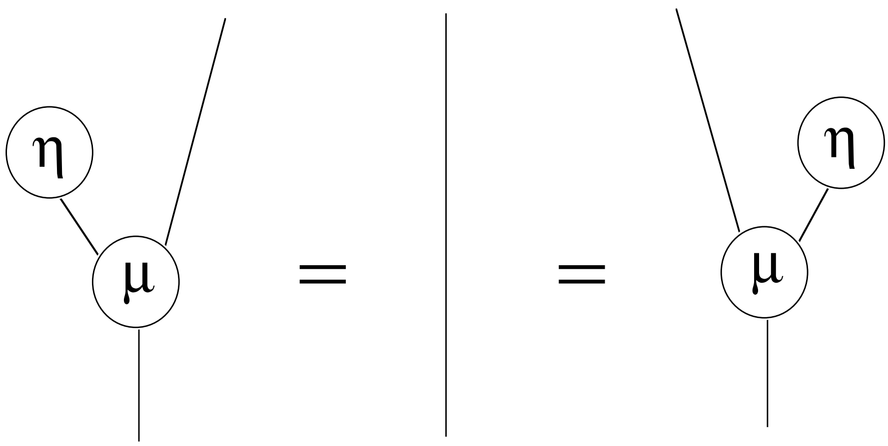
Monoids in a monoidal category

A *monoid* in $\mathcal{V}$ is an object A equipped with a "multiplication" $\mu : A \otimes A \longrightarrow A$ and a "unit" $\eta : I \longrightarrow A$ satisfying

the associativity condition:



and the unit condition:



Here all strings are labelled by A .

Examples

- A monoid in the category Set of sets, where the tensor product is cartesian product, is a monoid in the usual sense.

- If we use the coproduct (disjoint union) in Set as tensor product, every set has a unique monoid structure.

- A monoid in Vect_k, with the usual tensor product of vector spaces, is precisely a k-algebra; monoids in monoidal k-linear categories are also sometimes called algebras.

- A monoid in the dual category Vect_k^{op}, with the usual tensor product of vector spaces, is precisely a k-*coalgebra*.

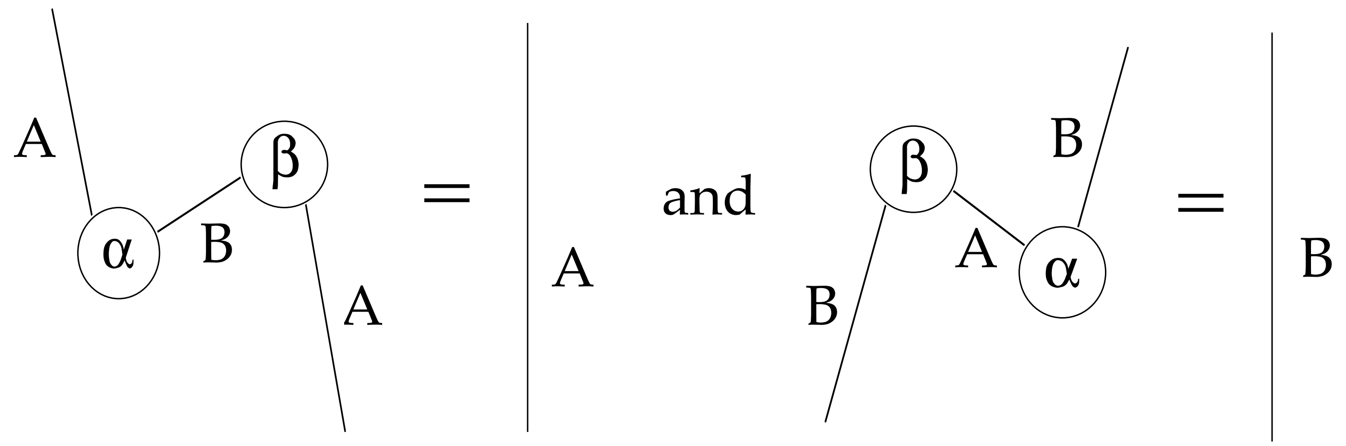
- A monoid in the category Cat of categories (where the morphisms are functors and the tensor product is cartesian product) is a *strict* monoidal category.

Duality within a monoidal category

A *duality* $A \dashv B$ between two objects A and B in a monoidal category $\mathcal{V}$ is a pair of morphisms

$$\alpha : A \otimes B \longrightarrow I \quad \text{and} \quad \beta : I \longrightarrow B \otimes A$$

called the *counit* and *unit*, respectively, such that



A monoidal category is called *autonomous* (*compact* or *rigid*) when for every object A there exist B and C with $C \dashv A \dashv B$.

Example

We have $A \dashv B$ in Vect_k for some B if and only if A is finite dimensional; in this case,

$$A^* \dashv A \dashv A^*$$

where $A^* = \mathrm{Vect}_k(A, k)$ is the space of linear functions from A to k.

Frobenius monoids in a monoidal category

Theorem *Suppose* A *is a monoid in* $\mathcal{V}$ *and* $\varepsilon : A \longrightarrow I$ *is a morphism. The following six conditions are equivalent and define* Frobenius monoid:

(a) there exists $\rho : I \longrightarrow A \otimes A$ *such that*

$$(A \otimes \mu) \circ (\rho \otimes A) = (\mu \otimes A) \circ (A \otimes \rho)$$

and $$(A \otimes \varepsilon) \circ \rho = \eta = (\varepsilon \otimes A) \circ \rho;$$

(b) there exists $\delta : A \longrightarrow A \otimes A$ *such that*

$$(A \otimes \mu) \circ (\delta \otimes A) = \delta \circ \mu = (\mu \otimes A) \circ (A \otimes \delta)$$

and $$(A \otimes \varepsilon) \circ \delta = 1_A = (\varepsilon \otimes A) \circ \delta;$$

(c) there exists $\delta : A \longrightarrow A \otimes A$ *such that* (A, ε, δ) *is a comonoid and*

$$(A \otimes \mu) \circ (\delta \otimes A) = \delta \circ \mu = (\mu \otimes A) \circ (A \otimes \delta);$$

(d) a counit $\sigma : A \otimes A \longrightarrow I$ *exists for a duality* $A \dashv A$ *with* $\sigma \circ (A \otimes \mu) = \sigma \circ (\mu \otimes A)$;

(e) $\sigma = \varepsilon \circ \mu$ *is a counit for* $A \dashv A$;

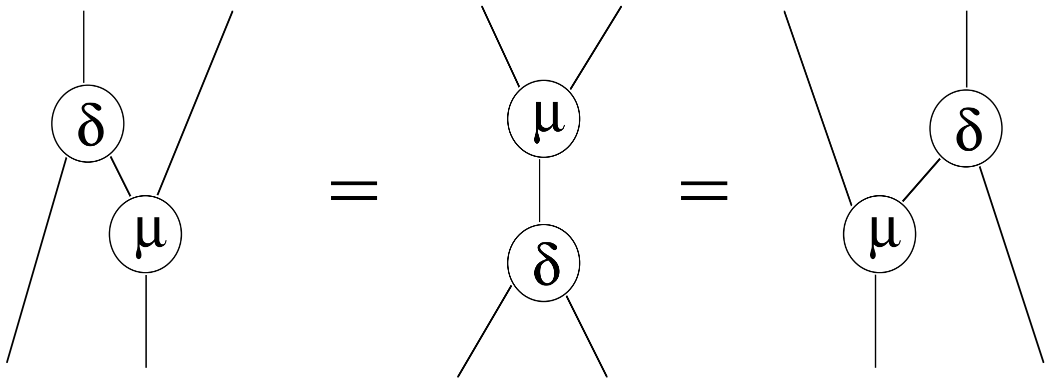
(f) the free functor $F : \mathcal{V} \longrightarrow \mathcal{V}^A$ *is right adjoint to the forgetful functor* $U : \mathcal{V}^A \longrightarrow \mathcal{V}$ *with counit* ε.

Example If $B \dashv A \dashv B$ then $A \otimes B$ is a Frobenius monoid in $\mathcal{V}$.

The self-dual nature of Frobenius monoid

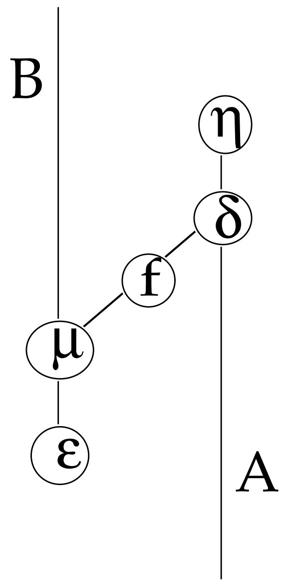
Part (c) of the Theorem:

A Frobenius algebra consists of a monoid and comonoid structure on A subject to the condition



Invertibility of Frobenius monoid morphisms

If $f : A \longrightarrow B$ is both a monoid and comonoid morphism then it has inverse represented by



Commutative Frobenius monoids

Assume $\mathcal{V}$ is braided.

A monoid A in $\mathcal{V}$ is *commutative* when

$$\begin{array}{ccc} A\otimes A & \xrightarrow{c_{A,A}} & A\otimes A \\ & {\scriptstyle \mu}\searrow \qquad \swarrow{\scriptstyle \mu} & \\ & A & \end{array} .$$

A comonoid A in $\mathcal{V}$ is *cocommutative* when

$$\begin{array}{ccc} & A & \\ & {\scriptstyle \delta}\swarrow \qquad \searrow{\scriptstyle \delta} & \\ A\otimes A & \xrightarrow{c_{A,A}} & A\otimes A \end{array} .$$

Proposition

For a Frobenius monoid, commutativity is equivalent to cocommutativity.

The group algebra

G finite group $\quad A = kG$

A	$A \otimes A \xrightarrow{\mu_1} A$ $g \otimes h \mapsto gh$	$A \xrightarrow{\delta_1} A \otimes A$ $g \mapsto g \otimes g$	cocom. Hopf
A^*	$A \xrightarrow{\delta_2} A \otimes A$ $g \mapsto \sum_h gh^{-1} \otimes h$	$A \otimes A \xrightarrow{\mu_2} A$ $g \otimes h \mapsto \begin{cases} g & \text{for } g = h \\ 0 & \text{otherwise} \end{cases}$	com. Hopf
	Frobenius	commutative and cocommutative Frobenius	

Moreover, the lower right square is a commutative and cocommutative Frobenius algebra in $\mathrm{Rep}_k G$.

Larson-Sweedler: Every finite-dimensional Hopf algebra admits a Frobenius structure.

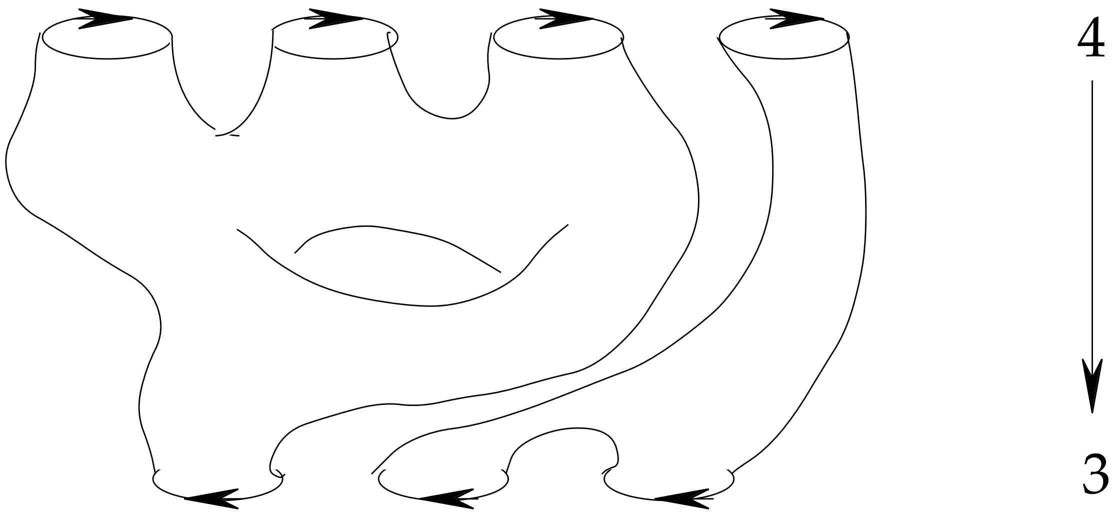
However: the coalgebra structure of the Frobenius structure is not that of the Hopf algebra.

2D Topological Quantum Field Theories

There is a symmetric monoidal category 2Cob of *2-cobordisms:*

objects are natural numbers;

a morphism $M : n \longrightarrow m$ is an oriented two-dimensional cobordism whose boundary consists of n circles with inward orientation and m circles with outward orientation, where two morphisms are identified when there is an orientation-preserving diffeomorphism between them.



composition is vertical stacking when target of one and source of other match;
tensoring is horizontal placement.

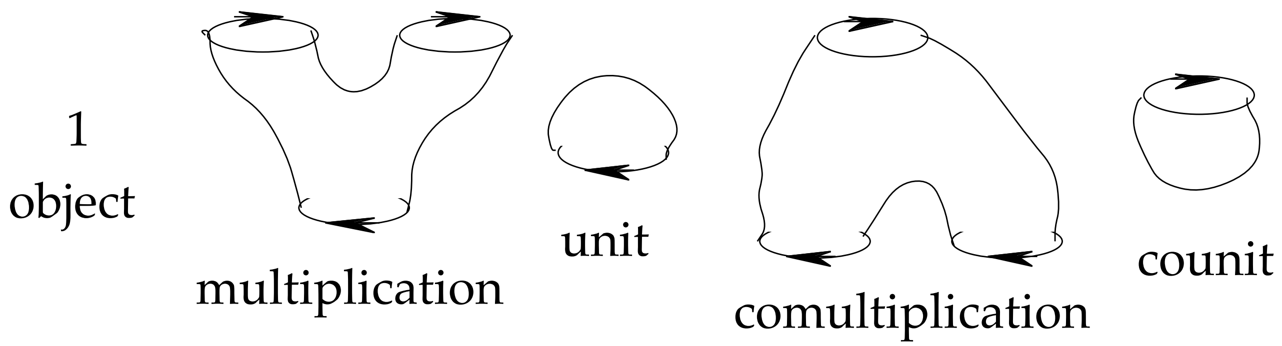
A *2D topological quantum field theory* is a symmetric strong monoidal functor

$$T : 2\mathrm{Cob} \longrightarrow \mathrm{Vect}_k .$$

A universal commutative Frobenius monoid

Theorem

In 2Cob *there is a commutative Frobenius algebra*



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Every commutative Frobenius monoid A *in any symmetric monoidal category* $\mathcal{V}$ *is the value* $A = T1$ *of an essentially unique symmetric strong monoidal functor* $T : 2\mathrm{Cob} \longrightarrow \mathcal{V}$.

Indeed, evaluation at 1 *determines an equivalence of groupoids*

$$\mathrm{SymmStMon}(2\mathrm{Cob}, \mathcal{V}) \simeq \mathrm{CommFrob}(\mathcal{V}).$$

Corollary

2D topological quantum field theories are determined up to isomorphism by commutative Frobenius algebras.

Modules

There is a monoidal bicategory $\mathrm{Vect_k}\text{-}\mathrm{Mod}$:

objects are k-linear categories $\mathcal{A}, \mathcal{B}, \ldots$;

morphisms $\mathcal{A} \xrightarrow{M} \mathcal{B}$ are k-linear functors

$$M : \mathcal{B}^{op} \otimes \mathcal{A} \longrightarrow \mathrm{Vect_k}$$

(called *modules from* $\mathcal{A}$ *to* $\mathcal{B}$);

2-cells are natural transformations;

composition of modules $\mathcal{A} \xrightarrow{M} \mathcal{B} \xrightarrow{N} \mathcal{C}$

has $(N \circ M)(C, A)$ defined as the coequalizer of

$$\prod_{B,B'} M(B', A) \otimes \mathcal{B}(B, B') \otimes N(C, B) \rightrightarrows \prod_{B,B'} M(B, A) \otimes N(C, B)$$

(called *tensor product over* $\mathcal{B}$);

tensor product $\mathcal{A} \otimes \mathcal{B}$ is defined by

$$\mathrm{ob}(\mathcal{A} \otimes \mathcal{B}) = \mathrm{ob}\mathcal{A} \times \mathrm{ob}\mathcal{B}$$

$$(\mathcal{A} \otimes \mathcal{B})((A, B), (A', B')) = \mathcal{A}(A, A') \otimes \mathcal{B}(B, B').$$

$\mathcal{A}^{op}$ behaves like a dual for vector spaces: there is an equivalence between modules

$\mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$ and modules $\mathcal{B} \longrightarrow \mathcal{A}^{op} \otimes \mathcal{C}$.

Frobenius monoidal categories

Just as we looked at monoids in monoidal categories, we look at pseudomonoids in monoidal bicategories. In Vect_k-Mod the pseudomonoids include monoidal k-linear categories such as Vect_k itself.

The Frobenius requirement is related to the notion of *star-autonomy* due to Michael Barr. Every rigid (autonomous, compact) monoidal category is star-autonomous. In particular, Vect_k is Frobenius.

Quantum groupoids provide further examples of Frobenius pseudomonoids. For further details:

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