Monoidal categories for the combinatorics of group representations

Ross Street

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Categories are known to be useful for organizing structural aspects of mathematics. However, they are also useful in finding out what structure can be dismissed (coherence theorems) and hence in aiding calculations. We want to illustrate this for finite set theory, linear algebra, and group representation theory.

We begin with some combinatorial set theory. Let $\mathbf{N}$ denote the set of natural numbers. We identify each $n \in \mathbf{N}$ with the finite set

$$n = \{ j \in \mathbf{N} : 0 \le j < n \}.$$

However, we must be careful to distinguish the cartesian product

$$m \times n = \{ (i, j) : 0 \le i < m,\ 0 \le j < n \}$$

from the isomorphic set mn. Let $\mathbf{S}$ denote the skeletal category of finite sets; explicitly, the objects are the $n \in \mathbf{N}$ and the morphisms are the functions between these sets.

We need to discuss the explicit construction of finite products in $\mathbf{S}$. Let

$$\pi_0^{m,n} : \ mn \longrightarrow m \quad \text{and} \quad \pi_1^{m,n} : \ mn \longrightarrow n$$

be the functions given by $\pi_0^{m,n}(k) = i$ and $\pi_1^{m,n}(k) = j$ where $k = i\,n + j$. That $\pi_0^{m,n}$ and $\pi_1^{m,n}$ are functions, and that $(mn, \pi_0^{m,n}, \pi_1^{m,n})$ is the categorical binary product of m and n in $\mathbf{S}$, follow from the fact that each natural number $0 \le k < mn$ has a unique expression in the form $k = i\,n + j$ with $0 \le i < m$ and $0 \le j < n$. Moreover, each natural number $0 \le h < mnp$ has a unique expression $h = i\,n\,p + j\,p + k$ with $0 \le i < m$, $0 \le j < n$ and $0 \le k < p$. So, referring to the diagram

$$\begin{tikzcd}
 & mnp \arrow[dl, "\pi_0^{mn,p}"'] \arrow[ddr, "\pi_1^{mn,p}"] & \\
mn \arrow[d, "\pi_0^{m,n}"'] \arrow[dr, "\pi_1^{m,n}"] & & \\
m & n & p
\end{tikzcd}
\qquad
\begin{tikzcd}
 & mnp \arrow[ddl, "\pi_0^{m,np}"'] \arrow[dr, "\pi_1^{m,np}"] & \\
 & & np \arrow[dl, "\pi_0^{n,p}"'] \arrow[d, "\pi_1^{n,p}"] \\
m & n & p
\end{tikzcd},$$

we see that $\pi_0^{m,n} \circ \pi_0^{mn,p} = \pi_0^{m,np}$, $\pi_1^{m,n} \circ \pi_0^{mn,p} = \pi_0^{n,p} \circ \pi_1^{m,np}$, $\pi_1^{mn,p} = \pi_1^{n,p} \circ \pi_1^{m,np}$. We denote these last three functions by $\pi_0^{m,n,p} : mnp \longrightarrow m$, $\pi_1^{m,n,p} : mnp \longrightarrow n$, $\pi_2^{m,n,p} : mnp \longrightarrow p$, to obtain a categorical ternary product $(mnp,\ \pi_0^{m,n,p},\ \pi_1^{m,n,p},\ \pi_2^{m,n,p})$ of m, n, p in $\mathbf{S}$. The universal property of product defines a functor $\otimes : \mathbf{S} \times \mathbf{S} \longrightarrow \mathbf{S}$ given on objects by $m \otimes n = mn$ and, on functions $f : m \longrightarrow m'$, $g : n \longrightarrow n'$, by

$$(f \otimes g)(i\,n + j) = f(i)n' + g(j)$$

for $i \in m$, $j \in n$. From the preceding remarks, it follows that the equalities

$$(m \otimes n) \otimes p = m \otimes (n \otimes p), \quad 1 \otimes m = m = m \otimes 1$$

are natural in each argument. So $\mathbf{S}$ becomes a strict monoidal category. There is a symmetry

$$c_{m,n} : m \otimes n \longrightarrow n \otimes m$$
given by $c_{m,n}(in + j) = jm + i$; it is certainly not the identity function (nor is it an involution) in general.

Of course **S** also has finite coproducts. Suppose we have a finite family $(m_i)_{i\in r}$ of objects m_i of **S**. For $0 \leq i < r$, we define a function $\iota_i : m_i \longrightarrow m_0 + m_1 + \ldots + m_{r-1}$ by
$$\iota_i(j) = m_0 + \ldots + m_{i-1} + j.$$
Then $(\iota_i)_{i\in r}$ is a family of coprojections for a categorical coproduct of the family $(m_i)_{i\in r}$. For any family $(f_i)_{i\in r}$ of functions $f_i : m_i \longrightarrow n_i$, we obtain a function
$$f_0 + f_1 + \ldots + f_{r-1} : m_0 + m_1 + \ldots + m_{r-1} \longrightarrow n_0 + n_1 + \ldots + n_{r-1}$$
given by $(f_0 + f_1 + \ldots + f_{r-1})(m_0 + \ldots + m_{i-1} + j) = n_0 + \ldots + n_{i-1} + f_i(j)$ for $j \in m_i$. This describes a functor $\Sigma_r : \mathbf{S}^r \longrightarrow \mathbf{S}$ which is the r-fold tensor product of another <u>strict</u> monoidal structure on **S**.

The product and coproduct monoidal structures on **S** are related by distributivity. In fact, for each $p \in \mathbf{S}$, the equality
$$(m_0 + \ldots + m_{r-1}) \otimes p = m_0 \otimes p + \ldots + m_{r-1} \otimes p$$
is natural in all the variables $m_0, \ldots, m_{r-1}$ and p, giving right distributivity. In particular, this implies (by taking $m_0 = \ldots = m_{r-1} = 1$) that
$$r \otimes g = g^{+r} : rp \longrightarrow rq,$$
where $g^{+r} = g + \ldots + g$ (r terms) and $g : p \longrightarrow q$ is any function. For left distributivity we have the composite
$$p \otimes (m_0 + \ldots + m_{r-1}) \xrightarrow{c_{p,m_0+\ldots+m_{r-1}}} (m_0 + \ldots + m_{r-1}) \otimes p = m_0 \otimes p + \ldots + m_{r-1} \otimes p$$
$$\xrightarrow{c_{m_0,p} + \ldots + c_{m_{r-1},p}} p \otimes m_0 + \ldots + p \otimes m_{r-1},$$
denoted by $\delta_{p,m_0,\ldots,m_{r-1}}$, which is not the identity function in general. We call a category with two monoidal structures, and distributivity constraints satisfying the axioms in [L] and [K], a *rigoid*. A rigoid is *right strict* when both of the monoidal structures are strict and the right distributivity constraint is an identity. The rigoids considered here will be even more special in that the tensor product usually written as a sum will always be a categorical coproduct.

The *equalizer* of two functions $f, g : m \longrightarrow n$ is the function $h : p \longrightarrow m$ where $h(0) < h(1) < \ldots < h(p-1)$ are the elements $h(i)$ of m with $f(h(i)) = g(h(i))$. We shall not discuss here coequalizers or function sets in **S** since they are not needed below.

Now we turn to combinatorial linear algebra over the field **C** of complex numbers. Write **Mat** for the skeletal category of matrices. That is, the objects are natural numbers, while the morphisms $a : m \longrightarrow m'$, called $m' \times m$-*matrices*, are functions $a : m' \times m \longrightarrow \mathbf{C}$; the composite $a' \circ a : m \longrightarrow m''$ of $a : m \longrightarrow m'$ and $a' : m' \longrightarrow m''$ is given by usual matrix multiplication:

$$(a' \circ a)_{i'', i} \;=\; \sum_{i' \in m'} a'_{i'', i'}\, a_{i', i} \;.$$

There is a functor $\Gamma : \mathbf{S} \longrightarrow \mathbf{Mat}$ defined to be the identity $\Gamma(n) = n$ on objects while, for each function $f : m \longrightarrow m'$, the matrix $\Gamma(f) : m \longrightarrow m'$ is such that $\Gamma(f)_{i', i} = 1$ if and only if $i' = f(i)$.

The binary *Kronecker product* $a \otimes b : mn \longrightarrow m'n'$ of matrices $a : m \longrightarrow m'$ and $b : n \longrightarrow n'$ is defined by

$$(a \otimes b)_{i'n'+j',\, in+j} \;=\; a_{i', i}\; b_{j', j} \,.$$

For example, the Kronecker product of the matrices $\begin{pmatrix} a \\ b \end{pmatrix} : 1 \longrightarrow 2$ and $\begin{pmatrix} w & x \\ y & z \end{pmatrix} : 2 \longrightarrow 2$ is the matrix $\begin{pmatrix} aw & ax \\ ay & az \\ bw & bx \\ by & bz \end{pmatrix} : 2 \longrightarrow 4$. This defines a functor $\otimes : \mathbf{Mat} \times \mathbf{Mat} \longrightarrow \mathbf{Mat}$ which is given on objects by $m \otimes n = mn$. Strict associativity $(a \otimes b) \otimes c \;=\; a \otimes (b \otimes c)$ is clear; indeed, we have the formula

$$(a \otimes b \otimes c)_{i'n'p' + j'p' + k',\, inp + jp + k} \;=\; a_{i', i}\; b_{j', j}\; c_{k', k}$$

for the ternary Kronecker product. With this structure **Mat** is a strict monoidal category. The Kronecker product is not the categorical product. In fact, **Mat** is an additive category, so direct sums provide both categorical product and categorical coproduct.

The *direct sum* $a \oplus b : m + n \longrightarrow m' + n'$ of matrices $a : m \longrightarrow m'$ and $b : n \longrightarrow n'$ is defined by

$$(a \oplus b)_{i', i} \;=\; \begin{cases} a_{i', i} & \text{for } \; i < m,\ i' < m' \\ b_{i'-m',\, i-m} & \text{for } \; i \geq m,\ i' \geq m' \\ 0 & \text{otherwise}\,. \end{cases}$$

This gives another strict monoidal structure $\oplus : \mathbf{Mat} \times \mathbf{Mat} \longrightarrow \mathbf{Mat}$ on **Mat**.

Proposition 1 *There is a symmetric right-strict rigoid structure on the category* **Mat** *made up of the Kronecker product and direct sum monoidal structures, and such that the functor* $\Gamma : \mathbf{S} \longrightarrow \mathbf{Mat}$ *is symmetric strict monoidal with respect to* $\otimes$, *strictly preserves the chosen finite coproducts, and preserves the distributivity isomorphisms.*

Proof After checking that $\Gamma(f \otimes g) = \Gamma(f) \otimes \Gamma(g)$ and $\Gamma(f \times g) = \Gamma(f) \oplus \Gamma(g)$, all that remains is to show is that the symmetry and distributivity isomorphisms on **Mat**, obtained as the values of Γ on the symmetry and distributivity isomorphisms on **S**, are actually natural with respect to all morphisms of **Mat**, and not only with respect to those in the image of Γ. We leave this calculation as an exercise for the reader, although we can see that it must hold for more conceptual reasons (indeed, we have arrived at our structure by transporting the rigoid structure on **Mat** from the classical one on the category **Vect** of finite-dimensional vector spaces.) ■

To find the *kernel* of a matrix $a : n \longrightarrow m$, we reduce it to row echelon form by Gaussian elimination, and find the columns $h(0) < h(1) < \ldots < h(p-1)$ which do not contain a leading 1 (these are the columns corresponding to non-echelon, or free, variables). Let $k : p \longrightarrow n$ be the matrix whose i-th column is the unique solution x to the homogeneous linear system $a\,x = 0$ with $x_{h(i)} = 1$ and $x_{h(j)} = 0$ for $j \neq i$.

The *joint kernel* of a pair of matrices $a : n \longrightarrow m$, $b : n \longrightarrow p$ is defined to be the kernel of the matrix $\begin{pmatrix} a \\ b \end{pmatrix} : n \longrightarrow m + p$. Since Gaussian elimination allows the interchange of rows, the joint kernel of the pair a, b is the same as the joint kernel of the pair b, a; in other words, joint kernel depends only on the set $\{a, b\}$, not on the order. It follows that we can define the *joint kernel* of any finite set of matrices with fixed source m.

Let $(-)^t : \mathbf{Mat}^{op} \longrightarrow \mathbf{Mat}$ be the functor which takes each matrix $a : n \longrightarrow m$ to its *transpose* $a^t : m \longrightarrow n$. Notice that $(a \otimes b)^t = a^t \otimes b^t$. We can use $(-)^t$ to calculate explicit coequalizers in **Mat** by taking the transpose of the equalizer of the transpose. We can also define the hom functor $\mathbf{mat} : \mathbf{Mat}^{op} \times \mathbf{Mat} \longrightarrow \mathbf{Mat}$ by

$$\mathbf{mat}\,(a : n \longrightarrow m\,,\; b : p \longrightarrow q) \;=\; (b \otimes a^t : pm \longrightarrow qn\,).$$

Then we have a natural bijection

$$\mathbf{Mat}(m \otimes n\,, p) \;\cong\; \mathbf{Mat}(n, \mathbf{mat}(m\,, p))$$

taking the matrix $a : mn \longrightarrow p$ to the matrix $\tilde{a} : n \longrightarrow pm$ with $\tilde{a}_{km+i,j} = a_{k,in+j}$.

For any categories C, X, we write X^C for the functor category (that is, the objects are functors $F : C \longrightarrow X$ and the morphisms are natural transformations between these. We shall identify a group G with the category whose only object is $*$ and whose morphisms $g : * \longrightarrow *$ are elements of G; composition is group multiplication. A *matrix representation* of a group G is a functor $\rho : G \longrightarrow \mathbf{Mat}$; an *intertwining operator* between representations is exactly a natural transformation between the functors. So $\mathbf{Mat}^G$ is the category of matrix representations of G. It is enriched in **Mat** by obtaining $\mathbf{mat}^G(\rho\,, \sigma)$ as the joint kernel of the set of matrices

$$\mathbf{mat}(\rho(g)\,, 1_{\sigma(e)}) - \mathbf{mat}(1_{\rho(e)}\,, \sigma(g)) \;:\; \mathbf{mat}(\rho(*)\,, \sigma(*)) \longrightarrow \mathbf{mat}(\rho(*)\,, \sigma(*)) \quad \text{for} \quad g \in G\,.$$

Proposition 2 *For each finite group* G *there is a finite set* $G^{\vee}$ *and an equivalence of categories*

$$\mathbf{Mat}^G \simeq \mathbf{Mat}^{G^{\vee}}.$$

Proof Take $G^\vee$ to be a set of irreducible representations of G, one for each isomorphism class. Let $\mathbf{i} : G^\vee \longrightarrow \mathbf{Mat}^G$ be the inclusion. Then we have the functor $\tilde{\mathbf{i}} : \mathbf{Mat}^G \longrightarrow \mathbf{Mat}^{G^\vee}$ given by $\tilde{\mathbf{i}}(\rho)(\lambda) = \mathbf{mat}^G(\lambda, \rho)$. Every representation ρ of G is a direct sum, in $\mathbf{Mat}^G$, of selected irreducibles. By Schur's Lemma, $\mathbf{mat}^G(\lambda, \rho)$ is the multiplicity of the irreducible λ in this direct sum. It follows that the functor $\tilde{\mathbf{i}}$ is fully faithful and essentially surjective. ■

It follows that $\mathbf{Mat}^{G^\vee}$ obtains a symmetric autonomous (= compact closed) monoidal structure transported across the equivalence from $\mathbf{Mat}^G$. This leads us to the study of symmetric closed monoidal structures on categories of the form $\mathbf{Mat}^\Lambda$ where Λ is a finite set (regarded as a discrete category). We wish to characterize such structures in terms of structure on the set Λ. This is a particular case of the problem solved by Day [D].

A set Λ is *rigged* when it is equipped with a distinguished element $1 \in \Lambda$ and, for all triples $\lambda, \mu, \nu \in \Lambda$, a natural number $\langle \lambda,\mu;\nu \rangle$, satisfying the following conditions:

$$\langle 1,\lambda;\nu \rangle = \langle \lambda,1;\nu \rangle = \begin{cases} 1 & \text{for } \lambda = \nu \\ 0 & \text{for } \lambda \neq \nu \end{cases},$$

$$\sum_{\xi \in \Lambda} \langle \lambda,\mu;\xi \rangle \langle \xi,\nu;\pi \rangle = \sum_{\xi \in \Lambda} \langle \mu,\nu;\xi \rangle \langle \lambda,\xi;\pi \rangle ,$$

for $\lambda, \mu \in \Lambda$, $\langle \lambda,\mu;\nu \rangle = 0$ for all but a finite number of ν.

The left-hand side of the first condition is denoted by $\langle \lambda;\nu \rangle$ and that of the second condition is denoted by $\langle \lambda,\mu,\nu;\pi \rangle$. Let $\mathbf{N}[\Lambda]$ denote the free commutative monoid on Λ; elements are functions $f : \Lambda \longrightarrow \mathbf{N}$ which are zero for all but a finite number of values; we identify each $\lambda \in \Lambda$ with the function which has value 1 at λ and 0 elsewhere. Rigged structures on Λ are in bijection with rig multiplications on the additive monoid $\mathbf{N}[\Lambda]$, for which the multiplicative unit is $1 \in \Lambda$. The multiplication for the rig $\mathbf{N}[\Lambda]$ relates to the rigged structure via the convolution formula

$$(f * g)(\nu) = \sum_{\lambda,\mu} \langle \lambda,\mu;\nu \rangle f(\lambda)\, g(\mu) .$$

The rigged set is called *commutative* when it satisfies $\langle \lambda,\mu;\nu \rangle = \langle \mu,\lambda;\nu \rangle$; this is precisely the condition that the rig $\mathbf{N}[\Lambda]$ should be commutative. A commutative rigged set is called $*$-*autonomous* when there is a function $(-)^* : \Lambda \longrightarrow \Lambda$ such that

$$\lambda^{**} = \lambda \quad \text{and} \quad \langle \lambda,\mu;\nu \rangle = \langle \lambda,\nu^*;\mu^* \rangle .$$

It is called *autonomous* when it is $*$-autonomous and $\langle \lambda,\mu;\nu \rangle = \langle \lambda^*,\mu^*;\nu^* \rangle$.

A *droup* is an autonomous commutative rigged set together with the following data:

non-zero complex numbers $\mathfrak{l}(\lambda)$, $\mathfrak{r}(\lambda)$ for all $\lambda \in \Lambda$, complex invertible matrices $\mathfrak{a}(\lambda,\mu,\nu;\pi)$ of size $\langle\lambda,\mu,\nu;\pi\rangle$ for all $\lambda, \mu, \nu, \pi \in \Lambda$,

complex invertible matrices $\mathfrak{c}(\lambda,\mu;\nu)$ of size $\langle\lambda,\mu;\nu\rangle$ for all $\lambda, \mu, \nu \in \Lambda$;

satisfying the following conditions (in which c denotes the symmetry for **Mat**):

(unity) $$\mathfrak{l}(\lambda)\,\mathfrak{a}(\lambda,1,\nu;\pi) = \mathfrak{r}(\nu)\,1_{\langle\lambda,\nu;\pi\rangle}$$

(3-cocyclicity)
$$\Big(\bigoplus_{\zeta} 1_{\langle\nu,\pi;\zeta\rangle}\otimes\mathfrak{a}(\lambda,\mu,\zeta;\sigma)\Big)\Big(\bigoplus_{\xi,\zeta} c_{\langle\lambda,\mu;\xi\rangle,\langle\nu,\pi;\zeta\rangle}\otimes 1_{\langle\xi,\zeta;\sigma\rangle}\Big)\Big(\bigoplus_{\xi} 1_{\langle\lambda,\mu;\xi\rangle}\otimes\mathfrak{a}(\xi,\nu,\pi;\sigma)\Big)$$
$$= \Big(\bigoplus_{\zeta}\mathfrak{a}(\mu,\nu,\pi;\zeta)\otimes 1_{\langle\lambda,\zeta;\sigma\rangle}\Big)\Big(\bigoplus_{\xi} 1_{\langle\mu,\nu;\xi\rangle}\otimes\mathfrak{a}(\lambda,\xi,\pi;\sigma)\Big)\Big(\bigoplus_{\zeta}\mathfrak{a}(\lambda,\mu,\nu;\zeta)\otimes 1_{\langle\zeta,\pi;\sigma\rangle}\Big)$$

(symmetry) $$\mathfrak{c}(\mu,\lambda;\nu)\,\mathfrak{c}(\lambda,\mu;\nu) = 1_{\langle\lambda,\mu;\nu\rangle}$$

(braiding)
$$\mathfrak{a}(\mu,\nu,\lambda;\pi)\Big(\bigoplus_{\xi} 1_{\langle\mu,\nu;\xi\rangle}\otimes\mathfrak{c}(\lambda,\xi;\pi)\Big)\mathfrak{a}(\lambda,\mu,\nu;\pi)$$
$$= \Big(\bigoplus_{\xi}\mathfrak{c}(\lambda,\nu;\xi)\otimes 1_{\langle\mu,\xi;\pi\rangle}\Big)\mathfrak{a}(\mu,\lambda,\nu;\pi)\Big(\bigoplus_{\xi}\mathfrak{c}(\lambda,\mu;\xi)\otimes 1_{\langle\xi,\nu;\pi\rangle}\Big)$$

(integrity) ????? is a natural number $\mathfrak{d}(\lambda)$.

For each droup Λ, we obtain a symmetric monoidal category $\mathbf{Mat}[\Lambda]$ whose objects are elements of $\mathbf{N}[\Lambda]$, whose morphisms $a : f \longrightarrow f'$ are families of matrices $a_\lambda : f(\lambda) \longrightarrow f'(\lambda)$, whose composition is given pointwise in **Mat**, and whose tensor product is given on objects by the convolution multiplication of $\mathbf{N}[\Lambda]$ and on morphisms $a : f \longrightarrow f'$, $b : g \longrightarrow g'$ by
$$(a * b)(\nu) = \bigoplus_{\lambda,\mu}\langle\lambda,\mu;\nu\rangle\, a_\lambda\otimes b_\mu \ .$$
The unit, associativity and commutativity constraints are induced in the obvious way by $\mathfrak{l}(\lambda)$, $\mathfrak{r}(\lambda)$, $\mathfrak{a}(\lambda,\mu,\nu;\pi)$, $\mathfrak{c}(\lambda,\mu;\nu)$.

Proposition 3 *For any droup* Λ, *the monoidal category* $\mathbf{Mat}[\Lambda]$ *is autonomous. There is a strict monoidal functor* $\omega : \mathbf{Mat}[\Lambda] \longrightarrow \mathbf{Mat}$ *determined by the condition* $\omega(\lambda) = \mathfrak{d}(\lambda)$.

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Macquarie University
N. S. W. 2109
AUSTRALIA
email: street@math.mq.edu.au

We now give a general result about monoidal structures. Suppose $\mathcal{V}$ is a monoidal category in the sense of [EK], [McL] and suppose $J : \mathcal{A} \longrightarrow \mathcal{V}$ is a fully faithful functor such that there is an object I' of $\mathcal{A}$ and an isomorphism $\phi_0 : I \cong JI'$, and, for all objects $A, B \in \mathcal{A}$, there exist an object $C \in \mathcal{A}$ and an isomorphism $JC \cong JA \otimes JB$. By making a choice $A \otimes B$ of object C and a choice of isomorphism $\phi_{2;A,B} : JA \otimes JB \cong J(A \otimes B)$ for each A, B, we obtain a unique monoidal structure on $\mathcal{A}$ such that I' is the unit object, $A \otimes B$ is the tensor product, and J, together with the isomorphisms ϕ_0 and the $\phi_{2;A,B}$, becomes a strong monoidal functor $J : \mathcal{A} \longrightarrow \mathcal{V}$. If the category $\mathcal{A}$ is skeletal (meaning that $A \cong B$ implies $A = B$), then the objects I' and $A \otimes B$ are uniquely determined, and the monoidal structure on $\mathcal{A}$ depends only on the choice of the isomorphisms ϕ_0 and $\phi_{2;A,B}$.