Relationship of Spanier's Quasi-topological Spaces to k-spaces.

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INTRODUCTION

Let $\mathcal{S}$ denote the category of sets and set maps. A cartesian closed category ([4], p.550) is essentially a category $\mathcal{V}$ with finite products, including a terminal object I , and an underlying-set functor $V: \mathcal{V} \to \mathcal{S}$ such that

(i) V is represented as $\mathcal{V}(I,-)$, and

(ii) $- \times B: \mathcal{V} \to \mathcal{V}$ has a coadjoint, denoted $(B,-)$, for each $B \in \mathcal{V}$.

Condition (ii) may be written as a natural isomorphism

$$\pi: \mathcal{V}(A \times B, C) \to \mathcal{V}(A, (B,C)) \text{ for all } A,B,C \in \mathcal{V}.$$

From this we obtain a natural isomorphism

$$p: (A \times B, C) \to (A, (B,C)) \text{ for all } A,B,C \in \mathcal{V} ,$$

with

$$\pi = Vp ,$$

as an easy consequence of the associativity of products and the Yoneda representation theorem (see, for example, [4] diagram (3·18), p.480).

The categories considered in this thesis are "simple" in the sense that their objects are sets with structure and their morphisms are set maps selected by these structures; for example $\mathcal{T}$ = the category of topological spaces and continuous maps between them. The structure on a set may, in turn, be redefinable in terms of a particular set or class of objects and morphisms; for example, if T denotes the topological space of two points $\{0,1\}$ with $\{0\}$ open and $\{1\}$ not open, then the set of open sets of any object $X \in \mathcal{T}$ is in 1-1 correspondence with the set $\mathcal{T}(X,T)$.

The above idea is employed in chapter 1 to construct the category $\mathcal{P}$ of quasi-topological bases or pre-spaces. The category $\mathcal{Q}$ of quasi-topological spaces and quasi-continuous maps (Spanier [7], §2) is then determined as a full reflective subcategory of $\mathcal{P}$. I have proved the existence of this adjoint functor relationship for two reasons; the first is to demonstrate (chapter 2) the induced quasi-topologies in $\mathcal{Q}$ ([7], §3) as consequences of the limit structure of $\mathcal{P}$, which is particularly simple, and the second is to obtain (chapter 3) the cartesian closure of $\mathcal{Q}$ ([7], theorem 4.1) as a result of the cartesian closure of $\mathcal{P}$. These observations show how Spanier's axioms (2·3) and (2·4) (denoted Q3 and Q4 in chapter 1) play an ancillary rôle in the above mentioned properties of $\mathcal{Q}$.

Theorem 3·3 of chapter 3 is a special form of a theorem developed, in collaboration with G. M. Kelly, to provide information on full reflective subcategories of $\mathcal{P}$ and similar categories. This theorem and the reflectivity of $\mathcal{Q}$ in $\mathcal{P}$, together with its use, are the only original contributions to the first three chapters of this thesis.

On the other hand, any inclusion of published material after chapter 3 is referred to explicitly.

In [3], theorem 3·3, R. Brown shows the category of hausdorff k-spaces to be a cartesian closed category. It turns out that a category $\mathcal{K}$ of topological spaces arises naturally from embedding $\mathcal{T}$ in $\mathcal{Q}$. Moreover the full sub-category determined by hausdorff spaces in $\mathcal{K}$ is precisely the category of hausdorff k-spaces. The definition of non-hausdorff k-spaces given in Kelley [6], p.230, is different to the definition of non-hausdorff objects in $\mathcal{K}$ and may not yield the same concept.

One aim of this thesis is to show that $\mathcal{K}$ does provide a suitable generalisation of the category of hausdorff k-spaces. In chapter 5, I give the definition of a function space topology which reduces to the ordinary compact-open topology when the domain is a hausdorff k-space. This definition allows the amount of actual "topological calculation" performed to be restricted to lemma 5·1. The cartesian closure of $\mathcal{K}$ follows from this lemma and, in turn, an alternative proof of R. Brown's result for hausdorff k-spaces is obtained.

The fundamental definitions and results of category theory — for example limits, the Yoneda representaion theorem, the theory of adjoint functors — are assumed from the outset. The notation employed for categories, functors and limits is sufficiently standard to warrant its use (see [4], I, §1), however several abbreviations will be noted:

(i) Given categories $\mathcal{M}$ and $\mathcal{A}$ with $\mathcal{M}$ small, and a functor $K: \mathcal{M} \to \mathcal{A}$, the limit of K in $\mathcal{A}$ (provided it exists) is denoted by

$$\lim K = \{A \underset{\alpha_M}{\to} KM \mid M \in \mathcal{M}\} .$$

This is abbreviated to $A \underset{\alpha_M}{\to} KM$ wherever the category $\mathcal{M}$ is indicated in the context. Similarly the colimit of K in $\mathcal{A}$ is denoted by

$$\operatorname{colim} K = KM \underset{M}{\overset{\alpha}{\to}} A$$

if it exists. For the case when $\mathcal{M} = \Lambda$ is a discrete category (that is, the only morphisms in $\mathcal{M}$ are the identity morphisms of objects) we put $A_\lambda = K(\lambda)$ for each $\lambda \in \Lambda$ so that colim K becomes the coproduct (or sum) in $\mathcal{A}$ of the set $\{A_\lambda \mid \lambda \in \Lambda\}$ of objects of $\mathcal{A}$, and is written

$$A_\lambda \underset{\alpha_\lambda}{\to} \sum_\Lambda A_\lambda .$$

Given a family $f_\lambda \in \mathcal{A}(A_\lambda, B)$ of morphisms indexed by Λ, the unique (!) morphism $f \in \mathcal{A}(\sum_\Lambda A_\lambda, B)$ such that

$$f\alpha_\lambda = f_\lambda \text{ for each } \lambda \in \Lambda$$

is denoted by the row vector

$$f = (\ldots, f_\lambda, \ldots)$$

of "length" card Λ whose λ-entry is f_λ. This notation is abbreviated to (f_λ).

(ii) The letter C is used to represent an arbitrary compact hausdorff topological space — that is, a proposition (or statement) involving C is assumed to be valid when C is any compact hausdorff space — unless stated otherwise.

(iii) Full embedding functors are not named and, in the case of a full reflective embedding $\mathcal{A} \to \mathcal{B}$, with reflection $S: \mathcal{B} \to \mathcal{A}$, the symbol "S" is also used to denote the composite $\mathcal{B} \xrightarrow{S} \mathcal{A} \to \mathcal{B}$. It is notationally convenient to avoid naming certain non-full embeddings — especially underlying-set functors — unless their presence needs to be emphasised.

Finally, very little of the theory of monoidal closed categories, as given in [4], is required for the following work. Apart from the definition of a monoidal closed category, one theorem of existence is used together with the section entitled "Cartesian Closed Categories" on page 550 of this reference.

CHAPTER I

Quasi-topological Spaces and Bases.

Let $\mathcal{C}$ denote the category of compact hausdorff topological spaces and continuous maps between them. The category $\mathcal{P}$ of quasi-topological bases is defined as follows. An object of $\mathcal{P}$ is a set X together with, for each $C \in \mathcal{C}$, a set $P(C,X)$ of set maps $C \to X$ satisfying the axioms:

Q1. For all $C \in \mathcal{C}$, $P(C,X)$ contains all constant maps.

Q2. $g \in \mathcal{C}(C_1, C_2)$ and $f \in P(C_2, X) \Rightarrow fg \in P(C_1, X)$.

A morphism between objects X and Y in $\mathcal{P}$ is a set map $f: X \to Y$ such that

$$g \in P(C,X) \Longrightarrow fg \in P(C,Y) \quad \text{for all } C \in \mathcal{C} \tag{1·1}$$

Such maps will be called $\mathcal{P}$-continuous.

There is a full embedding $\mathcal{C} \to \mathcal{P}$ which sends $B \in \mathcal{C}$ to its underlying set with $P(C,B) = \mathcal{C}(C,B)$ and sends $f \in \mathcal{C}(B,D)$ to itself. This functor is clearly faithful and

$$f \in \mathcal{P}(C,X) \Longrightarrow f.1_C \in P(C,X) \text{ by (1·1) because } 1_C \in \mathcal{C}(C,C) = P(C,C),$$

$$\Longrightarrow f \in P(C,X),$$

$$f \in P(C,X) \Longrightarrow fg \in P(D,X) \text{ for all } g \in P(D,C) = \mathcal{C}(D,C) \text{ and } D \in \mathcal{C}, \text{ by Q2.}$$

$$\Longrightarrow f \in \mathcal{P}(C,X) \text{ by (1·1).}$$

Hence $\mathcal{P}(C,X) = P(C,X)$ for all $C \in \mathcal{C}$, $X \in \mathcal{P}$. (1·2)

In particular, for $X = B \in \mathcal{C}$,

$$\mathcal{P}(C,B) = P(C,B) = \mathcal{C}(C,B) \quad . \qquad (1\cdot 3)$$

If C and B have the same underlying set and f is the identity set map, then f is $\mathcal{P}$-continuous iff f is continuous, hence C and B coincide in $\mathcal{P}$ iff they do in $\mathcal{C}$. These remarks show that the embedding $\mathcal{C} \to \mathcal{P}$ is full and object-injective.

Remark. In view of (1·2) the defining sets for a given $X \in \mathcal{P}$ will be denoted by $\mathcal{P}(C,X)$.

Proposition 1·1. The objects of $\mathcal{C}$ form a strong generating class for $\mathcal{P}$ in the sense that $f \in \mathcal{P}(X,Y)$ is an isomorphism in $\mathcal{P}$ iff $\mathcal{P}(1,f): \mathcal{P}(C,X) \to \mathcal{P}(C,Y)$ is an isomorphism in $\mathcal{S}$, for all $C \in \mathcal{C}$.

Proof. $\Longrightarrow$ is clear.

$\Longleftarrow$. Taking $C \in \mathcal{C}$ to be the space consisting of one point $*$ shows that f is a bijection by Q2. Taking $C \in \mathcal{C}$ general then shows that X and Y have identical $\mathcal{P}$-structures.

The category $\mathcal{Q}$ of quasi-topological spaces and quasi-continuous maps (Spanier [7], §2) is defined as the full subcategory of $\mathcal{P}$ determined by those $X \in \mathcal{P}$ which satisfy:

Q3. Given a finite (possibly empty) set $\{C_n \mid n \in N\}$ of objects $C_n \in \mathcal{C}$, with coproduct

$$C_n \xrightarrow{\alpha_n} \Sigma_N C_n \text{ in } \mathcal{C}, \text{ then}$$

$f \in \mathcal{P}(\Sigma_N C_n, X) \iff f_n = f\alpha_n \in \mathcal{P}(C_n, X)$ for each $n \in N$,

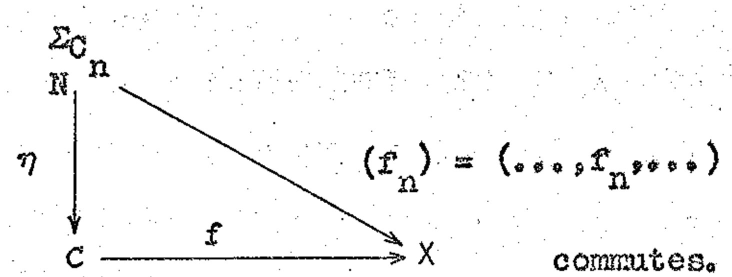
and

Q4. Given a surjection $\eta \in \mathcal{C}(D,C)$, and $f \in \mathcal{S}(C,X)$,

then $\quad f \in \mathcal{P}(C,X) \iff f\eta \in \mathcal{P}(D,X)$.

<u>Remark.</u> For $N = \emptyset$ (the empty set), Q3 states that the unique set map $\emptyset \to X$ is in $\mathcal{P}(\emptyset, X)$ for each $X \in \mathcal{Q}$.

<u>Definition.</u> For any $C \in \mathcal{C}$ and $X \in \mathcal{P}$ the set map $f: C \to X$ is <u>covered</u> if there exist a finite (possibly empty) set $\{C_n \mid n \in N\}$ of objects $C_n \in \mathcal{C}$, with coproduct $C_n \xrightarrow{\alpha_n} \Sigma_N C_n$ in $\mathcal{C}$, an $f_n \in \mathcal{P}(C_n, X)$ for each $n \in N$, and a surjection $\eta \in \mathcal{C}(\Sigma_N C_n, C)$ such that



commutes.

Such a covering of f is denoted by $(\Sigma_N C_n, f_n, \eta)$.

Clearly Q3 and Q4 are, together, equivalent to Q3'.

Q3'. $\quad f \in \mathcal{P}(C,X) \iff f \in \mathcal{S}(C,X)$ is covered.

<u>Remark.</u> On taking $C = \emptyset$, the unique set map $\emptyset \to X$ is always covered by taking $N = \emptyset$.

Observe that any $D \in \mathcal{C}$ is a quasi-topological space. This follows from

$f : C \to D$ covered by $(\sum_N C_n, f_n, \eta)$

$\implies f_n \in \mathcal{P}(C_n, D) = \mathcal{C}(C_n, D)$ for each $n \in N$, by definition of $\mathcal{C} \to \mathcal{P}$,

$\implies (f_n) = f\eta \in \mathcal{C}(\sum_N C_n, D)$,

$\implies f \in \mathcal{C}(C, D)$ because η is a continuous surjection between compact hausdorff spaces and is consequently a topological identification map,

$\implies f \in \mathcal{P}(C, D)$ by definition of $\mathcal{C} \to \mathcal{P}$,

as required for Q3'.

Proposition 1·2.

$\mathcal{Q}$ is a full reflective subcategory of $\mathcal{P}$.

Proof. $\mathcal{Q}$ is full in $\mathcal{P}$ by definition. Define the reflecting functor $F : \mathcal{P} \to \mathcal{P}$ on objects as

FX = set X with $f \in \mathcal{P}(C, FX)$ iff f is covered.

$f \in \mathcal{P}(C, X) \implies f$ covered by $(C, f, 1)$

$\implies f \in \mathcal{P}(C, FX)$,

i.e. $\mathcal{P}(C, X) \subset \mathcal{P}(C, FX)$ for all $C \in \mathcal{C}$, $X \in \mathcal{P}$. (1·4)

We require (a) $FX \in \mathcal{Q}$, and

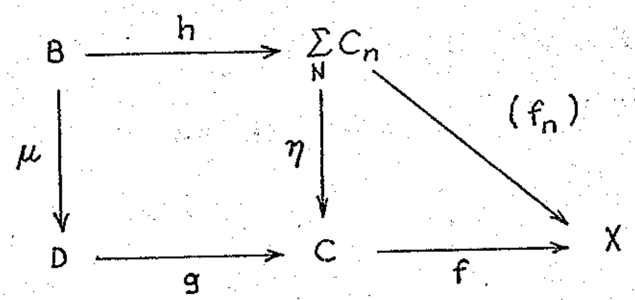
(b) $\mathcal{P}(FX, Y) = \mathcal{P}(X, Y)$ for all $X \in \mathcal{P}$, $Y \in \mathcal{Q}$, to hold.

(a) Q1. is satisfied by (1·4).

Q2. Given $g \in \mathscr{C}(D,C)$ and $(\sum_N C_n, f_n, \eta)$ covering $f: C \to X$, let

$$(h, \mu, B) = \text{pull-back } (g, \eta) \text{ in } \mathscr{C}$$

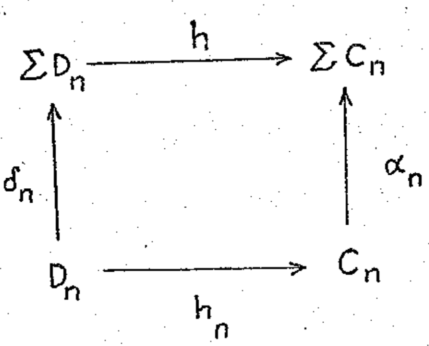
(this limit exists by the completeness of $\mathscr{C}$).



Then μ is a continuous surjection. Also B may be represented as a sum in $\mathscr{C}$; let $D_n = h^{-1}(C_n)$ with the subspace topology in B, then

$$B = \sum_N D_n \text{ in } \mathscr{S}.$$

Because h is continuous, each $h^{-1}(C_n)$ is open and closed in B whence $B = \sum_N D_n$ in $\mathscr{C}$. For each $n \in N$ let $h_n \in \mathscr{C}$ be the map defined by



commutes.

$(\alpha_n, \delta_n$ denote coproduct inclusions). Then, by Q2. for X,

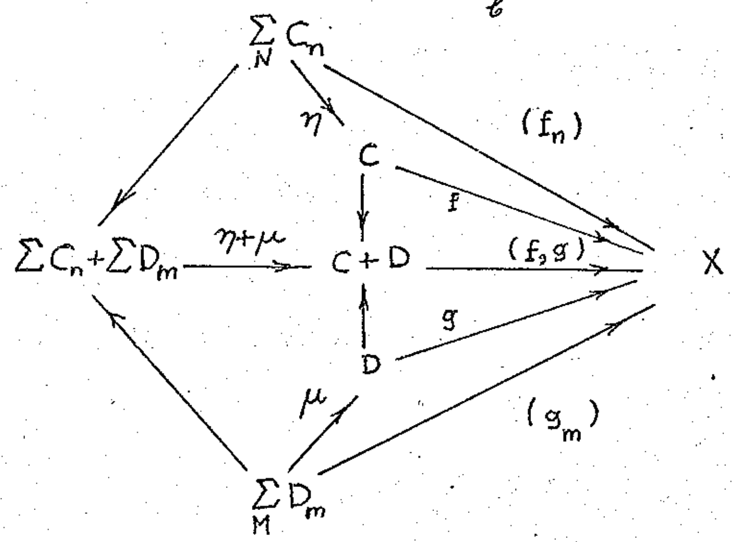
$f_n h_n \in \mathcal{P}(D_n, X)$ for all $n \in N$, so $(\sum_N D_n, f_n h_n, \mu)$ covers fg as required for

$$fg \in \mathcal{P}(D, FX) .$$

Q3. Given a cover $(\sum_N C_n, f_n, \eta)$ of $f: C \to X$ and

a cover $(\sum_M D_m, g_m, \mu)$ of $g: D \to X$

we obtain a cover for $(f,g): C \underset{\mathcal{C}}{+} D \to X$ from the diagram



where $\eta + \mu$ is a continuous surjection since both η and μ are (unmarked arrows denote coproduct inclusions in $\mathcal{C}$).

Also

$$(f,g)(\eta+\mu) = (f\eta, g\mu) = (\ldots, f_n, \ldots, g_m, \ldots)$$

as required for $(f,g) \in \mathcal{P}(C \underset{\mathcal{C}}{+} D, FX)$.

This verifies Q3 for the case of an indexing set of cardinal two which, together with the fact that $\emptyset \to X$ is always covered, clearly suffices for the general case.

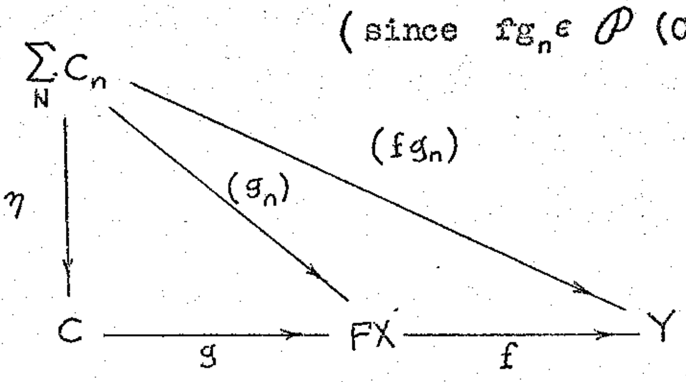
Q4. Suppose $f\eta$ is covered by $(\sum_N C_n, f_n, \mu)$ where $\eta \in \mathcal{C}(C,D)$ is a surjection and $f:D \to X$ is any set map. Then f is covered by $(\sum_N C_n, f_n, \eta\mu)$ and so is in $\mathcal{P}(D,FX)$.

(b) $$\mathcal{P}(FX,Y) = \mathcal{P}(X,Y) \text{ for all } X \in \mathcal{P},\ Y \in \mathcal{Q}.$$

$$\begin{aligned} f \in \mathcal{P}(FX,Y) &\Rightarrow fg \in \mathcal{P}(C,Y) \text{ for all } g \in \mathcal{P}(C,FX) \text{ by } (1\cdot1)\\ &\Rightarrow fg \in \mathcal{P}(C,Y) \text{ for all } g \in \mathcal{P}(C,X) \text{ by } (1\cdot4)\\ &\Rightarrow f \in \mathcal{P}(X,Y) \text{ by } (1\cdot1).\end{aligned}$$

$$\begin{aligned} f \in \mathcal{P}(X,Y) &\Rightarrow fh \in \mathcal{P}(C,Y) \text{ for all } h \in \mathcal{P}(C,X) \text{ by } (1\cdot1)\\ &\Rightarrow fg \in \mathcal{P}(C,Y) \text{ for all } g \in \mathcal{P}(C,FX),\end{aligned}$$

because $g \in \mathcal{P}(C,FX) \Rightarrow g$ is covered by $(\sum_N C_n, g_n, \eta)$
$\Rightarrow fg$ covered by $(\sum_N C_n, fg_n, \eta)$
(since $fg_n \in \mathcal{P}(C_n,Y)$ for all $n \in N$)



hence $fg \in \mathcal{P}(C,Y)$ by Q3' for $Y \in \mathcal{Q}$, as required by (1·1) for $f \in \mathcal{P}(FX,Y)$.

By (b), F is necessarily given on morphisms by

$$F_{XY} = \mathcal{P}(X,Y) \xrightarrow{\mathcal{P}(1,\beta_Y)} \mathcal{P}(X,FY) = \mathcal{P}(FX,FY) \text{ where}$$

$$\beta_Y \in \mathcal{P}(Y,FY) = \mathcal{P}(FY,FY) \text{ corresponds to } 1_{FY}.$$

That is, β_Y is the identity set map for all $Y \in \mathcal{P}$. (1·5)

By Q1 there exists a unique $\mathcal{P}$-structure on a single point $*$ and this corresponds to the generator of $\mathcal{C}$ under the inclusion $\mathcal{C} \to \mathcal{P}$. We may thus denote this object of $\mathcal{P}$ by $*$.

<u>Proposition 1·3.</u> The underlying-set functor $U: \mathcal{P} \to \mathcal{S}$

(a) is faithful,

(b) has an adjoint and a coadjoint and

(c) is represented by $*$.

<u>Proof.</u> (a) is trivial.

(b) An adjoint $D: \mathcal{S} \to \mathcal{P}$ to U is given by

$DX =$ set X with $\mathcal{P}(C,X) =$ all constant maps $C \to X$, for all $C \in \mathcal{C}$.

Clearly $DX \in \mathcal{P}$ and $\mathcal{P}(DX,Y) = \mathcal{S}(X,UY)$ for all $X \in \mathcal{S}$ and $Y \in \mathcal{P}$.

A coadjoint $T: \mathcal{S} \to \mathcal{P}$ to U is given by $TX =$ set X with $\mathcal{P}(C,X) = \mathcal{S}(C,X)$ for all $C \in \mathcal{C}$. Again it is clear that $TX \in \mathcal{P}$ and $\mathcal{S}(UY,X) = \mathcal{P}(Y,TX)$ for all $X \in \mathcal{S}$, $Y \in \mathcal{P}$.

(c) A natural isomorphism $\iota: U \to \mathcal{P}(*,-)$ is provided by

$$\mathcal{P}(*,Y) = \mathcal{P}(D*,Y) \quad \text{by definition of } * \text{ in } \mathcal{P}.$$

$$= \mathcal{S}(*,UY) \quad \text{on taking } X = * \in \mathcal{S} \text{ in (b)}.$$

$$\cong UY \quad \text{for all } Y \in \mathcal{P}.$$

<u>Corollary 1·4.</u>

The properties (a), (b), and (c) also hold for the composite $\mathcal{Q} \to \mathcal{P} \xrightarrow{U} \mathcal{S}$.

Proof. (a) and (c) are obvious from (a) and (c) for U and the fact that $* \in \mathcal{P}$ is an object of $\mathcal{Q}$.

(b) By proposition 1·2 and (b) for U we have that $\mathcal{S} \overset{D}{\to} \mathcal{P} \overset{F}{\to} \mathcal{Q}$ is adjoint to $\mathcal{Q} \to \mathcal{P} \overset{U}{\to} \mathcal{S}$. Because $TX \in \mathcal{Q}$ for all $X \in \mathcal{S}$ the coadjoint of $\mathcal{Q} \to \mathcal{P} \overset{U}{\to} \mathcal{S}$ is given simply as the factorisation of $\mathcal{S} \overset{T}{\to} \mathcal{P}$ through $\mathcal{Q} \to \mathcal{P}$.

Remark. There are two $\mathcal{P}$-structures on the empty set $\emptyset$. They are

$\emptyset_o = D\emptyset$ given by $\mathcal{P}(C, \emptyset_o) = \emptyset$ for all $C \in \mathcal{C}$,

and

$\emptyset_1$ given by $\mathcal{P}(C, \emptyset_1) = \emptyset$ for all $C \neq \emptyset$ in $\mathcal{C}$,

$= 1_\emptyset$ for $C = \emptyset$ in $\mathcal{C}$.

However $F\emptyset_o = \emptyset_1$ because $\emptyset \to X$ is covered for all $X \in \mathcal{P}$, as previously remarked.

No confusion will arise if we resume the omission of the symbol U.

CHAPTER 2.

Limits and Colimits.

Limits and colimits in the category $\mathcal{P}$ are described by the following propositions.

Proposition 2·1

For an arbitrary small category $\mathcal{A}$ and functor $K: \mathcal{A} \to \mathcal{P}$ let $X \xrightarrow[\alpha_A]{} KA$ be the limit of the composite $\mathcal{A} \xrightarrow[K]{} \mathcal{P} \to \mathcal{S}$. Then $X \xrightarrow[\alpha_A]{} KA$ is the limit of $K: \mathcal{A} \to \mathcal{P}$ if X is given in $\mathcal{P}$ by

$$f \in \mathcal{P}(C,X) \iff \alpha_A . f \in \mathcal{P}(C,KA) \quad \text{for all } C \in \mathcal{C} \text{ and } A \in \mathcal{A} \qquad (2.1)$$

Proof. Clearly $X, \alpha_A \in \mathcal{P}$ for all $A \in \mathcal{A}$. Suppose $Y \xrightarrow[\beta_A]{} KA$ is a family of morphisms in $\mathcal{P}$, indexed by obj $\mathcal{A}$, such that $Kf.\beta_A = \beta_B$ for all $A, B \in \mathcal{A}$, $f \in \mathcal{A}(AB)$. Then, because X is the limit of $\mathcal{A} \xrightarrow[K]{} \mathcal{P} \to \mathcal{S}$, $\exists!\, g \in \mathcal{S}(Y,X)$ such that

$$\alpha_A . g = \beta_A \quad \text{for all } A \in \mathcal{A}.$$

Moreover $g \in \mathcal{P}(Y,X)$ because $C \in \mathcal{C}$ and $h \in \mathcal{P}(C,Y)$

$\Rightarrow \beta_A h \in \mathcal{P}(C,KA)$ all $A \in \mathcal{A}$ since $\beta_A \in \mathcal{P}$

$\Rightarrow \alpha_A gh \in \mathcal{P}(C,KA)$ since $\alpha_A g = \beta_A$ for all $A \in \mathcal{A}$.

$\Rightarrow gh \in \mathcal{P}(C,X)$ by (2·1).

Hence $\lim K = X \xrightarrow[\alpha_A]{} KA$.

<u>Proposition 2.2.</u>

For an arbitrary small category $\mathcal{A}$ and functor $K:\mathcal{A}\to\mathcal{P}$ let $KA \xrightarrow[\alpha_A]{} X$ be the colimit of the composite $\mathcal{A}\to\mathcal{P}\to\mathcal{S}$. Then $KA \xrightarrow[\alpha_A]{} X$ is the colimit of $K:\mathcal{A}\to\mathcal{P}$ if X is defined in $\mathcal{P}$ by

$$f \in \mathcal{P}(C,X) \iff f \text{ is constant or } \exists\, A \in \mathcal{A},\ g \in \mathcal{P}(C,KA), \text{ such that } f = \alpha_A \cdot g\,, \tag{2·2}$$

for all $C \in \mathcal{C}$.

<u>Proof.</u> Clearly $X \in \mathcal{P}$ and $\alpha_A \in \mathcal{P}$ for all $A \in \mathcal{A}$. If $KA \xrightarrow[\beta_A]{} Y$ is a family of morphisms in $\mathcal{P}$, indexed by obj $\mathcal{A}$ and such that $\beta_B \cdot Kf = \beta_A$ for all $A,B \in \mathcal{A}$ and $f \in \mathcal{A}(A,B)$, then $\exists!\ h \in \mathcal{S}(X,Y)$ such that $h.\alpha_A = \beta_A$ for each $A \in \mathcal{A}$ because X is the colimit of $\mathcal{A}\xrightarrow[K]{}\mathcal{P}\to\mathcal{S}$. Moreover $h \in \mathcal{P}(X,Y)$ because

$$\begin{aligned} f \in \mathcal{P}(C,X) &\Rightarrow f \text{ is constant or } \exists\, A \in \mathcal{A},\ g \in \mathcal{P}(C,KA) \\ &\qquad \text{with } \alpha_A \cdot g = f\,, \text{ by } (2·2), \\ &\Rightarrow hf \text{ is constant or } hf = h\alpha_A \cdot g \\ &\qquad\qquad = \beta_A \cdot g \in \mathcal{P}(C,Y) \\ &\qquad \text{since } \beta_A \in \mathcal{P}(KA,Y)\,, \\ &\Rightarrow hf \in \mathcal{P}(C,Y)\,, \text{ for all } C \in \mathcal{C}. \end{aligned}$$

Hence $KA \xrightarrow[\alpha_A]{} X$ becomes the colimit of K.

The following general theorem of category theory is recorded (without proof) for future reference.

Theorem 2·3

Suppose $\mathcal{A}$ is a full reflective subcategory of $\mathcal{B}$ (the inclusion $\mathcal{A} \to \mathcal{B}$ having adjoint S with corresponding natural transformation

$$\beta: 1 \to S: \mathcal{B} \to \mathcal{B}) \text{ and}$$

$K: \mathcal{M} \to \mathcal{A}$ is a functor with $\mathcal{M}$ a small category. Then

(a) $\mathcal{B}$ complete $\Rightarrow$ $\mathcal{A}$ complete, with

$$\lim K \text{ in } \mathcal{A} = \lim K \text{ in } \mathcal{B}, \text{ and}$$

(b) $\mathcal{B}$ cocomplete $\Rightarrow$ $\mathcal{A}$ cocomplete, with

$$\text{colim } K \text{ in } \mathcal{A} = KM \xrightarrow[\alpha_M]{} X \xrightarrow[\beta_X]{} SX$$

where $KM \xrightarrow[\alpha_M]{} X$ = colim K in $\mathcal{B}$.

This theorem will now be employed to describe two types of colimit in $\mathcal{Q}$. By proposition 2·2 the coproduct in $\mathcal{P}$ of a set $\{X_\lambda | \lambda \in \Lambda\}$ of $\mathcal{P}$-objects is $X_\lambda \xrightarrow[\alpha_\lambda]{} \underset{\Lambda}{\Sigma} X_\lambda$, where

$$\left.\begin{array}{l} \underset{\Lambda}{\Sigma} X_\lambda = \text{the disjoint union of the sets } X_\lambda \\ \alpha_\lambda = \text{the } \lambda\text{-inclusion map,} \end{array}\right\} \qquad (2\cdot3)$$

together with $f \in \mathcal{P}(C, \underset{\Lambda}{\Sigma} X_\lambda) \Leftrightarrow f = \alpha_\lambda g$ for some $\lambda \in \Lambda$ and $g \in \mathcal{P}(C, X_\lambda)$ (2·4)

(this condition includes the case where f is constant).

In this manner $\underset{\Lambda}{\Sigma} X_\lambda$ becomes a $\mathcal{P}$-object which we denote by $\underset{\mathcal{P}}{\Sigma} X_\lambda$.

By theorem 2·3(b) the coproduct in $\mathcal{Q}$ of a set $\{X_\lambda | \lambda \in \Lambda\}$ of $\mathcal{Q}$-objects is given by $X_\lambda \xrightarrow[\alpha_\lambda]{} \underset{\Lambda}{\Sigma} X_\lambda$, with $\underset{\Lambda}{\Sigma} X_\lambda$ and α_λ as in (2·3), together with

$$f \in \mathcal{Q}(C, \underset{\Lambda}{\Sigma} X_\lambda) \iff \exists \text{ a cover } (\underset{N}{\Sigma} C_n, f_n, \eta) \text{ of } f \text{ with}$$
$$f_n \in \mathcal{P}(C_n, \underset{\mathcal{P}}{\Sigma} X_\lambda) \text{ for each } n \in N .$$

Hence $\underset{\Lambda}{\Sigma} X_\lambda$ becomes a $\mathcal{Q}$-object which we denote by $\underset{\mathcal{Q}}{\Sigma} X_\lambda$.

<u>Proposition 2·4.</u>

$$f \in \mathcal{Q}(C, \underset{\mathcal{Q}}{\Sigma} X_\lambda) \iff \exists \text{ a cover } (\underset{N}{\Sigma} C_n, \alpha_n f_n, 1_C) \text{ of } f \text{ with}$$
$$N \subset \Lambda \text{ and } f_n \in \mathcal{Q}(C_n, X_n) \text{ for each } n \in N .$$

<u>Proof.</u>

$\Leftarrow$ is clear because $f_n \in \mathcal{Q}(C_n, X_n) = \mathcal{P}(C_n, X_n)$ hence

$$\alpha_n \in \mathcal{P}(X_n, \underset{\mathcal{P}}{\Sigma} X_\lambda) \Rightarrow \alpha_n f_n \in \mathcal{P}(C_n, \underset{\mathcal{P}}{\Sigma} X_\lambda) \text{ by (1·1), for each } n \in N,$$
$$\Rightarrow f \in \mathcal{Q}(C, \underset{\mathcal{Q}}{\Sigma} X_\lambda) \text{ by Q3'.}$$

$\Rightarrow$. $f \in \mathcal{Q}(C, \underset{\mathcal{Q}}{\Sigma} X_\lambda) \Rightarrow \exists$ a cover $(\underset{M}{\Sigma} D_m, h_m, \eta)$ of f with

$$h_m \in \mathcal{P}(D_m, \underset{\mathcal{P}}{\Sigma} X_\lambda) \text{ for each } m \in M .$$

To reduce this cover to the given form, let $B_n = \Sigma D_\ell$ summed in $\mathcal{C}$ over all $\ell \in M$ for which

$$h_\ell = D_\ell \xrightarrow[h_{\ell n}]{} X_n \xrightarrow[\alpha_n]{} \underset{\mathcal{P}}{\Sigma} X_\lambda \qquad (2·5)$$

for some $h_{\ell n} \in \mathcal{Q}(D_\ell, X_n)$ — by (2·4), any $h_\ell \in \mathcal{P}(D_\ell, \underset{\mathcal{P}}{\Sigma} X_\lambda)$ has such a factorisation for a unique $n \in \Lambda$. $\underset{M}{\Sigma} D_m$ can now be written as $\sum_N B_n$ for a finite range $N \subset \Lambda$.

Let $C_n = \eta(B_n)$, with the subspace topology in C, and $\eta_n : B_n \to C_n$ be the corresponding surjection induced by η. Then $C = \sum_N C_n$ in $\mathcal{C}$ because

(i) η is a surjection so $C = \bigcup_N \eta(B_n)$,

(ii) the $\eta(B_n)$ are disjoint in C for,

$$x \in \eta(B_n) \cap \eta(B_r) , \; r \neq n \in N$$

$$\Rightarrow x = \eta(b_n) = \eta(b_r) \text{ for some } b_n \in B_n, \; b_r \in B_r ,$$

$$\Rightarrow f\eta(b_n) = f\eta(b_r) \text{ in } \sum_\Lambda X_\lambda$$

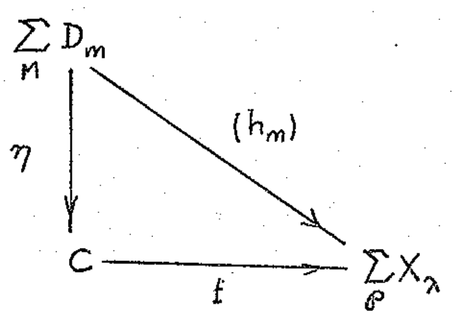
contrary to the definition of B_n in (2.5) because $X_n \cap X_r = \emptyset$ for $n \neq r \in N \subset \Lambda$,

(iii) $\eta(B_n)$ is closed in C for each $n \in N$, since B_n compact, η continuous

$$\Rightarrow \eta(B_n) \text{ compact in } C ,$$

$$\Rightarrow \eta(B_n) \text{ closed in } C \text{ since } C \text{ is hausdorff.}$$

Again, $f(C_n) = f\eta(B_n) \subset X_n$ by (2.5) and



commutes by definition of cover, so f induces a set map $f_n : C_n \to X_n$ such that

$$\begin{array}{ccc} B_n = \Sigma D_\ell & & \\ \eta_n \downarrow & \searrow^{(h_{\ell n})} & \\ C_n & \xrightarrow[f_n]{} & X_n \end{array}$$

commutes for each $n \in N$.

This means f_n is covered by $(\Sigma D_\ell, h_{\ell n}, \eta_n)$ whence $f_n \in \mathcal{Q}(C_n, X_n)$ by Q3' for $X_n \in \mathcal{Q}$. Moreover $f = (\alpha_n f_n) : C = \sum_N C_n \to \sum_{\mathcal{P}} X_\lambda$ by definition of f_n, so $(\sum_N C_n, \alpha_n f_n, 1)$ is a cover of the required form.

The covers needed to describe $\sum_{\mathcal{Q}} X_\lambda$ may thus be reduced to a canonical form. This is also true for coequalisers in $\mathcal{Q}$.

<u>Proposition 2.5.</u>

A map $\eta \in \mathcal{P}(X,Y)$ is a coequaliser in $\mathcal{P}$ iff

(a) it is a surjection, and

(b) $f \in \mathcal{P}(C,Y) \iff f = \eta g$ for some $g \in \mathcal{P}(C,X)$.

<u>Proof.</u>

(i) Given $h, k \in \mathcal{P}(Z,X)$, let $Y = \text{coequ.}(h,k)$ in $\mathcal{S}$, then η is a surjection.

$$Z \underset{k}{\overset{h}{\rightrightarrows}} X \xrightarrow{\eta} Y .$$

By proposition 2·2, coequ. (h,k) in $\mathcal{P}$ is the set Y with $f \in \mathcal{P}(C,Y) \iff f$ factors in $\mathcal{P}$ through η (this includes the cases where f is constant and where f factors in $\mathcal{P}$ through $\eta h = \eta k$).

(ii) Given $\eta \in \mathcal{P}(X,Y)$ satisfying (a) and (b), let (Z,h,k) be the discriminant of η in $\mathcal{P}$. By proposition 2·1 this discriminant both exists and is the discriminant of η in $\mathcal{S}$, hence

$$\eta = \text{coequ. } (h,k) \text{ in } \mathcal{S} \text{ by (a)}$$

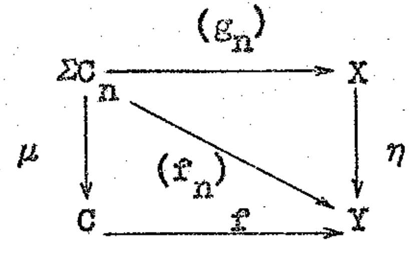
which implies η = coequ. (h,k) in $\mathcal{P}$ by (b), together with part (i) of this proof.

Coequalisers in $\mathcal{Q}$ may thus be described, using theorem 2·3(b), as maps $\eta \in \mathcal{Q}(X,Y)$ such that

(a) η is a surjection, and

(b) $f \in \mathcal{Q}(C,Y) \iff \exists$ a cover $(\sum_N C_n, f_n, \mu)$ of f with $f_n = \eta g_n$ for some $g_n \in \mathcal{Q}(C_n, X)$ and each $n \in N$.

The latter condition entails the commutativity of both triangles in



consequently (b) becomes:

$$f \in \mathcal{Q}(C,Y) \Longleftrightarrow \exists \text{ a } D \in \mathcal{C}\text{, a surjection } \mu \in \mathcal{C}(D,C)\text{,}$$
$$\text{and a } g \in \mathcal{Q}(D,X) \text{ such that}$$

$$\begin{array}{ccc} D & \xrightarrow{\;g\;} & X \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\eta} \\ C & \xrightarrow[\;f\;]{} & Y \end{array} \qquad \text{commutes.}$$

This means that a map $\eta \in \mathcal{Q}(X,Y)$ is a coequaliser in $\mathcal{Q}$ iff η is a surjection and Y has the quotient quasi-topology induced by η (Spanier [7], §3).

In concluding this chapter we emphasise a type of limit in $\mathcal{P}$ which is of importance in the sequel. The product of two objects X_1 and X_2 in $\mathcal{P}$ is, by proposition 2·1, the set (or cartesian) product

$$X_1 \xleftarrow[\alpha_1]{} X_1 \times X_2 \xrightarrow[\alpha_2]{} X_2$$

of the underlying sets, together with

$$f \in \mathcal{P}(C, X_1 \times X_2) \Longleftrightarrow \alpha_1 f \in \mathcal{P}(C, X_1) \text{ and } \alpha_2 f \in \mathcal{P}(C, X_2)\ .$$

Putting $f_1 = \alpha_1 f$ and $f_2 = \alpha_2 f$, this can be expressed as

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{P}(C, X_1 \times X_2) \Longleftrightarrow f_1 \in \mathcal{P}(C, X_1) \text{ and } f_2 \in \mathcal{P}(C, X_2)\ .$$

CHAPTER 3.

Cartesian Closure of the Category of Bases.

Given $X \in \mathcal{S}$ and $Y, Z \in \mathcal{P}$, define the component

$$\kappa_{XYZ} : \mathcal{S}(X, \mathcal{P}(Y,Z)) \to \mathcal{S}(X \times Y, Z)$$

of a natural transformation κ by

$$\kappa_{XYZ}(f) = X \times Y \xrightarrow{f \times 1} \mathcal{P}(Y,Z) \times Y \xrightarrow{e_{YZ}} Z$$

for all $f \in \mathcal{S}(X, \mathcal{P}(Y,Z))$, where the <u>evaluation map</u> e_{YZ} is given as $e_{YZ}(h,y) = h(y)$ for all $h \in \mathcal{P}(Y,Z)$ and $y \in Y$ and is natural in each variable.

<u>Definition.</u> An internal hom-functor

$$(-,-) : \mathcal{P}^* \times \mathcal{P} \to \mathcal{P}$$

is defined for $\mathcal{P}$ as follows. Given objects $X, Y \in \mathcal{P}$ let (X,Y) be the set $\mathcal{P}(X,Y)$ together with

$$f \in \mathcal{P}(C,(X,Y)) \iff \kappa(f) \in \mathcal{P}(C \times X, Y) \quad \text{for all } C \in \mathcal{C} \qquad (3\cdot1)$$

and for morphisms $f \in \mathcal{P}^*(X,X')$ and $g \in \mathcal{P}(Y,Y')$ let $(f,g) = \mathcal{P}(f,g) : (X,Y) \to (X',Y')$ as a set map. Then

(i) we obtain $(X,Y) \in \mathcal{P}$ from

Q2. given any $g \in \mathcal{C}(C,D)$ and $f \in \mathcal{P}(D,(X,Y))$ then $fg \in \mathcal{P}(C,(X,Y))$ since

$$\kappa(fg) = C \times X \xrightarrow{g \times 1} D \times X \xrightarrow{f \times 1} \mathcal{P}(X,Y) \times X \xrightarrow{e} Y$$

$$= C \times X \xrightarrow{g \times 1} D \times X \xrightarrow{\kappa(f)} Y \text{ is in } \mathcal{P}(C \times X, Y)$$

because $\kappa(f) \in \mathcal{P}(D\times X, Y)$, and

Q1. for any $f:* \to (X,Y)$, $\kappa(f) \in \mathcal{P}(*\times X, Y)$ corresponds to $f(*) \in \mathcal{P}(X,Y)$, under the obvious bijection. This, together with Q2., gives Q1.

(ii) $(f,g):(X,Y) \to (X',Y')$ is $\mathcal{P}$-continuous because

$$h \in \mathcal{P}(C,(X,Y)) \Rightarrow \kappa(h) \text{ is } \mathcal{P}\text{-continuous by } (3\cdot 1)$$

$$\Rightarrow \kappa((f,g)\cdot h) = C\times X' \xrightarrow[1\times f]{} C\times X \xrightarrow[\kappa(h)]{} Y \xrightarrow[g]{} Y'$$

is $\mathcal{P}$-continuous.

$$\Rightarrow (f,g)\cdot h \in \mathcal{P}(C,(X',Y')) \text{ by } (3\cdot 1),$$

for all $C \in \mathcal{C}$.

<u>Proposition 3·1</u>. $\mathcal{P}$ is a cartesian closed category, as defined by Eilenberg and Kelly in [4], IV, §2.

<u>Proof</u>. The evaluation map $e: \mathcal{P}(X,Y) \times X \to Y$ is a $\mathcal{P}$-continuous map $e:(X,Y)\times X \to Y$ (3·2)

because, for all $C \in \mathcal{C}$,

$$\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{P}(C,(X,Y)\times X) \Rightarrow f \in \mathcal{P}(C,(X,Y)) \text{ and } g \in \mathcal{P}(C,X)$$

$$\Rightarrow \kappa(f) \in \mathcal{P}(C\times X, Y) \text{ , by } (3\cdot 1), \text{ and } \begin{pmatrix} 1 \\ g \end{pmatrix} \in \mathcal{P}(C, C\times X)$$

$$\Rightarrow \kappa(f)\begin{pmatrix} 1 \\ g \end{pmatrix} \in \mathcal{P}(C,Y)$$

$$\Rightarrow e\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{P}(C,Y) \text{ because}$$

$e\begin{pmatrix} f \\ g \end{pmatrix} = \kappa(f)\begin{pmatrix} 1 \\ g \end{pmatrix}$ by definition of κ .

The map $r : X \to (Y, X\times Y)$ defined by $r(x)(y) = (x,y)$ for all $x \in X$, $y \in Y$ is $\mathcal{P}$-continuous. (3·3)

This follows from

$$f \in \mathcal{P}(C,X) \Rightarrow \kappa(rf) = f\times 1 \in \mathcal{P}(C\times Y, X\times Y) \text{ for all } C \in \mathcal{C}$$
$$\Rightarrow rf \text{ is } \mathcal{P}\text{-continuous by (3·1).}$$

Results (3·2) and (3·3) allow a natural isomorphism

$$\pi : \mathcal{P}(X\times Y, Z) \longrightarrow \mathcal{P}(X, (Y,Z)) \qquad (3·4)$$

to be defined as

$\pi(h) = X \overset{r}{\to} (Y, X\times Y) \overset{(1,h)}{\to} (Y,Z)$ for all $h \in \mathcal{P}(X\times Y, Z)$

with inverse $\pi^{-1}(f) = X \times Y \overset{f\times 1}{\to} (Y,Z)\times Y \overset{e}{\to} Z$ for all $f \in \mathcal{P}(X,(Y,Z))$.

$\pi(\pi^{-1}(f)) = f : X \to (Y,Z)$ because

$$\begin{aligned}(\pi(\pi^{-1}(f)))(x)(y) &= ((1,\pi^{-1}(f))\circ r)(x)(y)\\ &= (\pi^{-1}(f)\circ r(x))(y)\\ &= \pi^{-1}(f)(x,y)\\ &= f(x)(y) \text{ for all } x \in X,\ y \in Y.\end{aligned}$$

Similarly $\pi^{-1}\pi = 1$.

Because $\mathcal{P}$ is complete, it has a cartesian monoidal structure ([4], IV, 82) which is normalised ([4], p.491) by $U : \mathcal{P} \to \mathcal{S}$ together with the natural isomorphism $\iota : U \to \mathcal{P}(*,-)$ given in proposition 1·3(c). The final result may now be obtained from [4], theorem II.5.5.

A natural isomorphism

$$p: (X \times Y, Z) \cong (X, (Y, Z)) \qquad (3\cdot5)$$

is given by [4], II, (3·18) such that $\pi = Up$.

Corollary 3·2.

If $\mathcal{A}$ is a full subcategory of $\mathcal{P}$ such that

$(AB) \in \mathcal{A}$ for all $A, B \in \mathcal{A}$.

$* \in \mathcal{A}$

$A \underset{\mathcal{A}}{\times} B = A \underset{\mathcal{P}}{\times} B$ in $\mathcal{P}$, for all $A, B \in \mathcal{A}$,

then $\mathcal{A}$ has a directly induced cartesian closed structure from $\mathcal{P}$.

Proof. The above data may be completed by the full embedding of $\mathcal{A}$ in $\mathcal{P}$ together with proposition 3·1. Any faithful functor reflects commuting diagrams so the axioms for $\mathcal{A}$ follow from those for $\mathcal{P}$.

Theorem 3·3. Suppose $\mathcal{A}$ is a full reflective subcategory of $\mathcal{P}$, with reflecting functor $S: \mathcal{P} \to \mathcal{P}$, and with corresponding $\beta_X \in \mathcal{P}(X, SX)$ the identity set map for all $X \in \mathcal{P}$ (hence $\mathcal{P}(SX, A) = \mathcal{P}(X, A)$ for all $A \in \mathcal{A}$ and $X \in \mathcal{P}$). Then the following are equivalent:

(i) $X \in \mathcal{P}$, $A \in \mathcal{A} \Longrightarrow (X, A) \in \mathcal{A}$.

(ii) $C \in \mathcal{C}$, $A \in \mathcal{A} \Longrightarrow (C, A) \in \mathcal{A}$.

(iii) $X \in \mathcal{P}$, $A \in \mathcal{A} \Longrightarrow (SX, A) = (X, A)$.

(iv) $X, Y \in \mathcal{P} \Rightarrow S(X \times Y) = SX \times SY$.

(v) $C \in \mathcal{C}, X \in \mathcal{P} \Rightarrow S(C \times X) = C \times SX$.

Proof.

(i) $\Rightarrow$ (ii) trivially.

(ii) $\Rightarrow$ (iii) . For all $C \in \mathcal{C}$, $A \in \mathcal{A}$, $X \in \mathcal{P}$,

$$\mathcal{P}(C,(SX,A)) \cong \mathcal{P}(SX,(C,A)) \quad \text{by proposition } 3{\cdot}1$$
$$= \mathcal{P}(X,(C,A)) \quad \text{because } (C,A) \in \mathcal{A}.$$
$$\cong \mathcal{P}(C,(X,A)) \quad \text{by proposition } 3{\cdot}1.$$

This composite is clearly the identity set map so $(SX,A) = (X,A)$ by proposition 1·1.

(iii) $\Rightarrow$ (iv) . For all $X, Y \in \mathcal{P}$ and $A \in \mathcal{A}$

$$\mathcal{P}(S(X \times Y),A) = \mathcal{P}(X \times Y,A)$$
$$\cong \mathcal{P}(X,(Y,A)) \quad \text{by } (3{\cdot}4)$$
$$= \mathcal{P}(X,(SY,A)) \quad \text{by (iii)}$$
$$\cong \mathcal{P}(SY,(X,A)) \quad \text{by proposition } 3{\cdot}1$$
$$= \mathcal{P}(SY,(SX,A)) \quad \text{by (iii)}$$
$$\cong \mathcal{P}(SX,(SY,A)) \quad \text{by proposition } 3{\cdot}1$$
$$\cong \mathcal{P}(SX \times SY,A) \quad \text{by } (3{\cdot}4) .$$

Because $\beta_X : X \to SX$ is the identity set map, for all $X \in \mathcal{P}$, it is again obvious that the above composite of isomorphisms is the identity map. Hence

$$\mathcal{A}(S(X \times Y),A) = \mathcal{A}(SX \times SY,A) \quad \text{for all } A \in \mathcal{A}$$

because $\mathcal{A}$ is a full reflective subcategory of $\mathcal{P}$. By the representation theorem, the canonical map

$S(X \times Y) \to SX \times SY$ is thus an isomorphism in $\mathcal{A}$, hence in $\mathcal{P}$.

(iv) $\Rightarrow$ (i). For all $X \in \mathcal{P}$ and $A \in \mathcal{A}$

$$\beta_{(X,A)} : (X,A) \to S(X,A) \text{ is } \mathcal{P}\text{-continuous.}$$

Its inverse $\beta^{-1}_{(X,A)}$ is also $\mathcal{P}$-continuous because, being the identity set map, it is the image of $1_{(X,A)}$ under the following composition of isomorphisms:

$$\begin{aligned} \mathcal{P}((X,A),(X,A)) &\cong \mathcal{P}((X,A)\times X,A) \quad \text{by } (3\cdot 4) \\ &= \mathcal{P}(S((X,A)\times X),A) \\ &= \mathcal{P}(S(X,A) \times SX,A) \quad \text{by (iv)} \\ &= \mathcal{P}(S^2(XA) \times SX,A) \quad \text{because } S^2 = S \end{aligned}$$

for a full reflective embedding,

$$\begin{aligned} &= \mathcal{P}(S(S(X,A)\times X),A) \text{ by (iv)} \\ &= \mathcal{P}(S(X,A)\times X,A) \\ &\cong \mathcal{P}(S(X,A),(X,A)) \quad \text{by } (3\cdot 4). \end{aligned}$$

Finally (iv) $\Rightarrow$ (v) trivially, and

(v) $\Rightarrow$ (iii). For all $C \in \mathcal{C}$, $A \in \mathcal{A}$, $X \in \mathcal{P}$,

$$\begin{aligned} \mathcal{P}(C,(SX,A)) &\cong \mathcal{P}(C\times SX,A) \quad \text{by } (3\cdot 4) \\ &= \mathcal{P}(S(C\times X),A) \text{ by (v)} \\ &= \mathcal{P}(C\times X,A) \\ &\cong \mathcal{P}(C,(X,A)) \quad \text{by } (3\cdot 4). \end{aligned}$$

This composite is the identity set map again. Hence

$(SX,A) = (X,A)$ by proposition 1·1.

<u>Proposition 3·4.</u>

The category $\mathcal{A}$ of theorem 3·3 is, a fortiori, a cartesian closed category if it satisfies condition (i).

<u>Proof.</u> Let $X = B \in \mathcal{A}$ in (i), then $(B,A) \in \mathcal{A}$ for all $B, A \in \mathcal{A}$. By theorem 2·3 (a) we have

$$A \times_{\mathcal{A}} B = A \times_{\mathcal{P}} B \text{ for all } A, B \in \mathcal{A} \text{ and}$$
$$* = \text{the terminal object of } \mathcal{P}, \text{ in } \mathcal{A}.$$

The result follows from corollary 3·2.

<u>Corollary 3·5.</u>

$\mathcal{Q}$ is a cartesian closed category.

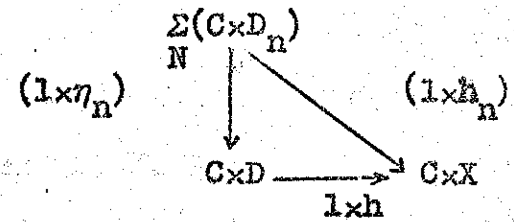
<u>Proof.</u> Taking $\mathcal{A} = \mathcal{Q}$ and $S = F$ in theorem 3·3, it suffices by proposition 3·4 to verify condition (v). The canonical map

$F(C \times X) \to C \times FX$ is trivially $\mathcal{P}$-continuous by definition of $C \times FX$ in $\mathcal{P}$ (proposition 2·1). This map is the identity on sets, and its inverse is $\mathcal{P}$-continuous because $\binom{g}{h} \in \mathcal{P}(D, C \times FX) \Rightarrow g \in \mathcal{C}(D,C)$ and $h \in \mathcal{P}(D, FX)$

$$\Rightarrow g \in \mathcal{C}(D,C) \text{ and } \begin{array}{ccc} \Sigma_N D_n & & \\ {\scriptstyle \eta=(\eta_n)}\downarrow & \searrow {\scriptstyle (h_n)} & \\ D & \xrightarrow[h]{} & X \end{array}$$

commutes for some cover $(\Sigma_N D_n, h_n, \eta)$ of h.

$\Rightarrow \quad g \in \mathcal{C}(D, C)$ and



commutes, where $(1 \times \eta_n)$ is a continuous surjection because $C \times - : \mathcal{C} \to \mathcal{C}$ preserves finite coproducts and all epimorphisms in $\mathcal{C}$,

$\Rightarrow \quad g \in \mathcal{C}(D, C)$ and $1 \times h \in \mathcal{P}(C \times D, F(C \times X))$
since $1 \times h$ is covered,

$\Rightarrow \quad (1 \times h)\begin{pmatrix} g \\ 1 \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix} \in \mathcal{P}(D, F(C \times X))$, for all $D \in \mathcal{C}$,

as required by (1·1). This completes the proof.

CHAPTER 4.

$\mathcal{K}$-spaces.

Henceforth $\mathcal{T}$ will denote the category of topological spaces X, Y, Z, ... and continuous maps, while general objects of $\mathcal{P}$ will be denoted by letters P, Q, ... The letter T will denote the set $\{0,1\}$ of two points with the non-extremal topology — $\{0\}$ open and $\{1\}$ not open. Clearly

$$f \in \mathcal{T}(X,Y) \Longleftrightarrow hf \in \mathcal{T}(X,T) \text{ for all } h \in \mathcal{T}(Y,T) \qquad (4\cdot1)$$

$\mathcal{T}$ can be embedded (non-fully) in $\mathcal{Q}$, hence in $\mathcal{P}$, by taking $X \in \mathcal{T}$ to the set X with

$$\mathcal{P}(C,X) = \mathcal{T}(C,X) \quad \text{for all } C \in \mathcal{C} \qquad (4\cdot2)$$

and $f \in \mathcal{T}(X,Y)$ to the set map f which is clearly $\mathcal{P}$-continuous (as in Spanier [7], §2).

<u>Definition.</u> A full subcategory $\mathcal{K}$ of $\mathcal{T}$ is defined by the topological spaces K for which

$$\mathcal{P}(K,T) = \mathcal{T}(K,T) \qquad (4\cdot3)$$

Such topological spaces will be called $\mathcal{K}$-spaces.

By (4·2) the category $\mathcal{C}$ of compact hausdorff spaces forms a full subcategory of $\mathcal{K}$. Using the obvious correspondence between the open (respt. closed) sets of a topological space X and the elements of $\mathcal{T}(X,T)$, we can rephrase (4·3) as: a topological space K is a $\mathcal{K}$-space iff

$V \subset K$ is open (respt. closed) in $K \iff$
for each $C \in \mathcal{C}$ and $g \in \mathcal{J}(C,K)$, the set (4·3)'
$g^{-1}(V)$ is open (respt. closed) in C.

Proposition 4·1.

$K \in \mathcal{J}$ is a $\mathcal{K}$-space iff $\mathcal{P}(K,X) = \mathcal{J}(K,X)$ for all $X \in \mathcal{J}$.

Proof.

$\Leftarrow$. Take $X = T$.

$\Rightarrow$. For all $X \in \mathcal{J}$, $\mathcal{J}(K,X) \subset \mathcal{P}(K,X)$ by the embedding $\mathcal{J} \to \mathcal{P}$, and

$f \in \mathcal{P}(K,X) \Rightarrow hf \in \mathcal{P}(K,T)$ by (1·1), for all $h \in \mathcal{J}(X,T) \subset \mathcal{P}(X,T)$

$\Rightarrow hf \in \mathcal{J}(K,T)$ by (4·3), for all $h \in \mathcal{J}(X,T)$

$\Rightarrow f \in \mathcal{J}(K,X)$ by (4·1), as required.

Theorem 4·2.

$\mathcal{K}$ is a full reflective subcategory of $\mathcal{P}$.

Proof. Take $X = L \in \mathcal{K}$ in proposition 4·1. Then $\mathcal{P}(K,L) = \mathcal{J}(K,L) = \mathcal{K}(K,L)$, so $\mathcal{K}$ is a full subcategory of $\mathcal{P}$.

Given $P \in \mathcal{P}$ define a topological space SP, with the same underlying set as P, by

$$\mathcal{J}(SP,T) = \mathcal{P}(P,T). \qquad (4\cdot4)$$

This simply states that $V \subset SP$ is open iff $g^{-1}(V)$ is open in C for all $C \in \mathcal{C}$ and $g \in \mathcal{P}(C,P)$. SP is a topological space because $\emptyset$ and SP are clearly open and if $\{V_n | n \in N\}$ is a finite set of open sets $V_n \subset SP$ then

$$g^{-1}(\bigcap_N V_n) = \bigcap_N g^{-1}(V_n)$$

is open in C for all $C \in \mathcal{C}$ and $g \in \mathcal{P}(C,P)$. Similarly arbitrary unions of open sets are open in SP.

$$\mathcal{J}(SP,X) = \mathcal{P}(P,X) \text{ for all } P \in \mathcal{P} \text{ and } X \in \mathcal{J} \qquad (4\cdot5)$$

because

$f \in \mathcal{J}(SP,X)$, $C \in \mathcal{C}$, $g \in \mathcal{P}(C,P)$ and $h \in \mathcal{J}(X,T)$

$\implies hf \in \mathcal{J}(SP,T)$

$\implies hf \in \mathcal{P}(P,T)$ by (4·4)

$\implies hfg \in \mathcal{P}(C,T)$ by (1·1)

$\implies hfg \in \mathcal{J}(C,T)$ by (4·2).

Hence $fg \in \mathcal{J}(C,X) = \mathcal{P}(C,X)$ by (4·1) and (4·2), for all $C \in \mathcal{C}$ and $g \in \mathcal{P}(C,P)$. This implies $f \in \mathcal{P}(P,X)$ by (1·1).

Conversely $f \in \mathcal{P}(P,X)$ and $h \in \mathcal{J}(X,T)$

$\implies hf \in \mathcal{P}(P,T)$ by (1·1), because $\mathcal{J}(X,T) \subset \mathcal{P}(X,T)$

$\implies hf \in \mathcal{J}(SP,T)$ by (4·4).

Hence $f \in \mathcal{J}(SP,X)$ by (4·1).

By (4·5), $S: \mathcal{P} \to \mathcal{J}$ is adjoint to the embedding $\mathcal{J} \to \mathcal{P}$ with $\beta_P \in \mathcal{P}(P, SP)$ the identity set map obtained from

$$1_{SP} \in \mathcal{J}(SP, SP) = \mathcal{P}(P, SP) \tag{4·6}$$

on putting $X = SP$ in (4·5). This means

$$\mathcal{P}(SP, Q) \subset \mathcal{P}(P, Q) \quad \text{for all } P, Q \in \mathcal{P}.$$

In particular, for $Q = X$ in $\mathcal{J}$,

$$\begin{aligned} \mathcal{P}(P, X) &= \mathcal{J}(SP, X) \quad \text{by (4·5)} \\ &\subset \mathcal{P}(SP, X) \\ &\subset \mathcal{P}(P, X) \quad \text{so that} \end{aligned}$$

$P \in \mathcal{P}$, $X \in \mathcal{J} \Rightarrow \mathcal{P}(P, X) = \mathcal{P}(SP, X) = \mathcal{J}(SP, X)$. It follows from Proposition 4·1 that

$$SP \in \mathcal{K} \quad \text{for all } P \in \mathcal{P} \tag{4·7}$$

whence, by (4·5), $\mathcal{K}$ is a full reflective subcategory of $\mathcal{P}$. By (4·6) the associated natural transformation $\beta: 1 \to S: \mathcal{P} \to \mathcal{P}$ is the identity set map in each component.

The class of objects and morphisms in $\mathcal{P}$ arising from the above embedding of $\mathcal{K}$ in $\mathcal{P}$ will be identified with $\mathcal{K}$. If a symbol had been assigned to the non-full embedding

$$\mathcal{J} \to \mathcal{P}$$

the notation of the theory would be cumbersome. However, because this symbol was omitted, care must be taken to state whether an equality (or, more generally, an isomorphism) of objects $X, Y \in \mathcal{J}$ occurs in $\mathcal{J}$ or in $\mathcal{P}$. For example, $SX = X$ in $\mathcal{J}$ iff $X \in \mathcal{K}$ (see Proposition 4·3). On the other hand, $1_X \in \mathcal{P}(X,X) = \mathcal{J}(SX,X)$ on taking $P = X \in \mathcal{J}$ in (4·5) and $\beta_X \in \mathcal{P}(X,SX)$ is, by (4·6), the identity set map hence

$$SX = X \text{ in } \mathcal{P} \text{ for all } X \in \mathcal{J}. \tag{4·8}$$

This means, in addition, that the category $\mathcal{K}$ considered in $\mathcal{P}$ is precisely the full subcategory determined by embedding $\mathcal{J}$ in $\mathcal{P}$.

Proposition 4·3.

$\mathcal{K}$ is a full coreflective subcategory of $\mathcal{J}$.

Proof. By (4·8)

$$\mathcal{P}(P,SX) = \mathcal{P}(P,X) \quad \text{for all } P \in \mathcal{P},\ X \in \mathcal{J} \tag{4·9}$$

Letting $P = K \in \mathcal{K}$

$\mathcal{K}(K,SX) = \mathcal{P}(K,SX)$ since $SX \in \mathcal{K}$ by (4·7) and $\mathcal{K}$ is a full subcategory of $\mathcal{P}$

$= \mathcal{P}(K,X)$ by (4·9)

$= \mathcal{J}(K,X)$ by Proposition 4·1, for all $X \in \mathcal{J}$. Hence

$$\mathcal{K}(K,SX) = \mathcal{J}(K,X) \quad \text{for all } K \in \mathcal{K} \text{ and } X \in \mathcal{J}. \tag{4·10}$$

In other words, the composite $\mathcal{J} \to \mathcal{P} \xrightarrow{S} \mathcal{K}$ is the required coreflection.

For the moment the composite

$$\mathcal{J} \to \mathcal{P} \overset{S}{\to} \mathcal{K}$$

will be denoted by R to enable (4·10) to be written as

$\mathcal{K}(K,RX) = \mathcal{J}(K,X)$ for all $K \in \mathcal{K}$ and $X \in \mathcal{J}$.

Putting K = RX we see that

$$1_{RX} \in \mathcal{K}(RX,RX) = \mathcal{J}(RX,X)$$

yields the component $\alpha_X \in \mathcal{J}(RX,X)$ of a natural transformation $\alpha : R \to 1 : \mathcal{J} \to \mathcal{J}$. Equivalently, using the usual notation for duals ([4], p.424), $\mathcal{K}^*$ is a full reflective subcategory of $\mathcal{J}^*$ with reflector R^* and corresponding natural transformation $\alpha^* : 1 \to R^* : \mathcal{J}^* \to \mathcal{J}^*$. Applying theorem 2·3(a) to this result shows that the sum in $\mathcal{K}$ (= product in $\mathcal{K}^*$) of a set-indexed collection of $\mathcal{K}$-spaces is their sum in $\mathcal{J}$, while the coequaliser of a pair of morphisms in $\mathcal{K}$ is obtained by forming their coequaliser in $\mathcal{J}$. It is well-known that a map is a coequaliser in $\mathcal{J}$ iff it is a topological identification map. The consequent closure of $\mathcal{K}$ in $\mathcal{J}$ under topological identification means that we can speak of an identification map in $\mathcal{K}$ whenever the domain of the map is a $\mathcal{K}$-space.

By theorem 2·3(b) the product in $\mathcal{K}$ of a set-indexed collection $\{K_\lambda | \lambda \in \Lambda\}$ of $\mathcal{K}$-spaces is

$$RX \overset{\alpha_X}{\to} X \underset{p_\lambda}{\to} K_\lambda \qquad (4·11)$$

where $X \underset{p_\lambda}{\to} K_\lambda$ is their product in $\mathcal{T}$. The equaliser in $\mathcal{K}$ of a pair of morphisms $f, g \in \mathcal{K}(K,L)$ is

$$RA \underset{\alpha_A}{\to} A \underset{i}{\to} K$$

where $A \underset{i}{\to} K$ is their equaliser in $\mathcal{T}$. Equalisers in $\mathcal{T}$ correspond to subspace maps, so i is an injection and A has the subspace topology in K. The topology on RA is strictly finer than the subspace topology unless $A \in \mathcal{K}$.

Proposition 4·4.

A closed subspace of a $\mathcal{K}$-space is a $\mathcal{K}$-space.

Proof. Given $K \in \mathcal{K}$ and $A < K$ closed, with the subspace topology, we require by (4·3)' that $V < A$ be open in A whenever (i) $g^{-1}(V)$ is open in C for all $C \in \mathcal{C}$ and $g \in \mathcal{T}(C,A)$.

Suppose $V < A$ satisfies (i) and $C \in \mathcal{C}$, $f \in \mathcal{T}(C,K)$ are arbitrary. $f^{-1}(A)$ is closed in C because A is closed in K and f is continuous so $f^{-1}(A)$, with the subspace topology in C, is a compact hausdorff space.

$$\begin{array}{ccc} C & \xrightarrow{f} & K \\ \cup & & \cup \\ f^{-1}(A) & \xrightarrow[g]{} & A \end{array}$$

$f \in \mathcal{T}(C,K)$ induces a continuous map $g: f^{-1}(A) \to A$, hence $g^{-1}(V) = f^{-1}(V)$ is open in $f^{-1}(A)$ by (i).

Because $f^{-1}(A)$ has the subspace topology in C, $f^{-1}(V) = f^{-1}(A) \cap W$ for some open set $W < C$.
consequently

$$\begin{aligned} f^{-1}(V \cup \tilde{A}) &= f^{-1}(V) \cup f^{-1}(\tilde{A}) \\ &= f^{-1}(V) \cup \widetilde{f^{-1}(A)} \\ &= (f^{-1}(A) \cap W) \cup \widetilde{f^{-1}(A)} \\ &= W \cup \widetilde{f^{-1}(A)} \end{aligned}$$

(where $\tilde{}$ denotes complements in K and in C) is open in C since $f^{-1}(A)$ is closed in C. But K is a $\mathcal{K}$-space and $C \in \mathcal{C}$, $f \in \mathcal{J}(C,K)$ are arbitrary so $V \cup \tilde{A}$ is open in K by (4·3)'. Hence $V = A \cap (V \cup \tilde{A})$ is open in A by definition of subspace topology, as required.

Having completed the topological description of the colimit and limit structure of $\mathcal{K}$, we omit the symbol R. The effect of R on a topological space X is again denoted by SX and is called the $\mathcal{K}$-space associated with X. The following two propositions point out that a variety of topological spaces are $\mathcal{K}$-spaces.

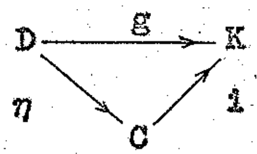
Proposition 4·5.

A hausdorff topological space K is a $\mathcal{K}$-space iff

$V \subset K$ is open (respt. closed) in $K \iff$ (4·12)
$C \cap V$ is open (respt. closed) in C for all compact subspaces C of K.

<u>Proof</u>. (4·12) $\Rightarrow$ (4·3)'. Let $V \subset K$ be such that $g^{-1}(V)$ is open in C for all $C \in \mathcal{C}$ and $g \in \mathcal{J}(C,K)$. This is so, in particular, whenever g is a subspace map in which case $g^{-1}(V) = C \cap V$. Because K is hausdorff all the compact subspaces of K are hausdorff, whence V is open in K by (4·12). This verifies (4·3)'.

(4·3)' $\Rightarrow$ (4·12). Suppose $V \subset K$ is such that $C \cap V$ is open in C for all compact subspaces C in K. Given any $D \in \mathcal{C}$ and $g \in \mathcal{J}(D,K)$ there exists a canonical factorisation of g



in $\mathcal{J}$ with i a subspace map and η a surjection, which implies that C is a compact subspace of K. $i^{-1}(V) = C \cap V$ is then open in C by hypothesis, so $g^{-1}(V) = \eta^{-1}i^{-1}(V)$ is open in D by the continuity of η. Because $D \in \mathcal{C}$ and $g \in \mathcal{J}(D,K)$ are arbitrary we have that V is open in K by (4·3)'. Hence (4·12) is established. The argument is exactly the same for 'closed' replacing 'open'.

This means that hausdorff $\mathcal{K}$-spaces are precisely (hausdorff) k-spaces as defined by R. Brown in [2], §2. Consequently $\mathcal{K}$ contains the examples of this reference:

hausdorff topological spaces which are either locally compact, first countable or CW-complexes. Also, by Proposition 4·5 and [2], Proposition 2·5, there exist topological spaces which are not $\mathcal{K}$-spaces.

Proposition 4·6.

Any trivial topological space X — the only open sets are $\emptyset$ and X — is a $\mathcal{K}$-space.

Proof. If $V \subset X$ is neither $\emptyset$ nor X let C be the closed interval of real numbers (usual topology) from 0 to 2,

$A = [0,1[$ be the interval 0 to 1, closed at 0 and open at 1,

$B = [1,2]$ be the interval 1 to 2, closed at both ends, and

$g : C \to X$ be defined by

$$g(A) = x \in X - V, \quad g(B) = y \in V.$$

Then g is continuous because $X \in \mathcal{J}$ is trivial, and $g^{-1}(V) = B$ is not open in C. Hence $g^{-1}(V)$ open in C for all $C \in \mathcal{C}$ and $g \in \mathcal{J}(C,X)$ implies $V = \emptyset$ or X, as required.

Similarly $T = \{0,1\}$ is a $\mathcal{K}$-space — take C, A, B as above, with $g: C \to T$ given by $g(A) = 0$, $g(B) = 1$. Then g is continuous and $g^{-1}(1) = B$ is not open in C.

<u>Note 4·7.</u> Let $\mathcal{J}_2$ and $\mathcal{K}_2$ denote the full subcategories determined by the hausdorff objects in $\mathcal{J}$ and in $\mathcal{K}$ respectively. It is well-known that the "forgetful" embedding $\mathcal{J}_2 \to \mathcal{J}$ has an adjoint H (see, for example, Freyd [5], Chapter 3, exercise K) and it can be shown that, for each $X \in \mathcal{J}$, HX is obtained by making an identification in X (pointed out by G.M. Kelly in lectures on the existence of adjoints, 1966). Assuming this result we obtain a natural isomorphism

$$\mathcal{K}_2(HX,Y) \cong \mathcal{K}(X,Y) \text{ for all } X \in \mathcal{K},\ Y \in \mathcal{K}_2 \qquad (4\cdot13)$$

by the fullness of the embedding $\mathcal{K} \to \mathcal{J}$ and the closure of $\mathcal{K}$ in $\mathcal{J}$ under identification.

<u>Note 4·8.</u> In [7], lemma 5·5, E. Spanier shows that there exists a quasi-topological space P with $P \ncong X$ in $\mathcal{Q}$ for any $X \in \mathcal{J}$; this implies that $P \ncong X$ in $\mathcal{P}$ for any $X \in \mathcal{J}$. Because $X = SX$ in $\mathcal{P}$ by (4·8), $\mathcal{K} \to \mathcal{P}$ is not a dense functor and thus is not a category equivalence.

CHAPTER 5.

Cartesian Closure of $\mathcal{K}$.

For any quasi-topological base P and topological space X a function space topology is definable on $\mathcal{P}(P,X)$ which enables theorem 3·3(1) to be established when $\mathcal{A} = \mathcal{K}$.

Definition. Given $P \in \mathcal{P}$, $X \in \mathcal{J}$, $C \in \mathcal{C}$ and $g \in \mathcal{P}(C,P)$, $h \in \mathcal{J}(X,T)$ let

$$V(g,h) = \{f \in \mathcal{P}(P,X) \mid hfg(C) = 0\} \qquad (5{\cdot}1)$$

The topology generated on $\mathcal{P}(P,X)$ by the open subbase consisting of all these sets defines an object $[P,X] \in \mathcal{J}$.

Remark. When $P = K$ is a hausdorff $\mathcal{K}$-space, $[K,X]$ is clearly the set $\mathcal{P}(K,X)$ ($= \mathcal{J}(K,X)$ by proposition 4·1) together with the ordinary compact-open topology (as defined in Kelley [6] p.221); for each $C \in \mathcal{C}$, every $g \in \mathcal{P}(C,K)$ is continuous by (4·2) and factors canonically in $\mathcal{J}$ through a unique compact hausdorff image in K. In particular, when $K = C \in \mathcal{C}$, $[C,T]$ is the set $\mathcal{J}(C,T)$ ($\cong$ the set of closed sets of C) with the topology generated by an open subbase comprising all subsets

$$V_A = \{f \in \mathcal{J}(C,T) \mid f(A) = 0\} \qquad (5{\cdot}2)$$

where A is a compact (= closed) subset of C.

Lemma 5·1 $(C,T) = [C,T]$ in $\mathcal{P}$ for all $C \in \mathcal{C}$.

Proof. We first show

$$\mathcal{J}(D\times C,T) \underset{\rightarrow}{\cong} \mathcal{J}(D,[C,T]) \quad \text{for all } D \in \mathcal{C} \tag{5·3}$$

under the map $G \mapsto g$ given by

$$g(b)(c) = G(b,c) \quad \text{for all } b \in D \text{ and } c \in C .$$

By (5·2), (5·3) is proved by showing that $W = G^{-1}(0)$ is open in $D \times C$ (that is, G is continuous) iff

(a) given any $b \in D$, the set $g(b)^{-1}(0) = \{c \in C | g(b)(c)=0\}$
$= \{c \in C | (b,c) \in W\}$

is open in C (that is, $g(D) \subset \mathcal{J}(C,T)$), and

(b) given any compact $A \subset C$, the set

$$\begin{aligned} g^{-1}(V_A) &= \{b \in D | g(b) \in V_A\} \\ &= \{b \in D | g(b)(A) = 0\} \\ &= \{b \in D | G(b\times A) = 0\} \\ &= \{b \in D | b \times A \subset W\} \end{aligned}$$

is open in D (that is, g is continuous).

$\Longrightarrow$. (a) follows immediately from the continuity of the map $c \mapsto (b,c)$ for each $b \in D$. Any point $b_0 \in D$ is compact. Kelley [6], theorem 5·12 states that there then exist $U \subset D$ open and $V \subset C$ open such that $b_0 \times A \subset U \times V \subset W$, because A is compact and W is open. So $b_0 \in U \subset \{b \in D | b \times A \subset W\}$ with U open in D, whence (b).

$\Leftarrow$. For all $(b_o, c_o) \in W$ the set $\{c \in C | (b_o, c) \in W\}$ is open in C by (a) and contains c_o. Because $C \in \mathcal{C}$, C is a regular topological space so there exist V open and A closed (hence compact) in C such that

$$c_o \in V \subset A \subset \{c \in C | (b_o, c) \in W\} .$$

Thus $b_o \in U = \{b \in D | b \times A \subset W\}$, which is open by (b). This means $(b_o, c_o) \in U \times V \subset W$ whence W is open as required.

$$\mathcal{J}(D \times C, T) = \mathcal{P}(D \times C, T) \quad \text{and}$$

$$\mathcal{J}(D, [C, T]) = \mathcal{P}(D, [C, T]) \quad \text{by } (4\cdot 2) \quad \text{hence ,}$$

by definition of $\mathcal{P}(D, (C, T))$ (see (3·1)),

$$\mathcal{P}(D, (C, T)) = \mathcal{P}(D, [C, T]) \quad \text{for all } D \in \mathcal{C} .$$

The result follows from proposition 1·1.

Lemma 5·2. For each $C \in \mathcal{C}$, $g \in \mathcal{P}(C, P)$ and $h \in \mathcal{J}(X, T)$ the map $\mathcal{P}(g, h) : [P, X] \to [C, T]$ is continuous.

Proof. Take a subbasic open set V_A of $[C, T]$ for some compact $A \subset C$. Then

$$\mathcal{P}(g, h)^{-1}(V_A) = \{f \in \mathcal{P}(P, X) | \mathcal{P}(g, h)f = hfg \in V_A\}$$

$$= \{f \in \mathcal{P}(P, X) | hfgi(A) = 0\} \quad \text{where}$$

$i : A \to C$ denotes the inclusion $A \subset C$

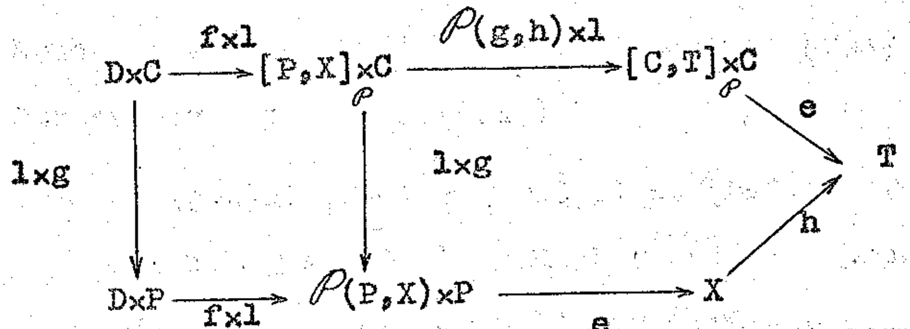
$$= V(gi, h) \quad \text{by } (5\cdot 1).$$ Consequently

$\mathcal{P}(g, h)^{-1}(V_A)$ is open in $[P, X]$ by definition.

<u>Theorem 5·3.</u>

$$(P,X) = [P,X] \text{ in } \mathcal{P} \text{ for all } P \in \mathcal{P},\ X \in \mathcal{J}.$$

<u>Proof.</u> For all $C,D \in \mathcal{C}$, $g \in \mathcal{P}(C,P)$, $h \in \mathcal{J}(X,T)$ and $f \in \mathcal{S}(D,\mathcal{P}(P,X))$ the diagrams



commute,

where e denotes the relevant evaluation map. (5·4)

Moreover $e:[C,T] \underset{\mathcal{P}}{\times} C \to T$ is $\mathcal{P}$-continuous (5·5)

by lemma 5·1 which states that $[C,T] = (C,T)$, and (3·2) which states that $e:(C,T) \times C \to T$ is $\mathcal{P}$-continuous.

Hence, for each $C,D \in \mathcal{C}$, $g \in \mathcal{P}(C,P)$, $h \in \mathcal{J}(X,T)$ and $f \in \mathcal{P}(D,(P,X))$,

$\kappa(f) = D\times P \underset{f\times 1}{\to} \mathcal{P}(P,X)\times P \underset{e}{\to} X \in \mathcal{P}(D\times P, X)$ by (3·1)

$\Longrightarrow \kappa(\mathcal{P}(g,h)f) = e(\mathcal{P}(g,h)\times 1)(f\times 1) \in \mathcal{P}(D\times C, T)$ by (5·4)

$\Longrightarrow \mathcal{P}(g,h)f \in \mathcal{P}(D,(C,T))$ by (3·1)

$\Longrightarrow \mathcal{P}(g,h)f \in \mathcal{J}(D,[C,T])$ by lemma 5·1 and (4.2)

$\Longrightarrow f^{-1}(V(g,h))$ is open in D because

(i) $\mathcal{P}(g,h)^{-1}(V_C) = \{f \in \mathcal{P}(P,X) \mid \mathcal{P}(g,h)f \in V_C\}$

$= \{f \in \mathcal{P}(P,X) \mid hfg(C) = 0\}$ by (5.2)

$= V(g,h)$ by (5·1), and

(ii) the point V_C is open in $[C,T]$ by definition.

Hence $f \in \mathcal{J}(D,[P,X])$ by definition of $[P,X]$.

Conservely $f \in \mathcal{J}(D,[P,X])$

$\Longrightarrow$ $e(\mathcal{P}(g,h)\times 1)(f\times 1) \in \mathcal{P}(D\times C,T) = \mathcal{J}(D\times C,T)$ for all $C \in \mathcal{C}$, $g \in \mathcal{P}(C,P)$ and $h \in \mathcal{J}(X,T)$, because

(i) $e \in \mathcal{P}([C,T] \underset{\mathcal{P}}{\times} C,T)$ by (5·5), and

(ii) $\mathcal{P}(g,h) \in \mathcal{J}([P,X],[C,T])$ by lemma 5·2.

$\Longrightarrow$ $he(f\times 1)(1\times g) \in \mathcal{J}(D\times C,T)$ for all $C \in \mathcal{C}$, $g \in \mathcal{P}(C,P)$ and $h \in \mathcal{J}(X,T)$ by (5·4)

$\Longrightarrow$ $e(f\times 1)(1\times g) \in \mathcal{J}(D\times C,X)$ for all $C \in \mathcal{C}$ and $g \in \mathcal{P}(C,P)$ by (4·1)

$\Longrightarrow$ $e(f\times 1)(1\times g)\binom{\ell}{1} = e(f\times 1)\binom{\ell}{g} \in \mathcal{J}(C,X)$ for all $C \in \mathcal{C}$, $g \in \mathcal{P}(C,P)$ and $\ell \in \mathcal{C}(C,D)$

$\Longrightarrow$ $\kappa(f) = e(f\times 1) \in \mathcal{P}(D\times P,X)$ by (1·1)

$\Longrightarrow$ $f \in \mathcal{P}(D,(P,X))$ by (3·1).

Thus $\mathcal{P}(D,(P,X)) = \mathcal{J}(D,[P,X])$

$= \mathcal{P}(D,[P,X])$ by (4·2), for

all $D \in \mathcal{C}$, whence

$(P,X) = [P,X]$ by proposition 1·1.

Proposition 5·4.

$\mathcal{K}$ is a cartesian closed category.

Proof. By theorem 4·2 and proposition 3·4, it suffices to verify:

(i) $P \in \mathcal{P},\ L \in \mathcal{K} \implies (P,L) \in \mathcal{K}$.

Taking $X = L \in \mathcal{K}$ in theorem 5·3, we obtain

$$\begin{aligned}(P,L) &= [P,L] \text{ in } \mathcal{P}\\ &= S[P,L] \text{ in } \mathcal{P} \text{ by } (4\cdot 8)\\ &\in \mathcal{K} \text{ by } (4\cdot 7), \text{ for all } P \in \mathcal{P}, \text{ as}\end{aligned}$$

required.

Alternative proof. The result also follows directly from lemma 5·1; by theorems 3·3 and 4·2, together with proposition 3·4, it suffices to verify:

(v) $C \in \mathcal{C},\ P \in \mathcal{P} \implies C \underset{\mathcal{P}}{\times} SP = S(C \underset{\mathcal{P}}{\times} P)$ in $\mathcal{P}$.

By definition of S (see (4·4)), $C \underset{\mathcal{P}}{\times} SP$ and $S(C \underset{\mathcal{P}}{\times} P)$ have identical underlying sets and

$$\mathcal{J}(S(C \underset{\mathcal{P}}{\times} P),T) = \mathcal{P}(C \underset{\mathcal{P}}{\times} P,T) .$$

Also $\mathcal{P}(C \underset{\mathcal{P}}{\times} P,T) \cong \mathcal{P}(P \underset{\mathcal{P}}{\times} C,T)$ by the natural commutativity of products

$$\begin{aligned}&\cong \mathcal{P}(P,(C,T)) \text{ by proposition } 3\cdot 1\\ &= \mathcal{P}(P,[C,T]) \text{ by lemma } 5\cdot 1\\ &= \mathcal{J}(SP,[C,T]) \text{ by } (4\cdot 5)\\ &= \mathcal{P}(SP,[C,T]) \text{ by proposition } 4\cdot 1 \text{ since } SP \in \mathcal{K}\end{aligned}$$

$= \mathscr{P}(SP,(C,T))$ by lemma 5·1

$\cong \mathscr{P}(SP \times_{\mathscr{P}} C, T)$ by proposition 3·1

$\cong \mathscr{P}(C \times_{\mathscr{P}} SP, T)$ by the natural commutativity of products

$= \mathscr{J}(C \times_{\mathscr{P}} SP, T)$ by (4·3) because $C \times_{\mathscr{P}} SP \in \mathscr{K}$ by theorem 2·3(a).

Hence $\mathscr{J}(S(C \times_{\mathscr{P}} P), T) = \mathscr{J}(C \times_{\mathscr{P}} SP, T)$ which means that $S(C \times_{\mathscr{P}} P)$ and $C \times_{\mathscr{P}} SP$ have the same open sets, as required.

The reason for intboducing $[P,X]$ for each $P \in \mathscr{P}$ and $X \in \mathscr{J}$, and thenproving theorem 5·3, is to demonstrate the relationship of $(P,X) \in \mathscr{P}$ to the compact-open topology when $P = K \in \mathscr{K}_2$.

Proposition 5·5.

$\mathscr{K}_2$ is a cartesian closed category.

Proof. For each $Y \in \mathscr{J}$ we have

(i) $SY \in \mathscr{K}$ by (4·7), and

(ii) $1_Y \in \mathscr{J}(SY,Y)$ on taking $P = Y$ in (4·5).

Hence $Y \in \mathscr{J}_2 \Rightarrow SY \in \mathscr{K}_2$ (5·6)

Let $P = K \in \mathscr{K}_2$ in $[P,X]$. It has already been remarked that $[K,X]$ is the ordinary compact-open topology on the set $\mathscr{J}(K,X)$. Kelley [6], theorem 7·4, states, among other things, that $[K,X]$ is hausdorff whenever the range space X is hausdorff. Hence, on taking $X = L \in \mathscr{K}_2$,

$$
\begin{aligned}
(K,L) &= [K,L] \text{ in } \mathcal{P} \text{ by theorem } 5{\cdot}3\\
&= S[K,L] \text{ in } \mathcal{P} \text{ by } (4{\cdot}8)\\
&\in \mathcal{K}_2 \qquad\qquad \text{by } (5{\cdot}6).
\end{aligned}
$$

It follows from theorem 4·2 and (4·13) that $\mathcal{K}_2$ is a full reflective subcategory of $\mathcal{P}$ hence, by theorem 2·3(a), $K \underset{\mathcal{P}}{\times} L \in \mathcal{K}_2$ for all $K, L \in \mathcal{K}_2$ and $* \in \mathcal{K}_2$. Thus $\mathcal{K}_2$ is a cartesian closed category by corollary 3·2.

Proposition 5·5 is obtained by R. Brown in [3], theorem 3·3, by direct topological calculation. This chapter is concluded with a useful consequence of proposition 5·4.

<u>Corollary 5·6.</u> The product in $\mathcal{K}$ of two identification maps $\eta \in \mathcal{K}(K,A)$ and $\mu \in \mathcal{K}(L,B)$ is an identification map.

<u>Proof.</u> Products in any category are commutative (provided they exist); that is, we have natural isomorphisms

$$c_{AL}: A \times L \to L \times A$$

$$\text{and} \quad c_{BA}: B \times A \to A \times B \quad \text{in } \mathcal{K}.$$

Each of the functors $- \times L$ and $- \times A : \mathcal{K} \to \mathcal{K}$ has a coadjoint, by proposition 5·4, hence preserves coequalisers in $\mathcal{K}$ = identification maps in $\mathcal{K}$ (see Chapter 4). $\eta \times \mu$ is thus the composite

$$K \times L \xrightarrow{\eta \times 1} A \times L \xrightarrow{c_{AL}} L \times A \xrightarrow{\mu \times 1} B \times A \xrightarrow{c_{BA}} A \times B$$

of identification maps. Since this is again an identification map, the result follows.

CHAPTER 6.

Concluding Remarks.

1. $\mathcal{K}$-space Groups.

The aim of the preceding chapters was to demonstrate the cartesian closure of the subcategories $\mathcal{K}_2$, $\mathcal{K}$ and $\mathcal{Q}$ of $\mathcal{P}$ and, for $\mathcal{K}$ and $\mathcal{Q}$, this was achieved by establishing the results of theorem 3·3. The stronger consequences of this theorem will be indicated briefly below, for the category $\mathcal{K}$.

Let $\mathcal{A}$ denote any one of the categories $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{K}$, $\mathcal{K}_2$, $\mathcal{J}$. An <u>$\mathcal{A}$-group</u> is defined to be an object $A \in \mathcal{A}$ together with a group structure — in the ordinary sense — on the underlying set of A such that the maps

(i) multiplication $\mu_A : A \underset{\mathcal{A}}{\times} A \to A$,

(ii) inversion $\iota_A : A \to A$, and

(iii) identity $\varepsilon_A : * \to A$,

determined by this group structure, are morphisms in $\mathcal{A}$. Given $\mathcal{A}$-groups A and B, an <u>$\mathcal{A}$-group homomorphism</u> from A to B is a map $f \in \mathcal{A}(A,B)$ such that

$$\begin{array}{ccc} A \underset{\mathcal{A}}{\times} A & \xrightarrow{\mu_A} & A \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ B \underset{\mathcal{A}}{\times} B & \xrightarrow[\mu_B]{} & B \end{array}$$

commutes.

$\mathcal{A}$-groups and $\mathcal{A}$-group homomorphisms form a subcategory Gp$\mathcal{A}$ of $\mathcal{A}$; for example, Gp$\mathcal{J}$ is the category of topological groups and continuous homomorphisms between them. There may exist many $\mathcal{A}$-group structures on a given object $A \in \mathcal{A}$, hence the "forgetful" embedding Gp$\mathcal{A} \to \mathcal{A}$ is not object-injective.

If any functor $\mathcal{A} \to \mathcal{B}$ between two of the categories $\mathcal{P}, \mathcal{Q}, \mathcal{K}, \mathcal{K}_2, \mathcal{J}$, preserves all finite products (including the empty product $*$) of objects in $\mathcal{A}$ then it induces an obvious functor Gp$\mathcal{A} \to$ Gp$\mathcal{B}$. In particular, the full embedding functors $\mathcal{K}_2 \to \mathcal{K}$ and $\mathcal{K} \to \mathcal{P}$ of chapter 4 are reflective, hence induce full embeddings Gp$\mathcal{K}_2 \to$ Gp$\mathcal{K}$ and Gp$\mathcal{K} \to$ Gp$\mathcal{P}$ respectively.

Proposition 6·1.

The full embedding Gp$\mathcal{K} \to$ Gp$\mathcal{P}$ is reflective.

Proof. $\mathcal{K}$ was shown to be cartesian closed (see proposition 5·4) by verifying that $\mathcal{K}$, together with the adjoint functor $S : \mathcal{P} \to \mathcal{K}$ and natural transformation $\beta : 1 \to S$, satisfied the conditions of theorem 3·3. Condition (iv) of this theorem states that S preserves all finite products of objects in $\mathcal{P}$ (clearly S preserves the terminal object $*$ of $\mathcal{P}$ since it already lies in $\mathcal{K}$) hence S induces a functor $S' : Gp\mathcal{P} \to$ Gp$\mathcal{K}$.

Moreover $\beta_P : P \to SP$ is a $\mathcal{P}$-group homomorphism for each $P \in Gp\mathcal{P}$, because it is the identity set map by (4·6). Thus

$$Gp\mathcal{P}(S'P,L) = Gp\mathcal{P}(P,L)$$

is immediate from

$$\mathcal{P}(\beta_P,1) : \mathcal{P}(SP,L) \cong \mathcal{P}(P,L) ,$$

for all $P \in Gp\mathcal{P}$ and $L \in Gp\mathcal{K}$, as required for S' to be adjoint to $Gp\mathcal{K} \to Gp\mathcal{P}$.

This result is useful because several concepts of group theory can be simply imitated in $Gp\mathcal{P}$. For a $\mathcal{K}$-group K to be a topological group, it is clearly sufficient that

$$K \times_{\mathcal{K}} K = K \times_{\mathcal{J}} K \text{ in } \mathcal{J} .$$

We thus obtain, as examples of $\mathcal{K}$-groups, topological groups which are hausdorff and either locally compact or first countable (by [2], theorem 2·11, where R. Brown denotes $K \times_{\mathcal{J}} K$ by $K \times_c K$) and groups with the trivial topology.

2. A Representation of $\mathcal{P}$.

$\mathcal{P}$ admits a representation as a category of functors. Let $[\mathcal{C}^*, \mathcal{S}]$ denote the "category" whose objects are functors from $\mathcal{C}^*$ to $\mathcal{S}$ and whose morphisms are natural transformations between such functors (this is not a proper category in our sense of the word, because the natural transformations between two functors may not form a set). For convenience, axiom Q1 will be used to identify $\mathcal{P}(*,P)$ with the set P for each $P \in \mathcal{P}$.

An embedding $E: \mathcal{P} \to [\mathcal{C}^*, \mathcal{S}]$ is given by

$$EP(C) = \mathcal{P}(C,P) ,$$

$$EP(g) = \mathcal{P}(g,1): \mathcal{P}(D,P) \to \mathcal{P}(C,P) , \text{ and}$$

$$Ef(C) = \mathcal{P}(1,f): \mathcal{P}(C,P) \to \mathcal{P}(C,Q) , \text{ for}$$

all $C,D \in \mathcal{C}$, $P,Q \in \mathcal{P}$, $g \in \mathcal{C}(C,D)$ and $f \in \mathcal{P}(P,Q)$. E is faithful by axiom Q1. Moreover E is full: let $\alpha: EP \to EQ$ be a natural transformation for some P and $Q \in \mathcal{P}$, and $f = \alpha_*: P \to Q$ as a set map. Then, for all $C \in \mathcal{C}$, $a \in C = \mathcal{P}(*,C)$, and $g \in \mathcal{P}(C,P)$, we have

$$\begin{aligned} fg(a) &= \alpha_* g(a) \\ &= \alpha_* \mathcal{P}(a,1)(g) \\ &= \mathcal{P}(a,1)\alpha_C(g) \quad \text{by the naturality of } \alpha \\ &= \alpha_C(g)(a) . \end{aligned}$$

Thus $fg = \alpha_C(g) \in \mathcal{P}(C,Q)$ for all $C \in \mathcal{C}$ and $g \in \mathcal{P}(C,P)$, whence $f \in \mathcal{P}(P,Q)$ by (1·1) and $\alpha_C = \mathcal{P}(1_C,f)$ for all $C \in \mathcal{C}$, as required.

The symbol E will now be omitted. The trivial base on a set X is defined to be TX where T is the coadjoint of $U: \mathcal{P} \to \mathcal{S}$ (see proposition 1·3(b)) .

<u>Proposition 6·2.</u> A functor $S \in [\mathcal{C}^*, \mathcal{S}]$ is a quasi-topological base iff it is a subfunctor of a trivial base.

<u>Proof.</u> Clearly any base P is a subfunctor of the associated trivial base TUP .

Conversely, let $S: \mathcal{C}^* \to \mathcal{S}$ be a functor, P a trivial base, and

$$\alpha : S \to \mathcal{P}(-,P) = \mathcal{S}(-,P)$$

be a natural transformation with all components α_C injections.

In the natural map

$$S_{CD} : \mathcal{C}(D,C) \to \mathcal{S}(SC,SD) \,,$$

which defines the functor S on morphisms, take $D = * \in \mathcal{C}$ and define

$$s_C : SC \to \mathcal{S}(C, S*)$$

as $s_C(f)(a) = S_{C*}(a)(f)$ for all $f \in SC$ and $a \in C$.
It follows from the naturality of S_{CD} in each variable that s_C is natural in C . By the naturality of α

$$\begin{array}{ccc} SC & \xrightarrow{\ \alpha_C\ } & \mathcal{S}(C,P) \\ {\scriptstyle S_{C*}(a)}\downarrow & & \downarrow{\scriptstyle \mathcal{S}(a,1)} \\ S* & \xrightarrow[\ \alpha_*\ ]{} & \mathcal{S}(*,P) \end{array}$$

commutes for all $a \in C$. (6·1)

For all $f \in SC$ and $a \in C$

$$(\mathcal{S}(1,\alpha_*)s_C(f))(a) = \alpha_*(s_C(f)(a))$$
$$= \alpha_*(S_{C*}(a)(f)) \quad \text{by}$$

definition of s_C ,

$$= (\alpha_* \cdot s_{C*}(a))(f)$$

$$= \mathcal{S}(a,1)\alpha_C(f) \quad \text{by } (6\cdot 1)$$

$$= \alpha_C(f)(a) \ .$$

In other words

$$\begin{array}{ccc} SC & \xrightarrow{\alpha_C} & \mathcal{S}(C,P) \\ & \underset{s_C}{\searrow} & \uparrow \mathcal{S}(1,\alpha_*) \\ & & \mathcal{S}(C,S^*) \end{array}$$

commutes, whence s_C is an injection for each $C \in \mathcal{C}$ because each α_C is an injection by hypothesis.

A quasi-topological base is now defined to be the set S^* together with $\mathcal{P}(C,S^*) = SC \subset \mathcal{S}(C,S^*)$ for each $C \in \mathcal{C}$.

Axiom Q2. holds because, given any $f \in \mathcal{C}(C,D)$ and $g \in SD = \mathcal{P}(D,S^*)$

$$\begin{array}{ccc} SD & \xrightarrow{s_D} & \mathcal{S}(D,P) \\ Sf \downarrow & & \downarrow \mathcal{S}(f,1) \\ SC & \xrightarrow[s_C]{} & \mathcal{S}(C,P) \end{array}$$

commutes by the naturality of s, hence $gf \in SC = \mathcal{P}(C,S^*)$ by definition.

Q1. follows from Q2. since

$$\mathcal{S}(*,S^*) = S^* = \mathcal{P}(*,S^*) \quad \text{by definition.}$$

By definition of $\mathcal{P} \to [\mathcal{C}^*, \mathcal{S}]$, S^* with this $\mathcal{P}$-structure is the required base.

It follows that the functors from $\mathcal{C}^*$ to $\mathcal{S}$ which represent quasi-topological spaces are those which

(i) are subfunctors of trivial bases, and

(ii) preserve all finite limits in $\mathcal{C}^*$.

3. Topological Spaces.

It is apparent from §2 that the process by which $\mathcal{P}$ is constructed from $\mathcal{C}$ depends to a very limited extent on special properties of the category $\mathcal{C}$. A similar process could be applied to any category which, like $\mathcal{C}$,

(i) is a category of sets-with-structure and morphisms which are set maps, and

(ii) has a terminal generator which represents the underlying-set functor.

Two additional examples will be described.

Let $\mathcal{F}$ denote the category of finite sets and set maps and $\bar{\mathcal{F}}$ denote the category of simplicial complexes and simplicial maps ([4], IV, §7). $\bar{\mathcal{F}}$ may be defined as follows. An object of $\bar{\mathcal{F}}$ is a set U together with a selected set $\bar{F}(n,U)$ of set maps $n \to U$ for each $n \in \mathcal{F}$, satisfying:

$\bar{F}1.$ $$\bar{F}(*,U) = \mathcal{S}(*,U)$$

$\bar{F}2.$ $$f \in \mathcal{F}(m,n) \text{ and } g \in \bar{F}(n,U) \Longrightarrow gf \in \bar{F}(m,U) .$$

An element of $\bar{\mathcal{F}}(U,V)$ is a set map $f:U \to V$ such that $g \in \bar{F}(n,U) \Longrightarrow fg \in \bar{F}(n,V)$ for all $n \in \mathcal{F}$. $\bar{\mathcal{F}}$ is then cartesian closed in precisely the same manner in which $\mathcal{P}$ is .

A second example — suggested by G. M. Kelly — arises from two deficiencies, the first of which is well-known.

<u>Proposition 6·3.</u>

$\mathcal{T}$ is not a cartesian closed category.

<u>Proof.</u> Let $\mathbb{Q}$ be the rational numbers as a subspace of the real line (with the usual topology). $- \times \mathbb{Q} : \mathcal{T} \to \mathcal{T}$ does not preserve the coequaliser (identification map) $\mathbb{Q} \xrightarrow{\eta} A$ induced by identifying the integers $\mathbb{Z}$ to a single point in $\mathbb{Q}$ (Bourbaki [1] , p. 151, exercise 6). Thus $- \times \mathbb{Q}$ does not have a coadjoint.

Secondly the embedding $\mathcal{T} \to \mathcal{P}$ is seen (chapter 4) to be of no value in applications where a space $X \in \mathcal{T}$ is not allowed to be identified with its associated $\mathcal{K}$-space SX .

Suppose $\mathcal{C}$ is replaced by $\mathcal{T}$ itself and $\bar{\bar{\mathcal{T}}}$ is constructed in place of $\mathcal{P}$. Denote the sets defining an object $A \in \bar{\bar{\mathcal{T}}}$, and satisfying the analogues of axioms Q1 and Q2, by $\bar{\bar{T}}(X, A)$ for each $X \in \mathcal{T}$. Then $\mathcal{T}$ is fully embedded in a natural way in $\bar{\bar{\mathcal{T}}}$ which, like $\mathcal{P}$, is cartesian closed.

Can a smaller cartesian closed category $\mathcal{V}$, in which $\mathcal{T}$ has a full limit preserving embedding, be obtained from $\bar{\bar{\mathcal{T}}}$? A suitable process for selecting such a full subcategory of $\bar{\bar{\mathcal{T}}}$ may be based on the way $\mathcal{Q}$ is obtained from $\mathcal{P}$. Consider the class of those colimits in $\mathcal{J}$ which are preserved by $- \times X : \mathcal{J} \to \mathcal{J}$ for each $X \in \mathcal{J}$. This contains all the set indexed coproducts in $\mathcal{J}$, together with a class $\mathcal{E}$ of identification maps in $\mathcal{J}$. Take $\mathcal{V}$ to be the full subcategory of $\bar{\bar{\mathcal{J}}}$ determined by those $A \in \bar{\bar{\mathcal{J}}}$ which satisfy:

V3. If $X_\lambda \xrightarrow[\alpha_\lambda]{} \sum_{\Lambda} X_\lambda$ is a set indexed coproduct in $\mathcal{J}$ then

$$f \in \bar{\bar{T}}(\sum_{\Lambda} X_\lambda, A) \iff f\alpha_\lambda \in \bar{\bar{T}}(X_\lambda, A) \quad \text{for each } \lambda \in \Lambda .$$

V4. If $\eta \in \mathcal{J}(X, Y)$ belongs to $\mathcal{E}$ then

$$f \in \bar{\bar{T}}(Y, A) \iff f\eta \in \bar{\bar{T}}(X, A) .$$

The reflectivity of $\mathcal{V}$ in $\bar{\bar{\mathcal{J}}}$ and the consequent cartesian closure of $\mathcal{V}$ have recently been proved in a more general setting. The reflecting functor is not however given explicitly, as it is for the case $F: \mathcal{P} \to \mathcal{Q}$ in chapter 1, and the proof of existence will not be given here.

BIBLIOGRAPHY

[1] Bourbaki, N., *Éléments de mathématique*, Livre III (*Topologie générale*), Ch. I, 3^{me} Éd. (Hermann and Cie. Paris 1961).

[2] Brown, R., Ten topologies for $X \times Y$, *Quart. J. Math. Oxford* (2), 14 (1963), 303-319.

[3] _______, Function spaces and product topologies, *Quart. J. Math. Oxford* (2), 15 (1964), 238-250.

[4] Eilenberg, S. and Kelly, G. M., Closed Categories, *Proceedings of the Conference on Categorical Algebra*, La Jolla 1965 (Springer 1966), 421-562.

[5] Freyd, P., *Abelian Categories* (Harper and Row, New York 1964).

[6] Kelley, J. L., *General Topology* (D. Van Nostrand Co., New York 1955).

[7] Spanier, E. H., Quasi-topologies, *Duke Math. J.*, 30 (1963), 1-14.