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AN INTRODUCTION TO TANNAKA DUALITY AND QUANTUM GROUPS

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Introduction

The goal of this paper is to give an account of classical Tannaka duality [C] in such a way as to be accessible to the general mathematical reader, and to provide a key for entry to more recent developments [SR, DM] and quantum groups [D1]. Expertise in neither representation theory nor category theory is assumed.

Naively speaking, Tannaka duality theory is the study of the interplay which exists between a group and the category of its representations. The early duality theorems of Tannaka-Krein [Ta, Kr] concentrate on the problem of reconstructing a compact group from the collection of its representations. In the abelian case, this problem amounts to reconstructing the group from its character group, and is the content of the Pontrjagin duality theorem. A good exposition of this theory can be found in the book by Chevalley [C]. In these early developments, there was little or no use of categorical concepts, partly because they did not exist at the time. Moreover, the mathematical community was not yet familiar with category theory, and it was possible to avoid it [BtD].

To Grothendieck we owe the understanding that the process of Tannaka duality can be reversed. In his work to solve the Weil conjectures, he constructed the category of motives as the universal recipient of a Weil cohomology [Kl]. By using a fiber functor from his category of motives to vector spaces, he could construct a pro-algebraic group G. He also conjectured that the category of motives could be recaptured as the category of representations of G. This group is called the Grothendieck Galois group, since it is an extension of the Galois group of $\overline{\mathbf{Q}}/\mathbf{Q}$. The work spreading from these ideas can be found in [SR, DM]. For other aspects of this question, see [Cb].

An entirely different development came from mathematical physicists working on superselection principles in quantum field theory [DHR] where it was discovered that the superselection structure could be described in terms of a category whose objects are certain endomorphisms of the C^*-algebra of local observables, and whose arrows are intertwining operators. Reversing the duality process, they succeeded in constructing a compact group whose representations can be identified with their superselection category [DR].

Another impulse to the development of Tannaka duality comes from the theory of quantum groups. These new mathematical objects were discovered by Jimbo [J] and Drinfel'd [D1] in connection with the work of L.D. Faddeev and his collaborators on the quantum inverse scattering method. V.V. Lyubashenko [Ly] initiated the use of Tannaka duality in the construction of quantum groups; also see K.-H. Ulbrich [U]. We should also mention S.L. Woronowicz [W] in the case of compact quantum groups. Recently, S. Majid [M3] has shown that one can use Tannaka-Krein duality for constructing the quasi-Hopf algebras introduced by Drinfel'd [D2] in connection with the solution of the Knizhnik-Zamolodchikov equation.

The theory of angular momentum in Quantum Physics [BL1] might also provide some extra motivation for studying Tannaka duality. The Racah-Wigner algebra, the $9-j$ and $3-j$ symbols, and, the Racah and Wigner coefficients, all seem to be about the explicit description of the structures which exist on the category of representations of some compact groups like SU(2) or SU(3) [BL2]. Here the theory of orthogonal polynomials and special functions comes into play [AK]. Also, the q-analogues of the classical orthogonal polynomials show up as spherical functions on quantum groups [Ko].

Another essential aspect of the picture that should be mentioned is the connection between knot theory, Feynman diagrams, category theory, and quantum groups. It was discovered by Turaev [T] that the new invariants found by Jones [Jn1] could be constructed from Yang-Baxter operators. The method was formalised by showing that certain categories constructed geometrically from tangles of strings or ribbons could be given a simple algebraic presentation [T, FY2, JS1]. It is remarkable that notation, introduced by Penrose [Pn] for calculating with ordinary tensors, has the right degree of generality to express correctly the calculations in any tensor category [JS2]. What is emerging here is a new symbiosis between algebra, geometry and physics, the consequences of which are not yet fully under-stood. A good review of the recent research on the whole subject is given by P. Cartier [Ct].

After a brief review of Pontrjagin duality and Fourier transforms for locally compact abelian groups, we give a treatment in Section 1 of the classical Tannaka theory for compact (non-abelian) groups. In the compact case, representations must be considered, because characters are no longer sufficient to recapture the compact group. There is also a notion of Fourier transform applying, and we examine this in detail. Where possible we work with a general topological monoid M. A central object of our analysis is the algebra R(M) of *representative* complex-valued functions on M. In Section 2, we show that R(M) is a bialgebra and compute it in the case where M is the unitary group U(n).

Section 3 and Section 4 begin the modern treatment of Tannaka reconstruction, motivated at each stage by the example of a topological monoid. Instrumental here is the *Fourier cotransform* which can be seen as the continuous predual of the Fourier transform. In fact, the Fourier cotransform provides an isomorphism between the reconstructed object and R(M).

Section 5 is an introduction to Tannaka duality for homogeneous spaces [IS]. The proof of the duality theorem of Section 5 is independent of the proof of the one appearing in Section 1.

Sections 6 and 7 are devoted to the characterization of the category of comodules over a coalgebra. Sections 8 and 9 study extra structure which is possessed by the coalgebra $\mathrm{End}^{\vee}(X)$ of Section 4.

Section 10 introduces the concept of braided tensor categories and Yang-Baxter operators at the appropriate level of generality. Section 11 is a brief description of the categorical axiomatization of the geometry of knots and tangles.

Section 12 is a too brief introduction to quantum groups. There are many important aspects of the theory of quantum groups which could be mentioned, such as the work of Lusztig [Lu], Kashdan-Lusztig [KL], Rosso [R], and Deligne [D]. Our goal here is a modest one (the paper is already much longer than our editors expected), and we apologize to those whose work we have not mentioned.

Contents:

§1. Classical Tannaka duality.

Before describing Tannaka duality, we briefly recall Pontrjagin duality. Let G be a commutative locally-compact group. A *character* χ of G is a continuous homomorphism

$$\chi : G \longrightarrow \mathbf{T}$$

where $\mathbf{T}$ is the multiplicative group of complex numbers of modulus 1. The characters form a group $G^\vee$ which is given the topology of uniform convergence on compact subsets of G. It turns out that $G^\vee$ is locally compact. There is a canonical pairing

$$\langle\, ,\rangle : \quad G^\vee \times G \longrightarrow \mathbf{T},$$

and we obtain a canonical homomorphism

$$i : G \longrightarrow (G^\vee)^\vee.$$

Theorem 1 (Pontrjagin). *The canonical homomorphism* i *is an isomorphism of topological groups.*

The group G is compact if and only if the dual group $G^\vee$ is discrete. We have

$$\mathbf{T}^\vee = \mathbf{Z}, \qquad \mathbf{Z}^\vee = \mathbf{T}, \qquad \mathbf{R}^\vee = \mathbf{R}, \qquad (\mathbf{Z}/n)^\vee = \mathbf{Z}/n.$$

Many groups are self dual, such as the additive groups of the local fields.

Pontrjagin duality goes hand-in-hand with the theory of Fourier transforms which we briefly describe. There is a positive measure dx on G called the Haar measure. It is the unique (up to scalar multiple) Borel measure which is invariant under translations. Using it, we can define the spaces $L^1(G)$ and $L^2(G)$ of integrable and square integrable functions. The Fourier transform

$$\mathcal{F} : L^1(G) \cap L^2(G) \longrightarrow L^2(G^\vee)$$

is defined as follows

$$(\mathcal{F}f)(s) = \int_G f(x) \langle s, x\rangle \, dx .$$

The set $L^1(G) \cap L^2(G)$ is a dense subspace of $L^2(G)$ and we have:

Theorem 2 (Plancherel). *With correct normalisation of the Haar measure* ds *on* $G^\vee$, *the mapping* $f \longmapsto \mathcal{F}f$ *extends uniquely to an isometry*

$$\mathcal{F} : L^2(G) \xrightarrow{\ \sim\ } L^2(G^\vee).$$

The inverse transformation $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}(g)(x) = \int_{G^\vee} g(s)\, \overline{\langle s, x\rangle}\, ds .$$

The measure ds on $G^\vee$ which produces the isometry is unique, and is called the measure associated to dx. When G is compact, we often choose dx so that the total mass of G is 1. In this case, the corresponding associated measure ds on the discrete group $G^\vee$ assigns mass 1 to singletons.

It is an open problem to formulate and prove a general duality theorem for non-commutative locally compact groups such as Lie groups. Even the case of simple algebraic groups is not well understood despite the enormous accumulating knowledge on their irreducible representations. However, when the group is compact, there is a good duality theory due to H. Peter, H. Weyl and T. Tannaka. In this case, the dual object $G^\vee$ is discrete and so belongs to the realm of algebra. In order to describe this theory, it will be convenient to introduce some of the needed concepts in a more general setting.

Let M be a topological monoid. *A (finite) representation of* M consists of a finite-dimensional complex vector space V together with a continuous homomorphism

$$\pi_V : M \longrightarrow \mathrm{End}(V)$$

into the monoid $\mathrm{End}(V)$ of linear endomorphisms of V. *Real representations* are defined using real vector spaces. We denote by $\mathcal{Rep}(M,\mathbf{C})$ [respectively, $\mathcal{Rep}(M,\mathbf{R})$] the category of all complex [respectively, real] finite representations of M. There is a forgetful functor

$$\mathbf{U} : \mathcal{Rep}(M, \mathbf{C}) \longrightarrow \mathcal{Vect}_{\mathbf{C}}$$

where $\mathcal{Vect}_{\mathbf{C}}$ denotes the category of all complex vector spaces. Recall that a natural transformation $u : \mathbf{U} \longrightarrow \mathbf{U}$ is a family of maps $u_V : V \longrightarrow V$ indexed by $V \in \mathcal{Rep}(M, \mathbf{C})$ such that the square

$$\begin{tikzcd} V \arrow[r, "u_V"] \arrow[d, "h"'] & V \arrow[d, "h"] \\ W \arrow[r, "u_W"'] & W \end{tikzcd}$$

commutes for all morphisms $h : V \longrightarrow W$ of representations ("intertwining operators"). Clearly, each element $x \in M$ produces such a natural transformation $\pi(x) : \mathbf{U} \longrightarrow \mathbf{U}$ whose V component is the element $\pi_V(x) : V \longrightarrow V$.

There is a topology on the set $\mathrm{End}(\mathbf{U})$ of natural transformations from $\mathbf{U}$ to $\mathbf{U}$. It is the coarsest topology rendering all the projections $u \longmapsto u_V \in \mathrm{End}(V)$ continuous. Composition and addition of natural transformations turns $\mathrm{End}(\mathbf{U})$ into a topological algebra. There is a *conjugation operation*

$$\mathrm{End}(\mathbf{U}) \longrightarrow \mathrm{End}(\mathbf{U}), \qquad u \longmapsto \bar{u},$$

given by

$$\bar{u}_V(x) = \overline{u_{\bar{V}}(\bar{x})}$$

where $\bar{V}$ denotes the conjugate representation of V (the elements of the vector space $\bar{V}$ are the same as those of V but the identity map $V \longrightarrow \bar{V}$, denoted $x \longmapsto \bar{x}$, is an anti-linear isomorphism).

We would like to characterize the natural transformations of the form $\pi(x)$. Recall that a natural transformation u is *tensor preserving* (or "monoidal") when we have

$$u_{V\otimes W} = u_V \otimes u_W \qquad \text{and} \qquad u_I = 1_I$$

where I denotes the trivial 1-dimensional representation of M. We say that u is *self-conjugate* when $u = \bar{u}$. The set of tensor preserving self-conjugate transformations is a closed subset of $\mathrm{End}(\mathbf{U})$ which is also closed under composition. We call this subset the *Tannaka monoid* of M and denote it by $\mathcal{T}(M)$. For any $x \in M$, the natural transformation $\pi(x) \in \mathrm{End}(\mathbf{U})$ belongs to $\mathcal{T}(M)$. We have a continuous homomorphism

$$\pi : M \longrightarrow \mathcal{T}(M).$$

Proposition 3. $\mathcal{T}(M)$ *is a topological group if* M *is.*

Proof. Suppose that M is a topological group. Then any representation

$$\pi_V : M \longrightarrow GL(V)$$

has a dual, or "contragredient", representation

$$\pi_{V^\vee} : M \longrightarrow GL(V^\vee)$$

where $V^\vee$ is the dual vector space of V and

$$\pi_{V^\vee}(x) = {}^t\pi_V(x)^{-1}.$$

We have a morphism of representations $\varepsilon : V^\vee \otimes V \longrightarrow I = \mathbf{C}$ defined by $\varepsilon(s \otimes x) = \langle s, x\rangle$. From the naturality of $u \in \mathcal{T}(M)$, we have the equality

$$\varepsilon\, u_{V^\vee \otimes V} = u_I\, \varepsilon\, .$$

Using the equalities

$$u_{V^\vee \otimes V} = u_{V^\vee} \otimes u_V \qquad \text{and} \qquad u_I = 1_I\, ,$$

we see that

$$\langle\, u_{V^\vee}(s), u_V(x)\,\rangle = \langle\, s, x\,\rangle.$$

This shows that $u_{V^\vee}$ is the contragredient transformation of u_V ,

$$u_{V^\vee} \circ {}^t u_V = 1_{V^\vee}\ ,$$

and implies that u_V is invertible. **qed**

When $M = G$ is a compact group, the algebra $\mathrm{End}(\mathbf{U})$ has a particularly simple structure. Let $G^\vee$ be the set of isomorphism classes of irreducible representations of G. Let us choose a representation (V_λ, π_λ) in each such class λ. For any $u \in \mathrm{End}(\mathbf{U})$, let us write u_λ for the V_λ component of u, and put

$$q(u) = (\, u_\lambda \mid \lambda \in G^\vee\,).$$

Proposition 4. *The map*

$$q \ :\ \mathrm{End}(\mathbf{U}) \longrightarrow \prod_{\lambda \in G^\vee} \mathrm{End}(V_\lambda)$$

is an isomorphism of topological algebras.

Proof. Using naturality, we have the commutative diagram

$$\begin{tikzcd}
V \arrow[r, "u_V"] \arrow[d] & V \arrow[d] \\
V \oplus W \arrow[r, "u_{V\oplus W}"'] & V \oplus W \\
W \arrow[r, "u_W"'] \arrow[u] & W \arrow[u]
\end{tikzcd}$$

for any $V, W \in \mathcal{R}ep(G, \mathbf{C})$, which shows that $u_{V\oplus W} = u_V \oplus u_W$. This implies that u is entirely determined by $q(u)$ since any representation of G decomposes as a direct sum of irreducibles.

To prove that q is surjective, let $t = (t_\lambda \mid \lambda \in G^\vee)$ be a family of elements $t_\lambda \in \mathrm{End}(V_\lambda)$. We want to construct an element $u \in \mathrm{End}(\mathbf{U})$ so that $u_\lambda = t_\lambda$ for every $\lambda \in G^\vee$. Let $S \in \mathcal{R}ep(G, \mathbf{C})$. There is a unique decomposition of S as a direct sum of isotypical components

$$S = \sum_{\lambda \in G^\vee} S_\lambda .$$

Moreover, for each $\lambda \in G^\vee$, the canonical map

$$\psi_\lambda : V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, S_\lambda) \longrightarrow S_\lambda$$

is an isomorphism of G-modules. We put

$$u_{S_\lambda} = \psi_\lambda \circ (t_\lambda \otimes 1) \circ \psi_\lambda^{-1}$$

and

$$u_S = \sum_{\lambda \in G^\vee} u_{S_\lambda} .$$

It is not difficult to verify that the family

$$u = (u_S \mid S \in \mathcal{R}ep(G, \mathbf{C}))$$

defines a natural transformation $u : \mathbf{U} \longrightarrow \mathbf{U}$ having the required property. The continuity of q^{-1} is a consequence of the fact that the topology on $\mathrm{End}(\mathbf{U})$ is the coarsest rendering continuous all the projections $u \longmapsto u_\lambda$, $\lambda \in G^\vee$. **qed**

Remark. The proof of Proposition 4 is based on the fact that the category $\mathcal{R}ep(G, \mathbf{C})$ is the closure under direct sums of its subcategory of irreducible representations.

Proposition 5. *$\mathcal{T}(G)$ is compact if G is a compact group.*

Proof. Any representation V of a compact group admits a positive definite invariant hermitian form $g : V \otimes V \longrightarrow \mathbf{C}$. We can view g as a $\mathbf{C}$-linear pairing

$$h : \bar{V} \otimes V \longrightarrow \mathbf{C}, \qquad h(\bar{x}, y) = g(x, y).$$

For any $u \in \mathrm{End}(\mathbf{U})$, we have

$$h \circ u_{\bar{V} \otimes V} = u_I \circ h.$$

But, if u is tensor preserving and self conjugate, we have

$$u_{\bar{V} \otimes V} = u_{\bar{V}} \otimes u_V, \quad u_I = 1, \quad u_{\bar{V}}(\bar{x}) = \overline{u_V(x)}.$$

This means that

$$h(\overline{u_V(x)}, u_V(y)) = h(\bar{x}, y) ;$$

that is,

$$g(u_V(x), u_V(y)) = g(x, y),$$

which means that u_V belongs to the unitary group $U(V, g)$. It follows from this that $\mathcal{T}(G)$ is a closed subgroup of a product of compact groups. **qed**

To better understand the structure of $\mathcal{T}(G)$ we need to take account of the $*$–involution

$$(\ )^* : \mathrm{End}(\mathbf{U}) \longrightarrow \mathrm{End}(\mathbf{U}).$$

Its definition can be given in terms of two other involutions:

$$u^* \;=\; (\bar{u})^\vee \;=\; \overline{(u^\vee)}$$

where, by definition, $\bar{u}$ is the conjugation operation considered earlier and

$$(u^\vee)_V \;=\; (u_{V^\vee})^\vee .$$

A better description, valid only for compact groups, is to say that it transports across the isomorphism q of Proposition 4 to the *canonical* C^*–algebra structure on each $\mathrm{End}(V_\lambda)$ for $\lambda \in G^\vee$: the adjoint h^* of an element $h \in \mathrm{End}(V_\lambda)$ is defined by the equation

$$g(h^*(x), y) \;=\; g(x, h(y))$$

and does not depend on the choice of g since g is unique up to a scalar multiple (by Schur's Lemma applied to the irreducible representation V_λ).

Remark. The algebra $\mathrm{End}(V_\lambda)$ does not depend on the choice of V_λ in the class λ. For, it follows from Schur's Lemma that the algebras $\mathrm{End}(V_1)$ and $\mathrm{End}(V_2)$ are *canonically* isomorphic for any two isomorphic irreducible representations V_1, V_2.

An element $u \in \mathrm{End}(\mathbf{U})$ is *unitary* if $u^*u = uu^* = 1$. The group of unitary elements is isomorphic to the product

$$\prod_{\lambda \in G^\vee} U(d_\lambda)$$

where $U(d_\lambda) \subset \mathrm{End}(V_\lambda)$ is a unitary group of dimension $d_\lambda = \dim V_\lambda$. We have proved that $\mathcal{T}(G)$ is a closed subgroup of this product.

The theory of Fourier transforms on compact groups can now be described. The *Fourier transform* of a continuous map $f : G \longrightarrow \mathbf{C}$ is an element $\mathcal{F}f \in \mathrm{End}(\mathbf{U})$. It is defined by the integral

$$(\mathcal{F}f)_V \;=\; \int_G f(x)\, \pi_V(x)\, dx \ .$$

It is easy to see that

$$\mathcal{F}(f * g) \;=\; (\mathcal{F}f)\,(\mathcal{F}g),$$

where $f * g$ is the convolution of f and g given by

$$(f * g)(x) = \int_G f(x\, y^{-1})\; g(y)\; dy \ .$$

Also, we have

$$\mathcal{F}f^* = (\mathcal{F}f)^*$$

where f^* is defined by $f^*(x) = \overline{f}(x^{-1})$.

It follows from Proposition 4 that $\mathcal{F}f$ is entirely determined by its effect on irreducible representations

$$(\mathcal{F}f)(\lambda) \;=\; \int_G f(x)\, \pi_\lambda(x)\, dx \ ,$$

so that $\mathcal{F}$ defines a map

$$\mathcal{F}: \mathcal{C}(G, \mathbf{C}) \longrightarrow \prod_{\lambda\in G^\vee} \mathrm{End}(V_\lambda)$$

on the algebra $\mathcal{C}(G, \mathbf{C})$ of complex-valued continuous functions on G. We would like to describe an inverse to $\mathcal{F}$. It should be noted that $\mathcal{F}$ cannot be surjective because we have

$$\| \mathcal{F}f(\lambda) \| \leq \| f \|_\infty$$

where the norm on the C*–algebra $\mathrm{End}(V_\lambda)$ is used on the left hand side. This shows that $\mathcal{F}$ lands in the set of elements of bounded norm

$$\| u \| = \sup_{\lambda\in G^\vee} \| u_\lambda \| < +\infty .$$

This set forms a genuine C*–algebra

$$\prod_{\lambda\in G^\vee}^{\mathrm{bded}} \mathrm{End}(V_\lambda) \subset \prod_{\lambda\in G^\vee} \mathrm{End}(V_\lambda) .$$

We begin by describing a partial inverse $\mathcal{F}^{-1}$ to $\mathcal{F}$:

$$\mathcal{F}^{-1}: \sum_{\lambda\in G^\vee} \mathrm{End}(V_\lambda) \longrightarrow \mathcal{C}(G, \mathbf{C})$$

$$(\mathcal{F}^{-1}g)(x) = \sum_{\lambda\in G^\vee} \mathrm{Tr}(g(\lambda)\, \pi_\lambda(x)^{-1})\ d_\lambda .$$

But, to prove anything about this inverse, we need the orthogonality relations.

Lemma 6. *If* $\lambda, \delta \in G^\vee$ *and* $A\in \mathrm{Hom}(V_\delta, V_\lambda)$ *then*

$$\int_G \pi_\lambda(x) A\, \pi_\delta(x)^{-1}\, dx = \begin{cases} \frac{\mathrm{Tr}(A)}{d_\lambda}\mathrm{id} & \text{if } \lambda = \delta \\ 0 & \text{otherwise} . \end{cases}$$

Proof. This is a classical lemma, but we give the proof for completeness. The left hand side is an operator obtained by averaging the function $x \longmapsto \pi_\lambda(x)\ A\ \pi_\delta(x)^{-1}$ over the group. It is therefore an invariant map from V_δ to V_λ. By Schur's lemma, it must be equal to 0 if $\lambda \neq \delta$ and a scalar multiple of the identity if $\lambda = \delta$. To check the equality, we just need to take the trace on both sides. **qed**

Lemma 7. *If* $\lambda, \delta \in G^\vee$ *and* $v\in V_\lambda$, $w\in V_\delta$ *then*

$$\int_G \pi_\lambda(x)(v) \otimes \pi_\delta(x)^{-1}(w)\ dx = \begin{cases} \frac{w\otimes v}{d_\lambda} & \text{if } \lambda = \delta \\ 0 & \text{otherwise} . \end{cases}$$

Proof. It is enough to prove that, for any linear form $\phi : V_\lambda \longrightarrow \mathbf{C}$, we have

$$\int_G \pi_\lambda(x)(v)\ \phi(\pi_\delta(x)^{-1}(w))\ dx = \begin{cases} \frac{w\phi(v)}{d_\lambda} & \text{if } \lambda = \delta \\ 0 & \text{otherwise} . \end{cases}$$

This equality is a special case of Lemma 6 with the operator A given by $A(y) = \phi(y)\, v$. **qed**

Proposition 8 (Orthogonality Relations). *If* $\lambda, \delta \in G^\vee$ *and* $A\in \mathrm{End}(V_\lambda)$, $B\in \mathrm{End}(V_\delta)$ *then*

$$\text{(i)} \int_G \mathrm{Tr}(A\ \pi_\lambda(x))\,\mathrm{Tr}(B\ \pi_\delta(x^{-1}))\ dx \quad = \begin{cases} \dfrac{\mathrm{Tr}(AB)}{d_\lambda} & \text{if } \lambda = \delta \\ 0 & \text{otherwise} \end{cases} .$$

$$\text{(ii)} \int_G \mathrm{Tr}(A\ \pi_\lambda(x))\,\overline{\mathrm{Tr}(B\ \pi_\delta(x))}\ dx \quad = \begin{cases} \dfrac{\mathrm{Tr}(AB^*)}{d_\lambda} & \text{if } \lambda = \delta \\ 0 & \text{otherwise} \end{cases} .$$

Proof. It suffices to verify relation (i) when both A and B are of rank 1, since the identity is bilinear in A, B and the rank 1 operators span all the operators. Put $A(x) = \phi(x)\,v$ and $B(x) = \psi(x)\,v$. The result follows from application of the linear form $\phi \otimes \psi : V_\lambda \otimes V_\delta \longrightarrow \mathbf{C}$ to the equality in Lemma 7. The second relation follows from the first and

$$\overline{\mathrm{Tr}(B\pi_\delta(x))} \quad = \quad \mathrm{Tr}(B^*\pi_\delta(x)^*) \quad = \quad \mathrm{Tr}(B^*\pi_\delta(x)^{-1}) \ ._{\mathbf{qed}}$$

Corollary 9. *If* $\lambda, \delta \in G^\vee$ *and* $A \in \mathrm{End}(V_\lambda)$ *then the following identity holds:*

$$d_\lambda \int_G \mathrm{Tr}(A\,\pi_\lambda(x))\ \pi_\delta(x^{-1})\ dx \ = \begin{cases} A & \text{for } \lambda = \delta \\ 0 & \text{otherwise} \end{cases} .$$

Proof. Since the pairing $\langle A,B\rangle = \mathrm{Tr}(AB)$ is exact, it suffices to check the equation after applying $\mathrm{Tr}(B\ -)$ on both sides. But then the orthogonality relations give the result. **qed**

On $\mathcal{C}(G, \mathbf{C})$ we introduce the inner product

$$\langle\, f, g\,\rangle \ = \ \int_G \bar{f}(x)g(x)dx \ .$$

Note that we have

$$\langle\, f, g\,\rangle = \varepsilon(f^* * g)$$

where $\varepsilon : \mathcal{C}(G, \mathbf{C}) \longrightarrow \mathbf{C}$ is the functional "evaluate at $e \in G$" : $\varepsilon(f) = f(e)$.

On $\sum_{\lambda\in G^\vee} \mathrm{End}(V_\lambda)$ we introduce the inner product

$$\langle\, g, h\,\rangle = \sum_{\lambda \in G^\vee} \mathrm{Tr}(g^*(\lambda)\,h(\lambda))d_\lambda \ .$$

Proposition 10. *The following equalities hold:*

(a) $\mathcal{F}\mathcal{F}^{-1}g = g$,
(b) $\mathcal{F}^{-1}(g\,h) = (\mathcal{F}^{-1}g) * (\mathcal{F}^{-1}h)$,
(c) $\mathcal{F}^{-1}(g^*) = (\mathcal{F}^{-1}g)^*$,
(d) $\langle\, \mathcal{F}^{-1}g, \mathcal{F}^{-1}h\,\rangle = \langle\, g, h\,\rangle$.

Proof. We have

$$(\mathcal{F}(\mathcal{F}^{-1}g))(\delta) \ = \sum_{\lambda \in G^\vee} \int_G d_\lambda\ \mathrm{Tr}(g(\lambda)\pi_\lambda(x)^{-1})\ \pi_\delta(x)\ dx \ .$$

But, by Corollary 9, we have

$$d_\lambda \int_G \mathrm{Tr}(g(\lambda)\,\pi_\lambda(x^{-1}))\,\pi_\delta(x)dx = \begin{cases} g(\lambda) & \text{for } \lambda = \delta \\ 0 & \text{otherwise} \end{cases} .$$

The equality $\mathcal{F}\mathcal{F}^{-1}g = g$ follows. Similarly for (b). The equality (c) is trivial. To prove the last, we note that

$$\varepsilon(\mathcal{F}^{-1}g) = \sum_{\lambda\in G^\vee} \mathrm{Tr}(g(\lambda))d_\lambda ,$$

and therefore

$$\begin{aligned}\langle \mathcal{F}^{-1} g, \mathcal{F}^{-1} h \rangle &= \varepsilon((\mathcal{F}^{-1} g)^* * (\mathcal{F}^{-1} h)) \\ &= \varepsilon\ \mathcal{F}^{-1}(g^* h) \\ &= \sum_{\lambda\in G^\vee} \mathrm{Tr}(g^*(\lambda)h(\lambda))\,d_\lambda \\ &= \langle g, h\rangle . \end{aligned}$$ **qed**

The domain of $\mathcal{F}^{-1}$ can be completed by using the norm

$$\| u \|^2 = \langle u, u\rangle .$$

Its completion is a Hilbert space isomorphic to the Hilbert sum

$$\sum_{\lambda\in G^\vee}^{\text{hilbert}} \mathrm{End}\,(V_\lambda)$$

where $\mathrm{End}(V_\lambda)$ is given the Hilbert space metric defined by

$$\| A \|^2 = d_\lambda \mathrm{Tr}(A^* A).$$

The continuous extension of $\mathcal{F}^{-1}$ defines an isometric embedding

$$\mathcal{F}^{-1} : \sum_{\lambda\in G^\vee}^{\text{hilbert}} \mathrm{End}\,(V_\lambda) \longrightarrow L^2(G) .$$

Definition. Let M be a topological monoid. A function $f : M \longrightarrow \mathbf{C}$ is said to be *representative* of $(V, \pi_V)\in \mathcal{Rep}(M,\mathbf{C})$ when there is a linear form

$$\phi : \mathrm{End}(V) \longrightarrow \mathbf{C}$$

such that $f = \phi \circ \pi_V$.

Equivalently, f is representative of (V, π_V) when there is some $A \in \mathrm{End}(V)$ such that $f(x) = \mathrm{Tr}(A\,\pi_V(x))$ for every $x \in M$. This means that f is a linear combination of the coefficients $\pi_{ij}(x)$ of π_V in some basis of V. We denote by $R(V, \pi_V)$, or $R(V)$, the set of functions that are representative of (V, π_V). Let $R(M)$ be the set of functions on M which are representative of some representation (V, π_V).

Proposition 11. $R(M)$ *is a subalgebra of* $\mathcal{C}(M, \mathbf{C})$ *which is closed under conjugation* $f \longmapsto \bar{f}$.

Proof. The result follows from the easy observations below.

$$R(V_1 \oplus V_2) = R(V_1) + R(V_2), \quad R(V_1)R(V_2) \subseteq R(V_1 \otimes V_2)$$

$$R(\overline{V}) = \overline{R(V)}$$

$$R(I) = \{\text{ constant functions }\}$$ **qed**

Theorem 12 (Peter-Weyl). *For any compact group* G *and any* $x\in G$, $x \neq e$, *there exists a finite dimensional representation* (V, π_V) *such that* $\pi_V(x) \neq \mathrm{id}$.

Corollary 13. *On a compact group any continuous function can be uniformly approximated by representative functions.*

Proof. Let $x, y \in G$, $x \neq y$. Theorem 12 says that there exists a representation π_V such that $\pi_V(x\,y^{-1}) \neq \mathrm{id}$; that is, $\pi_V(x) \neq \pi_V(y)$. But there exists a linear form $\phi : \mathrm{End}(V) \longrightarrow \mathbf{C}$ such that

$$\phi\,(\pi_V(x)) \neq \phi\,(\pi_V(y)).$$

This shows that the subalgebra $R(G)$ separates points; and since $R(G)$ is closed under conjugation, the result follows from the Stone-Weierstrass theorem. **qed**

Theorem 14 (Plancherel theorem for compact groups). *The Fourier transform* $\mathcal{F}$ *can be extended continuously to an isometry*

$$\mathcal{F} \;:\; L^2(G) \xrightarrow{\;\sim\;} \sum_{\lambda \in G^\vee}^{\text{hilbert}} \mathrm{End}\,(V_\lambda) \quad .$$

Proof. We have already defined an isometric embedding

$$\mathcal{F}^{-1} \;:\; \sum_{\lambda \in G^\vee}^{\text{hilbert}} \mathrm{End}\,(V_\lambda) \longrightarrow L^2(G)\,.$$

The subspace $\mathrm{Im}(\,\mathcal{F}^{-1}\,)$ is closed since it is complete. But it contains the dense subspace $R(G)$; so $\mathrm{Im}(\,\mathcal{F}^{-1}\,) = L^2(G)$, and the theorem is proved. **qed**

The group G acts on the left and right of $L^2(G)$ via the equalities:

$$(x \bullet f)\,(y) = f\,(x^{-1}\,y)\,, \qquad (f \bullet x)\,(y) = f\,(y\,x^{-1}).$$

Also, G acts on both sides of each of the Hilbert spaces $\mathrm{End}(V_\lambda)$:

$$x \bullet A = \pi_\lambda(x)\,A, \qquad A \bullet x = A\,\pi_\lambda(x)\,.$$

The Fourier transform respects these actions:

$$\mathcal{F}(x \bullet f) = x \bullet (\,\mathcal{F}f)\,, \qquad \mathcal{F}(f \bullet x) = (\,\mathcal{F}f) \bullet x\,.$$

A function $f \in \mathcal{C}(G, \mathbf{C})$ is a *class* function if it is constant on conjugacy classes of G; or equivalently, if it is invariant under conjugation:

$$f = x \bullet f \bullet x^{-1}.$$

The class functions are exactly the members of the *centre* of the algebra $\mathcal{C}(G, \mathbf{C})$ with respect to the convolution product.

Proposition 15. *An element* $u \in \mathrm{End}(\mathbf{U})$ *is central if and only if, for every* $V \in \mathcal{R}ep(G, \mathbf{C})$, *the linear map* $u_V : V \longrightarrow V$ *is a* G-*homomorphism.*

Proof. If u is central then, for every $x \in G$, we have $u\,\pi(x) = \pi(x)\,u$, and so

$$u_V\,\pi_V(x) = \pi_V(x)\,u_V$$

for every V. Conversely, if u_V is a G-homomorphism then, for every $w \in \mathrm{End}(\mathbf{U})$, we have the commutative square below showing that $uw = wu$; so u is central. **qed**

$$\begin{tikzcd} V \arrow[r, "u_V"] \arrow[d, "w_V"'] & V \arrow[d, "w_V"] \\ V \arrow[r, "u_V"'] & V \end{tikzcd}$$

We would like to restrict the Fourier transform to its central part. If $f \in \mathcal{C}(G, \mathbf{C})$ is a class function then $x \bullet f = f \bullet x$, and therefore

$$\pi(x)\,(\mathcal{F}f) = (\mathcal{F}f)\,\pi(x),$$

which shows that $\mathcal{F}f$ is central.

For any $\lambda \in G^\vee$, the map $(\mathcal{F}f)(\lambda) : V_\lambda \longrightarrow V_\lambda$ is a G-homomorphism. According to Schur's lemma, it must be a scalar multiple of the identity. Denoting this scalar by $(Tf)(\lambda)$, we have

$$(Tf)(\lambda) = \frac{1}{d_\lambda}\,\mathrm{Tr}(Tf)(\lambda) = \frac{1}{d_\lambda}\int_G f(x)\,\chi_\lambda(x)\,dx$$

where $\chi_\lambda(x) = \mathrm{Tr}(\pi_\lambda(x))$ is the character of the irreducible representation V_λ. From the relation $(\mathcal{F}f)(\lambda) = (Tf)(\lambda)\,\mathrm{id}_\lambda$, we deduce the relations

$$T(f * g) = (Tf)\,(Tg),$$
$$T(f^*) = \overline{Tf}\,.$$

The centre of each algebra $\mathrm{End}(V_\lambda)$ is $\mathbf{C} \bullet \mathrm{id}_\lambda$. According to Propositions 4 and 15, the centre of the algebra $\mathrm{End}(\mathbf{U})$ is equal to the product

$$\prod_{\lambda \in G^\vee} \mathbf{C} \cdot \mathrm{id}_\lambda \;\cong\; \mathbf{C}^{G^\vee}.$$

We define the inverse Fourier transform of any function $g : G^\vee \longrightarrow \mathbf{C}$ of finite support by

$$(T^{-1}g)(x) = \sum_{\lambda \in G^\vee} g(\lambda)\overline{\chi_\lambda(x)}\,d_\lambda\,.$$

It is a consequence of Proposition 10 that:

(a) $T\,T^{-1} g = g$,
(b) $T^{-1}(g\,h) = (T^{-1} g) * (T^{-1} h)$,
(c) $T^{-1}(g^*) = (T^{-1} g)^*$,
(d) $\langle T^{-1} g, T^{-1} h \rangle = \langle g, h \rangle$.

In this last equality, the inner product on the right hand side is defined by

$$\langle g, h \rangle = \sum_{\lambda \in G^\vee} \bar{g}(\lambda)\,h(\lambda)\,d_\lambda^2\,.$$

The *spectral measure* $d\lambda$ on $G^\vee$ assigns weight d_λ^2 to the singletons $\{\lambda\}$. The Hilbert space $L^2(G^\vee)$ is the space of square summable functions with respect to the spectral measure. We write $C(G)$ for the space of conjugacy classes of G. It is the orbit space of G acting on itself by conjugation. There is a canonical measure on $C(G)$ obtained by taking the image of the Haar measure along the projection $G \longrightarrow C(G)$. Obviously, $L^2(C(G))$ is isomorphic to the subspace of $L^2(G)$ consisting of square integrable class functions.

Theorem 16. *The Fourier transforms* T *and* T^{-1} *have continuous extensions to mutually inverse isometries*

$$L^2(C(G)) \xleftarrow{\;\sim\;}\!\!\!\rightarrow L^2(G^\vee)\,.$$

Proof. This is just a description of the restriction of the Fourier transforms $\mathcal{F}$ and $\mathcal{F}^{-1}$ to the central parts. **qed**

A collection $\mathcal{X}$ of representations of G is *closed* when it contains
(i) π_1 if π_1 is isomorphic to some $\pi_2 \in \mathcal{X}$,

(ii) π_1 if π_1 is a subrepresentation of $\pi_2 \in \mathcal{X}$,
(iii) $\pi_1 \oplus \pi_2$ if $\pi_1, \pi_2 \in \mathcal{X}$,
(iv) $\pi_1 \otimes \pi_2$ if $\pi_1, \pi_2 \in \mathcal{X}$,
(v) $\bar{\pi}_1$ if $\pi_1 \in \mathcal{X}$,
(vi) I.

The set of representative functions of the members of $\mathcal{X}$ is a subalgebra $R(\mathcal{X})$ of the algebra $R(G)$.

Lemma 17. *Let $\mathcal{X}$ be a closed collection of representations of G. Suppose that, for every $x \in G$, $x \neq e$, there exists a representation $\pi_V \in \mathcal{X}$ such that $\pi_V(x) \neq e$. Then $\mathcal{X} = \mathcal{Rep}(G, \mathbf{C})$.*

Proof. If $\mathcal{X} \neq \mathcal{Rep}(G, \mathbf{C})$, there will be an element $\lambda \in G^\vee$ such that $\pi_\lambda \notin \mathcal{X}$. The orthogonality relations then imply that the representative functions of π_λ are orthogonal to the elements of $R(\mathcal{X})$. In particular, χ_λ is orthogonal to $R(\mathcal{X})$. On the other hand, the hypothesis implies that the subalgebra $R(\mathcal{X})$ separates the points of G and is closed under conjugation. According to the Stone-Weierstrass theorem, $R(\mathcal{X})$ is dense in $\mathcal{C}(G, \mathbf{C})$. This is a contradiction. **qed**

We would like to prove that, for any compact group G, the map

$$\pi : G \longrightarrow \mathcal{T}(G)$$

induces, via restriction, an isomorphism of algebras

$$\pi^* : R(\mathcal{T}(G)) \longrightarrow R(G).$$

We shall obtain this from the next result.

Lemma 18. *The restriction functor*

$$\pi^* : \mathcal{Rep}(\mathcal{T}(G), \mathbf{C}) \longrightarrow \mathcal{Rep}(G, \mathbf{C})$$

is an equivalence of categories.

Proof. We define an extension functor

$$e : \mathcal{Rep}(G, \mathbf{C}) \longrightarrow \mathcal{Rep}(\mathcal{T}(G), \mathbf{C})$$

as follows. For any $V \in \mathcal{Rep}(G, \mathbf{C})$, the map $u \longmapsto u_V$ is a representation of $\mathcal{T}(G)$ on V. The commutative triangle below shows that $\pi^*(p_V) = \pi_V$. We put $e(V, \pi_V) = (V, p_V)$. For any morphism $h : (V, \pi_V) \longrightarrow (W, \pi_W)$ in $\mathcal{Rep}(G, \mathbf{C})$ and any $u \in \mathcal{T}(G)$, we have $h \circ u_V = u_W \circ h$. This shows that h is also a morphism $h : (V, p_V) \longrightarrow (W, p_W)$ in $\mathcal{Rep}(\mathcal{T}(G), \mathbf{C})$.

$$\begin{tikzcd} G \arrow[rr, "\pi_V"] \arrow[dr, "\pi"'] & & GL(V) \\ & \mathcal{T}(G) \arrow[ur, "p_V"'] & \end{tikzcd}$$

It is easy to see that e preserves direct sums, tensor products and conjugate representations. Moreover, from the same commutative triangle (above), if (V, π_V) is irreducible then $e(V, \pi_V)$ is irreducible. From this it follows that the image of the functor e is a full subcategory whose objects constitute a closed collection $\mathcal{X}$ of representations of $\mathcal{T}(G)$. We prove that $\mathcal{X}$ separates the elements of $\mathcal{T}(G)$. Let $u \in \mathcal{T}(G)$, $u \neq e$. There exists a representation V of G such that $u_V \neq id_V$. This means that $p_V(u) \neq id_V$ and thus p_V separates u from e. By Lemma 17, the collection $\mathcal{X}$ is all of $\mathcal{R}ep(\mathcal{T}(G), \mathbf{C})$, and so e is an equivalence of categories. Finally, π^* is an equivalence since $\pi^* \circ e = id$. **qed**

Lemma 19. *The restriction map*

$$\pi^* : R(\mathcal{T}(G)) \longrightarrow R(G)$$

is an isomorphism of algebras.

Proof. We define an inverse $e : R(G) \longrightarrow R(\mathcal{T}(G))$ in the following manner. Any $f \in R(G)$ has a unique representation in the form

$$f(x) = \sum_{\lambda \in G^\vee} Tr(g(\lambda)\, \pi_\lambda(x))\, d_\lambda$$

where $g \in \sum_{\lambda \in G^\vee} End(V_\lambda)$. (In fact, $g = \mathcal{F} f$.) We define

$$e(f)(u) = \sum_{\lambda \in G^\vee} Tr(g(\lambda)\, p_\lambda(u))\, d_\lambda .$$

According to Lemma 18, we have $G^\vee = \mathcal{T}(G)^\vee$ from which the bijectivity of e follows. Finally, π^* is bijective since $\pi^* \circ e = id$. **qed**

Theorem 20 (Tannaka-Krein). *For any compact group* G, *the canonical map*

$$\pi : G \longrightarrow \mathcal{T}(G)$$

is an isomorphism.

Proof. We first prove the injectivity of π. According to Peter-Weyl (Theorem 12), for any $x \in G$, $x \neq e$, there is a representation (V, π_V) such that $\pi_V(x) \neq id$. But then $\pi(x) \neq e$ since $p_V(\pi(x)) = \pi_V$. To prove the surjectivity of π, we first prove

$$(*) \qquad \int_{\mathcal{T}(G)} f(u)\, du = \int_G f(\pi(x))\, dx$$

for any $f \in \mathcal{C}(\mathcal{T}(G), \mathbf{C})$. It is enough to prove this equality when $f \in R(\mathcal{T}(G))$ since $R(\mathcal{T}(G))$ is dense in $\mathcal{C}(\mathcal{T}(G), \mathbf{C})$ by Corollary 13. Using Lemma 19, we see that a function $f \in R(\mathcal{T}(G))$ has a unique representation

$$f(u) = \sum_{\lambda \in G^\vee} Tr(g(\lambda)\, p_\lambda(u))\, d_\lambda ,$$

so that

$$f(\pi(x)) = \sum_{\lambda \in G^\vee} Tr(g(\lambda)\, \pi_\lambda(x))\, d_\lambda .$$

But the orthogonality relations give

$$\int_G Tr(g(\lambda)\pi_\lambda(u))\, du = \begin{cases} g(I) & \text{for } \lambda = I \\ 0 & \text{otherwise} \end{cases} .$$

Similarly,

$$\int_{T(G)} Tr(g(\lambda) p_\lambda(u))\, du = \begin{cases} g(I) & \text{for } \lambda = I \\ 0 & \text{otherwise} \end{cases} .$$

Equality (*) follows.

To prove the surjectivity, let us suppose that $\mathrm{Im}(\pi) \neq \mathcal{T}(G)$. Let f be a positive function whose support is contained in the complement of the closed subset $\mathrm{Im}(\pi)$. Then we have

$$\int_{\mathcal{T}(G)} f(u)\, du > 0 \quad \text{and} \quad \int_G f(\pi(x))\, dx = 0$$

which is a contradiction. **qed**

§2. The bialgebra of representative functions.

This Section provides a more complete description of the algebra of representative functions on a compact group and, more generally, on a topological monoid. We shall see that R(M) is a bialgebra, and we shall compute it in some special cases. We begin by giving a simple characterization of representative functions on locally compact monoids.

Let M be a locally compact monoid. For each $y \in M$, consider the translation operators $\lambda_y(x) = yx$ and $\rho_y(x) = xy$. We enrich $\mathcal{C}(M, \mathbf{C})$ with the topology of uniform convergence on compact subsets of M. The maps

$$(y, f) \longmapsto f \circ \rho_y \qquad \text{and} \qquad (f, y) \longmapsto f \circ \lambda_y$$

define continuous left and right actions of M on $\mathcal{C}(M, \mathbf{C})$.

Proposition 1. *A continuous complex-valued function* f *on a locally compact monoid* M *is representative if and only if the linear subspace of* $\mathcal{C}(M, \mathbf{C})$ *spanned by any one of the three subsets*

$$\{ f \circ \lambda_y \mid y \in M \}, \qquad \{ f \circ \rho_y \mid y \in M \}, \qquad \{ f \circ \lambda_y \circ \rho_z \mid y, z \in M \}$$

is finite dimensional.

Proof. Let $\pi_V : M \longrightarrow \mathrm{End}(V)$ be a representation of M. The set $R(V, \pi_V)$ of representative functions for π_V is closed under left and right translation; for, if $f(x) = \mathrm{Tr}(A\, \pi_V(x))$, we have

$$f \circ \lambda_y \circ \rho_z(x) = \mathrm{Tr}(A\, \pi_V(yxz)) = \mathrm{Tr}(\pi_V(z)\, A\, \pi_V(y)\, \pi_V(x)).$$

Conversely, let us assume, for example, that $\{ f \circ \rho_y \mid y \in M \}$ generates a finite dimensional vector subspace V. The continuous left action of M on $\mathcal{C}(M, \mathbf{C})$ restricts to a continuous left action of M on V. We obtain in this way a continuous representation $\pi_V : M \longrightarrow \mathrm{End}(V)$ of M. Let $\varepsilon : V \longrightarrow \mathbf{C}$ be the linear form $\varepsilon(h) = h(e)$ where $e \in M$ is the unit. We have

$$f(x) = \varepsilon(f \circ \rho_x) = \varepsilon(\pi_V(x)(f)) = \mathrm{Tr}(A\, \pi_V(x))$$

where $A \in \mathrm{End}(V)$ is the rank one linear function $h \longmapsto \varepsilon(h)\, f$. **qed**

For any topological monoid M, there is a bialgebra structure on R(M) which we now describe. Let

$$\Delta : \mathcal{C}(M, \mathbf{C}) \longrightarrow \mathcal{C}(M \times M, \mathbf{C})$$

be the algebra homomorphism defined by

$$(\Delta f)(x, y) = f(xy).$$

We have canonical inclusions

$$R(M) \otimes R(M) \subset \mathcal{C}(M, \mathbf{C}) \otimes \mathcal{C}(M, \mathbf{C}) \subset \mathcal{C}(M \times M, \mathbf{C}).$$

Lemma 2. *For all* $f \in R(M)$, $\quad \Delta f \in R(M) \otimes R(M)$.

Proof. It is enough to prove the result when $f(x) = \pi_{ij}(x)$ where $\pi = (\pi_{ij})$ is some representation of M. In this case we have

$$\pi_{ij}(xy) = \Sigma_k \pi_{ik}(x)\,\pi_{kj}(y)\,;$$

that is,

$$\Delta\pi_{ij} = \Sigma_k \pi_{ik} \otimes \pi_{kj}\,. \ \mathbf{qed}$$

Hence Δ gives a comultiplication for $R(M)$. The counit $\varepsilon : R(M) \longrightarrow \mathbf{C}$ of the coalgebra is given by $\varepsilon(f) = f(e)$.

When the monoid M is a topological group, the bialgebra $R(M)$ becomes a Hopf algebra. The antipode $\nu : R(M) \longrightarrow R(M)$ is defined by

$$\nu(f)(x) = f(x^{-1})\,.$$

To see that $\nu(f) \in R(M)$, suppose that $f(x) = \mathrm{Tr}(\pi(x)A)$. Then we have

$$f(x^{-1}) = \mathrm{Tr}(\pi(x^{-1})A) = \mathrm{Tr}({}^tA\,{}^t\pi(x^{-1})) = \mathrm{Tr}(\pi^\vee(x)\,{}^tA)$$

where $\pi^\vee(x) = {}^t\pi(x^{-1})$ is the contragredient representation of π.

The algebra $R(G)$ is easy to describe when G is a compact abelian group. In this case, $R(G)$ is the linear subspace of $\mathcal{C}(G, \mathbf{C})$ spanned by the set $G^\vee$ of characters of G. However, the orthogonality relations imply that $G^\vee$ is a linearly independent subset of $\mathcal{C}(G, \mathbf{C})$. This shows that $R(G)$ is isomorphic to the enveloping algebra $\mathbf{C}[G^\vee]$ (usually called the group algebra) of the dual group $G^\vee$. For any $\chi \in G^\vee$, the relation

$$\chi(xy) = \chi(x)\,\chi(y)$$

implies the relation

$$\Delta\chi = \chi \otimes \chi\,.$$

This shows that the coalgebra structure $\Delta : \mathbf{C}[G^\vee] \longrightarrow \mathbf{C}[G^\vee] \otimes \mathbf{C}[G^\vee]$ is simply the linear extension of the diagonal map $\Delta : G^\vee \longrightarrow G^\vee \times G^\vee$. The antipode ν is the linear extension of the inverse operation $G^\vee \longrightarrow G^\vee$. It is also worthwhile to identify the operation corresponding to conjugation $f \longmapsto \overline{f}$ in $R(G)$: it is the *antilinear* extension of the inverse operation.

If G is the circle group $\mathbf{T}$ then $R(G)$ is the ring of finite Fourier series:

$$R(\mathbf{T}) = \mathbf{C}[\,\mathbf{Z}\,] = \mathbf{C}[\,z, z^{-1}\,]$$

where $z = e^{i\theta}$ and $\bar{z} = z^{-1}$.

When G is not abelian, the following result is useful in identifying $R(G)$. For any representation π_V of G, let $\det(\pi_V)$ be the one-dimensional representation obtained as the composite of $\pi_V : G \longrightarrow GL(V)$ and $\det : GL(V) \longrightarrow GL(1)$.

Proposition 3. *Let* G *be a compact group, and let* π_V *be a faithful representation of* G. *The algebra* $R(G)$ *is generated by the coefficients of* π_V *together with* $\det(\pi_V)^{-1}$.

Proof. Let A be the subalgebra of $R(G)$ generated by the coefficients of π_V and $\det(\pi_V)^{-1}$. Let us verify first that A is closed under conjugation. But we have

$$\overline{\det(\pi_V)}^{-1} = \det(\pi_V)\,,$$

and also

$$\overline{R(\pi_V)} = R(\pi_{\overline{V}}) = R(\pi_{V^\vee})\,.$$

It is therefore sufficient to verify that the entries of the contragredient representation are contained in A. This is true because of the familiar formula expressing the entries of an inverse matrix as cofactors divided by the determinant.

To finish the proof, let $\mathcal{X}$ be the collection of representations whose coefficients belong to A. Then $\mathcal{X}$ is a closed collection, and, since π_V is faithful, we can apply Lemma 16. This gives $\mathcal{X} = \mathcal{R}ep(G, \mathbf{C})$ so that $A = R(G)$. **qed**

We would like to apply Proposition 3 to compact Lie groups, but to do this we need the following result.

Proposition 4. *Each compact Lie group* G *admits a faithful representation.*

Proof. We shall use the fact that the poset of closed submanifolds of a given compact manifold is artinian. This means that any non-empty collection of submanifolds contains a minimal element; or equivalently, that any decreasing sequence of submanifolds must

stop. For any $V \in \mathcal{R}ep(G, \mathbf{C})$, the kernel of π_V is a closed submanifold of G. Let $\ker(\pi_W)$ be a minimal element in this collection. According to the Peter-Weyl Theorem, for any $x \neq e$, there exists $V \in \mathcal{R}ep(G, \mathbf{C})$ such that $x \notin \ker(\pi_V)$. But we have

$$\ker(\pi_V \oplus \pi_W) = \ker(\pi_W) \cap \ker(\pi_V),$$

so that $\ker(\pi_W) \subsetneq \ker(\pi_V)$, and so $x \notin \ker(\pi_W)$. This proves that $\ker(\pi_W) = \{e\}$. **qed**

The last two Propositions have the following immediate consequence.

Corollary 5. *The algebra of representative functions on a compact Lie group is finitely generated.*

We shall now give a complete description of R(G) for the compact Lie group $G = U(n)$ of unitary $n \times n$ complex matrices. Let $\mathbf{C}[(z_{ij})]$ be the ring of polynomials in the n^2 indeterminates z_{ij} $(1 \leq i, j \leq n)$ and let $d = \det(z_{ij})$. If we map z_{ij} into the coefficient ρ_{ij} of the standard representation

$$\rho : U(n) \hookrightarrow \mathrm{End}(\mathbf{C}^{n \times n}),$$

we obtain a homomorphism

$$\iota : \mathbf{C}[(z_{ij}), d^{-1}] \longrightarrow R(U(n))$$

sending d^{-1} to $\det(\rho)^{-1}$.

Proposition 6. *The homomorphism* ι *is an algebra isomorphism, so that*

$$R(U(n)) \cong \mathbf{C}[(z_{ij}), d^{-1}].$$

Proof. Proposition 3 implies that ι is surjective. To show that ι is injective, let us first remark that, for any $P(z) \in \mathbf{C}[(z_{ij}), d^{-1}]$, the function $\iota(P(z))$ on $U(n)$ is exactly the function $u \longmapsto P(u)$. Therefore, if $\iota(P(z)) = 0$ then $P(u) = 0$ for every $u \in U(n)$. We have to show that this implies that $P = 0$. This property is sometimes formulated by saying that $U(n)$ is Zariski dense in $GL(n, \mathbf{C})$ (it is the famous "unitarian trick" of Hermann Weyl which he used to reduce the theory of rational representations of $GL(n, \mathbf{C})$ to the easier case of continuous representations of $U(n)$). Although this is very classical, we give a brief proof. The *Cauchy transformation* $Z = C(A)$ of a matrix $A \in \mathrm{End}(\mathbf{C}^n)$ is defined by the formula

$$Z = (I + A)(I - A)^{-1}.$$

Its inverse is given by

$$A = (Z - I)(Z + I)^{-1}.$$

The Cauchy transformation is a birational correspondence which is well defined between a neighbourhood of $A = 0$ and a neighbourhood of $Z = I$ in $\mathbf{C}^{n \times n}$. When A is anti-Hermitian, Z is unitary, and conversely. As well as the Cauchy transformation, we shall use the invertible linear transformation $A = L(B)$ defined by

$$A = (B - {}^tB) + i\,(B + {}^tB)\,,$$

$$B = \frac{A - {}^tA}{2} + \frac{A + {}^tA}{2\,i}\,.$$

When B is real, A is anti-Hermitian, and conversely. By hypothesis, the function

$$Q(B) = P(C(L(B)))$$

is a rational function of B which vanishes identically when B is real in some neighbourhood of 0. This clearly implies that $Q = 0$ identically; and therefore $P = 0$ identically since

$$P(Z) = Q(L^{-1}(C^{-1}(Z)))\,. \quad \textbf{qed}$$

The Hopf algebra structure on $\mathbf{C}[(z_{ij}), d^{-1}]$ which corresponds to the one on $R(U(n))$ is easy to identify. The comultiplication is given by

$$\Delta z_{ij} = \Sigma_k\, z_{ik} \otimes z_{kj}$$

and the counit by

$$\varepsilon z_{ij} = \delta_{ij} \text{ (the Kronecker delta)}\,.$$

The antipode is given by

$$\nu z = z^{-1}$$

where $z = (z_{ij})$ and $\nu z = (\nu z_{ij})$. With this notation, we have

$$\Delta z = z \otimes z$$

where $\Delta z = (\Delta z_{ij})$ and $z \otimes z = (\Sigma_k\, z_{ik} \otimes z_{kj})$.

We should also identify the conjugation operation $(\overline{\ \ })$ on $\mathbf{C}[(z_{ij}), d^{-1}]$ which corresponds to the conjugation $f \longmapsto \overline{f}$ on $R(U(n))$. It is easy to see that $(\overline{\ \ })$ is the unique antilinear ring homomorphism such that

$$\overline{z} = {}^tz^{-1}$$

where $\overline{z} = (\overline{z}_{ij})$.

For any commutative $\mathbf{C}$-algebra A, we define its *spectrum* $\mathrm{Spec}(A)$ as the set of all algebra homomorphisms $\chi : A \longrightarrow \mathbf{C}$. We give $\mathrm{Spec}(A)$ the topology of pointwise convergence. When A is finitely generated, say $A = \mathbf{C}[a_1, \ldots, a_n]$, the mapping $\chi \longmapsto (\chi(a_1), \ldots, \chi(a_n))$ is a homeomorphism between $\mathrm{Spec}(A)$ and a closed algebraic subset of $\mathbf{C}^n$. When A is enriched with an antilinear involution $a \longmapsto \overline{a}$, we can define a conjugation operation on $\mathrm{Spec}(A)$ by putting, for all $a \in A$,

$$\overline{\chi}(a) = \overline{\chi(\overline{a})}\,.$$

The *real spectrum* $\mathrm{Spec}_{\mathbf{R}}(A)$ of A is the set of $\chi \in \mathrm{Spec}(A)$ such that $\overline{\chi} = \chi$.

The spectrum of a finitely generated commutative Hopf algebra is a complex algebraic group. If the Hopf algebra is enriched with an antilinear involution (respecting the Hopf algebra structure) then its real spectrum is a real algebraic group. In the example above we see that the spectrum of $R(U(n))$ is the algebraic group $GL(n, \mathbf{C})$ and its real spectrum is $U(n)$. This is a special case of a more general result. For any compact group G, consider the canonical map

$$G \longrightarrow \mathrm{Spec}_{\mathbf{R}}(R(G))$$

sending $x \in G$ into the homomorphism χ_x given by $\chi_x(f) = f(x)$.

Theorem 7. *For any compact group* G, *there is a canonical homeomorphism*

$$G \xrightarrow{\ \sim\ } \mathrm{Spec}_{\mathbf{R}}(R(G))\,.$$

The proof of the above Theorem 7 will be delayed until the next Section. It follows from Tannaka reconstruction. We note immediately this striking consequence:

Corollary 8. *Every compact Lie group is (real) algebraic.*

Proof. If G is a compact Lie group then, according to Corollary 5, the Hopf algebra $R(G)$ is finitely generated. So $\mathrm{Spec}(R(G))$ is therefore a complex algebraic group whose real part is equal to G. **qed**

To end this Section, we prove the following result needed in Section 8. Let M and M' be topological monoids. We have an obvious canonical map

$$i \;:\; R(M) \otimes R(M') \longrightarrow R(M \times M').$$

Proposition 9. *The canonical map* i *(above) is an isomorphism.*

Proof. Clearly i is injective. We prove it surjective. It suffices to prove that the matrix entries π_{rs} of any representation π_V of $M \times M'$ belongs to the image of i. We have

$$\pi_{rs}(x, y) \;=\; \Sigma_t\, \pi_{rt}(x, e)\, \pi_{ts}(e, y).$$

This shows that

$$\pi_{rs} \;=\; i\,(\Sigma_t\, \pi^1{}_{rt} \otimes \pi^2{}_{ts})$$

where $\pi^1{}_V(x) = \pi_V(x, e)$ and $\pi^2{}_V(y) = \pi_V(e, y)$. **qed**

§3. The Fourier cotransform.

In this Section we shall describe a transformation which has the Fourier transform as its dual. It is more fundamental than the Fourier transform in the sense that it behaves better algebraically. Starting with a category $\mathcal{C}$ and a functor $X : \mathcal{C} \longrightarrow \mathcal{V}ect_{\mathbf{C}}$ with values in finite dimensional vector spaces, we shall construct a certain vector space $\mathrm{End}^\vee(X)$ whose dual is the algebra $\mathrm{End}(X) = \mathrm{Hom}(X, X)$ of natural transformations from X to X. When $\mathcal{C} = \mathcal{R}ep(M, \mathbf{C})$ and $X = \mathbf{U}$ is the forgetful functor, the Fourier cotransform defines an isomorphism

$$\mathcal{F}^\vee \;:\; \mathrm{End}^\vee(\mathbf{U}) \xrightarrow{\;\sim\;} R(M)$$

whose transpose is the (generalised) Fourier transform. In Section 4 we shall show that $\mathrm{End}^\vee(\mathbf{U})$ supports a natural coalgebra structure and that $\mathcal{F}^\vee$ is a coalgebra isomorphism.

Let $\mathcal{C}$ be a small category. For any pair of functors $X, Y : \mathcal{C} \longrightarrow \mathcal{V}ect_{\mathbf{C}}$, let $\mathrm{Hom}(X, Y)$ be the set of natural transformations between X and Y. Now $\mathrm{Hom}(X, Y)$ is a vector space which can be described as follows. For any arrow $f : A \longrightarrow B$ in $\mathcal{C}$, let

$$p_f\,,\, q_f \;:\; \prod_{C \in \mathcal{C}} \mathrm{Hom}(X(C), Y(C)) \longrightarrow \mathrm{Hom}(X(A), Y(B))$$

be the maps out of the product whose values at $u \;=\; (\,u_C \mid C \in \mathcal{C}\,)$ are given by

$$p_f(u) \;=\; Y(f)\, u_A\;, \qquad\qquad q_f(u) \;=\; u_B\, X(f)\;.$$

Then clearly, $\mathrm{Hom}(X, Y)$ is the common equalizer of the pairs (p_f, q_f) :

$$u \in \mathrm{Hom}(X, Y) \qquad \Longleftrightarrow \qquad \text{for all } f \in \mathcal{C},\;\; p_f(u) \;=\; q_f(u)\;.$$

This construction can be dualised when $X(C)$ and $Y(C)$ are finite dimensional for every $C \in \mathcal{C}$ (actually we just need $Y(C)$ to be finite dimensional for every $C \in \mathcal{C}$, but here we shall not consider this more general situation). More precisely, we want to construct a vector space $\mathrm{Hom}^\vee(X,Y)$ whose dual will be $\mathrm{Hom}(X,Y)$ (it is therefore suggestive to think of $\mathrm{Hom}^\vee(X,Y)$ as a *predual* of $\mathrm{Hom}(X,Y)$). By definition, $\mathrm{Hom}^\vee(X,Y)$ is the common coequalizer of the maps

$$ {}^{\mathbf{t}}p_f \,,\, {}^{\mathbf{t}}q_f \;:\; \mathrm{Hom}(X(A),\, Y(B))^* \longrightarrow \sum_{C\,\in\,\mathcal{C}} \mathrm{Hom}(X(C),\, Y(C))^* $$

into the direct sum. But, for any pair of finite dimensional vector spaces V, W, we have canonical isomorphisms

$$ \mathrm{Hom}(V,\ W)^* \cong (V^*\otimes W)^* \cong W^*\otimes V \cong \mathrm{Hom}(W,\ V)\,. $$

The pairing between Hom(V, W) and Hom(W, V) is explicitly given by

$$ \langle S,\ T\rangle = \mathrm{Tr}(ST) = \mathrm{Tr}(TS)\,. $$

Taking this duality into consideration, we see that $\mathrm{Hom}^\vee(X,\ Y)$ can be defined as the common coequalizer of the maps

$$ i_f \,,\, j_f \;:\; \mathrm{Hom}(Y(B),\, X(A)) \longrightarrow \sum_{C\,\in\,\mathcal{C}} \mathrm{Hom}(Y(C),\, X(C)) $$

where, for any $S\in\mathrm{Hom}(Y(B),\ X(A))$,

$$ i_f(S) = (\,S\circ Y(f)\,,\,A)\,, \qquad j_f(S) = (\,X(f)\circ S\,,\,B)\,. $$

In this description, the second component of a pair $(S\circ Y(f), A)$ indicates to which component of the direct sum this element belongs. A handier description of $\mathrm{Hom}^\vee(X,\ Y)$ is the following. For any $C\in\mathcal{C}$ and any $S\in\mathrm{Hom}(Y(C),\ X(C))$, let us write $[S]$ for the image of S under the canonical map

$$ \mathrm{Hom}(Y(C),\ X(C)) \longrightarrow \mathrm{Hom}^\vee(X,\ Y)\,. $$

We have that $\mathrm{Hom}^\vee(X,\ Y)$ is generated as a vector space by the symbols $[S]$ subject to the relations

(i) $[\alpha S+\beta T] = \alpha[S]+\beta[T]$ for $S, T\in\mathrm{Hom}(Y(C),\ X(C))$,

(ii) $[S\circ Y(f)] = [X(f)\circ S]$ for $f: A \longrightarrow B$ and $S\in\mathrm{Hom}(Y(B),\ X(A))$.

It is clear by construction that $\mathrm{Hom}(X,\ Y)$ is the linear dual of $\mathrm{Hom}^\vee(X,\ Y)$. The explicit pairing is

$$ \mathrm{Hom}\,(X, Y)\otimes\mathrm{Hom}^\vee(X,\ Y) \longrightarrow \mathbf{C}\,, \qquad \langle u, [S]\rangle = \mathrm{Tr}(u_C\circ S) $$

where $S\in\mathrm{Hom}(Y(C),\ X(C))$.

Before continuing the general study of $\mathrm{Hom}^\vee(X,\ Y)$, let us compute $\mathrm{Hom}^\vee(X, X) = \mathrm{End}^\vee(X)$ in the special case where $\mathcal{C} = \mathcal{Rep}(M,\ \mathbf{C})$ and $X=\mathbf{U}$ is the forgetful functor. We define the *Fourier cotransform*

$$ \mathcal{F}^\vee \;:\; \mathrm{End}^\vee(\mathbf{U}) \longrightarrow R(M), $$

as follows:

$$ \mathcal{F}^\vee(z)\,(x) = \langle \pi\,(x)\,,\, z\rangle\,. $$

More explicitly, if $z = [A]$ where $A\in\mathrm{End}(V)$ and $\pi_V: M\longrightarrow\mathrm{End}(V)$ then

$$ \mathcal{F}^\vee([A])\,(x) = \langle \pi\,(x)\,,\,[A]\rangle = \mathrm{Tr}(\pi_V(x)\,A)\,. $$

Theorem 1. *For any topological monoid* M, *the Fourier cotransform is an isomorphism*

$$\mathcal{F}^\vee \ : \ \mathrm{End}^\vee(\mathbf{U}) \xrightarrow{\ \sim\ } R(M) \ .$$

To prove Theorem 1, we shall need two Lemmas.

Lemma 2. *If* $V_1, V_2 \in \mathcal{Rep}(M, \mathbf{C})$, *and if* $A \in \mathrm{End}(V_1 \oplus V_2)$ *is written as*

$$A \ = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

then

$$[A] = [A_{11}] + [A_{22}].$$

Proof. Let $j_i : V_i \longrightarrow V_1 \oplus V_2$ and $p_i : V_1 \oplus V_2 \longrightarrow V_i$ $(i = 1, 2)$ be the canonical inclusions and projections. We have

$$\begin{aligned} [A] &= [A\,(j_1 p_1 + j_2 p_2)] \\ &= [A\, j_1 p_1] + [A\, j_2 p_2] \\ &= [p_1 A\, j_1] + [p_2 A\, j_2] \\ &= [A_{11}] + [A_{22}]. \end{aligned}$$ **qed**

Lemma 3. *Any* $z \in \mathrm{End}^\vee(\mathbf{U})$ *can be represented as* $z = [\phi \otimes v]$ *for some* $v \in V$ *and* $\phi \in V^*$ *where* $\pi_V : M \longrightarrow \mathrm{End}(V)$. *Moreover, this can be done in such a way that* v *generates* V *as an* M-*module.*

Proof. Each element $z \in \mathrm{End}^\vee(\mathbf{U})$ can be represented as $z = [A]$ for some $A \in \mathrm{End}(W)$ and $W \in \mathcal{Rep}(M, \mathbf{C})$. But we can express A as a sum of matrices of rank one:

$$A \ = \sum_{i \in I} \phi_i \otimes v_i \ ;$$

so that,

$$z \ = \sum_{i \in I} [\phi_i \otimes v_i] \ .$$

Let V be the representation obtained as the direct sum of I copies of W, and let

$$v \ = (v_i \mid i \in I) \ , \qquad\qquad \phi \ = \sum_{i \in I} \phi_i p_i$$

where $p_i : V \longrightarrow W$ is the i-th projection. Using Lemma 2, we see that $z = [\phi \otimes v]$.

To prove the second sentence, let V' be the submodule of V generated by v. The inclusion $j : V' \longrightarrow V$ is an arrow in the category $\mathcal{Rep}(M, \mathbf{C})$. Let B be the composite

$$V \xrightarrow{\ \phi\ } \mathbf{C} \xrightarrow{\ v\ } V' \ .$$

Then we have

$$[\phi \otimes v] = [j B] = [B j] = [(\phi \mid V') \otimes v].$$ **qed**

Proof of Theorem 1. The surjectivity of $\mathcal{F}^\vee$ is immediate from the definition of representative function. Suppose that $\mathcal{F}^\vee(z) = 0$. According to Lemma 3, we can suppose that $z = [\phi \otimes v]$ and that v generates the M-module V. Then we have

$$\phi\,(\pi_V(x)(v)) \ = \ \mathcal{F}^\vee(z)(x) \ = \ 0$$

for every $x \in M$. This shows that the submodule generated by v is contained in $\ker(\phi)$. Therefore $\phi = 0$ and $z = 0$. **qed**

At this point it might be worthwhile to explain why we call $\mathcal{F}^\vee$ the "Fourier cotransform". Suppose G is a compact group. The continuous dual of the Banach space $\mathcal{C}(G, \mathbf{C})$ is the space $\mathcal{M}(G, \mathbf{C})$ of bounded measures on G. The Fourier transform of $\mu \in \mathcal{M}(G, \mathbf{C})$ is the element $\mathcal{F}\mu \in \mathrm{End}(\mathbf{U})$ given by

$$(\mathcal{F}\mu)_V = \int_G \pi_V(x)\, d\mu(x)\,.$$

The Fourier transform of a function $f \in \mathcal{C}(G, \mathbf{C})$ is simply the Fourier transform of $f\,dx$ where dx denotes the Haar measure on G.

Proposition 4. *If* $z \in \mathrm{End}^\vee(\mathbf{U})$ and $\mu \in \mathcal{M}(G, \mathbf{C})$ *then*

$$\langle \mathcal{F}\mu , z \rangle = \langle \mu , \mathcal{F}^\vee z \rangle.$$

Proof. If $z = [A]$ where $A \in \mathrm{End}(V)$ and $\pi_V : G \longrightarrow \mathrm{End}(V)$ then we have

$$\begin{aligned} \langle \mathcal{F}\mu , [A] \rangle &= \mathrm{Tr}\,((\mathcal{F}\mu)_V A) \\ &= \mathrm{Tr}\,(\int_G \pi_V(x)\, A\, d\mu(x)) \\ &= \int_G \mathrm{Tr}\,(\pi_V(x)\, A)\, d\mu(x) \\ &= \int_G \mathcal{F}^\vee([A])(x)\, d\mu(x) \\ &= \langle \mu, \mathcal{F}^\vee([A]) \rangle. \end{aligned}$$ **q e d**

The above Proposition 4 shows that the Fourier transform

$$\mathcal{F} : \mathcal{M}(G, \mathbf{C}) \longrightarrow \mathrm{End}(\mathbf{U})$$

is the continuous dual of the linear map

$$\mathcal{F}^\vee : \mathrm{End}^\vee(\mathbf{U}) \longrightarrow R(G) \subset \mathcal{C}(G, \mathbf{C}).$$

At this point, it becomes possible to extend the domain of the Fourier transform. We have a canonical inclusion

$$\mathcal{M}(G, \mathbf{C}) \subset R(G)^*$$

since R(G) is dense in the Banach space $\mathcal{C}(G, \mathbf{C})$ (Section 1 Corollary 13). We shall extend $\mathcal{F}$ to the full linear dual of R(G):

$$\mathcal{F} : R(G)^* \longrightarrow \mathrm{End}(\mathbf{U}).$$

Let us write $\langle h, f \rangle$ for the evaluation pairing between $R(G)^*$ and R(G). The Fourier transform of $h \in R(G)^*$ can be defined by the formula

$$(\mathcal{F}h)_V = \langle h , \pi_V \rangle$$

by which we mean that, if (π_{ij}) is the matrix of π_V for some basis of V then $(\langle h, \pi_{ij} \rangle)$ is the matrix of $(\mathcal{F}h)_V$ for the same basis.

One might think of the elements of $R(G)^*$ as generalised distributions on G. More precisely, when G is a compact Lie group, it can be proved that R(G) is a dense subspace of the space $\mathcal{C}^\infty(G, \mathbf{C})$ of smooth functions with the smooth topology [Sc]. The continuous dual of $\mathcal{C}^\infty(G, \mathbf{C})$ is the space $\mathcal{D}(G, \mathbf{C})$ of distributions on G. We thus have a canonical inclusion

$$\mathcal{D}(G, \mathbf{C}) \subset R(G)^*,$$

showing that the Fourier transform is defined on distributions. Recall that there is an algebra structure on $\mathcal{D}(G, \mathbf{C})$ given by the convolution product of distributions. There is also an algebra structure on $R(G)^*$ (and more generally on each $R(M)^*$) which is the dual of the coalgebra structure on $R(G)$. It is easy to verify that this product on $R(G)^*$ extends the convolution product of distributions. In particular, the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of the group G is a subalgebra of $R(G)^*$, since it is equal to the subalgebra of $\mathcal{D}(G, \mathbf{C})$ consisting of distributions whose supports are concentrated at the unit element $e \in G$. The Lie algebra $\mathfrak{g} \subset U(\mathfrak{g}) \subset R(G)^*$ corresponds to the ε-derivations

$$D : R(G) \longrightarrow \mathbf{C};$$

that is, the linear maps such that

$$D(f\,g) = D(f)\,\varepsilon(g) + \varepsilon(f)\,D(g)$$

where $\varepsilon : R(G) \longrightarrow \mathbf{C}$ is evaluation at $e \in G$.

In defining the generalised Fourier transform there was no need to restrict to compact Lie groups. Quite generally, we have:

Proposition 5. *For any topological monoid* M, *the (generalised) Fourier transform is an isomorphism of topological algebras*

$$\mathcal{F} : R(M)^* \xrightarrow{\ \sim\ } \mathrm{End}(\mathbf{U}).$$

Proof. As in the proof of Proposition 4, it can be shown that, for any $z \in \mathrm{End}^{\vee}(\mathbf{U})$ and $h \in R(M)^*$, we have

$$\langle \mathcal{F}h, z \rangle = \langle h, \mathcal{F}^{\vee} z \rangle.$$

This proves that $\mathcal{F}$ is an isomorphism since it is the transpose of $\mathcal{F}^{\vee}$ and we have Theorem 1. Also this shows that $\mathcal{F}$ is bicontinuous since the topology on both sides is the usual topology for linear duals (namely, pointwise convergence for linear functionals). To finish the proof, we must verify that $\mathcal{F}$ is an algebra homomorphism. By definition of the convolution product, we have, for all $h, k \in R(M)^*$ and $f \in R(M)$,

$$\langle h * k, f \rangle = \sum_{r=1}^{n} \langle h, f_r \rangle \langle k, g_r \rangle$$

where

$$\Delta f = \sum_{r=1}^{n} f_r \otimes g_r .$$

If we apply this formula to the entries π_{ij} of the matrix of a representation π_V of M (relative to some basis of V), we obtain

$$\langle h * k, \pi_{ij} \rangle = \sum_{r=1}^{n} \langle h, \pi_{ir} \rangle \langle k, \pi_{rj} \rangle ,$$

which means exactly that

$$\mathcal{F}(h * k)_V = (\mathcal{F}h)_V (\mathcal{F}k)_V \,.\ \textbf{qed}$$

When G is a compact group, we can combine the last Proposition 5 with Section 1 Proposition 4 to obtain:

Corollary 6. *For any compact group* G, *the Fourier transform defines an isomorphism of* *topological algebras*

$$\mathcal{F} : R(G)^* \xrightarrow{\ \sim\ } \prod_{\lambda \in G^\vee} \mathrm{End}(V_\lambda)\ .$$

We now return to the more general situation of a topological monoid M. Recall that we have a canonical map

$$j\ :\ M \longrightarrow R(M)^*$$

defined by $\langle j(x), f \rangle = f(x)$. We might say that $j(x)$ is the Dirac measure concentrated at $x \in M$. We have a commutative triangle

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[dr, "\pi"'] & R(M)^* \arrow[d, "\mathcal{F}"] \\ & \mathrm{End}(\mathbf{U}) \end{tikzcd}$$

since, for every $V \in \mathcal{R}ep(M, \mathbf{C})$,

$$\mathcal{F}(j(x))_V = \langle j(x), \pi_V \rangle = \pi_V(x) = \pi(x)_V\ .$$

Let $\mathrm{End}^{\otimes}(\mathbf{U})$ denote the submonoid of $\mathrm{End}(\mathbf{U})$ consisting of the tensor-preserving natural transformations (the product is composition). The Tannaka monoid $\mathcal{T}(M)$ of M is the submonoid of $\mathrm{End}^{\otimes}(\mathbf{U})$ consisting of the self-conjugate natural transformations.

Proposition 7. *The Fourier transform $\mathcal{F}$ induces the following isomorphisms of topological monoids:*

$$\mathrm{Spec}(R(M)) \xrightarrow{\ \sim\ } \mathrm{End}^{\otimes}(\mathbf{U}), \qquad \mathrm{Spec}_{\mathbf{R}}(R(M)) \xrightarrow{\ \sim\ } \mathcal{T}(M).$$

Moreover, the following triangle commutes.

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[dr, "\pi"'] & \mathrm{Spec}_{\mathbf{R}}(R(M)) \arrow[d, "\wr"'] \\ & \mathcal{T}(M) \end{tikzcd}$$

Proof. For any representations $V, W \in \mathcal{R}ep(M, \mathbf{C})$, let (π_{ij}) and (ρ_{rs}) be the matrices of π_V and π_W in some basis of V and W, respectively. If we express the equality

$$(\mathcal{F}k)_{V \otimes W} = (\mathcal{F}k)_V \otimes (\mathcal{F}k)_W$$

in matrix form, we obtain the equality

$$\langle k, \pi_{ij}\, \rho_{rs} \rangle = \langle k, \pi_{ij} \rangle \langle k, \rho_{rs} \rangle$$

for all i, j, r, s. This proves that k is an algebra homomorphism if and only if $\mathcal{F}k \in \mathrm{End}^{\otimes}(\mathbf{U})$ (we are using here the fact that $R(M)$ is the linear span of the matrix entries of the representations of M). This establishes the first bijection. The second bijection is a consequence of the formula

$$\mathcal{F}\bar{k} = \overline{\mathcal{F}k}$$

where the conjugate $\bar{k}$ of an element $k \in R(M)^*$ is defined by

$$\langle \bar{k}, f \rangle = \overline{\langle k, \bar{f} \rangle}\ .$$

Bicontinuity of the bijections follows from that of the Fourier transform. qed

Corollary 8. *For any compact group* G*, there is a commutative triangle of compact group isomorphisms:*

$$\begin{tikzcd} G \arrow[rr, "\sim", "j"'] \arrow[dr, "\pi"', "\sim"] & & \mathrm{Spec}_{\mathbf{R}}(R(G)) \arrow[dl, "\mathcal{F}", "\sim"'] \\ & \mathcal{T}(G) & \end{tikzcd}$$

Proof. According to Tannaka-Krein (Theorem 20), π is an isomorphism. Hence, $j = \mathcal{F}^{-1} \circ \pi$ is also an isomorphism. $_{\mathbf{qed}}$

§4. The coalgebra $\mathrm{End}^\vee(X)$.

In this Section, we shall introduce a coalgebra structure on $\mathrm{End}^\vee(X)$ for any functor $X : \mathcal{C} \longrightarrow \mathcal{V}ect_{\mathbf{C}}$ whose values are finite dimensional vector spaces. More generally, under the same hypotheses on functors Y, Z, we shall describe a map

$$\Delta : \mathrm{Hom}^\vee(X, Z) \longrightarrow \mathrm{Hom}^\vee(Y, Z) \otimes \mathrm{Hom}^\vee(X, Y)$$

which dualises the usual composition map. We shall also show that there is a coaction

$$\gamma : X \longrightarrow Y \otimes \mathrm{Hom}^\vee(X, Y)$$

dualising the usual evaluation action

$$\mathrm{Hom}(X, Y) \otimes X \longrightarrow Y.$$

We begin by giving another description of $\mathrm{Hom}^\vee(X, Y)$ useful for many purposes. It is based on the concept of *tensor product* [F] of a contravariant functor $S : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{V}ect_{\mathbf{C}}$ with a covariant functor $T : \mathcal{C} \longrightarrow \mathcal{V}ect_{\mathbf{C}}$. The behaviour of this tensor product is similar to that of the tensor product of right and left modules. To stress this analogy, we write *on the right* the action of $f : A \longrightarrow B$ on an element $x \in S(B)$:

$$S(f)(x) = x \bullet f$$

and similarly, we shall write *on the left* the action of f on $y \in T(A)$:

$$T(f)(y) = f \bullet y .$$

The required tensor product is based on the notion of *bilinear pairing*

$$q : S \times T \longrightarrow V$$

where V is a vector space. This q is a family $(q_C \mid C \in \mathcal{C})$ of bilinear pairings

$$q_C : S(C) \times T(C) \longrightarrow V$$

which respect the actions of S and T. More precisely, it is required that we have

$$q_A(x \bullet f , y) = q_B(x , f \bullet y)$$

for all $f : A \longrightarrow B$ in $\mathcal{C}$, $x \in S(B)$, $y \in T(A)$. The universal recipient of such a pairing is a vector space called the *tensor product* over $\mathcal{C}$ of S with T, and is denoted by $S \otimes_{\mathcal{C}} T$. We now formulate one of its fundamental properties. Note that a family $(q_C \mid C \in \mathcal{C})$ of bilinear pairings

$$q_C : S(C) \times T(C) \longrightarrow V$$

corresponds to a family $(q'_C \mid C \in \mathcal{C})$ of linear maps

$$q'_C : T(C) \longrightarrow \mathrm{Hom}(S(C), V) .$$

Let us denote by $\mathrm{Hom}(S, V)$ the *covariant* functor whose value at $C \in \mathcal{C}$ is the vector space $\mathrm{Hom}(S(C), V)$.

Proposition 1. *The correspondence described above determines a bijection between linear* *maps* $S \otimes_{\mathcal{C}} T \longrightarrow V$ *and natural transformations* $T \longrightarrow \mathrm{Hom}(S, V)$.

Suppose now that the functors $X, Y : \mathcal{C} \longrightarrow \mathcal{V}ect_{\mathbb{C}}$ take their values in finite dimensional vector spaces. For any object $C \in \mathcal{C}$, the map

$$[\ ,\ ]\ :\ Y(C)^* \otimes X(C) \longrightarrow \mathrm{Hom}^{\vee}(X, Y)\,, \qquad [\phi, x] = [\phi \otimes x]$$

is a bilinear pairing between the contravariant functor Y^* and the covariant functor X since we have, for all $f : A \longrightarrow B$ in $\mathcal{C}$, $x \in X(A)$, $\phi \in Y(B)^*$,

$$[\phi\, Y(f) \otimes x] = [\phi \otimes X(f)\, x].$$

Using this pairing, we obtain a canonical map

$$Y^* \otimes_{\mathcal{C}} X \longrightarrow \mathrm{Hom}^{\vee}(X, Y).$$

The verification of the next result is left to the reader.

Proposition 2. *The canonical map described above is an isomorphism*

$$Y^* \otimes_{\mathcal{C}} X \xrightarrow{\ \sim\ } \mathrm{Hom}^{\vee}(X, Y)\,.$$

We can now apply Proposition 1 to this situation using the canonical isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(Y^*, V) \xrightarrow{\ \sim\ } Y \otimes V\,.$$

We see that there is a bijection between the linear maps $\mathrm{Hom}^{\vee}(X, Y) \longrightarrow V$ and the linear maps $X \longrightarrow Y \otimes V$. To describe this bijection explicitly, we need the natural transformation

$$\gamma\ :\ X \longrightarrow Y \otimes \mathrm{Hom}^{\vee}(X, Y)$$

defined as follows. For any $C \in \mathcal{C}$ and any basis $e_1, \ldots, e_n$ of $Y(C)$, we put

$$\gamma_C(x) \;=\; \sum_{i=1}^{n} e_i \otimes [e_i^* \otimes x]\,.$$

We leave to the reader the verification that γ is natural.

$$\begin{tikzcd}
X \arrow[r, "\gamma"] \arrow[dr, "n"'] & Y \otimes \mathrm{Hom}^{\vee}(X, Y) \arrow[d, "1 \otimes \tilde{n}"] \\
& Y \otimes V
\end{tikzcd}$$

Proposition 3. *For any natural transformation* $n : X \longrightarrow Y \otimes V$, *there exists a unique linear map* $\tilde{n}\ :\ \mathrm{Hom}^{\vee}(X, Y) \longrightarrow V$ *such that the above triangle commutes.*

We can now describe the map

$$\Delta\ :\ \mathrm{Hom}^{\vee}(X, Z) \longrightarrow \mathrm{Hom}^{\vee}(Y, Z) \otimes \mathrm{Hom}^{\vee}(X, Y)$$

whose dual is the usual composition map

$$\mathrm{Hom}(Y, Z) \otimes \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(X, Z)\,.$$

We have $\Delta = \tilde{n}$ where n is the composite

$$X \xrightarrow{\ \gamma\ } Y \otimes \mathrm{Hom}^{\vee}(X, Y) \xrightarrow{\ \gamma \otimes 1\ } Z \otimes \mathrm{Hom}^{\vee}(Y, Z) \otimes \mathrm{Hom}^{\vee}(X, Y)\,.$$

A short calculation gives

$$\Delta\,[\phi\otimes x] = \sum_{i=1}^{n} [\phi\otimes y_i]\otimes[y_i^*\otimes x]$$

where $x\in X(C), \phi\in Z(C)^*$ and $\{y_1, \ldots, y_n\}, \{y_1^*, \ldots, y_n^*\}$ are dual bases of $Y(C), Y(C)^*$. In particular, Δ defines a coalgebra structure on $\mathrm{End}^\vee(X) = \mathrm{Hom}^\vee(X, X)$:

$$\Delta \;:\; \mathrm{End}^\vee(X) \longrightarrow \mathrm{End}^\vee(X)\otimes \mathrm{End}^\vee(X)\,.$$

To compute Δ explicitly, recall that, for any vector space V, there is a coalgebra structure on $\mathrm{End}(V)$ since $\mathrm{End}(V)$ is self dual and is itself an algebra. If $e_1, \ldots, e_n$ is a basis of V then the coalgebra structure $\delta : \mathrm{End}(V) \longrightarrow \mathrm{End}(V)\otimes\mathrm{End}(V)$ is given by

$$\delta(e_{ij}) = \sum_{k=1}^{n} e_{ik}\otimes e_{kj}$$

where $e_{ij} = e_i^*\otimes e_j$; the counit is the trace map $\mathrm{Tr} : \mathrm{End}(V) \longrightarrow \mathbf{C}$. The canonical map

$$\sum_{C\in \mathcal{C}} \mathrm{End}(X(C)) \longrightarrow \mathrm{End}^\vee(X)$$

expresses the coalgebra $\mathrm{End}^\vee(X)$ as a quotient of the direct sum of coalgebras. This implies that we have

$$\Delta[e_{ij}] = \sum_{k=1}^{n} [e_{ik}]\otimes[e_{kj}] \quad ,$$

and that the counit $\varepsilon : \mathrm{End}^\vee(X) \longrightarrow \mathbf{C}$ is given by $\varepsilon\,[A] = \mathrm{Tr}\,A$.

The following result supplements Section 3 Theorem 1.

Proposition 4. *For any topological monoid* M, *the Fourier cotransform*

$$\mathcal{F}^\vee \;:\; \mathrm{End}^\vee(\mathbf{U}) \longrightarrow R(M)$$

is an isomorphism of coalgebras.

Proof. Let $V\in \mathcal{R}ep(M, \mathbf{C})$ and take $A\in\mathrm{End}(V)$. The calculation

$$\varepsilon\, \mathcal{F}^\vee[A] = \mathrm{Tr}(\pi_V(e)A) = \mathrm{Tr}(A) = \varepsilon[A]$$

shows that $\mathcal{F}^\vee$ preserves counits. If $e_1, \ldots, e_n$ is a basis of V then we have

$$\mathcal{F}^\vee([e_{ij}])(x) = \mathrm{Tr}(\pi_V(x)\,e_{ij}) = e_i^*(\pi_V(x)\,e_j) = \pi_{ij}(x)\,.$$

This shows that $\mathcal{F}^\vee$ preserves Δ since

$$\Delta\pi_{ij} = \sum_{k=1}^{n} \pi_{ik}\otimes\pi_{kj}\,, \quad \Delta[e_{ij}] = \sum_{k=1}^{n} [e_{ik}]\otimes[e_{kj}]\,,$$

and $\mathrm{End}^\vee(\mathbf{U})$ is the linear span of the elements $[e_{ij}]$ as V runs over the representations of M. **qed**

When G is a compact group, we have a canonical map of coalgebras

$$i \;:\; \sum_{\lambda\in G^\vee} \mathrm{End}(V_\lambda) \longrightarrow \mathrm{End}^\vee(\mathbf{U})\,.$$

It is an isomorphism since ${}^t i$ is the isomorphism q of Section 1 Proposition 4. We shall use this map to identify $\mathrm{End}^\vee(\mathbf{U})$ with $\Sigma_\lambda \mathrm{End}(V_\lambda)$. With this convention, the following result is the dual of Section 3 Corollary 6.

Corollary 5. *For any compact group* G, *the Fourier cotransform provides an isomorphism of coalgebras:*

$$\mathcal{F}^{\vee} : \quad \sum_{\lambda \in G^{\vee}} \mathrm{End}(V_{\lambda}) \xrightarrow{\ \sim\ } R(G) \quad .$$

It is of interest to see the description of the inverse cotransform

$$\mathcal{F}^{\vee\,-1} : \ R(G) \longrightarrow \sum_{\lambda \in G^{\vee}} \mathrm{End}(V_{\lambda}) \ .$$

The orthogonality relations imply that, for any $f \in R(G)$, we have

$$\mathcal{F}^{\vee\,-1}(f)(\lambda) \ = \ d_{\lambda} \int_{G} \pi_{\lambda}(x)^{-1} \ f(x)\, dx \ .$$

Recall that a (right) comodule over a coalgebra E is a vector space V equipped with a coaction

$$\alpha \ : \ V \longrightarrow V \otimes E$$

which is associative and unitary; that is, the following diagrams commute.

$$\begin{tikzcd}
V \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & V \otimes E \arrow[d, "\alpha \otimes E"] \\
V \otimes E \arrow[r, "V \otimes \delta"'] & V \otimes E \otimes E
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
V \arrow[r, "\alpha"] \arrow[dr, equal] & V \otimes E \arrow[d, "V \otimes \varepsilon"] \\
& V
\end{tikzcd}$$

When V is finite dimensional, there is a bijection between comodule structures $\alpha : V \longrightarrow V \otimes E$ and coalgebra maps

$$\tilde{\alpha} : \mathrm{End}(V) \longrightarrow E \ .$$

If $e_1, \ldots, e_n$ is a basis for V then we have

$$\alpha(e_i) = \sum_{j=1}^{n} e_j \otimes \alpha_{ij}$$

where (α_{ij}) is a matrix with entries in E. Putting $e_{ij} = e_i^{*} \otimes e_j$, we then see that the coalgebra map determined by α is given by the equations

$$\tilde{\alpha}\,(e_{ij}) \ = \ \alpha_{ij} \ .$$

We have the formulas

$$\tilde{\alpha}\,(\phi \otimes v) \ = \ (\phi \otimes 1)\,\alpha(v) \ , \qquad \alpha(v) \ = \ \sum_{i=1}^{n} e_i \otimes \tilde{\alpha}\,(e_i^{*} \otimes v)$$

valid for all $v \in V$ and $\phi \in V^{*}$. Expressed in terms of the matrix (α_{ij}), the associativity and unitarity conditions are very simply:

$$\delta\,\alpha \ = \ \alpha \,\underline{\otimes}\, \alpha \qquad \text{and} \qquad \varepsilon\,\alpha \ = \ \mathrm{id}$$

where

$$\delta\,(\alpha_{ij}) \ = \ (\,\delta\,\alpha_{ij}\,)\,, \quad (\alpha_{ij}) \,\underline{\otimes}\, (\alpha_{ij}) \ = \ (\,\textstyle\sum_k \alpha_{ik} \otimes \alpha_{kj}\,)\,, \quad \varepsilon\,(\alpha_{ij}) \ = \ (\,\varepsilon\,\alpha_{ij}\,)\,, \quad \mathrm{id} \ = \ (\delta_{ij})\,.$$

The vector space V is canonically a comodule over the coalgebra $\mathrm{End}(V)$ via the coaction

$$c : V \longrightarrow V \otimes \mathrm{End}(V), \qquad c(x) = \Sigma_i\ e_i \otimes (e_i^* \otimes x);$$

the corresponding coalgebra map

$$\tilde{c} : \mathrm{End}(V) \longrightarrow \mathrm{End}(V)$$

is the identity map of End(V).

Warning. The dual E^* of a coalgebra is naturally an algebra. However, the reader should note that a right comodule over E has a natural structure of a *left* E^*-module (but not of a *right* E^*-module). If $\alpha : V \longrightarrow V \otimes E$ is the coaction then, for all $\phi \in V^*$ and $v \in V$, we have

$$\phi \bullet v = (1 \otimes \phi)\, \alpha(v).$$

A *morphism* $f : (V, \alpha) \longrightarrow (W, \beta)$ of E-comodules is a linear mapping $f : V \longrightarrow W$ such that the following square commutes.

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "f"'] & V \otimes E \arrow[d, "f \otimes E"] \\ W \arrow[r, "\beta"'] & W \otimes E \end{tikzcd}$$

The category of (right) E-comodules will be designated by $\mathcal{Comod}$E, and its full subcategory of finite dimensional comodules will be denoted by $\mathcal{Comod}_f$E.

It is instructive to compute the finite dimensional comodules over the coalgebra R(M) for any topological monoid M. First, for any $V \in \mathcal{Rep}(M, \mathbf{C})$, if we compose the map $\gamma_V : V \longrightarrow V \otimes \mathrm{End}^\vee(\mathbf{U})$ with the Fourier cotransform, we obtain a map $c_V : V \longrightarrow V \otimes R(M)$ which is a coaction. To see this, pick a basis $e_1, \dots, e_n$ of V. Then a short computation gives

$$c_V(e_i) = \sum_{j=1}^{n} e_j \otimes \pi_{ij}$$

where (π_{ij}) is the matrix of π_V in the basis $e_1, \dots, e_n$. The identities

$$\pi_{ij}(x\,y) = \sum_{k=1}^{n} \pi_{ik}(x)\, \pi_{kj}(y), \qquad \pi_{ij}(e) = \delta_{ij}$$

mean that $\delta(\pi) = \pi \underline{\otimes} \pi$, $\varepsilon(\pi) = \mathrm{id}$, and therefore that c_V is a coaction. We have defined a functor

$$\mathbf{U}^\sim : \mathcal{Rep}(M, \mathbf{C}) \longrightarrow \mathcal{Comod}_f R(M).$$

Proposition 6. *For any topological monoid* M, *the functor* $\mathbf{U}^\sim$ *is an equivalence of categories.*

Proof. It suffices to describe an inverse $\mathbf{U}^{\sim -1}$. Let (V, α) be a finite dimensional comodule over R(M). We have

$$\alpha(e_i) = \sum_{j=1}^{n} e_j \otimes \alpha_{ij}$$

where $e_1, \dots, e_n$ is a basis for V. The relations $\delta(\alpha) = \alpha \underline{\otimes} \alpha$, $\varepsilon(\alpha) = \mathrm{id}$ mean that the matrix (α_{ij}) determines a representation π of M which is continuous since the functions α_{ij} are. We put $\mathbf{U}^{\sim -1}(V, \alpha) = (V, \pi)$. $_{\mathbf{qed}}$

Let $\mathcal{C}$ be a category. An E-*comodule structure* on a functor $X : \mathcal{C} \longrightarrow \mathcal{V}ect_{\mathbb{C}}$ is a natural transformation

$$\alpha : X \longrightarrow X \otimes E$$

such that α_C is a comodule structure on $X(C)$ for all $C \in \mathcal{C}$. Using α, we obtain a functor

$$(X, \alpha) : \mathcal{C} \longrightarrow \mathcal{C}omod(E)$$

by putting $(X, \alpha)(C) = (X(C), \alpha_C)$. This functor fits into a commutative triangle

$$\begin{tikzcd} & \mathcal{C}omod(E) \arrow[d, "U"] \\ \mathcal{C} \arrow[ur, "(X,\alpha)"] \arrow[r, "X"'] & \mathcal{V}ect_{\mathbb{C}} \end{tikzcd}$$

where $\mathbf{U}$ is the forgetful functor. What is more, it is clear that any functor $X^{\sim} : \mathcal{C} \longrightarrow \mathcal{C}omod(E)$ such that $\mathbf{U}\,X^{\sim} = X$ is of the form (X, α) for a unique comodule structure α on X. When X takes its values in finite dimensional vector spaces, the natural transformation $\gamma : X \longrightarrow X \otimes End^{\vee}(X)$ defines an $End^{\vee}(X)$-comodule structure on X. This follows from the commutativity of the triangles

$$\begin{tikzcd} X(C) \arrow[rr, "\gamma_C"] \arrow[dr, "c"'] & & X(C) \otimes End^{\vee}(X) \\ & X(C) \otimes End(X(C)) \arrow[ur, "X(C) \otimes [\ ]"'] & \end{tikzcd}$$

where c is the $End(X(C))$-comodule structure on $X(C)$ and where $[\ \ ] : End(X(C)) \longrightarrow End^{\vee}(X)$ is the canonical morphism of coalgebras.

Proposition 7. *Let* $X : \mathcal{C} \longrightarrow \mathcal{V}ect_{\mathbb{C}}$ *be a functor whose values are finite dimensional vector spaces. For any coalgebra* E *and any comodule structure* $n : X \longrightarrow X \otimes E$*, there is precisely one morphism* $\tilde{n} : End^{\vee}(X) \longrightarrow E$ *of coalgebras such that the following triangle commutes.*

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[dr, "n"'] & & X \otimes End^{\vee}(X) \arrow[dl, "X \otimes \tilde{n}"] \\ & X \otimes E & \end{tikzcd}$$

Proof. To prove this Proposition, it suffices to verify that the linear map $\tilde{n}$, whose existence and uniqueness are assured by Proposition 3, is a map of coalgebras. The verific-ation is left to the reader. $_{\mathbf{qed}}$

§5. Tannaka duality for homogeneous spaces.

In this part we show how Tannaka duality can be used to obtain results on homogeneous spaces. The proof given here is independent of the proof in Section 1. As in Section 2, the results proved here can be used to show that any homogeneous space over a compact Lie group is a (real) algebraic variety. This is the basis for the construction of quantum homogeneous spaces such as quantum spheres [Pd].

Let G be a compact group. By a *(left)* G-*space* we mean a topological space X equipped with a continuous (left) action $G \times X \longrightarrow X$. A G-space is *homogeneous* when it is non-empty and the action is transitive. For each closed subgroup $H \subset G$, the space G/H of orbits for the action of H on the right of G is a homogeneous G-space. If $K \subset G$ is another subgroup, we shall write $K \backslash G/H$ for the space of orbits for the action of K on the left of G/H.

Let X be a homogeneous G-space. For any representation V of G, we shall write V^X for the vector space $\mathrm{Hom}_G(X, V)$ of G-invariant maps from X to V. If we choose a basepoint $x_0 \in X$ and H is the stabilizer of x_0, then the map $g \longmapsto g x_0$ induces an isomorphism between G/H and X, while the map $a \longmapsto a(x_0)$ defines an isomorphism between V^X and the subspace $V^H \subset V$ consisting of the H-invariant vectors. For all $V \in \mathcal{R}ep(G, \mathbf{C})$, putting

$$\mathbf{U}^X(V) = V^X \quad \text{and} \quad \mathbf{U}^H(V) = V^H,$$

we obtain two isomorphic functors

$$\mathbf{U}^X,\ \mathbf{U}^H : \mathcal{R}ep(G, \mathbf{C}) \longrightarrow \mathcal{V}ect_{\mathbf{C}}\,.$$

Let Y be another homogeneous G-space. We define the *Fourier transform* of a G-invariant function $f \in \mathcal{C}(Y \times X, \mathbf{C})$ to be the natural transformation

$$\mathcal{F}f : \mathbf{U}^X \longrightarrow \mathbf{U}^Y$$

whose value at $V \in \mathcal{R}ep(G, \mathbf{C})$ is the linear map

$$(\mathcal{F}f)_V : V^X \longrightarrow V^Y$$

given by the formula

$$(\mathcal{F}f)_V(a)(y) = \int_X f(y, x)\, a(x)\, dx$$

where dx is the normalised Haar measure on X. As in Section 1, it can be proved that the projection

$$\mathrm{Hom}(\mathbf{U}^X, \mathbf{U}^Y) \longrightarrow \prod_{\lambda \in G^\vee} \mathrm{Hom}\,(V_\lambda^X, V_\lambda^Y)$$

is an isomorphism, so that we can view the Fourier transform as a map

$$\mathcal{F} : \mathcal{C}(Y \times X, \mathbf{C})^G \longrightarrow \prod_{\lambda \in G^\vee} \mathrm{Hom}\,(V_\lambda^X, V_\lambda^Y)\,.$$

We now want to show that $\mathcal{F}$ defines an isometry of Hilbert spaces. Let $L^2(Y \times X)$ denote the space of square integrable functions on $Y \times X$ with respect to the product $dx\,dy$ of the normalised Haar measures on X and Y. Also, let $L^2(Y \times X/G)$ denote the space of square integrable functions on $Y \times X/G$ with respect to the projection onto $Y \times X/G$ of the product measure $dx\,dy$. The space $L^2(Y \times X/G)$ is isometric to the subspace $L^2(Y \times X)^G$ of G-invariant elements of $L^2(Y \times X)$ in which $\mathcal{C}(Y \times X, \mathbf{C})^G$ is dense. On the other hand, for any $\lambda \in G^\vee$, there is a canonical Hilbert space structure on $\mathrm{Hom}(V_\lambda{}^X, V_\lambda{}^Y)$ that we shall now describe. More generally, any G-invariant inner product on X induces an inner product on V^X given by

$$\langle a,b\rangle = \langle\ a(x_0), b(x_0)\rangle$$

where the value of the right hand side does not depend on the choice of the basepoint $x_0 \in X$ since we have

$$\langle a(\sigma x_0), b(\sigma x_0)\rangle = \langle \sigma\ a(x_0), \sigma b(x_0)\rangle = \langle\ a(x_0), b(x_0)\rangle .$$

Using inner products on $V_\lambda{}^X$ and $V_\lambda{}^X$ which are induced by some G-invariant metric on V_λ, we can define the adjoint $f^* : V_\lambda{}^Y \longrightarrow V_\lambda{}^X$ of any map $f : V_\lambda{}^X \longrightarrow V_\lambda{}^Y$. This adjoint does not depend on the particular choice of the G-invariant metric on V_λ. On the vector space $\mathrm{Hom}(V_\lambda{}^X, V_\lambda{}^Y)$ we shall use the inner product

$$\langle f, g\rangle = \ d_\lambda \mathrm{Tr}(f^* g)$$

where d_λ denotes (as before) the dimension of V_λ.

Theorem 1. *The Fourier transform extends continuously to an isometry*

$$\mathcal{F} \ :\ L^2(Y \times X / G) \xrightarrow{\ \sim\ } \sum_{\lambda \in G^\vee}^{\text{hilbert}} \mathrm{Hom}(V_\lambda^X, V_\lambda^X) \ .$$

Proof. We can suppose that $X = G/H$ and $Y = G/K$. Let $p_H : V \longrightarrow V^H$ be the averaging operator defined for any $V \in \mathcal{R}ep(G, \mathbf{C})$ by the formula

$$p_H(v) \ = \ \int_H h\, v\, dh$$

where dh is the normalised Haar measure on H. If $i_K : V^K \hookrightarrow V$ denotes the inclusion, it is easy to see that the map $f \longmapsto i_K\ f\ p_H$ defines an isometric embedding of $\mathrm{Hom}(V_\lambda{}^H, V_\lambda{}^K)$ into $\mathrm{Hom}(V_\lambda, V_\lambda)$. We obtain in this way an isometric embedding

$$\alpha \ :\ \sum_{\lambda \in G^\vee}^{\text{hilbert}} \mathrm{Hom}(V_\lambda^H, V_\lambda^K) \ \longrightarrow \ \sum_{\lambda \in G^\vee}^{\text{hilbert}} \mathrm{Hom}(V_\lambda, V_\lambda)$$

whose image consists of the elements which are fixed under the actions of K on the left and H on the right. On the other hand, the map $g \longmapsto (y_0\ , \ g\, x_0)$ induces a measure preserving homeomorphism between $K\backslash G/H$ and $Y \times X / G$ (the measure on $K\backslash G/H$ is obtained by projecting the Haar measure on G). This defines an isometric embedding

$$\beta \ :\ L^2(Y \times X\ /\ G) \longrightarrow L^2(G)$$

whose image consists of the elements fixed by K (on the left) and by H (on the right). The Theorem then follows from the relation

$$\mathcal{F} \circ \beta \ = \ \alpha \circ \mathcal{F}$$

and the fact that the Fourier transform of Section 1 Theorem 14 respects the actions of G on the left and on the right on both its domain and codomain. **qed**

When $Y = G$ acting on itself by left translation, we have $Y \times X / G \cong X$. We can define the Fourier transform of a function $f \in \mathcal{C}(X, \mathbf{C})$ to be the natural transformation

$$\mathcal{F}f \ :\ \mathbf{U}^X \longrightarrow \mathbf{U}$$

whose value at $V \in \mathcal{R}ep(G, \mathbf{C})$ is given by the formula

$$(\ \mathcal{F}f\ )_V(a) \ = \ \int_X f(x)\, a(x)\, dx\ .$$

Corollary 2. *The Fourier transform can be extended continuously to an isometry*

$$\mathcal{F} \ :\ L^2(X) \xrightarrow{\ \sim\ } \sum_{\lambda \in G^\vee}^{\text{hilbert}} \mathrm{Hom}(V_\lambda^X, V_\lambda) \ .$$

At this point we want to have similar results for the Fourier transform of generalised distributions on X. We shall let $R(X)$ denote the set of elements of $\mathcal{C}(X, \mathbf{C})$ whose orbits generate a finite dimensional subspace of $\mathcal{C}(X, \mathbf{C})$. According to Section 2 Proposition 1, this $R(G)$ is the set of representative functions on G (as before). When $X = G/H$, we have $R(G/H) = R(G)^H$. Let us see that, for all $V \in \mathcal{R}ep(G, \mathbf{C})$, $a \in V^X$, and $\phi \in V^*$, the function $x \longmapsto \phi(a(x))$ belongs to $R(X)$. But, if $x_0 \in X$, then the function

$$g \longmapsto \phi(a(g x_0)) = \phi(\pi_V(g)\, a(x_0))$$

belongs to $R(G)$ and is invariant under right translation of the stabilizer H of x_0; this proves the claim. We now define the Fourier transform $\mathcal{F}h$ of an element $h \in R(X)^*$. For $V \in \mathcal{R}ep(G, \mathbf{C})$, let $e_1, \ldots, e_n$ be a basis for V. Every element $a \in V^X$ can be written as

$$a(x) = \sum_{i=1}^{n} a_i(x)\, e_i$$

where $a_i(x) \in R(X)$ since $a_i(x) = e_i^*(a(x))$. We put

$$(\mathcal{F}h)_V(a) \;=\; \sum_{i=1}^{n} \langle h, a_i \rangle\, e_i\,.$$

Proposition 3. *The Fourier transform defines an isomorphism of topological vector spaces*

$$\mathcal{F} \;:\; R(X)^* \xrightarrow{\sim} \prod_{\lambda \in G^\vee} \mathrm{Hom}(V_\lambda^X, V_\lambda)\,.$$

Proof. We can suppose that $X = G/H$ and consider the averaging operation

$$p_H \;:\; R(G) \longrightarrow R(G/H)$$

defined by

$$p_H(f) \;= \int_H f(xh)\, dh\,.$$

Composition with p_H determines an embedding

$$\beta \;:\; R(G/H)^* \longrightarrow R(G)^*$$

whose image is the set of (generalized) distributions which are invariant under right translation by the elements of H. Similarly, if we compose with the averaging operators $p_H : V_\lambda \longrightarrow V_\lambda{}^H$, we obtain a map

$$\alpha \;:\; \prod_{\lambda \in G^\vee} \mathrm{Hom}(V_\lambda^H, V_\lambda) \longrightarrow \prod_{\lambda \in G^\vee} \mathrm{Hom}(V_\lambda, V_\lambda)$$

whose image consists of the elements which are invariant under right translation by elements of H. The result then follows from the identity

$$\mathcal{F} \circ \beta \;=\; \alpha \circ \mathcal{F}$$

and Section 3 Corollary 6. **qed**

The next thing to do is to define the Fourier transform of G-invariant generalized distributions on $Y \times X$. We shall do this without choosing basepoints in X and Y.

Definition. Let G be a compact group acting continuously on a topological vector space V. An element $v \in V$ is G-*finite* when its orbit generates a finite dimensional subspace of V.

By definition, $R(X)$ is the set of G-finite elements of $\mathcal{C}(X, \mathbf{C})$. Let us see that $R(X)$ is also equal to the G-finite elements of $R(X)^*$. To do this, let $j : R(X) \longrightarrow R(X)^*$ be the map which associates to a function $f \in R(X)$ the measure $f(x)dx$ on X. For any $g \in R(X)$, we have

$$\langle j(f), g \rangle = \int_X f(x)\,g(x)\,dx\ .$$

The image of j consists of G-finite elements since j is G-equivariant and every element of $R(X)$ is G-finite. Let $R(X)^\circ$ denote the set of G-finite elements of $R(X)^*$.

Proposition 4. *The mapping* j *determines an isomorphism*

$$R(X) \xrightarrow{\sim} R(X)^\circ.$$

Moreover, $R(X)^\circ$ *is dense in* $R(X)^*$.

Proof. We shall prove the result when $X = G$; the general result is proved similarly. If we use the isomorphism

$$\mathcal{F} \,:\, R(G)^* \xrightarrow{\sim} \prod_{\lambda \in G^\vee} \mathrm{Hom}(V_\lambda, V_\lambda)\ .$$

This reduces the problem to proving that, if an element

$$g \in \prod_{\lambda \in G^\vee} \mathrm{End}(V_\lambda)$$

is G-finite (for the left action of G) then all except a finite number of its components are zero. For each $\lambda \in G^\vee$, let $p_\lambda = (\, p_\lambda(\delta) \mid \delta \in G^\vee\,)$ be the projection operator, where

$$p_\lambda(\delta) = \begin{cases} \mathrm{id} & \text{if } \delta = \lambda \\ 0 & \text{otherwise}\ . \end{cases}$$

Then the orthogonality relations imply that $p_\lambda = \mathcal{F}(d_\lambda\, \bar{\chi}_\lambda)$ where χ_λ is the character of V_λ. We have the formula

$$p_\lambda g = d_\lambda \int_G \overline{\chi_\lambda}\,(x)\,\pi(x) g\,dx$$

which shows that, if g is G-finite, then all its components $p_\lambda g$ belong to the finite dimensional subspace generated by the orbit Gg. This proves that $p_\lambda g = 0$ except for a finite number of $\lambda \in G^\vee$. **qed**

Let $R(Y\times X)$ be the set of $G\times G$-finite elements of $\mathcal{C}(Y\times X, \mathbf{C})$. We have a canonical isomorphism

$$R(Y\times X) \cong R(X) \otimes R(Y)\ .$$

For all elements $h \in R(X)^{*\,Y}$, the pairing

$$(g, f) \longmapsto \int_Y g(y)\,\langle h(y), f\rangle\,dy$$

is a G-invariant bilinear form $\rho(h)$ on $R(Y)\times R(X)$. Let us say that a generalized distribution $t \in R(Y\times X)^*$ is *right regular* when it is of the form $\rho(h)$ for some $h \in \mathcal{C}(Y, R(X)^*)$.

Proposition 5. *Every G-invariant generalized distribution is regular. More precisely, the map* ρ *defines an isomorphism*

$$\mathrm{Hom}_G(Y, R(X)^*) \cong \mathrm{Hom}_G(R(Y\times X), \mathbf{C}).$$

Proof. We describe an inverse to ρ. We compose the canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}_G(R(Y\times X), \mathbf{C}) &\cong \mathrm{Hom}_G(R(Y)\otimes R(X), \mathbf{C}) \cong \mathrm{Hom}_G(R(X), R(Y)^*) \\ &\cong \mathrm{Hom}_G(R(X), R(Y)^\circ) \cong \mathrm{Hom}_G(R(X), R(Y)) \end{aligned}$$

with the isomorphism

$$k \,:\, \mathrm{Hom}_G(R(X), R(Y)) \longrightarrow \mathrm{Hom}_G(Y, R(X)^*)$$

defined as follows. For any G-invariant operator $u : R(X) \longrightarrow R(Y)$, we put

$$\langle k(u)(y), f \rangle = u(f)(y).$$

We leave the reader to check that k is bijective and we have defined an inverse to ρ. **qed**

We can now describe the Fourier transform of $t \in R(Y\times X)^{*G}$. According to Proposition 5, t is right regular, so that $t = \rho(h)$ for some $h \in \mathrm{Hom}_G(Y, R(X)^*)$. For all $V \in \mathcal{Rep}(G, \mathbf{C})$, we want to define the Fourier transform

$$(\mathcal{F}t)_V : V^X \longrightarrow V^Y.$$

Let $e_1, \ldots, e_n$ be a basis for V. For all $a \in V^X$, we can write $a = \sum_i a_i e_i$ where $a_i \in R(X)$. Then we have

$$(\mathcal{F}t)_V(a)(y) = \sum_{i=1}^{n} \langle h(y), a_i \rangle e_i .$$

Proposition 6. *The Fourier transform defines an isomorphism of topological vector spaces*

$$R(X \times Y)^{*G} \cong \prod_{\lambda \in G^\vee} \mathrm{Hom}(V_\lambda^X, V_\lambda^Y) .$$

Proof. If we exponentiate the Fourier transform of Proposition 3 by Y, we obtain an isomorphism

$$R(X)^{*Y} \cong \prod_{\lambda \in G^\vee} \mathrm{Hom}(V_\lambda^X, V_\lambda^Y) .$$

Composing this with the isomorphism of Proposition 5, we have the Fourier transform as defined above. **qed**

To complete these results, we should describe the Fourier *cotransform*

$$\mathcal{F}^\vee : \mathrm{Hom}^\vee(\mathbf{U}^X, \mathbf{U}^Y) \longrightarrow R(Y\times X)^G.$$

We shall suppose that $Y = G$ and postpone the general case. We want to describe the cotransform

$$\mathcal{F}^\vee : \mathrm{Hom}^\vee(\mathbf{U}^X, \mathbf{U}) \longrightarrow R(X).$$

For all $V \in \mathcal{Rep}(G, \mathbf{C})$, all $\phi \in V^*$ and all $a \in V^X$, we put

$$\mathcal{F}^\vee([\phi \otimes a]) = \phi \circ a .$$

Proposition 7. *The Fourier cotransform* $\mathcal{F}^\vee$ *is an isomorphism*

$$\mathrm{Hom}^\vee(\mathbf{U}^X, \mathbf{U}) \xrightarrow{\sim} R(X).$$

We now want to use these results to study the spectrum of the algebra $R(X)$. Let us first remark that, for all $V, W \in \mathcal{Rep}(G, \mathbf{C})$, we have a canonical pairing

$$\otimes : V^X \times W^X \longrightarrow (V\otimes W)^X \qquad (a, b) \longmapsto a\otimes b$$

defined by $(a\otimes b)(x) = a(x)\otimes b(x)$. We also have a unit element $1 \in \mathbf{C}^X = \mathbf{C}$.

Definition. A natural transformation $u : \mathbf{U}^X \longrightarrow \mathbf{U}$ is *tensor preserving* when, for all V, $W \in \mathcal{Rep}(G, \mathbf{C})$, $a \in V^X$, $b \in W^X$, the following equations hold:

$$u_{V\otimes W}(a\otimes b) = u_V(a)\otimes u_W(b) \quad \text{and} \quad u_I(1) = 1.$$

Let $\mathrm{Hom}^\otimes(\mathbf{U}^X, \mathbf{U})$ denote the set of tensor-preserving natural transformations. There is also the notion of *self-conjugate* natural transformation $u : \mathbf{U}^X \longrightarrow \mathbf{U}$; this means that

$$u_{\overline{V}}(\overline{a}) = \overline{u_V(a)}$$

for all $V \in \mathcal{Rep}(G, \mathbf{C})$ and all $a \in V^X$. Let $\mathcal{T}(X)$ denote the set of self-conjugate tensor-preserving natural transformations $\mathbf{U}^X \longrightarrow \mathbf{U}$.

For all $x \in X$, we have an element $\pi(x) \in \mathcal{T}(X)$ defined by $\pi(x)_V(a) = a(x)$ for $a \in V^X$. There is also the canonical map $j : X \longrightarrow R(X)^*$ defined by $\langle j(x), f \rangle = f(x)$.

Proposition 8. *The Fourier transform* $\mathcal{F}$ *induces the following isomorphisms of topological spaces:*

$$\mathrm{Spec}(R(X)) \xrightarrow{\sim} \mathrm{Hom}^{\otimes}(\mathbf{U}^X, \mathbf{U}), \qquad \mathrm{Spec}_{\mathbf{R}}(R(X)) \xrightarrow{\sim} \mathcal{T}(X).$$

Moreover, the following triangle commutes.

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "\pi"'] & \mathrm{Spec}_{\mathbf{R}} R(X) \arrow[d, "\mathcal{F}"] \\ & \mathcal{T}(X) \end{tikzcd}$$

Proof. Let $h \in R(X)^*$ and let $V, W \in \mathcal{Rep}(G, \mathbf{C})$. Let us choose a basis $e_1, \ldots, e_n$ for V and a basis $f_1, \ldots, f_n$ for W. For any $a \in V^X$ and $b \in W^X$, we have $a = \sum_i a_i e_i$ and $b = \sum_j b_j f_j$. The equation

$$(\mathcal{F}h)_{V \otimes W}(a \otimes b) = (\mathcal{F}h)_V(a) \otimes (\mathcal{F}h)_W(b)$$

means exactly that we have

$$\langle h, a_i b_j \rangle = \langle h, a_i \rangle \langle h, b_j \rangle.$$

Similarly, the self-conjugacy condition for $\mathcal{F}h$ means exactly that we have

$$\langle h, \overline{a_i} \rangle = \overline{\langle h, a_i \rangle}.$$

This proves the Proposition since the coefficients a_i linearly generate $R(X)$. **qed**

Let $V \in \mathcal{Rep}(G, \mathbf{C})$ and let $g : V \otimes V \longrightarrow \mathbf{C}$ be a positive definite invariant hermitian form. Let us see that, for any $u \in \mathcal{T}(X)$, the map $u_V : V^X \longrightarrow V$ is an isometric embedding, where the metric on V^X is the one induced from g on V. We can view g as a $\mathbf{C}$-linear pairing $h : \overline{V} \otimes V \longrightarrow \mathbf{C}$ and we can repeat *mutatis mutandis* the proof of Section 1 Proposition 5. We obtain that

$$g(u_V(a), u_V(b)) = g(a, b),$$

which means that u_V is an isometric embedding. The collection of these embeddings is compact, since it is a homogeneous space over the unitary group $U(V, g)$. It follows that $\mathcal{T}(X)$ is a closed subspace of a product of compact spaces. This proves:

Proposition 9. $\mathcal{T}(X)$ *is compact.*

We next want to prove that $X \xrightarrow{\sim} \mathcal{T}(X)$. For this we need some preliminaries. Suppose that G acts continuously on some compact space S. Let $A \subset \mathcal{C}(S, \mathbf{C})$ be a subalgebra which separates points of S and which is closed under conjugation.

Lemma 10. *If* A *is stable under the action of* G *and every element of* A *is* G*-finite then* A^G *separates the orbits of* G *in* S.

Proof. Consider the averaging operator $E : \mathcal{C}(S, \mathbf{C}) \longrightarrow \mathcal{C}(S, \mathbf{C})$ defined by

$$E(f)(x) = \int_G f(g\,x)dg\ .$$

If $f\in A$ then $E(f)\in A$ since $E(f)$ belongs to the convex hull of the orbit Gf whose linear span is a finite dimensional subspace of A. Let $p : S \longrightarrow S/G$ be the projection, and let c, $c' \in S/G$ be any two distinct orbits of G in S. The space S/G is compact and so there is a continuous function $\phi : S/G \longrightarrow \mathbf{R}$ such that $\phi(c) = 0$ and $\phi(c') = 1$. It follows from the Stone-Weierstrass Theorem that A is dense in $\mathcal{C}(S, \mathbf{C})$; so we can find $f\in \mathcal{C}(S, \mathbf{C})$ such that

$$\| f - \phi\circ p \| \le \tfrac{1}{4}$$

where we use the uniform norm. We have

$$\|E(f) - \phi\circ p\| = \|E(f - \phi\circ p)\| \le \|f - \phi\circ p\| \le \tfrac{1}{4}\ .$$

This implies that, for all $x\in c$ and $y\in c'$,

$$|E(f)(x) - E(f)(y)| \ge \tfrac{1}{2}\ .$$

So we have found an element $E(f)\in A^G$ which separates the orbits c, c'. **qed**

Theorem 11. *The maps* π, j *provide two homeomorphisms*

$$\pi : X \xrightarrow{\ \sim\ } \mathcal{T}(X) \quad \textit{and} \quad j : X \xrightarrow{\ \sim\ } \mathrm{Spec}_{\mathbf{R}}(R(X)).$$

Proof. To prove that j is injective is equivalent to proving that $R(X)$ separates the points of X. But this follows from Lemma 10 since we have $X \cong G/H$, $R(X) = R(G)^H$ and $R(G)$ separates the points of G (Section 1 Corollary 13). Let $S = \mathrm{Spec}_{\mathbf{R}}(R(X))$. Then S is compact since $\mathcal{T}(X)$ is (Proposition 9) and Fourier transform gives a homeomorphism $S \cong \mathcal{T}(X)$. The group G acts on S since it acts on $R(X)$. This action is continuous since $S \subset R(X)^*$ and G acts continuously on $R(X)^*$. The algebra $A = R(X)$ becomes a subalgebra of $\mathcal{C}(S, \mathbf{C})$ if we put $f(\chi) = \chi(f)$ for all $f\in R(X)$ and $\chi\in S$. Clearly, A separates the points of S and the hypotheses of Lemma 10 are satisfied. This proves that A^G separates the points of S/G. But we have $A^G = R(X)^G = \mathbf{C}$, and so S/G must reduce to a singleton. This implies that $j : X \longrightarrow S$ is surjective since its image $j(X)$ is an orbit of G in S. We have proved that j is a homeomorphism. It follows that π is a homeomorphism, since $\pi = \mathcal{F}\circ j$. **qed**

§6. Minimal models.

A *model* of a representative function f on a topological monoid M is a pair $(\phi, v)\in V^*\times V$ such that $f(x) = \phi(\pi_V(x)v)$ where V is some representation of M. Such a pair is far from being unique. A model is *minimal* if $\dim V$ is minimal. It is easy to see that a given representative function has a minimal model which is unique up to a (unique) isomorphism. In this part, we shall extend the concept of minimal model by giving a description of the elements of the tensor product of any two functors which satisfy some exactness conditions. This description is the key technical tool for proving the Representation Theorem of Section 7.

Recall that a category $\mathcal{C}$ is $\mathbf{C}$-*additive* when each of its homsets $\mathcal{C}(A, B)$ has the structure of a complex vector space and composition is $\mathbf{C}$-bilinear. Recall also that a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between $\mathbf{C}$-additive categories is $\mathbf{C}$-*additive* when the maps

$$F : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(FA, FB)$$

are **C**-linear.

Let $\mathcal{C}$ be an abelian **C**-additive category. We shall suppose that $\mathcal{C}$ is artinian, meaning that each decreasing sequence of subobjects in $\mathcal{C}$ terminates. Let $T : \mathcal{C} \longrightarrow \mathcal{V}ect_{\mathbf{C}}$ be a left exact **C**-additive functor. We shall need the category $el(T)$ of *elements* of T. An object of $el(T)$ is a pair (A, x) consisting of $A \in \mathcal{C}$ and $x \in T(A)$. A arrow $f : (A, x) \longrightarrow (B, y)$ in $el(T)$ is an arrow $f : A \longrightarrow B$ such that $T(f)(x) = y$. We shall say that $x \in T(A)$ is *contained* in a subobject $B \rightarrowtail A$ when x is in the image of the map $T(B) \rightarrowtail T(A)$. From the left exactness of T we see that, if x is contained in the subobjects $B \rightarrowtail A$ and $B' \rightarrowtail A$, then it is contained in their intersection $B \wedge B' \rightarrowtail A$. Using the artinian property of the poset of subobjects of A, it follows that x is contained in a smallest subobject $\mathrm{Span}(x)$ of A. When $\mathrm{Span}(x) = A$, we say that x *generates* A. We also need the full subcategory $\mathrm{Span}(T)$ of $el(T)$ whose objects are the pairs (A, x) in which $A = \mathrm{Span}(x)$. Observe that, between any two objects of $\mathrm{Span}(T)$, there is at most one arrow. To see this, suppose $f, g : (A, x) \longrightarrow (B, y)$; so x is contained in $\mathrm{Ker}(f - g)$ since $T(f)(x) = y = T(g)(x)$. Therefore, $f = g$ since $A = \mathrm{Span}(x)$. Note also that, if $f : (A, x) \longrightarrow (B, y)$ is an arrow in $\mathrm{Span}(T)$, the map $f : A \longrightarrow B$ is epimorphic. This is because $T(f)(x)$ is contained in the image of f and therefore $\mathrm{Im}(f) = B$ since $T(f)(x) = y$ is generating.

We can now give a first description of the elements of the tensor product $S \otimes_{\mathcal{C}} T$ for any contravariant **C**-additive functor S. We suppose that the abelian category $\mathcal{C}$ is artinian and that the covariant functor T is left exact. We shall write $[\phi \otimes x]$ for the image of a pair (ϕ, x) by the canonical map

$$S(A) \times T(A) \longrightarrow S \otimes_{\mathcal{C}} T.$$

Proposition 1. *Under the above hypotheses, any element of* $S \otimes_{\mathcal{C}} T$ *is of the form* $[\phi \otimes x]$ *for some* $(\phi, x) \in S(A) \times T(A)$ *for which* $A = \mathrm{Span}(x)$. *Moreover, if* $(\phi, x) \in S(A) \times T(A)$ *and* $(\psi, y) \in S(B) \times T(B)$, *then the equality* $[\phi \otimes x] = [\psi \otimes y]$ *holds if and only if there exists an object* $(C, z) \in \mathrm{Span}(T)$ *and arrows*

$$(A, x) \xleftarrow{\ f\ } (C, z) \xrightarrow{\ g\ } (B, y)$$

such that $S(f)(\phi) = S(g)(\psi)$.

Proof. We shall use some standard results from category theory [ML]. Since T is left exact, the category $el(T)$ has finite limits and is therefore filtered (or codirected). If $p : el(T) \longrightarrow \mathcal{C}$ is the projection functor and $y : \mathcal{C}^{op} \longrightarrow \mathcal{V}ect_{\mathbf{C}}{}^{\mathcal{C}}$ is the Yoneda embedding then we have the canonical isomorphism

$$T \xleftarrow{\ \sim\ } \varinjlim \left(el(T)^{op} \xrightarrow{\ p\ } \mathcal{C}^{op} \xrightarrow{\ y\ } \mathcal{V}ect_{\mathbf{C}}{}^{\mathcal{C}} \right).$$

Tensoring this isomorphism with S and using the isomorphisms

$$S \otimes_{\mathcal{C}} y(A) \cong S(A),$$

we obtain a canonical isomorphism

$$S \otimes_{\mathcal{C}} T \xleftarrow{\ \sim\ } \varinjlim \left(el(T)^{op} \xrightarrow{\ p\ } \mathcal{C}^{op} \xrightarrow{\ S\ } \mathcal{V}ect_{\mathbf{C}} \right).$$

The result is then a consequence of the standard description of colimits of directed diagrams, and the fact that $\mathrm{Span}(T)$ is initial in $el(T)$. **qed**

To obtain a more complete description of the elements of $S \otimes_{\mathcal{C}} T$, we shall make additional assumptions on the category $\mathcal{C}$ and the functor S. More precisely, we shall suppose that C is *noetherian*, meaning that each increasing sequence of subobjects terminates; or equivalently, that C^{op} is artinian. We shall also suppose that S is *left exact,* meaning that S transforms right exact sequences in C into left exact sequences of linear maps. As before, for any $\phi \in S(A)$, there is a smallest quotient object Cospan(ϕ) *supporting* ϕ (we adopt this terminology to avoid confusion with the previous case of a covariant functor T). If Cospan(ϕ) = A, we say that ϕ *cogenerates* A. We shall say that a pair $(\phi, x) \in S(A) \times T(A)$ is a *model* of an element $z \in S \otimes_C T$ when $z \in [\phi \otimes x]$. A model (ϕ, x) is *minimal* when Cospan(ϕ) = A = Span(x). An *isomorphism* between $(\phi, x) \in S(A) \times T(A)$ and $(\psi, y) \in S(B) \times T(B)$ is an invertible map $f : A \longrightarrow B$ such that $S(f^{-1})(\phi) = \psi$ and $T(f)(x) = y$. In this case we have

$$[\phi \otimes x] = [S(f)(\psi) \otimes x] = [\psi \otimes T(f)(x)] = [\psi \otimes y].$$

If two minimal models $(\phi, x), (\psi, y)$ are isomorphic, the isomorphism is unique since x and y are generating.

Theorem 2. *Suppose that the abelian* **C***-linear category* C *is artinian and noetherian, and that the functors* S *and* T *are left exact. Then every element of* $S \otimes_C T$ *has a minimal model which is unique up to a unique isomorphism. Moreover, for any* $(\phi, x) \in S(A) \times T(A)$, *there is a minimal pair* $(\phi', x') \in S(A') \times T(A')$ *such that* $[\phi' \otimes x'] = [\phi \otimes x]$ *where* A' *is a subquotient of* A.

Proof. We begin by proving the last statement. Let

$$\mathrm{Span}(x) \xrightarrow{\ q\ } A' \xrightarrow{\ j\ } \mathrm{Cospan}(\phi)$$

be the image factorization of the composite

$$\mathrm{Span}(x) \xrightarrow{\ i\ } A \xrightarrow{\ p\ } \mathrm{Cospan}(\phi).$$

By definition, there exist x_1 and ϕ_1 such that $S(p)(\phi_1) = \phi$ and $T(i)(x_1) = x$. If we put $\phi' = S(j)(\phi_1)$ and $x' = T(q)(x_1)$ then we have

$$[\phi' \otimes x'] = [\phi' \otimes T(q)(x_1)] = [S(q)(\phi') \otimes x_1] = [S(jq)(\phi_1) \otimes x_1]$$

$$= [S(pi)(\phi_1) \otimes x_1] = [S(i)(\phi) \otimes x_1] = [\phi \otimes T(i)(x_1)] = [\phi \otimes x].$$

Moreover, x' generates A' since x_1 generates Span(x). Similarly, ϕ' cogenerates A'. This proves that the pair (ϕ', x') is a minimal model of $[\phi \otimes x]$. This proves the existence of a minimal model for each element of $S \otimes_C T$ since all elements of $S \otimes_C T$ are of the form $[\phi \otimes x]$ by Proposition 1. To prove the uniqueness, suppose that $(\phi, x) \in S(A) \times T(A)$ and $(\psi, y) \in S(B) \times T(B)$ are both minimal models of z. Let (C, z), f, g be as in Proposition 1. The map f is epimorphic since $T(f)(z) = x$ and z, x are both generating. If $\sigma = S(f)(\phi)$ then Cospan(σ) $\cong$ A since f is epimorphic and ϕ is cogenerating. Similarly, g is epimorphic and Cospan(σ) $\cong$ B since ψ is cogenerating. This proves that we have an isomorphism $i : A \xrightarrow{\sim} B$ such that $i f = g$ and $S(i)(\psi) = \phi$. We have also

$$T(i)(x) = T(i)T(f)(z) = T(i f)(z) = T(g)(z) = y.$$

Hence i is an isomorphism between the pairs (ϕ, x) and (ψ, y). We have already remarked that such an isomorphism must be unique. **qed**

Corollary 3. *For all* $(\phi, x) \in S(A) \times T(A)$, *the equality* $[\phi \otimes x] = 0$ *holds if and only if there* *exists a monomorphism* $j : B \rightarrowtail A$ *such that* $x \in \mathrm{Im}(T(j))$ *and* $S(j)(\phi) = 0$.

Proof. Let $(\phi', x') \in S(A') \times T(A')$ be a minimal model of $[\phi \otimes x]$ where A' is the image of the map $\mathrm{Span}(x) \rightarrowtail A \twoheadrightarrow \mathrm{Cospan}(\phi)$ as in the proof of the Theorem. If $[\phi' \otimes x'] = 0$ then $A' = 0$ by uniqueness of the minimal model. This shows that the composite $\mathrm{Span}(x) \rightarrowtail A \twoheadrightarrow \mathrm{Cospan}(\phi)$ is zero, and the result follows with $B = \mathrm{Span}(x)$. **qed**

To end this Section, we shall prove that, for any coalgebra E, we have an isomorphism $E \cong \mathrm{End}^\vee(U)$ where $U : \mathcal{C}omod_f E \longrightarrow \mathcal{V}ect_{\mathbf{C}}$ is the forgetful functor. We have an obvious coaction $\alpha : U \longrightarrow U \otimes E$ obtained by putting together all the coactions $\alpha_V : V \longrightarrow V \otimes E$ where $(V, \alpha) \in \mathcal{C}omod_f E$. Using Section 4 Proposition 5, we obtain a morphism of coalgebras

$$\tilde{\alpha} : \mathrm{End}^\vee(U) \longrightarrow E .$$

In order to prove this morphism is invertible, we need the following Lemma whose proof is left to the reader.

Lemma 4. *Let* (V, α) *be a finite dimensional comodule over a coalgebra* E. *For each* $\phi \in V^*$, *the comodule* $\mathrm{Cospan}(\phi)$ *is obtained by taking the image factorization of the map*

$$V \xrightarrow{\ \alpha\ } V \otimes E \xrightarrow{\ \phi \otimes 1\ } E .$$

Proposition 5. *For each coalgebra* E, *the map*

$$\tilde{\alpha} : \mathrm{End}^\vee(U) \longrightarrow E$$

is an isomorphism of coalgebras.

Proof. We easily see that, for all $x \in V$, $\phi \in V^*$ and $(V, \alpha) \in \mathcal{C}omod_f E$,

$$\tilde{\alpha}\,[\phi \otimes x] = (\phi \otimes 1)\, \alpha_V(x).$$

To prove surjectivity, take $x \in E$. There is a finite dimensional subcomodule $V \subset E$ such that $x \in V$. Let ϕ be the restriction of the counit $\varepsilon : E \longrightarrow \mathbf{C}$ to the subspace V. Then we have

$$\tilde{\alpha}\,[\phi \otimes x] = (\phi \otimes 1)\delta(x) = (\varepsilon \otimes 1)\,\delta(x) = x .$$

To prove injectivity, take $z \in \mathrm{End}^\vee(U)$ in the kernel, and let $(\phi, x) \in V^* \times V$ be a minimal model of z. We have

$$(\phi \otimes 1)\, \alpha_V(x) = \tilde{\alpha}\,(z) = 0 .$$

But Lemma 4 shows that the map $(\phi \otimes 1)\, \alpha_V$ is injective since ϕ is cogenerating. This shows that $x = 0$ and hence $z = 0$. **qed**

§7. The representation theorem.

In this Section, we shall prove a representation theorem for an abelian category $\mathcal{C}$ equipped with an exact faithful functor U with values in finite dimensional vector spaces. We prove that $\mathcal{C}$ is equivalent to the category of finite dimensional E-comodules, where E is the coalgebra $\mathrm{End}^\vee(U)$ constructed in Section 4. This result is basic in the theory of

Tannakian categories in which the category comes first, or is given, and the group comes second, or is constructed. There are many examples of Tannakian categories in nature, especially from algebraic geometry in Grothendieck's theory of motives [DM]. The representation theorem proved here is not needed in the rest of this paper.

We first recall a few properties of the category of comodules over a coalgebra C. For any vector space V, the tensor product $V \otimes C$ has a comodule structure obtained by left comultiplication

$$V \otimes \delta : V \otimes C \longrightarrow V \otimes C \otimes C.$$

It is the cofree comodule over the vector space V. If $\alpha : V \longrightarrow V \otimes C$ is a comodule structure on V then α is also a morphism of comodules from V to the cofree comodule $V \otimes C$. The set of subcomodules of V is closed under intersections and sums. If $f : V \longrightarrow W$ is a morphism of comodules then the direct and inverse images under f of subcomodules are subcomodules. For completeness we include a proof of the following classical result.

Proposition 1. *Every comodule is the directed union of its finite dimensional subcomodules. Every coalgebra is the directed union of its finite dimensional subcoalgebras.*

Proof. Let $E \subset V$ be a finite dimensional subspace of the comodule (V, α). There exists a finite dimensional subspace $F \subset V$ such that $\alpha(E) \subset F \otimes C$. We have $E \subset \alpha^{-1}(F \otimes C)$ and $\alpha^{-1}(F \otimes C)$ is a subcomodule since it is the inverse image of the subcomodule $F \otimes C$ of the cofree comodule $V \otimes C$. Moreover, we have $\alpha^{-1}(F \otimes C) \subset F$ since

$$F \otimes C \subset (1 \otimes \varepsilon)^{-1}(F) \quad \text{and} \quad \alpha^{-1}(1 \otimes \varepsilon)^{-1}(F) = ((1 \otimes \varepsilon)\alpha)^{-1}(F) = F.$$

This proves the first sentence of the Proposition, and the second sentence follows once we add this remark: the subcomodules of C, as a comodule over $C^{op} \otimes C$ via left and right comultiplication, are exactly the subcoalgebras of C (where C^{op} is the opposite coalgebra of C). **qed**

We need to prove some results on the category $\mathcal{C}omod_f C$ of finite dimensional comodules over a coalgebra C.

Definition. A full subcategory of an abelian category is *replete* if it is closed under finite direct sums, subobjects and quotients.

In this definition it is explicitly assumed that any object isomorphic to an object of the replete subcategory also belongs to it. For any subcoalgebra $C' \subset C$, we have an inclusion $\mathcal{C}omod_f C' \subset \mathcal{C}omod_f C$ since every comodule over C' is naturally a comodule over C. Note that $\mathcal{C}omod_f C'$ is a replete subcategory of $\mathcal{C}omod_f C$.

Proposition 2. *For any coalgebra* C, *the assignment* $C' \longmapsto \mathcal{C}omod_f C'$ *defines a bijection between the subcoalgebras of* C *and the replete subcategories of* $\mathcal{C}omod_f C$.

Proof. We shall produce the inverse assignment. For any C-comodule (V, α), let $\mathrm{Im}(V)$ be the image of the coalgebra morphism

$$\tilde{\alpha} : \mathrm{End}(V) \longrightarrow C .$$

So $\mathrm{Im}(V, \alpha)$ is a subcoalgebra of C. For any replete subcategory $\mathcal{D} \subset \mathcal{C}omod_f C$, let $\mathrm{Im}(\mathcal{D})$

be the sums of the subcoalgebras $\mathrm{Im}(V,\alpha)$ as (V,α) runs over the objects of $\mathcal{D}$. For any subcoalgebra $C' \subset C$, we obviously have $\mathrm{Im}(\mathcal{D}) \subset C'$ if and only if $\mathcal{D} \subset \mathit{Comod}_f C'$. This shows that

$$\mathrm{Im}(\mathit{Comod}_f C') \subset C' \quad \text{and} \quad \mathcal{D} \subset \mathit{Comod}_f(\mathrm{Im}(\mathcal{D})).$$

To prove that $C' \subset \mathrm{Im}(\mathit{Comod}_f C')$ we can suppose that $C' = C$. Using Proposition 1, it will be enough to show that every finite dimensional subcoalgebra V of C is contained in $\mathrm{Im}(\mathit{Comod}_f C)$. Now the map

$$\tilde{\delta} : \mathrm{End}(V) \longrightarrow V$$

corresponding to $\delta : V \longrightarrow V \otimes V$ is surjective since

$$\tilde{\delta}(\varepsilon \otimes v) \;=\; (\varepsilon \otimes 1)\,\delta(v) \;=\; v\ .$$

This shows that $\mathrm{Im}(V,\alpha) = V$ and hence $\mathrm{Im}(\mathit{Comod}_f C) \supset V$. To prove that $\mathit{Comod}_f(\mathrm{Im}(\mathcal{D})) \subset \mathcal{D}$, we can suppose that $\mathrm{Im}(\mathcal{D}) = C$. Then our task is to prove that $\mathcal{D} \subset \mathit{Comod}_f C$. Let us first remark that, for any comodule (V,α), the subcoalgebra $\mathrm{Im}(V,\alpha)$ is generated by the images of the maps

$$V \xrightarrow{\alpha} V \otimes C \xrightarrow{\phi \otimes 1} C\ ,$$

where ϕ runs over V^*. This is a direct consequence of the identity

$$\tilde{\alpha}\,(\phi \otimes x) \;=\; (\phi \otimes 1)\,\alpha(x)\ .$$

If $V \in \mathcal{D}$ then the images of the maps $(\phi \otimes 1)\,\alpha$ belong to $\mathcal{D}$ since $\mathcal{D}$ is replete. Therefore $\mathrm{Im}(V,\alpha)$ belongs to $\mathcal{D}$. By hypothesis, C is generated by the $\mathrm{Im}(V,\alpha)$ for $(V,\alpha) \in \mathcal{D}$. This shows that every finite dimensional submodule of C belongs to $\mathcal{D}$. Let us prove that each $(W,\beta) \in \mathit{Comod}_f C$ belongs to $\mathcal{D}$. If $\phi_1, \ldots, \phi_n$ is a basis for W^* then the images W_i of the maps

$$W \xrightarrow{\beta} W \otimes C \xrightarrow{\phi_i \otimes 1} C$$

belong to $\mathcal{D}$ since they are subcomodules of C. Combining the maps $W \longrightarrow W_i$ together, we obtain a map from W into the direct sum of the W_i. This map is easily seen to be monomorphic. So $W \in \mathcal{D}$.**qed**

Suppose now that $\mathcal{C}$ is a **C**-additive category and $U : \mathcal{C} \longrightarrow \mathcal{V}ect_{\mathbf{C}}$ is a **C**-additive functor with values in finite dimensional spaces. In Section 4 we saw that the natural transformation $\gamma : U \longrightarrow U \otimes \mathrm{End}^\vee(U)$ determines a lifting

$$U^\sim : \mathcal{C} \longrightarrow \mathit{Comod}_f \mathrm{End}^\vee(U)$$

of U up into the category of $\mathrm{End}^\vee(U)$-comodules.

Theorem 3. *If $\mathcal{C}$ is abelian and U is exact and faithful then $U^\sim$ is an equivalence of categories.*

Proof. Obviously $U^\sim$ is faithful. Before proving that $U^\sim$ is full, let us see that, for any object $A \in \mathcal{C}$ and any subcomodule $E \subset U^\sim(A)$, there exists a subobject $j : B \rightarrowtail A$ such that $E = \mathrm{Im}U(j)$. Let $e_1, \ldots, e_n$ be a basis for the vector space $U(A)$ chosen in such a way that $e_1, \ldots, e_k$ generate E. For any $x \in U(A)$, the coaction $\alpha : U(A) \longrightarrow U(A) \otimes \mathrm{End}^{\vee}(U)$ is given by the formula

$$\alpha(x) = \sum_{i=1}^{n} e_i \otimes [e_i^* \otimes x].$$

The hypothesis $\alpha(E) \subset E \otimes \mathrm{End}^{\vee}(U)$ means that, for all $i \leq k$ and $j > k$, we have

$$[e_j^* \otimes e_i] = 0 .$$

We can now apply Section 6 Corollary 3 since $\mathrm{End}^{\vee}(U) = U^* \otimes_{\mathcal{C}} U$ and both U and U^* are exact ($\mathcal{C}$ is both artinian and noetherian since U is faithful). This means that, for all $i \leq k$ and $j > k$, there is a subobject $B_{ij} \hookrightarrow A$ such that $e_i \in U(B_{ij})$ and $e_j^*(U(B_{ij})) = 0$. If we put

$$B_i = \bigcap_{j>k} B_{ij} \qquad \text{and} \qquad B = \bigcup_{i \leq k} B_i$$

then we have $e_i \in U(B)$ for every $i \leq k$, and $e_j^*(U(B)) = 0$ for every $j > k$. This shows that $E \subset U(B)$, and $U(B) \subset E$ since

$$E = \bigcap_{j>k} \mathrm{Ker}(e_j^*) .$$

To prove that $U^{\sim}$ is full, let $f : U^{\sim}(A) \longrightarrow U^{\sim}(B)$ be a morphism of comodules. Then the graph Γ_f of f is a subcomodule of $U^{\sim}(A) \oplus U^{\sim}(B) \cong U^{\sim}(A \oplus B)$. Therefore there is a subcomodule $C \subset A \oplus B$ whose image can be identified with Γ_f. We claim that C is the graph of an arrow $u : A \longrightarrow B$ whose image under U is f. To see this, let i be the composite of the inclusion $C \hookrightarrow A \oplus B$ and the first projection $p_1 : A \oplus B \longrightarrow A$. Then $U(i)$ is the isomorphism $\Gamma_f \subset U(A) \oplus U(B) \longrightarrow U(A)$. Since U is exact and faithful, it follows that $\mathrm{Ker}(i) = \mathrm{Coker}(i) = 0$; so i is an isomorphism. The arrow u is defined as the composite

$$A \xrightarrow{i^{-1}} C \hookrightarrow A \oplus B \xrightarrow{p_2} B .$$

Obviously $U(u) = f$ since U transforms this description of u into the description of f.

To finish the proof, we have to show that every object of $\mathit{Comod}_f \mathrm{End}^{\vee}(U)$ is isomorphic to a comodule in the image of $U^{\sim}$. If we let $\mathcal{D}$ be the full subcategory consisting of the comodules isomorphic to those in the image of $U^{\sim}$, we have to prove that $\mathcal{D} = \mathit{Comod}_f \mathrm{End}^{\vee}(U)$. But we have seen that the image of $U^{\sim}$, and therefore $\mathcal{D}$, is closed under subobjects. A similar argument shows that it is closed under quotients. Thus, $\mathcal{D}$ is a replete subcategory, and, from Proposition 2, we have $\mathcal{D} = \mathit{Comod}_f C'$ for some subcoalgebra $C' \subset \mathrm{End}^{\vee}(U)$. This implies that, for all $A \in \mathcal{C}$, the vector space $U(A)$ has the structure of a C'-comodule $\gamma'_A : U(A) \longrightarrow U(A) \otimes C'$ which 'lifts' the coaction of $\mathrm{End}^{\vee}(U)$. More precisely, we have the commutative triangle of natural transformations below.

$$\begin{tikzcd}
U \arrow[rr, "\gamma'"] \arrow[dr, "\gamma"'] & & U \otimes C' \arrow[dl, "U \otimes i"] \\
& U \otimes \mathrm{End}^{\vee}(U) &
\end{tikzcd}$$

Using Section 4 Proposition 5, we have a unique coalgebra map

$$\tilde{\gamma}' \ : \ \mathrm{End}^{\vee}(U) \longrightarrow C'$$

such that the following triangle commutes.

$$\begin{tikzcd} U \arrow[rr, "\gamma"] \arrow[dr, "\gamma'"'] & & U \otimes \mathrm{End}^{\vee}(U) \arrow[dl, "U \otimes \tilde{\gamma}'"] \\ & U \otimes C' & \end{tikzcd}$$

But then we have

$$i\, \tilde{\gamma}' = \mathrm{id}$$

by the uniqueness property of Section 4 Proposition 5. This proves that i is surjective, and therefore $C' = \mathrm{End}^{\vee}(U)$ since i is an inclusion.**qed**

For any coalgebra C, the dual algebra C^* acts on the left of any (right) C-comodule. For any $V \in \mathcal{Comod}(C)$, $\phi \in C^*$ and $x \in C$, we write $\underline{\phi\rfloor}\, x$ for the action of ϕ on x. We have

$$\underline{\phi\rfloor}\, x \ = \ (1_V \otimes \phi)\, \alpha_V(x)\, .$$

Each element $\phi \in C^*$ defines a natural transformation $\underline{\phi\rfloor}(\ )\ :\ \mathbf{U} \longrightarrow \mathbf{U}$ where $\mathbf{U}\ :\ \mathcal{Comod}(C) \longrightarrow \mathcal{Vect}_C$ is the forgetful functor.

Proposition 4. *For any coalgebra* C, *the map* $\phi \longmapsto \underline{\phi\rfloor}(\ )$ *is an isomorphism of algebras*

$$C^* \xrightarrow{\ \sim\ } \mathrm{Hom}(\mathbf{U}\, ,\, \mathbf{U}\,)\, .$$

Proof. Using the fact that any comodule is a direct colimit of finite dimensional comodules, we can replace the category $\mathcal{Comod}(C)$ by its subcategory $\mathcal{Comod}_f(C)$. In this case, we have defined a coalgebra isomorphism

$$\tilde{\alpha}\ : \mathrm{End}^{\vee}(\mathrm{U}\,) \longrightarrow C\ .$$

This induces an isomorphism $C^* \xrightarrow{\ \sim\ } \mathrm{End}^{\vee}(\mathbf{U})^*$. According to Section 4 Proposition 3, any natural transformation $\mathbf{U} \longrightarrow \mathbf{U}$ $(= \mathbf{U} \otimes \mathbf{C})$ is of the form $(1 \otimes \phi)\,\gamma \ = \underline{\phi\rfloor}(\ )$ for a unique $\phi \in \mathrm{End}^{\vee}(\mathbf{U})^*$; so the result follows.**qed**

§8. The bialgebra End$^{\vee}$(X) and tensor categories.

The tensor product $M \otimes N$ of two modules over an algebra A is an $A \otimes A$-module, but it is not in general an A-module. However, when A is a bialgebra, we can define an A-module structure on $M \otimes N$ by restricting the $A \otimes A$-module structure along the diagonal (= comultiplication) map $\delta\ :\ A \longrightarrow A \otimes A$. Similarly, the tensor product $M \otimes N$ of two comodules over a coalgebra C is a $C \otimes C$-comodule, and, when C is a bialgebra, we can corestrict this comodule structure along the multiplication map $\mu : C \otimes C \longrightarrow C$ to obtain a C-comodule structure on $M \otimes N$. The category $\mathcal{Comod}(C)$ of comodules over a bialgebra C is therefore a tensor category (also called a monoidal category). The main purpose of this section is to reverse this process: starting with a pair $(\mathcal{C}, X)$ consisting of a tensor category $\mathcal{C}$ and a tensor-product-preserving functor $X\ :\ \mathcal{C} \longrightarrow \mathcal{Vect}_f$, we shall show that the coalgebra $\mathrm{End}^{\vee}(X)$ can be enriched with the structure of a bialgebra.

In what follows, we let $\mathcal{V}ect_f$ denote the category of finite dimensional vector spaces. In Section 4 we saw how to construct a coalgebra $\mathrm{End}^\vee(X)$ from a pair $(\mathcal{C}, X)$ where $\mathcal{C}$ is a category and $X : \mathcal{C} \longrightarrow \mathcal{V}ect_f$ is a functor. It is easy to see in addition that a commutative triangle of functors

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "X"'] & & \mathcal{D} \arrow[dl, "Y"] \\ & \mathcal{V}ect_f & \end{tikzcd}$$

gives rise to a map of coalgebras $\mathrm{End}^\vee(X) \longrightarrow \mathrm{End}^\vee(Y)$ which we might call the *corestriction* along F (it is predual to the usual restriction map $\mathrm{End}(Y) \longrightarrow \mathrm{End}(X)$). When the functor F is an equivalence of categories, the corestriction map along F is an isomorphism of coalgebras. It is useful to formalise this process by introducing the category $\mathcal{C}at/\mathcal{V}ect_f$ of categories over $\mathcal{V}ect_f$. An object of $\mathcal{C}at/\mathcal{V}ect_f$ is a pair $(\mathcal{C}, X)$ where $\mathcal{C}$ is a (small) category and $X : \mathcal{C} \longrightarrow \mathcal{V}ect_f$ is a functor. A morphism $(F, \alpha) : (\mathcal{C}, X) \longrightarrow (\mathcal{D}, Y)$ consists of a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ and a natural isomorphism $\alpha : X \xrightarrow{\sim} YF$. Composition of morphisms is the obvious one. We have a covariant functor

$$\mathrm{End}^\vee : \mathcal{C}at/\mathcal{V}ect_f \longrightarrow \mathcal{C}oalg$$

with values in the category $\mathcal{C}oalg$ of coalgebras. We define the *(external) tensor product* of $X : \mathcal{C} \longrightarrow \mathcal{V}ect_f$ with $Y : \mathcal{D} \longrightarrow \mathcal{V}ect_f$ to be the functor

$$X \underline{\otimes} Y : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{V}ect_f$$

where, for $(A, B) \in \mathcal{C} \times \mathcal{D}$,

$$(X \underline{\otimes} Y)(A, B) = X(A) \otimes Y(B).$$

Proposition 1. *There is a canonical isomorphism*

$$\theta : \mathrm{End}^\vee(X) \otimes \mathrm{End}^\vee(Y) \xrightarrow{\sim} \mathrm{End}^\vee(X \underline{\otimes} Y).$$

Proof. For any $(A, B) \in \mathcal{C} \times \mathcal{D}$, $S \in \mathrm{End}(X(A))$, $T \in \mathrm{End}(Y(B))$, we have $S \otimes T \in \mathrm{End}(X(A) \otimes Y(B))$. We put $\theta([S] \otimes [T]) = [S \otimes T]$. The best way to prove that θ is well defined and an isomorphism is to see that it is a special case of the following canonical isomorphism between tensor products of functors

$$(H \otimes_{\mathcal{C}} X) \otimes (K \otimes_{\mathcal{D}} Y) \xrightarrow{\sim} (H \underline{\otimes} K) \otimes_{\mathcal{C} \times \mathcal{D}} (X \underline{\otimes} Y)$$

where H, K are contravariant functors on $\mathcal{C}, \mathcal{D}$, respectively. When $H = X^*$ and $K = Y^*$, we have $H \otimes_{\mathcal{C}} X = \mathrm{End}^\vee(X)$ and $K \otimes_{\mathcal{D}} Y = \mathrm{End}^\vee(Y)$. **qed**

Recall [ML] that a *tensor* (or "monoidal") *category* $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ consists of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ (called the *tensor product*), an object $I \in \mathcal{C}$ (called the *unit object*) and natural isomorphisms

$$a = a_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C),$$
$$l = l_A : I \otimes A \xrightarrow{\sim} A, \qquad r = r_A : A \otimes I \xrightarrow{\sim} A$$

(called the *associativity, left unit, right unit constraints,* respectively), such that, for all objects $A, B, C, D \in \mathcal{C}$, the following two *coherence* conditions hold:

$$a_{A,B,C\otimes D} \circ a_{A\otimes B,C,D} = (A\otimes a_{B,C,D}) \circ a_{A,B\otimes C,D} \circ (a_{A,B,C}\otimes D) \quad \text{and}$$
$$(A\otimes l_B) \circ a_{A,I,B} = r_A \otimes B.$$

It follows [ML] that all objects obtained by computing the tensor product of a sequence $A_1 \otimes A_2 \otimes \ . \ . \ . \ \otimes A_m$ by bracketing it differently and by cancelling units are coherently isomorphic to each other.

A tensor category is called *strict* when all the constraints $a_{A,B,C}$, l_A, r_A are identity arrows. Each tensor category $\mathcal{C}$ is equivalent to a *strict* tensor category $\mathrm{st}(\mathcal{C})$. The objects of $\mathrm{st}(\mathcal{C})$ are words $w = A_1 A_2 \ldots A_m$ in objects of $\mathcal{C}$. An arrow $f : w \longrightarrow w'$ is an arrow $f : [w] \longrightarrow [w']$ in $\mathcal{C}$ where we define

$$[\varnothing] = I, \quad [A] = A, \quad \text{and}$$
$$[A_1 A_2 \ldots A_{m+1}] = [A_1 A_2 \ldots A_m] \otimes A_{m+1} .$$

The tensor $\otimes$ for $\mathrm{st}(\mathcal{C})$ is given by $v \,\bar{\otimes}\, w = vw$ and by commutativity of the following square

$$\begin{tikzcd}
{[v] \otimes [w]} \arrow[r, "f \otimes g"] \arrow[d, "\wr"'] & {[v'] \otimes [w']} \arrow[d, "\wr"] \\
{[vw]} \arrow[r, "f \,\bar{\otimes}\, g"'] & {[v'w']}
\end{tikzcd} \quad .$$

An example of a tensor category is the category $\mathcal{C}at/\mathcal{V}ect_f$ with the external tensor product described above. The unit object I in $\mathcal{C}at/\mathcal{V}ect_f$ is the functor $\mathbf{C} : \mathbf{1} \longrightarrow \mathcal{V}ect_f$ where $\mathbf{1}$ is the category with a single object $*$ and a single arrow (the identity of $*$) and $\mathbf{C}$ denotes the functor assigning to $*$ the one-dimensional vector space $\mathbf{C} \in \mathcal{V}ect_f$.

Let $\mathcal{C}, \mathcal{D}$ denote tensor categories. Recall [ML] that a *tensor* (or "strong monoidal") *functor* $F = (F, \phi, \phi_0) : \mathcal{C} \longrightarrow \mathcal{D}$ consists of a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, a natural isomorphism

$$\phi = \phi_{A,B} : FA \otimes FB \xrightarrow{\sim} F(A \otimes B),$$

and an isomorphism $\phi_0 : I \xrightarrow{\sim} FI$, such that the following three equations hold (where we write as if $\mathcal{C}, \mathcal{D}$ were strict):

$$\phi_{A\otimes B,C} \circ (\phi_{A,B}\otimes FC) = \phi_{A,B\otimes C} \circ (FA\otimes \phi_{B,C}),$$
$$\phi_0 \otimes FA = \phi_{I,A} \circ F(\phi_0 \otimes A) \quad \text{and} \quad FA\otimes \phi_0 = \phi_{A,I} \circ F(A\otimes \phi_0) .$$

The tensor functor is strict when all the isomorphisms $\phi_{A,B}$, ϕ_0 are identities. One example of a tensor functor is the equivalence $\mathcal{C} \xrightarrow{\sim} \mathrm{st}(\mathcal{C})$ taking A to [A]. Another example is the functor $\mathrm{End}^\vee : \mathcal{C}at/\mathcal{V}ect_f \longrightarrow \mathcal{C}oalg$.

Recall that a *monoid* $M = (M, \mu, \eta)$ in a tensor category $\mathcal{C}$ consists of an object $M \in \mathcal{C}$ and arrows $\mu : M\otimes M \longrightarrow M$, $\eta : I \longrightarrow M$ such that the following diagrams commute.

$$\begin{tikzcd}
M\otimes M\otimes M \arrow[r, "\mu\otimes M"] \arrow[d, "M\otimes\mu"'] & M\otimes M \arrow[d, "\mu"] \\
M\otimes M \arrow[r, "\mu"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\eta\otimes M"] \arrow[dr, equal] & M\otimes M \arrow[d, "\mu"] & M \arrow[l, "M\otimes\eta"'] \arrow[dl, equal] \\
& M &
\end{tikzcd}$$

A *comonoid* is a monoid in $\mathcal{C}^{\mathrm{op}}$. For example, algebras are monoids in the category of vector spaces, coalgebras are comonoids in the same category, and bialgebras are monoids in the category of coalgebras. Monoids in $\mathcal{C}at$ (where the tensor product is cartesian product) are strict tensor categories. Monoids in $\mathcal{C}at/\mathcal{V}ect_f$ are the pairs $(\mathcal{C}, X)$ for which $\mathcal{C}$

is a strict tensor category and $X : \mathcal{C} \longrightarrow \mathcal{V}ect_f$ is a tensor functor (not necessarily strict). It follows from Proposition 1 that the coalgebra $\mathrm{End}^\vee(X)$ corresponding to such a pair has the structure of a bialgebra since it enherits a monoid structure in the category of coalgebras. More generally, if we have a pair $(\mathcal{C}, X)$ where $\mathcal{C}$ is a tensor category, not necessarily strict, and where X is a tensor functor, we also have a bialgebra structure on $\mathrm{End}^\vee(X)$. To see this, we can replace the pair $(\mathcal{C}, X)$ by a pair $(\mathcal{C}', X')$ where $\mathcal{C}' = \mathrm{st}(\mathcal{C})$ is a strict tensor category equivalent to $\mathcal{C}$. We use the fact that the corestriction map $\mathrm{End}^\vee(X') \longrightarrow \mathrm{End}^\vee(X)$ along an equivalence $\mathcal{C}' \longrightarrow \mathcal{C}$ is an isomorphism of coalgebras. However, we shall directly describe the algebra structure on $\mathrm{End}^\vee(X)$ without recourse to $(\mathcal{C}', X')$. For any A, $B \in \mathcal{C}$, $S \in \mathrm{End}(X(A))$ and $T \in \mathrm{End}(X(B))$, let us write $S \otimes T$ for the dashed arrow in the square

$$\begin{tikzcd} X(A \otimes B) \arrow[r, dashed] \arrow[d, "\wr"] & X(A \otimes B) \arrow[d, "\wr"] \\ X(A) \otimes X(B) \arrow[r, "S \otimes T"'] & X(A) \otimes X(B) \end{tikzcd}$$

Also, let us write $1 \in X(I)$ for the element corresponding to $1 \in \mathbf{C}$ under the isomorphism $\mathbf{C} \cong X(I)$. These notational abuses are harmless, not only because the context will dissipate the ambiguity, but also because tensor functors satisfy a coherence theorem [Le]. We can now specify the algebra structure on $\mathrm{End}^\vee(X)$. The product of the elements $[S]$ and $[T]$ of $\mathrm{End}^\vee(X)$ is given by the simple formula

$$[S][T] = [S \otimes T].$$

The unit element of $\mathrm{End}^\vee(X)$ is equal to $[1]$.

When $\mathcal{C} = \mathcal{R}ep(M, \mathbf{C})$ and X is the forgetful functor $\mathbf{U}$, we obtain a bialgebra structure on $\mathrm{End}^\vee(\mathbf{U})$. The meaning of the bialgebra structure is elucidated by the next result.

Proposition 2. *For any topological monoid* M, *the Fourier cotransform*

$$\mathcal{F}^\vee : \mathrm{End}^\vee(\mathbf{U}) \longrightarrow R(M)$$

is an isomorphism of bialgebras.

Proof. It remains to verify that $\mathcal{F}^\vee$ is a homomorphism of algebras. For all $V, W \in \mathcal{R}ep(M, \mathbf{C})$ and all $A \in \mathrm{End}(V)$, $B \in \mathrm{End}(W)$, we have

$$\mathcal{F}^\vee([A][B]) = \mathcal{F}^\vee([A \otimes B]) = \mathrm{Tr}(\pi_{V \otimes W} \circ (A \otimes B)) = \mathrm{Tr}((\pi_V \otimes \pi_W) \circ (A \otimes B))$$

$$= \mathrm{Tr}(\pi_V A \otimes \pi_W B) = \mathrm{Tr}(\pi_V A)\,\mathrm{Tr}(\pi_W B) = \mathcal{F}^\vee([A])\,\mathcal{F}^\vee([B]) \text{ and}$$

$$\mathcal{F}^\vee(1) = \mathrm{Tr}(\pi_I) = 1\,.$$ **qed**

The next thing we shall do is to characterize the algebra structure on $\mathrm{End}^\vee(X)$ by a universal property. More precisely, for all algebras A, we shall prove that the correspondence $n \longmapsto \tilde{n}$ is a bijection between tensor-preserving natural transformations $X \longrightarrow X \otimes A$ and algebra homomorphisms $\mathrm{End}^\vee(X) \longrightarrow A$. Our first task is to define the former. A *coaction* of the algebra A on a vector space V is a linear map $\alpha : V \longrightarrow V \otimes A$, or equivalently, a linear map

$$\tilde{\alpha} : \mathrm{End}(V) \longrightarrow A\,.$$

We define the *trace* of α as the value of the last linear map at the identity endomorphism of V. If $e_1, \ldots, e_n$ is a basis of V and

$$\alpha(e_i) = \sum_{j=1}^{n} e_j \otimes \alpha_{ij}$$

then

$$\mathrm{Tr}(\alpha) = \sum_{i=1}^{n} \alpha_{ii} .$$

We have the formula

$$\tilde{\alpha}(S) = \mathrm{Tr}(\alpha S)$$

which is valid for all $S \in \mathrm{End}(V)$. The *tensor product* $\alpha \underline{\otimes} \beta$ of coactions $\alpha : V \longrightarrow V \otimes A$, $\beta : W \longrightarrow W \otimes A$ is defined to be the composite

$$V \otimes W \xrightarrow{\alpha \otimes \beta} V \otimes A \otimes W \otimes A \xrightarrow{\sim} V \otimes W \otimes A \otimes A \xrightarrow{V \otimes W \otimes \mu} V \otimes W \otimes A$$

where $\mu : A \otimes A \longrightarrow A$ is the multiplication of the algebra A, and the middle isomorphism in the composite uses the symmetry map $A \otimes W \xrightarrow{\sim} W \otimes A$. If

$$\beta(f_r) = \sum_{s=1}^{m} f_s \otimes \beta_{rs}$$

gives a matrix for β then

$$(\alpha \underline{\otimes} \beta)(e_i \otimes f_r) = \sum_{j,\, s=1}^{n,\, m} (e_j \otimes f_s) \otimes (\alpha_{ij} \beta_{rs})$$

gives a matrix for $\alpha \underline{\otimes} \beta$.

We shall say that a natural transformation $u : X \longrightarrow X \otimes A$ is *tensor preserving* when, for all $C, D \in \mathcal{C}$, we have

$$u_{C \otimes D} = u_C \underline{\otimes} u_D \quad \text{and} \quad u_I = 1 .$$

Let $\mathcal{C}$ be a tensor category and $X : \mathcal{C} \longrightarrow \mathcal{V}ect_f$ be a tensor functor. In Section 4 Proposition 3, we defined a natural transformation $\gamma : X \longrightarrow X \otimes \mathrm{End}^{\vee}(X)$.

Proposition 3. *The natural transformation* γ *is tensor preserving. Moreover, for all algebras* A *and all tensor-preserving natural transformations* $n : X \longrightarrow X \otimes A$, *there is precisely one algebra homomorphism* $\tilde{n} : \mathrm{End}^{\vee}(X) \longrightarrow A$ *such that the following triangle commutes.*

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[dr, "n"'] & & X \otimes \mathrm{End}^{\vee}(X) \arrow[dl, "X \otimes \tilde{n}"] \\ & X \otimes A & \end{tikzcd}$$

Proof. To prove γ tensor preserving, take $C, D \in \mathcal{C}$. Choose a basis $e_1, \ldots, e_m$ of X(C) and a basis $f_1, \ldots, f_n$ of X(D). By definition of $\gamma_C \underline{\otimes} \gamma_D$, we have the equality

$$(\gamma_C \underline{\otimes} \gamma_D)(x \otimes y) = \sum_{i,j} e_i \otimes f_j \otimes [e_i^* \otimes x][f_i^* \otimes y] .$$

On the other hand, we have

$$\gamma_{C \otimes D}(x \otimes y) \ = \ \sum_{i,\, j} e_i \otimes f_j \otimes [e_i^* \otimes f_i^* \otimes x \otimes y]\,.$$

The equality $\gamma_C \otimes \gamma_D = \gamma_{C \otimes D}$ is now a consequence of the identity

$$[\phi \otimes x][\psi \otimes y] = [\phi \otimes \psi \otimes x \otimes y]$$

which holds in $\mathrm{End}^{\vee}(X)$. To prove the rest of the Proposition, let $n : X \longrightarrow X \otimes A$ be a tensor-preserving natural transformation. According to Section 4 Proposition 3, there exists precisely one linear map $\tilde{n} : \mathrm{End}^{\vee}(X) \longrightarrow A$ such that the triangle of the Proposition commutes. A straightforward computation shows that we have $\tilde{n}([S]) = \mathrm{Tr}(n_C S)$, where we are using the trace introduced above. Using this we have the following calculation:

$$\tilde{n}([S][T]) = \tilde{n}([S \otimes T]) = \mathrm{Tr}(n_{C \otimes D}(S \otimes T)) = \mathrm{Tr}(n_C \otimes n_D \circ (S \otimes T))$$

$$= \mathrm{Tr}(n_C S \otimes n_D T) = \mathrm{Tr}(n_C S)\, \mathrm{Tr}(n_D T) = \tilde{n}([S])\ \tilde{n}([T])\ . \ _{\mathbf{qed}}$$

When A is a bialgebra, a coaction $\alpha : V \longrightarrow V \otimes A$ defines a comodule structure on V if and only if it is associative and unitary. Also, the tensor product of two coactions α, β is a comodule structure if both of α, β are. The category $\mathcal{Comod}_f(A)$ of finite dimensional comodules is a tensor category. Clearly, if a natural transformation $n : X \longrightarrow X \otimes A$ is tensor preserving and defines a comodule structure on X then we obtain a functor

$$X' = (X, n) \ : \ \mathcal{C} \longrightarrow \mathcal{Comod}_f(A)$$

which is tensor preserving and renders commutative the following triangle, where $\mathbf{U}$ is the forgetful functor.

$$\begin{tikzcd} & \mathcal{Comod}_f(A) \arrow[d, "\mathsf{U}"] \\ \mathcal{C} \arrow[ur, "X'"] \arrow[r, "X"'] & \mathcal{Vect}_f \end{tikzcd}$$

Proposition 4. *Let* $\mathcal{C}$ *be a tensor category and* $X : \mathcal{C} \longrightarrow \mathcal{Vect}_f$ *be a tensor functor. For all bialgebras* A, *there is a bijection between tensor-preserving functors* $X' : \mathcal{C} \longrightarrow \mathcal{Comod}_f(A)$ *such that* $\mathbf{U}X' = X$ *(i.e.* X' lifts X *) and bialgebra homomorphisms* $\mathrm{End}^{\vee}(X) \longrightarrow A$.

Proof. Just combine Proposition 4 with Section 4 Proposition 7. $_{\mathbf{qed}}$

§9. Duality and Hopf algebras.

We begin by recalling the basic concepts of duality theory in a tensor category $\mathcal{C}$. Given $A, B \in \mathcal{C}$, we shall say that a pair of maps $\eta : I \longrightarrow B \otimes A$, $\varepsilon : A \otimes B \longrightarrow I$ form an *adjunction* between A and B if the following two composites are identities:

$$A \xrightarrow{A \otimes \eta} A \otimes B \otimes A \xrightarrow{\varepsilon \otimes A} A\ , \quad B \xrightarrow{\eta \otimes B} B \otimes A \otimes B \xrightarrow{B \otimes \varepsilon} B\,.$$

We call η the *unit* and ε the *counit* of the adjunction. We say that A (respectively, B) is *left adjoint* or *left dual* to B (respectively *right adjoint* or *right dual* to A). We also write $(\eta, \varepsilon) : A \dashv B$ to indicate that the pair (η, ε) is an adjunction between A and B.

The unit and counit of an adjunction determine each other. More precisely, let us say that a map $\varepsilon : A\otimes B \longrightarrow I$ (a 'pairing') is *exact* when, for all objects X, Y, the function

$$\varepsilon^{\#} : \mathcal{C}(X, B\otimes Y) \longrightarrow \mathcal{C}(A\otimes X, Y), \qquad f \longmapsto (\varepsilon\otimes Y)(A\otimes f)$$

is bijective.

Proposition 1. *A pairing* $\varepsilon : A\otimes B \longrightarrow I$ *is exact if and only if there exists a map* $\eta : I \longrightarrow B\otimes A$ *such that the pair* (η, ε) *is an adjunction between* A *and* B.

Proof. If ε is exact then take η to be the unique map such that $\varepsilon^{\#}(\eta)$ is the canonical isomorphism $A\otimes I \xrightarrow{\sim} A$. Conversely, if (η, ε) is an adjunction then the function $h \longmapsto (B\otimes h)(\eta\otimes X)$ is an inverse for $\varepsilon^{\#}$. **qed**

In the category of finite dimensional vector spaces, a pairing $\varepsilon : V\otimes W \longrightarrow \mathbf{C}$ is exact if and only if the corresponding map $x \longmapsto \varepsilon(x, -)$ from V to W^* is an isomorphism. When the pairing is exact, we can describe $\eta : \mathbf{C} \longrightarrow W\otimes V$ by giving the value $\eta(1) \in W\otimes V$. To any basis $e_1, \ldots, e_n$ of V there corresponds a dual basis $f_1, \ldots, f_n$ of W such that $\varepsilon(e_i, f_j) = \delta_{ij}$. We have

$$\eta(1) = f_1\otimes e_1 + \ldots + f_n\otimes e_n .$$

In a tensor category $\mathcal{C}$, let $\varepsilon : A\otimes B \longrightarrow I$ and $\varepsilon' : A'\otimes B' \longrightarrow I$ be two exact pairings. We shall say that a map $f : A \longrightarrow A'$ is *left adjoint* to $g : B' \longrightarrow B$ (or that g is *right adjoint* to f) when we have

$$\varepsilon'(f\otimes A) = \varepsilon(B\otimes g) .$$

For all f, there is a unique right adjoint g given by

$$g = (\eta'\otimes B)(B'\otimes f\otimes B)(B'\otimes \varepsilon) .$$

Similarly, for all g, there is a unique left adjoint f given by

$$f = (A'\otimes \eta)(A'\otimes g\otimes A)(\varepsilon'\otimes A) .$$

Applying this to the case where $A = A'$, we see that two right adjoints B, B' of A are canonically isomorphic. Similarly for left adjoints.

Definition. A tensor category $\mathcal{C}$ is *autonomous* when every object of $\mathcal{C}$ has both a left and a right adjoint.

When $\mathcal{C}$ is autonomous, we can choose, for each $C\in\mathcal{C}$, a pair of adjunctions

$$(\eta_C, \varepsilon_C) : C^{\ell} \dashv C \quad \text{and} \quad (\eta'_C, \varepsilon'_C) : C \dashv C^{r} .$$

We obtain in this way a pair of contravariant functors

$$(\ )^{\ell} : \mathcal{C}^{op} \longrightarrow \mathcal{C} \quad \text{and} \quad (\ )^{r} : \mathcal{C}^{op} \longrightarrow \mathcal{C} .$$

Obviously, for all $C\in\mathcal{C}$, we have canonical isomorphisms

$$(C^{r})^{\ell} \cong C \cong (C^{\ell})^{r}$$

making the functors $(\ )^{\ell}$, $(\ )^{r}$ mutually quasi-inverse (i.e. they give an equivalence of categories).

It is instructive to work out an example of an autonomous tensor category where right and left adjoint are different. For any algebra A, let us write $\mathcal{Co}_f(A)$ for the category whose objects are the coactions $\alpha : V \longrightarrow V\otimes A$ on finite dimensional vector spaces. In Section 8 we defined a tensor product of coactions, and so this category becomes a tensor category. We first identify the adjunctions within this category. Let $(V, \alpha), (W, \beta) \in \mathcal{Co}_f(A)$ and let

$$(\eta, \varepsilon) : (V, \alpha) \dashv (W, \beta)$$

be an exact pairing. Clearly the pairing (η, ε) defines an exact pairing between the vector spaces V and W (since the forgetful functor $\mathcal{Co}_f(A) \longrightarrow \mathcal{Vect}_f$ preserves tensor product). Let $e_1, \ldots, e_n$ be a basis of V and let $f_1, \ldots, f_n$ be a dual basis. We have

$$\alpha(e_i) = \sum_j e_j \otimes \alpha_{ji} \quad , \quad \beta(f_i) = \sum_j f_j \otimes \beta_{ji} \quad ,$$

$$\varepsilon(e_i \otimes f_j) = \delta_{ij} \quad , \quad \eta(1) = \sum_i f_i \otimes e_i \quad .$$

Expressing that ε is a morphism in $\mathcal{Co}_f(A)$, we obtain

$$\sum_k \alpha_{ki} \beta_{kj} = \delta_{ij} \ .$$

Similarly, expressing that η is a morphism, we obtain

$$\sum_k \beta_{ik} \alpha_{jk} = \delta_{ij} \ .$$

If $\alpha = (\alpha_{ij})$ and $\beta = (\beta_{ij})$, these equalities can be formulated as the matrix equations $({}^t\alpha)\beta = \beta\,({}^t\alpha) = \mathrm{id}$. In other words, the *right adjoint* α^r of the matrix α is equal to $({}^t\alpha)^{-1}$. Similarly, we obtain that the *left adjoint* α^ℓ of α is the matrix ${}^t(\alpha^{-1})$. If the algebra A is not commutative, there is in general no relationship between ${}^t(\alpha^{-1})$ and $({}^t\alpha)^{-1}$. One might exist and not the other. We can inductively define

$$\alpha^{(0)} = \alpha\,, \qquad \alpha^{(n+1)} = (\alpha^{(n)})^\ell \text{ for } n \geq 0\,, \qquad \text{and} \qquad \alpha^{(n-1)} = (\alpha^{(n)})^r \text{ for } n \leq 0.$$

Let us say that a matrix α is *totally invertible* when $\alpha^{(n)}$ exists for all $n \in \mathbf{Z}$. Clearly the coactions with totally invertible matrices form an example of an autonomous category for which left and right duals do not coincide in general.

We now give a brief review of the basic theory of Hopf algebras. Recall that, for any coalgebra C and any algebra A, the *convolution product* defines an algebra structure on the vector space Hom(C, A), where the convolution $\phi * \psi$ of $\phi : C \longrightarrow A$ with $\psi : C \longrightarrow A$ is the composite

$$C \xrightarrow{\ \delta\ } C \otimes C \xrightarrow{\ \phi \otimes \psi\ } A \otimes A \xrightarrow{\ \mu\ } A\ .$$

The unit of Hom(C, A) is the composite

$$C \xrightarrow{\ \varepsilon\ } \mathbf{C} \xrightarrow{\ \eta\ } A$$

where $\eta(\lambda) = \lambda.1$. If $f : C' \longrightarrow C$ is a morphism of coalgebras and $g : A \longrightarrow A'$ is a morphism of algebras then the assignment $\phi \longmapsto g\phi f$ is a morphism of algebras

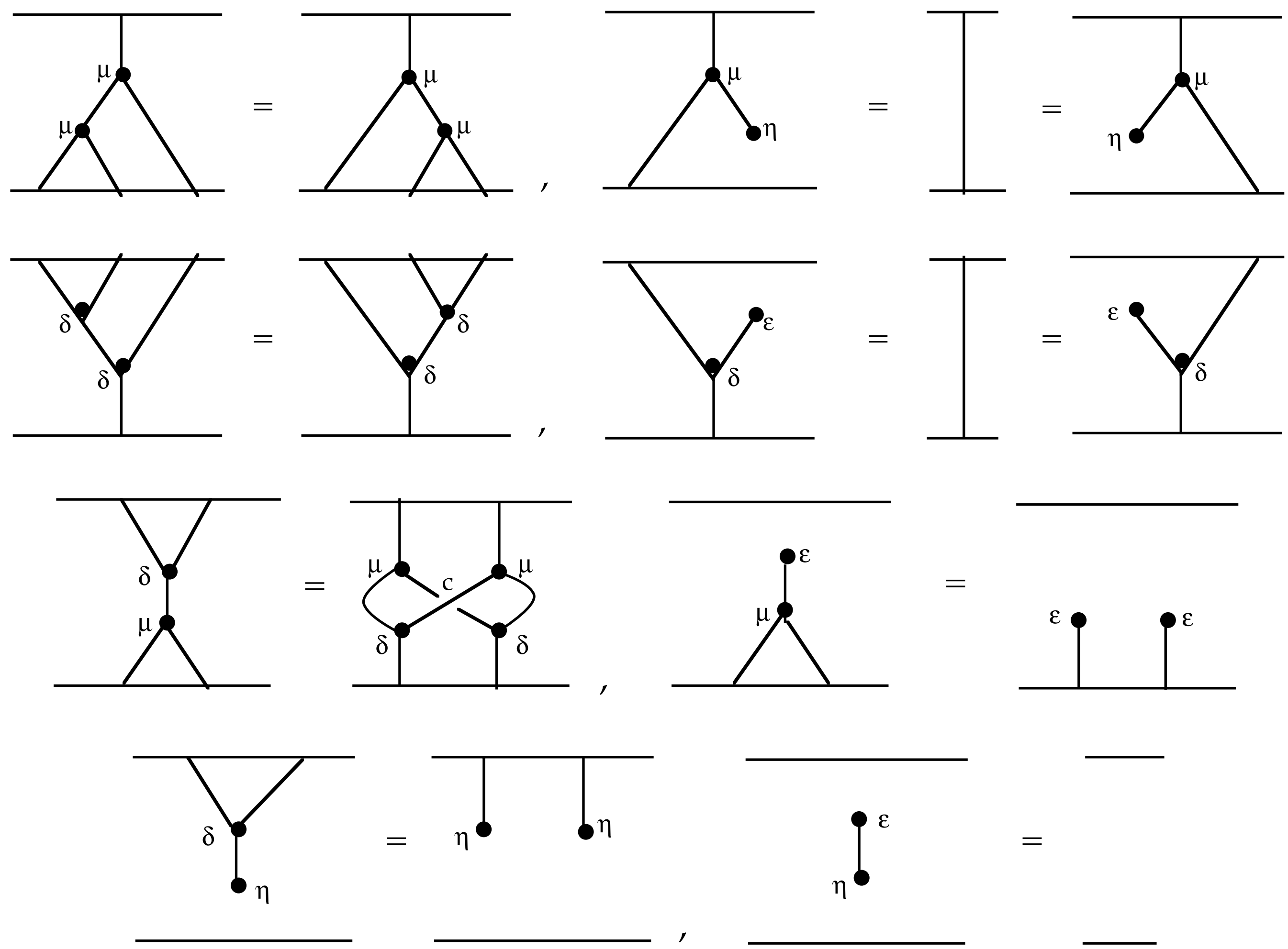
$$\mathrm{Hom}(C, A) \longrightarrow \mathrm{Hom}(C', A').$$

When $C = A = H$ is a bialgebra, we obtain an algebra structure on Hom(H, H). An *antipode* ν on a bialgebra H is a two-sided inverse for the identity map $1_H : H \longrightarrow H$ with respect to the convolution product. More explicitly, this means that the following two diagrams commute.

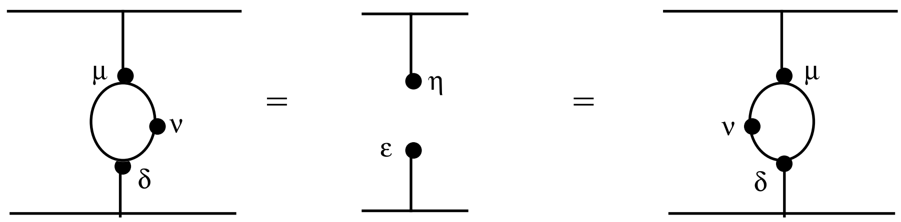
$$\begin{tikzcd}
 & H\otimes H \arrow[r, "\nu\otimes 1"] & H\otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\delta"] \arrow[dr, "\varepsilon"'] & & & H \\
 & \mathbf{C} \arrow[urr, "\eta"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H\otimes H \arrow[r, "1\otimes\nu"] & H\otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\delta"] \arrow[dr, "\varepsilon"'] & & & H \\
 & \mathbf{C} \arrow[urr, "\eta"'] & &
\end{tikzcd}$$

An antipode, when it exists, is unique. A bialgebra with an antipode is a *Hopf algebra*.

The translation of all the axioms on a bialgebra into pictorial notation [JS2] leads to the following diagrams.



Now we add to these the two axioms for an antipode ν .



Proposition 2. *In any Hopf algebra* H, *the antipode* $\nu : H \longrightarrow H$ *is an anti-endo-morphism of the algebra structure and of the coalgebra structure.*

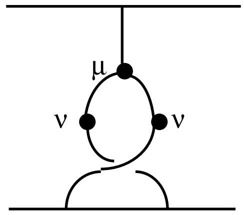
Proof. We will compute the convolution inverse of the multiplication $\mu : H \otimes H \longrightarrow H$ in the algebra $\mathrm{Hom}(H \otimes H, H)$. First, the map $\phi \longmapsto \phi\,\mu$ is an algebra morphism

$$\mathrm{Hom}(H, H) \longrightarrow \mathrm{Hom}(H \otimes H, H)$$

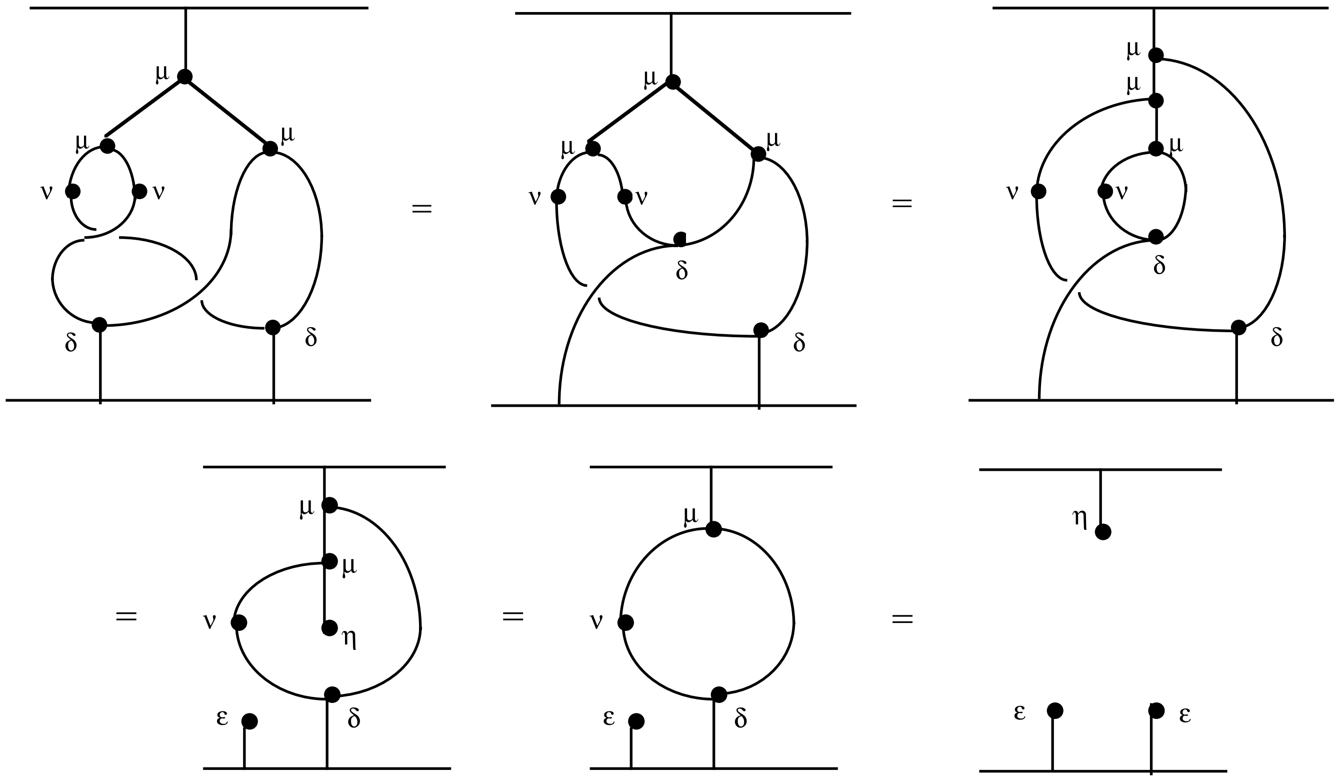
since $\mu : H \otimes H \longrightarrow H$ is a coalgebra morphism. This proves that $\nu\,\mu$ is the convolution inverse of $1_H\,\mu = \mu$. Secondly, we shall verify, by a direct pictorial computation, that the map

$$\mu' = (H \otimes H \xrightarrow{c} H \otimes H \xrightarrow{\nu \otimes \nu} H \otimes H \xrightarrow{\mu} H)$$

is a convolution inverse to μ. The diagram for μ' is as follows.



Starting with the diagram which expresses the product $\mu' * \mu$, we obtain the following sequence of diagrams having the same value:



The last picture is the unit of the algebra $\mathrm{Hom}(H \otimes H, H)$, so we have proved that μ' is the convolution inverse of μ. By uniqueness of convolution inverse, we obtain $\nu\,\mu = \mu'$, which means that ν is an anti-endomorphism of the algebra structure of H. Using a similar argument (invert the diagrams and replace ε, μ by η, δ), we obtain that ν is an anti-endomorphism of the coalgebra structure. **qed**

Now consider the bialgebra $H' = (H, \delta, \mu')$ obtained by reversing the multiplication μ; that is, $\mu'(x \otimes y) = \mu(y \otimes x)$.

Proposition 3. *For any Hopf algebra* H, *an antipode* ν' *exists for* H' *if and only if the antipode* ν *of* H *is bijective. In this case,* $\nu' = \nu^{-1}$.

Proof. If ν is bijective, composition with ν^{-1} defines an algebra morphism $\mathrm{Hom}(H,H) \longrightarrow \mathrm{Hom}(H,H')$ since $\nu^{-1} : H \longrightarrow H'$ is an algebra morphism. Therefore $1_H = \nu^{-1}\nu$ and $\nu^{-1} = \nu^{-1} 1_H$ are mutually inverse in $\mathrm{Hom}(H,H')$. This shows that ν^{-1} is an antipode for H'. Conversely, if ν' exists, composition with ν' gives an algebra morphism $\mathrm{Hom}(H,H) \longrightarrow \mathrm{Hom}(H,H')$, and therefore $\nu' = \nu' 1_H$ and $\nu'\nu$ are mutually inverse in $\mathrm{Hom}(H,H')$. By uniqueness of inverses, $\nu'\nu = 1_H$. A similar argument gives $\nu\nu' = 1_H$ which proves the result. **qed**

Recall that the category $\mathcal{C}omod_f(\mathrm{H})$ of finite dimensional (right) comodules over a bialgebra H is a tensor category.

Proposition 4. *Let* H *be a bialgebra. Then* H *has an antipode if and only if every finite dimensional (right)* H-*comodule has a left dual.*

Proof. Let $\alpha : V \longrightarrow V\otimes H$ be a finite dimensional comodule. If $e_1, \ldots, e_n$ is a basis of V then we have

$$\alpha(e_i) = \sum_{j=1}^{n} e_i \otimes \alpha_{ji}, \quad \delta(\alpha_{ij}) = \sum_{k=1}^{n} \alpha_{ik} \otimes \alpha_{kj}, \quad \varepsilon(\alpha_{ij}) = \delta_{ij}.$$

It follows that

$$\sum_{k=1}^{n} \nu(\alpha_{ik})\, \alpha_{kj} = \mu(\nu \otimes 1_H)\, \delta(\alpha_{ij}) = \mu(\varepsilon(\alpha_{ij})) = \delta_{ij},$$

and similarly that

$$\sum_{k=1}^{n} \alpha_{ik}\, \nu(\alpha_{kj}) = \delta_{ij}.$$

Hence $\alpha = (\alpha_{ij})$ is an invertible matrix. If we put $\beta = (\beta_{ij}) = (\nu\, \alpha_{ji}) = {}^t(\alpha^{-1})$ then we have a coaction $\beta : V^* \longrightarrow V^*\otimes H$ defined as

$$\beta(e_i^*) = \sum_{j=1}^{n} e_i^* \otimes \beta_{ji}.$$

We have seen earlier in this Section that such a coaction is *left dual* to α in the category $\mathcal{C}o_f(\mathrm{H})$. It remains to see that β provides a comodule structure. But we have

$$\begin{aligned} \delta\beta_{ij} &= \delta\nu\alpha_{ji} = c(\nu\otimes\nu)\,\delta\alpha_{ji} = c\textstyle\sum_k \nu(\alpha_{jk})\otimes\nu(\alpha_{ki}) \\ &= \textstyle\sum_k \nu(\alpha_{ki})\otimes\nu(\alpha_{jk}) = \textstyle\sum_k \beta_{ik}\otimes\beta_{kj}, \\ \varepsilon\beta_{ij} &= \varepsilon\nu\alpha_{ji} = \varepsilon\alpha_{ji} = \delta_{ji} = \delta_{ij}. \end{aligned}$$

The rest of this Proposition follows from the next Proposition 5. **qed**

Proposition 5. *Let* $\mathcal{C}$ *be a tensor category and* $X : \mathcal{C} \longrightarrow \mathcal{V}ect_f$ *be a tensor functor. If every object of* $\mathcal{C}$ *has a left dual then the coalgebra* $\mathrm{End}^\vee(X)$ *has an antipode.*

Proof. For all objects $C\in\mathcal{C}$, we have an exact pairing $\varepsilon : C^\ell\otimes C \longrightarrow I$. The vector space $X(C^\ell)$ is then a dual $X(C)^*$ of $X(C)$, since tensor functors preserve dual pairs. Therefore, for all $A\in\mathrm{End}(X(C))$, there is a transposed endomorphism ${}^tA\in\mathrm{End}(X(C^\ell))$. Let us put

$$\nu([A]) = [{}^tA].$$

Then ν is easily seen to be a well defined linear endomorphism of $\mathrm{End}^\vee(X)$. It remains to prove that ν is an antipode. Let $e_1, \ldots, e_n$ be a basis of $X(C)$ and let $e_1^*, \ldots, e_n^*$ be the dual basis of $X(C^\ell)$ (the pairing is $X(\varepsilon)$). Then, for all $\phi\in X(C^\ell)$ and $x\in X(C)$, we have

$$\delta[\phi\otimes x] = \textstyle\sum_i [\phi\otimes e_i]\otimes[e_i^*\otimes x] \quad \text{and} \quad \nu[\phi\otimes x] = [x\otimes\phi].$$

So that we have

$$\mu(\nu\otimes 1)\,\delta[\phi\otimes x] = \textstyle\sum_i [e_i\otimes\phi]\,[e_i^*\otimes x] = \textstyle\sum_i [(e_i\otimes e_i^*)\otimes(\phi\otimes x)] = [t\otimes(\phi\otimes x)]$$

where $t = \sum_i e_i \otimes e_i^*$ is the linear form $X(\varepsilon)$ on the space $X(C^\ell)\otimes X(C)$ (so that t is also just the trace map on $X(C)^* \otimes X(C)$). If $1 : \mathbf{C} \longrightarrow \mathbf{C}$ denotes the identity form, we have

$$[t\otimes (\phi \otimes x)] = [1\, X(\varepsilon)\otimes (\phi \otimes x)] = [1\otimes X(\varepsilon)(\phi \otimes x)] = [\phi(x)] = \eta_0 \varepsilon_0([\phi \otimes x])$$

where η_0, ε_0 here denote the unit and counit of H. **qed**

Definition. A Hopf algebra is *autonomous* when the antipode is a bijective map.

Proposition 6. *Let* H *be a bialgebra. The category* $\mathcal{Comod}_f(H)$ *is autonomous if and only if* H *is an autonomous Hopf algebra.*

Proof. Let H' be the bialgebra obtained by reversing the multiplication of H. Let $\mathcal{Comod}_f(H)'$ be the tensor category obtained by reversing the tensor product on $\mathcal{Comod}_f(H)$, so that $V\otimes' W = W\otimes V$. Then the map c given by $c(x\otimes y) = y\otimes x$ is an isomorphism between $V\otimes' W$ and $V\otimes W$ computed as H'-comodules. This shows that we have a canonical isomorphism of tensor categories between $\mathcal{Comod}_f(H')$ and $\mathcal{Comod}_f(H)'$. A right dual in $\mathcal{Comod}_f(H)$ is a left dual in $\mathcal{Comod}_f(H)'$. The result now follows from Propositions 3, 4, 5. **qed**

§10. Braidings and Yang-Baxter operators.

Recall [JS1] that a *braiding* for a tensor category $\mathcal{V}$ consists of a natural family of isomorphisms

$$c = c_{A,B} : A\otimes B \xrightarrow{\sim} B\otimes A$$

in $\mathcal{V}$ such that the following two diagrams commute (where the unnamed arrows are associativity constraints).

$$\begin{tikzcd}
 & (B\otimes A)\otimes C \arrow[r, "\sim"] & B\otimes(A\otimes C) \arrow[dr, "B\otimes c"] & \\
(A\otimes B)\otimes C \arrow[ur, "c\otimes C"] \arrow[dr, "\sim"'] & & & B\otimes(C\otimes A) \\
 & A\otimes(B\otimes C) \arrow[r, "c"'] & (B\otimes C)\otimes A \arrow[ur, "\sim"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A\otimes(C\otimes B) \arrow[r, "\sim"] & (A\otimes C)\otimes B \arrow[dr, "c\otimes B"] & \\
A\otimes(B\otimes C) \arrow[ur, "A\otimes c"] \arrow[dr, "\sim"'] & & & (C\otimes A)\otimes B \\
 & (A\otimes B)\otimes C \arrow[r, "c"'] & C\otimes(A\otimes B) \arrow[ur, "\sim"'] &
\end{tikzcd}$$

It follows from these axioms that $c_{A,I} : A\otimes I \xrightarrow{\sim} I\otimes A$ is equal to the canonical isomorphism $A\otimes I \xrightarrow{\sim} A \xrightarrow{\sim} I\otimes A$. Similarly for $c_{I,A} : I\otimes A \xrightarrow{\sim} A\otimes I$.

If c is a braiding then so is c' given by $c'_{A,B} = (c_{B,A})^{-1}$. A *symmetry* is a braiding for which $c = c'$.

A *braided tensor category* is a pair $(\mathcal{V}, c)$ consisting of a tensor category $\mathcal{V}$ and a braiding c.

Example 1. Let $\mathbf{B}_n$ be the Artin braid group. A presentation for $\mathbf{B}_n$ is given by the generators $s_1, \ldots, s_{n-1}$ and the relations

(A1) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \le i \le n-2$,
(A2) $s_i s_j = s_j s_i$ for $1 \le i < j-1 \le n-2$.

The *braid category* $\mathbf{B}$ is the disjoint union of the $\mathbf{B}_n$. More explicitly, the objects of $\mathbf{B}$ are the natural numbers $0, 1, 2, \ldots,$ the homsets are given by

$$\mathbf{B}(m, n) = \begin{cases} \mathbf{B}_n & \text{when } m = n \\ \varnothing & \text{otherwise} \end{cases},$$

and composition is the multiplication of the braid groups. The category $\mathbf{B}$ is equipped with a strict tensor structure defined by *addition of braids*

$$\oplus : \mathbf{B}_m \times \mathbf{B}_n \longrightarrow \mathbf{B}_{m+n}$$

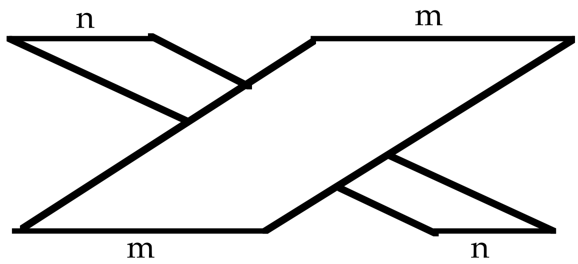
which is algebraically described by

$$s_i \oplus s_j = s_i \, s_{m+j} .$$

A braiding for $\mathbf{B}$ is given by the elements

$$c = c_{m,n} : m + n \longrightarrow n + m$$

illustrated by the following figure.



Theorem 1 [JS1]. $\mathbf{B}$ *is the free braided strict tensor category on one generating object.*

Example 2 [FY1]. Let G be an arbitrary fixed (discrete) group. A *crossed* G-*set* is a G-set together with a function

$$| \; | : X \longrightarrow G$$

satisfying the condition $|g\,x| = g\,|x|\,g^{-1}$. A *morphism* $f : X \longrightarrow Y$ of crossed G-sets is a function satisfying $f(g\,x) = g\,f(x)$ and $|f(x)| = |x|$. We have a category $\mathcal{Cr}(G)$ of crossed G-sets. This becomes a tensor category on taking the tensor product of crossed G-sets X, Y to be their cartesian product together with $|(x, y)| = |x|\,|y|$. A braiding $c = c_{X,Y}$: $X \otimes Y \xrightarrow{\sim} Y \otimes X$ for the tensor category $\mathcal{Cr}(G)$ is given by

$$c(x, y) = (|x|\,y, x) .$$

Example 3. Let R be a commutative ring. The category $\mathbf{Z}\mathcal{Mod}(R)$ of $\mathbf{Z}$-graded R-modules has a well-known tensor product:

$$(A \otimes B)_n = \sum_{p+q=n} A_p \underset{R}{\otimes} B_q .$$

For any invertible element $k \in R$, we can define a braiding via the formula

$$c(x \otimes y) = k^{pq} \, y \otimes x$$

for $x \in A_p$ and $y \in B_q$. When $k = -1$ we get the usual (anti-)symmetry on graded modules.

Example 4. The character group of the circle group $\mathbf{T} = U(1)$ is isomorphic to $\mathbf{Z}$. This shows that any representation $V \in \mathcal{Rep}(\mathbf{T}, \mathbf{C})$ splits as a direct sum of isotypical components

$$V = \sum_{k \in \mathbf{Z}} V_k$$

where the action of $z \in \mathbf{T}$ on $x \in V_k$ is equal to $z^k x$. The tensor category $\mathcal{Rep}(\mathbf{T}, \mathbf{C})$ is actually isomorphic to the category of $\mathbf{Z}$-graded $\mathbf{C}$-modules. This shows that, for any non-zero complex number $k \in \mathbf{C}$, we can define a braiding on $\mathcal{Rep}(\mathbf{T}, \mathbf{C})$ via the formula in the previous example.

Example 5. The centre of the unitary group $U(n)$ is a one-dimensional torus $\mathbf{T} = \{ zI :$

$z \in U(1)$ }. If we restrict to $\mathbf{T}$ the action of $U(n)$ on $V \in \mathcal{Rep}(U(n), \mathbf{C})$, we obtain a splitting

$$V = \sum_{k \in \mathbf{Z}} V_k$$

where the V_k are stable under the action of $U(n)$ since $\mathbf{T}$ is in the centre of $U(n)$. This shows that we can transfer the braiding of Example 4 to the category $\mathcal{Rep}(U(n), \mathbf{C})$.

Example 6. The centre of the group $SU(n)$ is the group $\mu_n \subset \Pi$ of n-th roots of unity. The dual group $\mu_n^\vee$ is isomorphic to $\mathbf{Z}/n$. For any $V \in \mathcal{Rep}(SU(n), \mathbf{C})$, we obtain a splitting

$$V = \sum_{k \in \mathbf{Z}/n} V_k \ .$$

If $k \in \mathbf{C}$ is an n-th root of unity, there is a braiding defined by

$$c(x \otimes y) = k^{pq}\ y \otimes x$$

where $p, q \in \mathbf{Z}/n$ are the degrees of x and y. When $n = 2$, the braiding is a symmetry. If we choose $k = -1$, the odd degree representations behave differently from the even ones under permutation symmetry. In theoretical physics, this is the mathematical structure which distinguishes fermions (odd representations) from bosons (even representations). When $n = 3$, we obtain a three-fold classification of the irreducible representations of $SU(3)$. If k is a primitive cubic root of unity, the braiding is not a symmetry and the braid group takes the place of the symmetric group as acting on tensor powers of $V^{\otimes n}$. In this case however, the operator

$$c = c_{V,V} : V \otimes V \xrightarrow{\sim} V \otimes V$$

satisfies the quadratic equation

$$c^2 + c + 1 = 0,$$

so that we are not too far away from the symmetric case (characterized by the equation $c^2 = 1$). The group $SU(3)$ is the one used by theoretical physicists for the theory of quarks. Is there any physical significance to the above braiding?

Example 7. Example 3 can be generalized as follows. For any abelian group A, let $A\mathcal{Mod}(R)$ be the category of A-graded R-modules. Let $k : A \times A \longrightarrow R^\times$ be a pairing of the (additive) group A into the (multiplicative) group $R^\times$ of invertible elements of R. We can define a braiding via the formula

$$c(x \otimes y) = k(p, q)\ y \otimes x$$

where x, y are homogeneous of degree p, q, respectively.

Example 8. Examples 4, 5 and 6 can be generalized by taking a compact group whose centre contains the dual group $A^\vee$ and where the pairing is $k : A \times A \longrightarrow \mathbf{C}^\times$.

Example 9. For any bialgebra A, the category $\mathcal{Mod}(A)$ of (left) A-modules is a tensor category. A braiding c on $\mathcal{Mod}(A)$ is completely determined by the element $\gamma = c_{A,A}(1 \otimes 1)$. To see this, let M, N be two A-modules. For any $x \in M$, $y \in N$, we have the commutative square below, in which $x : A \longrightarrow M$, $y : A \longrightarrow N$ send $1 \in A$ to $x \in M$, $y \in N$, respectively.

$$\begin{tikzcd}
A\otimes A \arrow[r, "c_{A\otimes A}"] \arrow[d, "x\otimes y"'] & A\otimes A \arrow[d, "y\otimes x"] \\
M\otimes N \arrow[r, "c_{M\otimes N}"'] & N\otimes M
\end{tikzcd}$$

This proves that

$$c_{M,N}(x\otimes y) = \gamma\ (y\otimes x)$$

where the right-hand $A\otimes A$-module structure on $N\otimes M$ is used. Of course, this element $\gamma\in A\otimes A$ must satisfy certain conditions. The first is that the assignment $x\otimes y \longmapsto \gamma\ (y\otimes x)$ should be A-linear. To express this condition, let us use the notation $\xi_{ijk\ldots}$ for the image of ξ under the *usual* canonical isomorphism

$$\sigma_{ijk\ldots}\ :\ M_1\otimes M_2\otimes M_3\otimes\ldots \xrightarrow{\ \sim\ } M_i\otimes M_j\otimes M_k\otimes\ldots$$

induced by the permutation $1\,2\,3\ldots \longmapsto i\,j\,k\ldots$. The formula for the braiding is then

$$c\,(x\otimes y) = \gamma\,.\,(x\otimes y)_{21}\ .$$

The A-linearity of c then amounts to the equation

$$\gamma\,.\,(\,\delta(a)\,.\,(x\otimes y)\,)_{21} = \delta(a)\,.\,\gamma\,.\,(x\otimes y)_{21}\,.$$

This equation is valid for all $a\in A$ and all x, y if and only if, for all $a\in A$,

$$\gamma\,.\,\delta(a)_{21} = \delta(a)\,.\,\gamma\ .$$

The reader might enjoy proving the following result: the braidings on $\mathcal{Mod}(A)$ are in bijection with the invertible elements $\gamma\in A\otimes A$ which satisfy the two equations below in addition to the one above.

$$(\delta\otimes 1_A)(\gamma) = (1\otimes\gamma)\,.\,(\gamma\otimes 1)_{132}\ ,\qquad (1_A\otimes\delta)(\gamma) = (\gamma\otimes 1)\,.\,(1\otimes\gamma)_{213}\ .$$

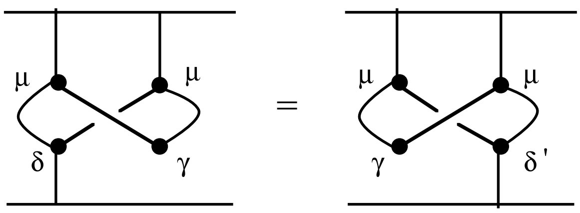
These equations can be written more elegantly as

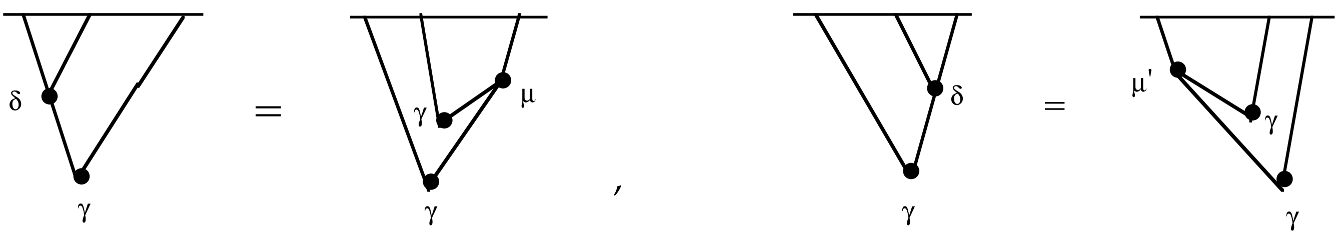
$$(\delta\otimes 1_A)(\gamma) = s_{23}(\gamma)\ s_{13}(\gamma)\,,\qquad (\delta\otimes 1_A)(1_A\otimes\delta)(\gamma) = s_{12}(\gamma)\ s_{13}(\gamma)$$

where we are using the insertion operators $s_{ij}\ :\ A\otimes A \longrightarrow A\otimes A\otimes A$ defined by

$$s_{12}(x\otimes y) = x\otimes y\otimes 1\,,\quad s_{23}(x\otimes y) = 1\otimes x\otimes y\,,\quad s_{13}(x\otimes y) = x\otimes 1\otimes y\,.$$

Translated into diagrammatic notation, the three equations for γ are as follows, where we put $\mu'(x\otimes y) = y\otimes x$ and $\delta' = \delta_{21}$.





To accommodate our next example, we need the following result which holds for any coalgebra C. Let $n \geq 1$ be an integer and let $\phi\in(C^{\otimes n})^*$ be a linear form. For any n-sequence $(V_1, \ldots, V_n)$ of C-comodules, we can define an operator

$$\underline{\phi\,|} \ : \ V_1 \otimes \ldots \otimes V_n \longrightarrow V_1 \otimes \ldots \otimes V_n$$

by using the fact that $V_1 \otimes \ldots \otimes V_n$ is a comodule over the coalgebra $C \otimes C \otimes \ldots \otimes C = C^{\otimes n}$. More precisely, for all $x_1 \in V_1, \ldots, x_n \in V_n$, we have

$$\underline{\phi\,|}\,(x_1 \otimes \ldots \otimes x_n) = (1 \otimes \phi)\ \alpha(x_1 \otimes \ldots \otimes x_n)$$

where we have used the $C^{\otimes n}$–comodule structure

$$\alpha \ : \ V_1 \otimes \ldots \otimes V_n \longrightarrow (V_1 \otimes \ldots \otimes V_n) \otimes C^{\otimes n}.$$

If $\mathbf{U}^{\otimes n} : \mathcal{Comod}(C)^n \longrightarrow \mathcal{Vect}_C$ denotes the functor

$$(V_1, \ldots, V_n) \longmapsto V_1 \otimes \ldots \otimes V_n$$

then we have defined a natural transformation

$$\underline{\phi\,|}\,(\ ) \ : \ \mathbf{U}^{\otimes n} \longrightarrow \mathbf{U}^{\otimes n} \ .$$

Proposition 2. *The assignment* $\phi \longmapsto \underline{\phi\,|}\,(\ )$ *is an algebra isomorphism*

$$(C^{\otimes n})^* \xrightarrow{\ \sim\ } \mathrm{Hom}(\mathbf{U}^{\otimes n}, \mathbf{U}^{\otimes n}) \ .$$

Proof. This is a consequence of Section 7 Proposition 4 and Section 8 Proposition 1. **qed**

For any bialgebra A, the category $\mathcal{Comod}(A)$ of (right) comodules is a tensor category. If c is a braiding on $\mathcal{Comod}(A)$, we obtain a linear form

$$\gamma \ = \ (A \otimes A \xrightarrow{c_{A,A}} A \otimes A \xrightarrow{\varepsilon \otimes \varepsilon} \mathbf{C}) \ .$$

To state the next Proposition we shall use the insertion operators $s_{ij} : (A{\otimes}A)^* \longrightarrow (A{\otimes}A{\otimes}A)^*$ defined by

$$s_{12} = {}^t(A{\otimes}A{\otimes}\varepsilon)\,, \quad s_{23} = {}^t(\varepsilon{\otimes}A{\otimes}A)\,, \quad s_{13} = {}^t(A{\otimes}\varepsilon{\otimes}A)\ .$$

Proposition 3. *The assignment* $c \longmapsto \gamma$ *described above is a bijection between braidings* c *on* $\mathcal{Comod}(A)$ *and linear forms* $\gamma \in (A{\otimes}A)^*$ *which are invertible for the convolution product* $*$ *and satisfy the following identities:*

$$\mu' * \gamma = \gamma * \mu\,, \quad \gamma\,(1_A \otimes \mu) = s_{13}(\gamma) * s_{12}(\gamma)\,, \ \ \gamma\,(\mu \otimes 1_A) = s_{13}(\gamma) * s_{23}(\gamma)\,.$$

Proof (Sketch). The braiding is obtained from γ by the formula

$$c\,(x \otimes y) = (\ \underline{\gamma\,|}\ x \otimes y)_{21} \ = \ \underline{\gamma_{21}|}\ y \otimes x \ .$$

The situation is then entirely dual to that of Example 9. We can obtain these equations by rotating the pictures in that Example through 180°. **qed**

The second and third equations of the above Proposition 3 can also be written:

$$\gamma(x, yz) = \sum_i \gamma(x'_i, z)\ \gamma(x''_i, y)\,, \quad \gamma(yz, x) = \sum_i \gamma(y, x'_i)\ \gamma(z, x''_i) \quad \text{where } \ \delta(x) = \sum_i x'_i \otimes x''_i\ .$$

Recall [J, D] that a *Yang-Baxter operator* on a vector space V is a linear isomorphism

$$R \ : \ V \otimes V \xrightarrow{\ \sim\ } V \otimes V$$

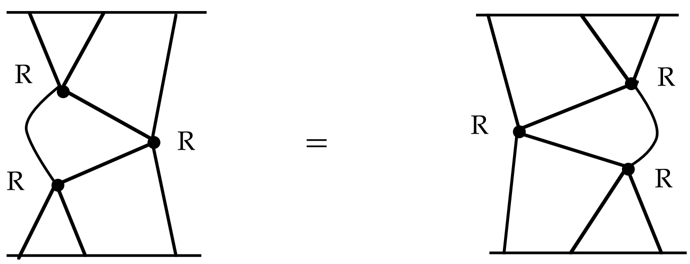
such that the following hexagon commutes.

$$\begin{tikzcd}
 & V\otimes V\otimes V \arrow[r, "R\otimes 1"] & V\otimes V\otimes V \arrow[dr, "1\otimes R"] & \\
V\otimes V\otimes V \arrow[ur, "1\otimes R"] \arrow[dr, "R\otimes 1"'] & & & V\otimes V\otimes V \\
 & V\otimes V\otimes V \arrow[r, "1\otimes R"'] & V\otimes V\otimes V \arrow[ur, "R\otimes 1"'] &
\end{tikzcd}$$

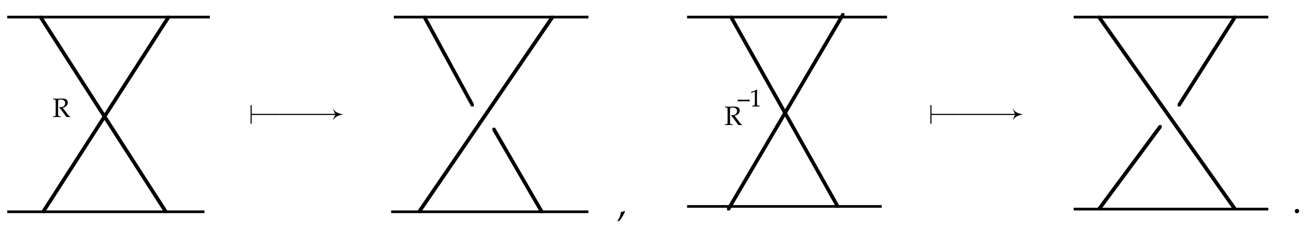
The equation

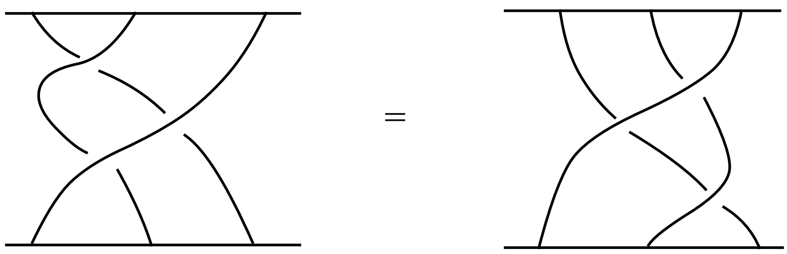
$$(R\otimes 1)(1\otimes R)(R\otimes 1) = (1\otimes R)(R\otimes 1)(1\otimes R)$$

is called the Yang-Baxter equation. The translation into pictures is:



The rule of the game is to replace (whenever possible) these planar string diagrams by 3-dimensional ones in which crossings replace the nodes labelled by R or R^{-1}, as indicated in the following picture. The Yang-Baxter equation is then depicted as the equality shown after that.





An example of a Yang-Baxter operator [J, T, JS3] is the following. Let $e_1, \ldots, e_n$ be a basis for V, and let $q \in \mathbf{C}$ be a non-zero complex number. We define $R = R_q : V\otimes V \xrightarrow{\sim} V\otimes V$ as follows:

$$R(e_i \otimes e_j) = \begin{cases} e_j \otimes e_i & \text{for } i > j \\ e_j \otimes e_i + (q - q^{-1}) e_i \otimes e_j & \text{for } i < j \\ q\, e_i \otimes e_i & \text{for } i = j \ . \end{cases}$$

The operator R satisfies the equation

$$(R - q)\,(R + q^{-1}) = 0.$$

Moreover, the inverse of R is given by

$$R^{-1}(e_i \otimes e_j) = \begin{cases} e_j \otimes e_i & \text{for } i < j \\ e_j \otimes e_i + (q^{-1} - q)\, e_i \otimes e_j & \text{for } i > j \\ q^{-1}\, e_i \otimes e_i & \text{for } i = j \end{cases} .$$

One can check directly that this R is a Yang-Baxter operator.

Given any Yang-Baxter operator R on V, we can define, for every $n \geq 0$, a representation π_R of the braid group $\mathbf{B}_n$ in the general linear group $GL(V^{\otimes n})$ by putting

$$\pi_R(s_i) = \overbrace{1 \otimes \ldots \otimes 1}^{i-1} \otimes R \otimes 1 \otimes \ldots \otimes 1$$

for each generator s_i of $\mathbf{B}_n$. Putting together these representations, we obtain a tensor functor

$$\pi_R : \mathbf{B} \longrightarrow \mathcal{V}ect_{\mathbb{C}} .$$

Proposition 4. *The correspondence* $R \longmapsto \pi_R$ *is a bijection between the Yang-Baxter operators on* V *and the (isomorphism classes of) tensor functors* $\pi : \mathbf{B} \longrightarrow \mathcal{V}ect_{\mathbb{C}}$ *such that* $\pi(1) = V$.

More generally, let $T : \mathcal{A} \longrightarrow \mathcal{V}$ be a functor from a category $\mathcal{A}$ to a tensor category $\mathcal{V}$.

Definition [JS3]. A Yang-Baxter operator on T is a natural family of isomorphisms

$$\mathbf{y} = \mathbf{y}_{A,B} : TA \otimes TB \xrightarrow{\sim} TB \otimes TA$$

such that the following hexagon commutes.

$$\begin{tikzcd}
 & TA \otimes TC \otimes TB \arrow[r, "\mathbf{y} \otimes 1"] & TC \otimes TA \otimes TB \arrow[dr, "1 \otimes \mathbf{y}"] & \\
TA \otimes TB \otimes TC \arrow[ur, "1 \otimes \mathbf{y}"] \arrow[dr, "\mathbf{y} \otimes 1"'] & & & TC \otimes TB \otimes TA \\
 & TB \otimes TA \otimes TC \arrow[r, "1 \otimes \mathbf{y}"'] & TB \otimes TC \otimes TA \arrow[ur, "\mathbf{y} \otimes 1"'] &
\end{tikzcd}$$

Any functor $T : \mathcal{A} \longrightarrow \mathcal{V}$ into a braided tensor category $\mathcal{V}$ comes equipped with a Yang-Baxter operator obtained from the braiding of $\mathcal{V}$:

$$\mathbf{y}_{A,B} = c_{TA,TB} : TA \otimes TB \xrightarrow{\sim} TB \otimes TA .$$

The importance of Yang-Baxter operators is partly explained by the following considerations. For any category $\mathcal{A}$, there is a braided tensor category $\mathbf{B} \int \mathcal{A}$ of braids having their strings labelled by arrows of $\mathcal{A}$. (The notation $\mathbf{B} \int \mathcal{A}$ is intended to indicate that it is a wreath product in a generalized sense [K, JS3].) The objects of $\mathbf{B} \int \mathcal{A}$ are finite sequences of objects of $\mathcal{A}$. An arrow

$$(\alpha, f_1, \ldots, f_n) : (A_1, \ldots, A_n) \longrightarrow (B_1, \ldots, B_n)$$

consists of $\alpha \in \mathbf{B}_n$ and $f_i \in \mathcal{A}(A_i, B_{\alpha(i)})$ where $i \longmapsto \alpha(i)$ is the permutation defined by α .

Composition of labelled braids is performed by composing the label on each string of the composite braid. The operation of addition of braids extends in the obvious way to labelled braids $\mathbf{B}\int\mathcal{A} \times \mathbf{B}\int\mathcal{A} \longrightarrow \mathbf{B}\int\mathcal{A}$, yielding a tensor structure on $\mathbf{B}\int\mathcal{A}$. There is an obvious braiding on $\mathbf{B}\int\mathcal{A}$ obtained from the braiding on $\mathbf{B}$. We have an inclusion functor

$$\mathrm{i} : \mathcal{A} \longrightarrow \mathbf{B}\int\mathcal{A}$$

identifying $\mathcal{A}$ with the labelled braids with a single string. The braiding on $\mathbf{B}\int\mathcal{A}$ defines a (formal) Yang-Baxter operator $\mathbf{z}$ on the functor i. The next Proposition explains the sense in which this $\mathbf{z}$ is universal.

Proposition 5 [JS3]. *The braided tensor category* $\mathbf{B}\int\mathcal{A}$ *is free on* $\mathcal{A}$. *Moreover, for any tensor category* $\mathcal{V}$ *and any pair* $(\mathbf{y}, \mathrm{T})$, *where* $\mathbf{y}$ *is a Yang-Baxter operator on* $\mathrm{T} : \mathcal{A} \longrightarrow \mathcal{V}$, *there exists a unique (up to a unique isomorphism) tensor functor* $\mathrm{T}' : \mathbf{B}\int\mathcal{A} \longrightarrow \mathcal{V}$ *such that* $\mathrm{T}'(\mathbf{z}) = \mathbf{y}$ *and the following triangle commutes.*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathrm{i}"] \arrow[dr, "\mathrm{T}"'] & \mathrm{B}\int\mathcal{A} \arrow[d, "\mathrm{T}'"] \\
 & \mathcal{V}
\end{tikzcd}$$

Example 10. For any algebra A, a Yang-Baxter operator on the forgetful functor

$$\mathrm{U} : \mathcal{M}od(\mathrm{A}) \longrightarrow \mathcal{V}ect_{\mathbb{C}}$$

is completely determined by the element

$$\gamma = \mathbf{y}_{\mathrm{A},\mathrm{A}}(1 \otimes 1) \in \mathrm{A}\otimes\mathrm{A} .$$

The operator is given by

$$\mathbf{y}_{\mathrm{M},\mathrm{N}}(x \otimes y) = \gamma\,(y \otimes x) .$$

Apart from invertibility, the only condition on γ is the equation

$$s_{23}(\gamma)\ s_{13}(\gamma)\ s_{12}(\gamma) = s_{12}(\gamma)\ s_{13}(\gamma)\ s_{23}(\gamma)$$

where $s_{ij} : \mathrm{A}\otimes\mathrm{A} \longrightarrow \mathrm{A}\otimes\mathrm{A}\otimes\mathrm{A}$ are the insertion operators. We shall say that an invertible element $\gamma \in \mathrm{A}\otimes\mathrm{A}$ satisfying these equations is a *Yang-Baxter element* of the algebra A. It should be distinguished from the operator y that it defines. More precisely, when $\mathrm{A} = \mathrm{End(V)}$ where V is a finite dimensional vector space, a Yang-Baxter element $\gamma \in \mathrm{End(V)}\otimes\mathrm{End(V)} \cong \mathrm{End(V\otimes V)}$ defines a Yang-Baxter operator $\mathrm{R} = \gamma \circ c$ where $c : \mathrm{V}\otimes\mathrm{V} \longrightarrow \mathrm{V}\otimes\mathrm{V}$ is the usual symmetry operator.

Example 11. For any coalgebra C, a Yang-Baxter operator $\mathbf{y}$ on the forgetful functor

$$\mathbf{U} : \mathcal{C}omod(\mathrm{C}) \longrightarrow \mathcal{V}ect_{\mathbb{C}}$$

is determined by the linear form

$$\gamma = (\mathrm{C} \otimes \mathrm{C} \xrightarrow{\ \mathrm{y}_{\mathrm{C},\mathrm{C}}\ } \mathrm{C} \otimes \mathrm{C} \xrightarrow{\ \varepsilon \otimes \varepsilon\ } \mathbf{C}) .$$

We have the formula

$$\mathbf{y}(x \otimes y) = (\,\underline{\gamma}\rfloor x \otimes y)_{21} = \underline{\gamma_{21}}\rfloor y \otimes x .$$

The linear form γ is invertible in the algebra $(\mathrm{C}\otimes\mathrm{C})^*$ and satisfies the following equation in the algebra $(\mathrm{C}\otimes\mathrm{C}\otimes\mathrm{C})^*$:

$$s_{12}(\gamma) * s_{13}(\gamma) * s_{23}(\gamma) = s_{23}(\gamma) * s_{13}(\gamma) * s_{12}(\gamma)$$

where the s_{ij} are the insertion operators defined before Proposition 3.

Definition [JS3]. Suppose $\mathcal{V}$ is a braided tensor category. A (full) *twist* for $\mathcal{V}$ is a natural family of isomorphisms

$$\theta = \theta_A : A \xrightarrow{\sim} A$$

such that $\theta_I = 1$ and the following square commutes. A tensor category equipped with a braiding and a twist is called a *balanced* (or *ribbon*) tensor category.

$$\begin{tikzcd} A \otimes B \arrow[r, "c_{A,B}"] \arrow[d, "\theta_{A\otimes B}"'] & B \otimes A \arrow[d, "\theta_B \otimes \theta_A"] \\ A \otimes B & B \otimes A \arrow[l, "c_{B,A}"] \end{tikzcd}$$

For any braided bialgebra (A, γ) (Example 9), the twists on $\mathcal{Mod}(A)$ are in bijection with invertible central elements $\tau \in A$ satisfying the equations

$$\varepsilon(\tau) = 1 \quad \text{and} \quad \delta(\tau)_{21} = \gamma_{21}(\tau \otimes \tau)\,\gamma\ .$$

We have $\tau = \theta_A(1)$ and $\theta(x) = \tau x$.

Similarly, for any cobraided bialgebra (A, γ), the twists on $\mathcal{C}omod(A)$ are in bijection with the invertible central elements $\tau \in A^*$ satisfying the equations

$$\tau(1) = 1 \quad \text{and} \quad \tau \circ \mu = \gamma * (\tau \otimes \tau) * \gamma_{21}\ .$$

Definition [JS3, Sh]. A tensor category is said to be *tortile* when it is balanced and each object A has a left dual A^* satisfying

$$\theta_{A^*} = {\theta_A}^*\ .$$

Definition. A *tortile* Hopf algebra (H, γ, τ) is a Hopf algebra H equipped with a braiding γ and a twist τ such that $\nu(\tau) = \tau$ where ν is the antipode.

Definition. A *cotortile* Hopf algebra (H, γ, τ) is a Hopf algebra equipped with a cobraiding γ and a cotwist τ such that $\tau \circ \nu = \nu$.

The category of finite dimensional comodules over a cotortile Hopf algebra is a tortile tensor category.

Proposition 6. *Every tortile tensor category is autonomous.*

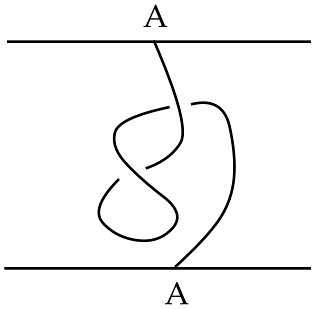
Proof. In any braided tensor category, if $(\eta, \varepsilon) : A^* \dashv A$ is an adjunction then $(\eta_1, \varepsilon_1) : A \dashv A^*$ is also an adjunction where $\varepsilon_1 = \varepsilon \circ c_{A,A^*}$ and $\eta_1 = (c_{A^*,A})^{-1} \circ \eta$. To see this, we can use the following abstract argument. On any tensor category $(\mathcal{C}, \otimes)$ there is a *reverse* tensor product $C \otimes' D = D \otimes C$. Clearly, if $(\eta, \varepsilon) : A^* \dashv A$ in $(\mathcal{C}, \otimes)$ then $(\eta, \varepsilon) : A \dashv A^*$ in $(\mathcal{C}, \otimes')$. When $\mathcal{C}$ is braided, we have a natural isomorphism

$$c = c_{C,D} : C \otimes D \xrightarrow{\sim} C \otimes' D$$

which is *coherent*: it is an isomorphism of the two tensor structures [JS3, JS4]. Using c we can transport the adjunction $(\eta, \varepsilon) : A \dashv A^*$ in $(\mathcal{C}, \otimes')$ to an adjunction $(\eta_1, \varepsilon_1) : A \dashv$

A^* in $(C,\otimes)$. That the formulas for η_1, ε_1 are as claimed is now clear. **qed**

Proposition 7. *In a tortile tensor category, the square of the twist* θ_A *is given by the following picture:*

$(\theta_A)^2 =$ 

Proof. We have the following commutative diagrams:

$$\begin{tikzcd} I \arrow[r, "\eta_A"] \arrow[d, "\theta_I = 1"'] & A\otimes A^* \arrow[r, "c_{A,A^*}"] \arrow[d, "\theta_{A\otimes A^*}"] & A^*\otimes A \arrow[d, "\theta_{A^*}\otimes\theta_A"] \\ I \arrow[r, "\eta_A"'] & A\otimes A^* & A^*\otimes A \arrow[l, "c_{A^*,A}"] \end{tikzcd} \qquad \begin{tikzcd} A\otimes A^* \arrow[r, "c_{A,A^*}"] \arrow[d, "\theta_A\otimes\theta_{A^*}"'] & A^*\otimes A \arrow[d, "\theta_{A^*}\otimes\theta_A"] \\ A\otimes A^* \arrow[r, "c_{A,A^*}"'] & A^*\otimes A \end{tikzcd}$$

which show that

$$c_{A,A^*}^{-1}\ c_{A^*,A}^{-1}\ \eta_A = (\theta_A\otimes\theta_{A^*})\ \eta_A$$

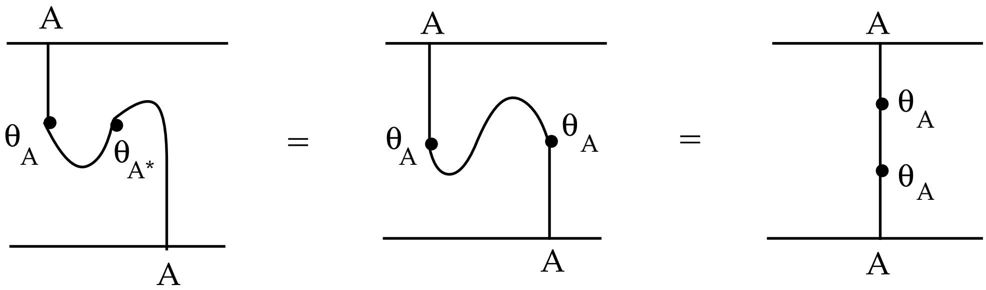
and therefore, tensoring this equality on the right with A and composing on the left with $A\otimes\varepsilon_A$, we have

$$(A\otimes\varepsilon_A)\,((c_{A,A^*}^{-1}\ c_{A^*,A}^{-1}\ \eta_A)\otimes A) = (A\otimes\varepsilon_A)\,(\theta_A\otimes\theta_{A^*}\otimes A)\,(\eta_A\otimes A)\,.$$

The left-hand side is equal to the value of the picture in the Proposition, so it remains to show that the right-hand side is equal to $(\theta_A)^2$. But we have

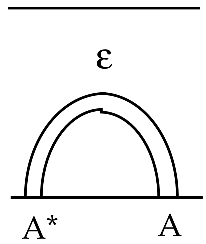
$$\varepsilon\,(\theta_{A^*}\otimes A) = \varepsilon\,(A\otimes\theta_A)$$

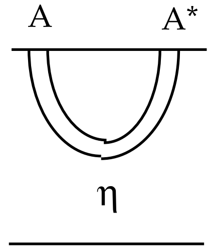
since $\theta_{A^*} = \theta_A{}^*$. The following sequence of pictures finishes the proof.



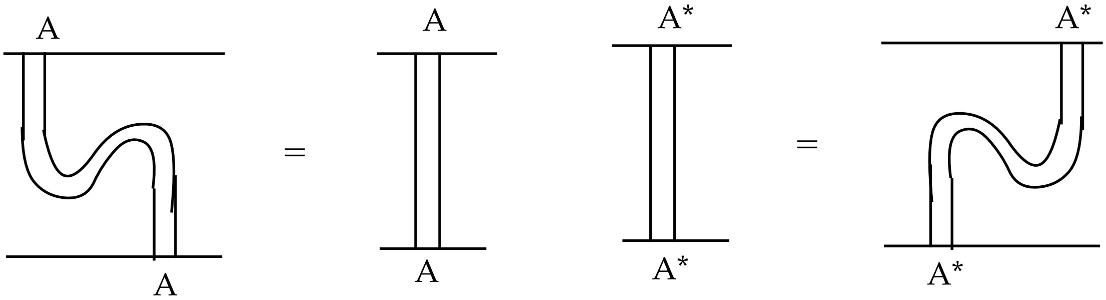
qed

The adjunction $(\eta_1, \varepsilon_1) : A \dashv A^*$ is not the appropriate one in a tortile tensor category. The reason is that, if (η_2, ε_2) denotes the pair obtained by a twofold application of the assignment $(\eta, \varepsilon) \longmapsto (\eta_1, \varepsilon_1)$, then we do not have $(\eta_2, \varepsilon_2) = (\eta, \varepsilon)$. Let us analyse the situation. For these pictures we shall use ribbons rather than strings. The twist θ will be represented by a full right hand screw turn of the ribbon. First, the pictures for ε and η are:

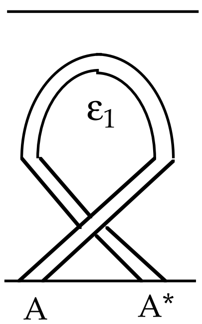


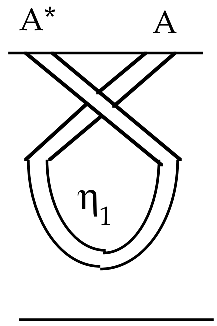


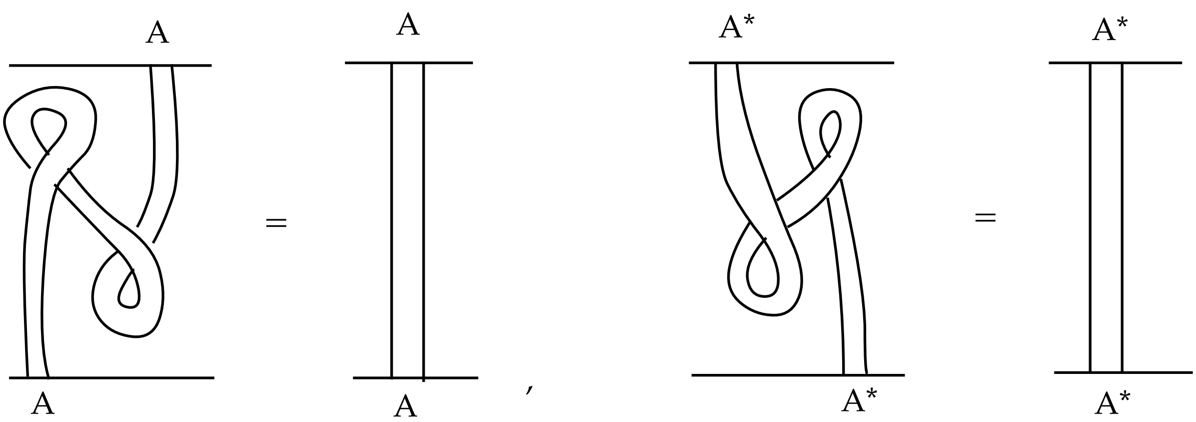
The adjunction equations are:



The pictures for ε_1, η_1 and the adjunction identities for them are shown below. By viewing the adjunction diagrams in 3-dimensional space, we can move the twisted ribbon on the left of these pictures and put it in the untwisted position on the right (the motion, technically called an isotopy, is restricted to a left-right sliding of the attached parts at the top and bottom).



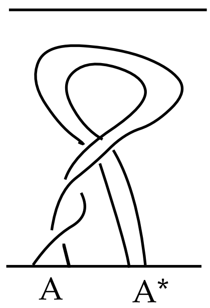


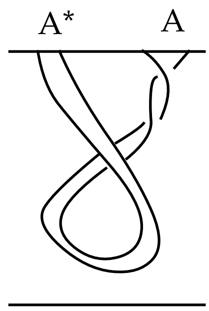


We would like to eliminate the looping in the pair (η_1, ε_1). One way to do this is to cancel the looping by a twist. If we put

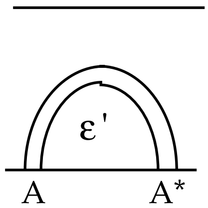
$$\varepsilon' = \varepsilon_1 \circ (\theta_A \otimes A^*) = \varepsilon \circ c_{A,\,A^*} \circ (\theta_A \otimes A^*), \qquad \eta' = (A^* \otimes \theta_A{}^{-1}) \circ \eta_1 = (A^* \otimes \theta_A{}^{-1}) \circ (c_{A^*,\,A})^{-1} \circ \eta$$

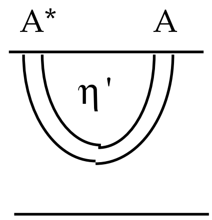
then the pictures for ε' and η' are:



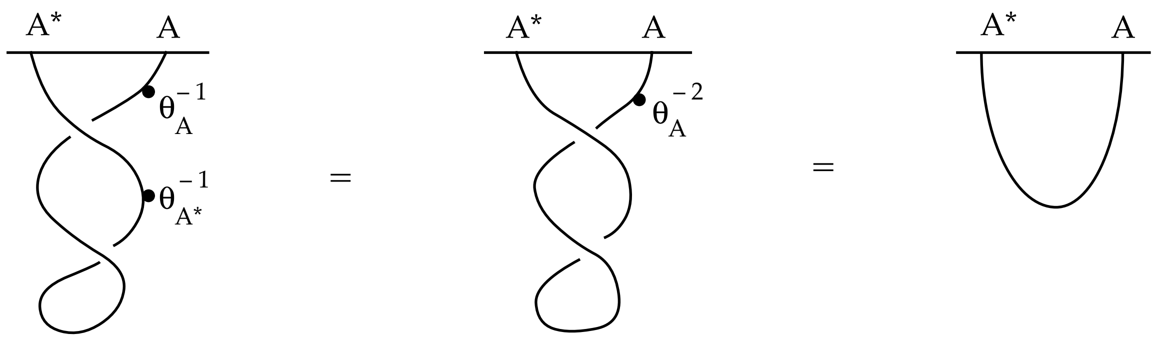


which should be redesigned to look like:





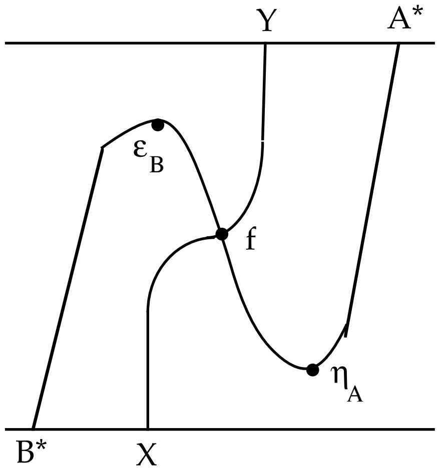
A formal verification that this is correct is as follows. If (η'', ε'') is the pair obtained by a two-fold application of the transformation $(\eta\,,\varepsilon\,) \longmapsto (\eta', \varepsilon')$ then we have $(\eta'', \varepsilon'') = (\eta\,,\varepsilon\,)$. To see this, look at the picture of η'' and compute (some steps of this calculation are missing and we invite the reader to fill in the gaps):



Suppose we have two adjunctions

$$(\eta_A\,, \varepsilon_A) \;:\; A^* \dashv A\,, \qquad (\eta_B\,, \varepsilon_B) \;:\; B^* \dashv B$$

in a tensor category. The *mate* of a map $f : X \otimes A \longrightarrow B \otimes Y$ is the map $f^{@} : B^* \otimes X \longrightarrow Y \otimes A^*$ described by the diagram

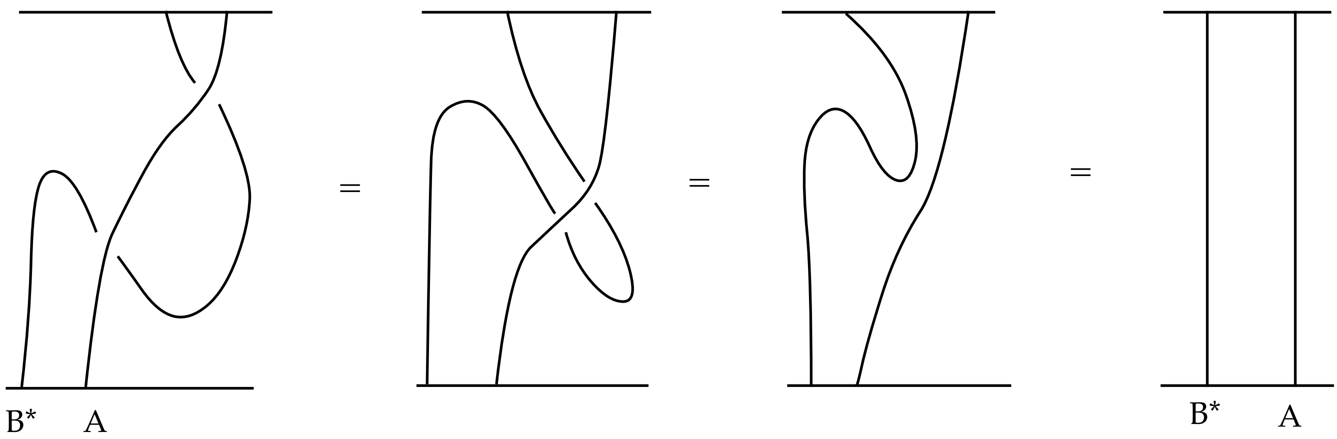


Equationally, this means that

$$f^{@} = (\varepsilon_B \otimes Y \otimes A^*)\,(B^* \otimes f \otimes A^*)\,(B^* \otimes X \otimes \eta_A)\,.$$

Proposition 8. *In a braided tensor category, the mate of* $c_{A,B} : A \otimes B \longrightarrow B \otimes A$ *is* $(c_{A,B^*})^{-1} : B^* \otimes A \longrightarrow A \otimes B^*$.

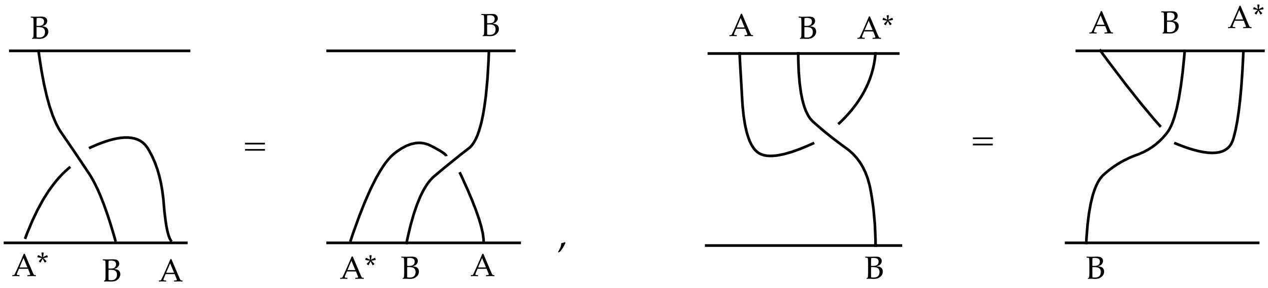
Proof. It suffices to prove that $(c_{A,B^*})\,(c_{A,B})^{@} = 1$, which is done by the following sequence of diagrams.



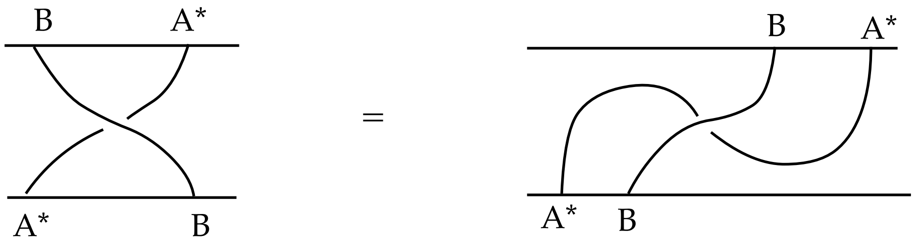
For the benefit of the reader, we also give the same proof written in the usual sequential notation:

$$
\begin{aligned}
(c_{A,B^*})\,(c_{A,B})^{@} &= (c_{A,B^*})\,(\varepsilon_B \otimes A \otimes B^*)\,(B^* \otimes c_{A,B} \otimes B^*)\,(B^* \otimes A \otimes \eta_B)\\
&= (\varepsilon_B \otimes c_{A,B^*})\,(B^* \otimes c_{A,B} \otimes B^*)\,(B^* \otimes A \otimes \eta_B)\\
&= (\varepsilon_B \otimes B^* \otimes A)\,(B^* \otimes B \otimes c_{A,B^*})\,(B^* \otimes c_{A,B} \otimes B^*)\,(B^* \otimes A \otimes \eta_B)\\
&= (\varepsilon_B \otimes B^* \otimes A)\,(B^* \otimes ((B \otimes c_{A,B^*}) \circ (c_{A,B} \otimes B^*))\,(B^* \otimes A \otimes \eta_B)\\
&= (\varepsilon_B \otimes B^* \otimes A)\,(B^* \otimes (c_{A,B\otimes B^*}))\,(B^* \otimes A \otimes \eta_B)\\
&= (\varepsilon_B \otimes B^* \otimes A)\,(B^* \otimes (c_{A,B\otimes B^*}\,(A \otimes \eta_B)))\\
&= (\varepsilon_B \otimes B^* \otimes A)\,(B^* \otimes ((\eta_B \otimes A) \circ c_{A,I}))\\
&= ((\varepsilon_B \otimes B^*) \circ (B^* \otimes \eta_B) \otimes A\\
&= B^* \otimes A\,. \ \textbf{qed}
\end{aligned}
$$

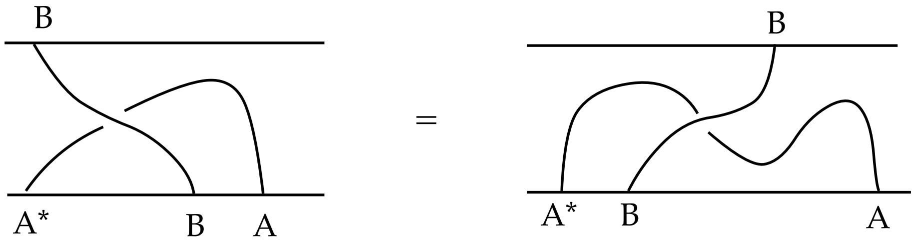
Corollary 9. *If* $(\eta\,,\varepsilon):\ A^* \dashv A$ *in a braided tensor category then the following equations hold:*



Proof. According to Proposition 8, we have the equality



which composes on the top with $B \otimes \varepsilon$ to yield the following equality.



This proves the result. **qed**

Definition [JS3, JS4] Let $T : \mathcal{A} \longrightarrow \mathcal{V}$ be a functor from a category $\mathcal{A}$ to a tensor category $\mathcal{V}$. A Yang-Baxter operator y on T is called *dualisable* when, for all $A \in \mathcal{A}$, the object TA has a left dual $(TA)^*$ and, for all $A, B \in \mathcal{A}$, the mates of $y_{A,B}\,,\,(y_{B,A})^{-1} : TA \otimes TB \longrightarrow TB \otimes TA$ are invertible.

It was shown by Proposition 8 that a braiding on a tensor category is a dualisable Yang-Baxter operator if every object has a left dual.

A dualisable Yang-Baxter operator y on a functor $T : \mathcal{A} \longrightarrow \mathcal{V}$ can be extended by duality to a Yang-Baxter operator y' on a functor $T' : \mathcal{A}' \longrightarrow \mathcal{V}$ where $\mathcal{A}'$ is the disjoint union $\mathcal{A} + \mathcal{A}^{\mathrm{op}}$ of the category $\mathcal{A}$ and its opposite $\mathcal{A}^{\mathrm{op}}$. To avoid ambiguities, we shall write A° and f° for the object and arrow in $\mathcal{A}^{\mathrm{op}}$ corresponding to A and f in $\mathcal{A}$. The extension of T is given as follows:

$$T'(A) = T(A),\quad T'(A^\circ) = T(A)^*,\quad T'(f) = T(f),\quad T'(f^\circ) = T(f)^*.$$

The extension of y is given as follows:

$$y'_{A,B} = y_{A,B}\,,\qquad y'_{A,B^\circ} = (y_{A,B}{}^{@})^{-1},\qquad y'_{A^\circ,B} = (y_{A,B}{}^{-1})^{@},\qquad y'_{A^\circ,B^\circ} = (y_{A,B})^*.$$

Proposition 10 [JS3] *The extension* y' *of a dualisable Yang-Baxter operator is a Yang-Baxter operator.*

The example of a Yang-Baxter operator $R = R_q : V \otimes V \longrightarrow V \otimes V$ previously given, on a finite dimensional vector space V and involving a non-zero $q \in \mathbf{C}$, is dualisable. A tedious straightforward calculation gives the following formulas, where we write $y_{V,V}$ for R and y_{V,V^*} for y_{V,V°:

$$y_{V,V}(e_i \otimes e_j) = \begin{cases} e_j \otimes e_i & \text{for } i > j \\ e_j \otimes e_i + (q - q^{-1}) e_i \otimes e_j & \text{for } i < j \\ q\, e_i \otimes e_i & \text{for } i = j \end{cases}$$

$$y_{V,V^*}(e_i \otimes e_j^{\,*}) = \begin{cases} e_j^* \otimes e_i & \text{for } i \neq j \\ q^{-1} e_i^* \otimes e_i + \sum\limits_{k>i} q^{2(i-k)} (q^{-1} - q)\, e_k^* \otimes e_k & \text{for } i = j \end{cases}$$

$$y_{V^*,V}(e_i^* \otimes e_j) = \begin{cases} e_j \otimes e_i^* & \text{for } i \neq j \\ q^{-1} e_i \otimes e_i^* + \sum\limits_{k<i} (q^{-1} - q)\, e_k \otimes e_k^* & \text{for } i = j \end{cases}$$

$$y_{V^*,V^*}(e_i^* \otimes e_j^*) = \begin{cases} e_j^* \otimes e_i^* & \text{for } i < j \\ e_j^* \otimes e_i^* + (q - q^{-1}) e_i^* \otimes e_j^* & \text{for } i > j \\ q\, e_i^* \otimes e_i^* & \text{for } i = j \quad . \end{cases}$$

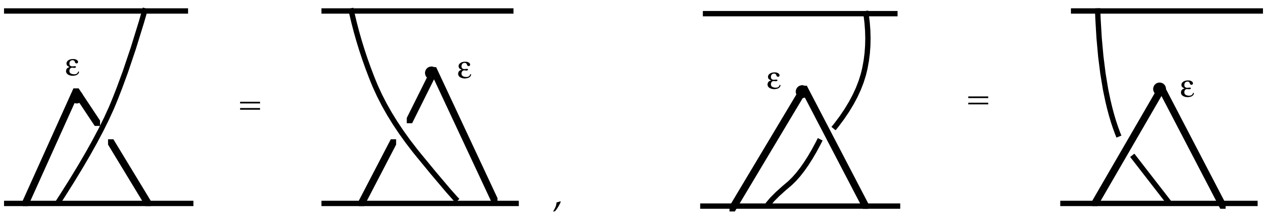
When $R : V \otimes V \longrightarrow V \otimes V$ is a dualisable Yang-Baxter operator on a finite dimensional vector space, we can use the extension R' to define a Yang-Baxter operator on $V \oplus V^*$. The vector space $V \oplus V^*$ is equipped with a non-degenerate symmetric pairing

$$\langle x \oplus \phi \mid y \oplus \psi \rangle = \phi(y) + \psi(x)$$

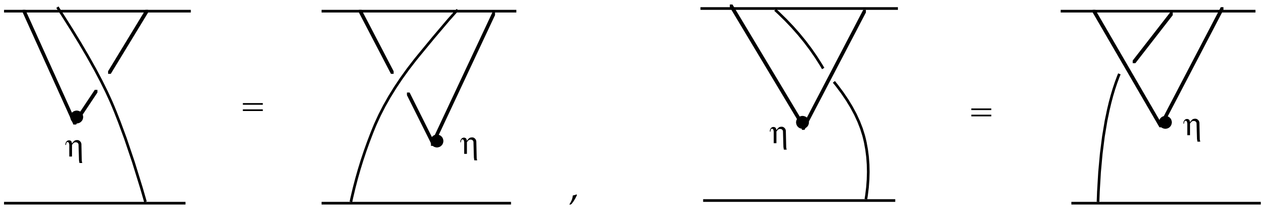
and also with a non-degenerate simplectic form

$$\omega(x \oplus \phi \mid y \oplus \psi) = \phi(y) - \psi(x).$$

Definition. Let R be a Yang-Baxter operator on an object Z in a tensor category, and let $\varepsilon : Z \otimes Z \longrightarrow I$ be an exact pairing. We say that R *respects* ε when we have the equations (in which the crossings are labelled by R and R^{-1} according to the convention previously explained):



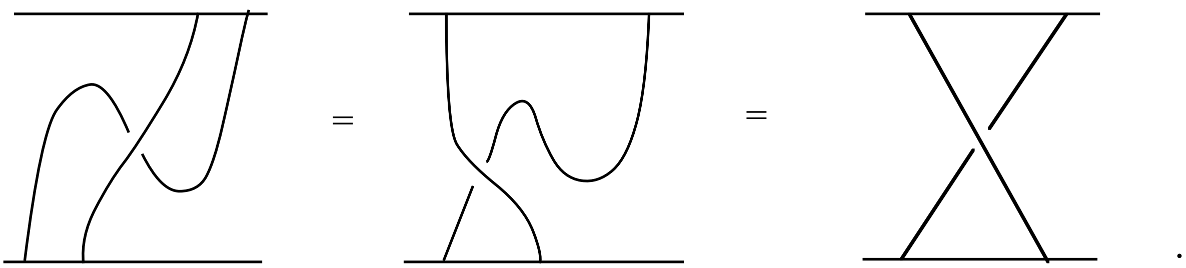
Proposition 11. *If* $(\eta, \varepsilon) : Z \dashv Z$ *and if* R *respects* ε *then the equations below hold:*



Proof. Exercise for the reader. qed

Proposition 12. *A Yang-Baxter operator* $R : Z \otimes Z \longrightarrow Z \otimes Z$ *which respects an exact pairing is dualisable.*

Proof. The mate of R is equal to R^{-1} by the diagram equalities:



Similarly, the mate of R^{-1} is equal to R. This proves that these mates are invertible. qed

Proposition 13. *If* R *is a dualisable Yang-Baxter operator on a finite dimensional vector space* V *then its extension* R' *to* $V \oplus V^*$ *respects both the canonical symmetric and the canonical symplectic pairings on* $V \oplus V^*$.

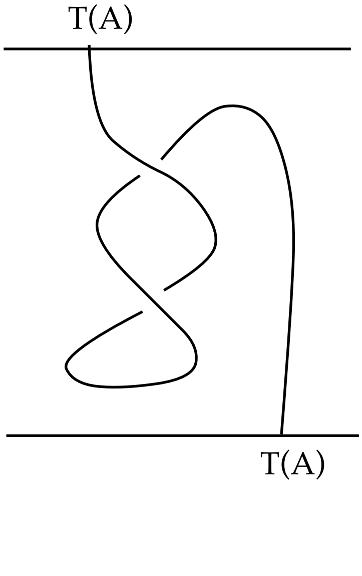
Proof. Exercise for the reader. qed

Suppose now that y is an arbitrary dualisable Yang-Baxter operator on a functor $T : \mathcal{A} \longrightarrow \mathcal{V}$. The picture below defines a canonical natural transformation

$$\theta' = \theta'_A : T(A) \longrightarrow T(A)$$

called the *double twist*. In the picture we use the extended Yang-Baxter operator y' to label the crossings.

$\theta'_A \; =$



Proposition 14. *For any dualisable Yang-Baxter operator* (y, T), *the double twist* θ' *is a natural isomorphism* $\theta' : T \xrightarrow{\sim} T$. *Moreover, the following equations hold:*

$$y(\theta' \otimes T) = (T \otimes \theta')\, y\,, \qquad y(T \otimes \theta') = (\theta' \otimes T)\, y\,.$$

Proof. The picture for the inverse of θ' is obtained from that for θ' by rotating through 180° and changing all the crossings; the rest is left to the reader. **qed**

Definition [JS3, JS4] Let y be a Yang-Baxter operator on a functor $T : : \mathcal{A} \longrightarrow \mathcal{V}$. A *twist* on y is a natural isomorphism $\theta : T \xrightarrow{\sim} T$ such that $y(\theta \otimes T) = (T \otimes \theta)\, y$, $y(T \otimes \theta) = (\theta \otimes T)\, y$. A *tortile* Yang-Baxter operator is a pair (y, θ) where y is a dualisable Yang-Baxter operator and θ is a a twist on y such that $\theta^2 = \theta'$ where θ' is the double twist defined by y.

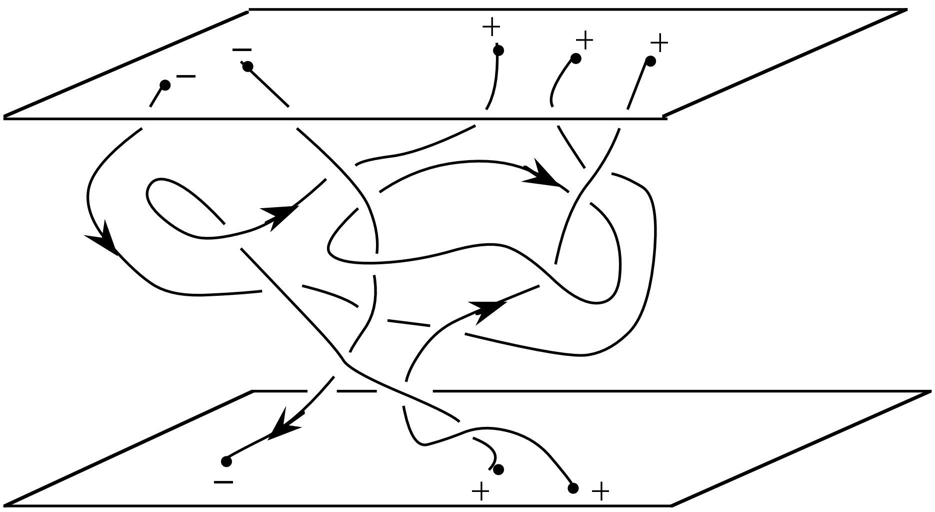
Example 11. In a tortile tensor category, the pair (c, θ) is a tortile Yang-Baxter operator since we have proved that $\theta^2 = \theta'$ (Proposition 7).

Example 12. A short calculation gives that the double twist on the operator R_q is equal to the map $x \longmapsto q^{2n} x$ on V of dimension n. If we put $\theta(x) = q^n x$, we obtain a tortile Yang-Baxter operator (R_q, θ).

§11. Knot invariants.

In this Section we provide a brief introduction to the method used by V.G. Turaev [T] to obtain knot invariants. We describe how Yang-Baxter operators can be used to produce tensor functors from the category of tangles of ribbons to vector spaces; see [MS] and [JS4]. If we apply the Tannaka duality machinery to these functors, we obtain quantum groups. This will be the subject of Section 12.

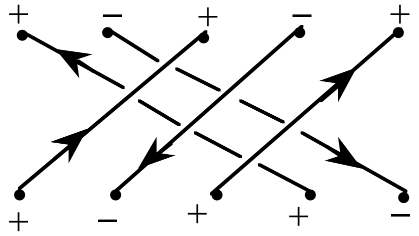
Let $\mathbf{P}$ be a Euclidean plane. A *geometric tangle* T is a compact 1-dimensional oriented submanifold of $[0,1] \times \mathbf{P}$ which is tamely embedded and whose boundary ∂T is equal to $T \cap \partial([0,1] \times \mathbf{P})$. We suppose that T meets $\partial([0,1] \times \mathbf{P})$ transversally. The *target* of T is the subset $\partial T \cap (\{1\} \times \mathbf{P})$ as an oriented submanifold. The *source* of T is the subset $\partial T \cap (\{0\} \times \mathbf{P})$, but with orientation reversed. A geometric tangle can be depicted:



A *tangle* is an isotopy class of geometric tangles where the isotopies keep the boundaries fixed. The source and target of a tangle are regarded as signed subsets of $\mathbf{P}$. Let 1, 2, 3, . . . denote equally spaced collinear points in the plane $\mathbf{P}$.

Now we can define the *autonomous braided tensor category* $\mathcal{T}$ *of tangles* [FY1&2, T2]. The objects are functions $A : \{1, 2, 3, \ldots, n\} \longrightarrow \{+, -\}$ for $n \geq 0$, called *signed sets.*

The arrows are the tangles which have these signed sets as sources and targets. Composition and tensor are as for braids. The braiding is illustrated by the following figure.

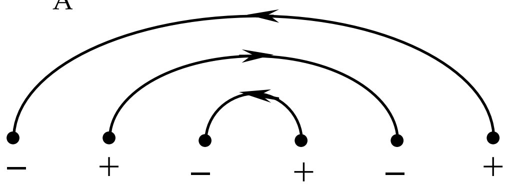


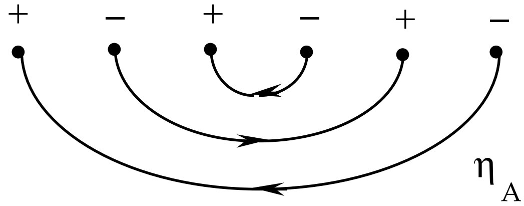
$c_{(+,-,+),\,(+,-)}$

The left dual A^* of a signed set A is given by reversing the order and the signs of the points. The arrows η_A and ε_A are illustrated in the following figure:

$A = (+, -, +)$

ε_A



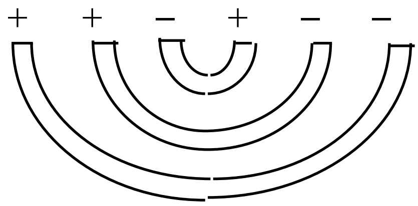


η_A

Let $\mathbf{S}^1$ be the unit circle in the tangent space of the Euclidean plane $\mathbf{P}$. A *framing* on a geometric tangle T is a continuous function $f : T \longrightarrow \mathbf{S}^1$. If $\varepsilon > 0$ is small enough then the set

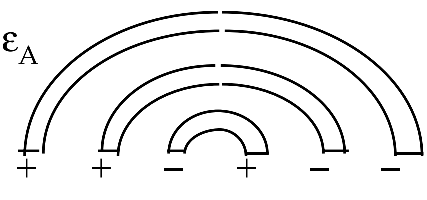
$$T_\varepsilon = \{\, x + \alpha\, f(x) \;:\; x \in T,\ 0 \le \alpha \le \varepsilon \,\}$$

is an embedded surface in $\mathbf{P} \times [0,1]$ which is called a *tangle of ribbons*. We should think of the pair (T, f) as a tangle of ribbons with arbitrary small width. The *source* of (T, f) is the source of T equipped with the framing induced by f. Similarly for the target. The direction of the line on which $1, 2, 3, \ldots$ are listed will be called the *eastward* framing. The opposite direction is the *westward* framing. We can now define the *tortile tensor category* $\mathcal{T}\sim$ of tangles on ribbons. The *objects* of $\mathcal{T}\sim$ are the signed subsets of $1, 2, 3, \ldots$. We suppose that the framing of a positive element of a signed subset is eastward, while the framing of a negative element is westward. The *arrows* are the (isotopy classes of) tangles of ribbons having these framed signed subsets as sources and targets. Composition and tensor are as in $\mathcal{T}$. The braiding is as in $\mathcal{T}$. The units and counits of an adjunction $(\eta_A, \varepsilon_A) : A^* \dashv A$ are illustrated in the following figure for $A = (+, -, -)$.

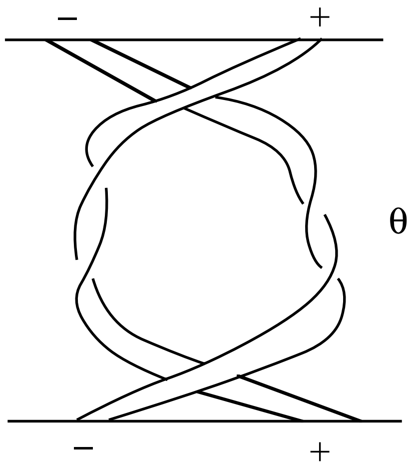


η_A

ε_A



The twist θ is illustrated by the picture:



Theorem 1 [Sh] *The category $\mathcal{T}\sim$ of tangles on ribbons is the free tortile tensor category generated by a single object.*

This means that, given any tortile tensor category $\mathcal{V}$ and any object $V \in \mathcal{V}$, there exists a tensor functor $F : \mathcal{T}\sim \longrightarrow \mathcal{V}$, which preserves the braidings and the twists, such that $F(+) = V$; moreover, F is unique up to a unique isomorphism of functors. In this theorem, the functor F is entirely determined by the *object* V if everything else is kept fixed. For example, if $\mathcal{V} = \mathcal{V}ect_f$ is the category of finite dimensional vector spaces, then the functor F depends only (up to isomorphism) on the dimension of the vector space $V = F(+)$. We obtain, in this way, poor invariants of tangles and knots. To obtain better invariants, we need to complement this theorem by another one.

Recall that a tortile Yang-Baxter operator in a tensor category C is a triple (V, R, θ) where $R : V\otimes V \xrightarrow{\sim} V\otimes V$ is a dualisable Yang-Baxter operator and $\theta : V \xrightarrow{\sim} V$ is a twist on R such that $\theta^2 = \theta'$, where θ' is the canonical double twist defined by R (as defined before Section 10 Proposition 14).

Theorem 2 [T, JS4] *The category $\mathcal{T}\sim$ of tangles on ribbons is free on a tortile Yang-Baxter operator.*

This means that, in order to define a tensor functor $F : \mathcal{T}\sim \longrightarrow C$, it suffices to select a tortile Yang-Baxter operator (V, R, θ) in C. The tensor functor F is the only (up to a unique isomorphism) one for which $V = F(+)$, $R = F(c_{+,+})$ and $\theta = F(\theta_+)$.

This is the method used by Turaev to obtain knot invariants like the Jones polynomial. In this case C is the category of vector spaces. Using the tortile Yang-Baxter operator (R_q, θ) defined in Section 10, we can associate a number $P(q) = F(K)$ to any framed knot K (considered as a morphism $K : I \longrightarrow I$ in the tensor category $\mathcal{T}\sim$). This number depends on q, and is, in fact, a Laurent polynomial in q.

§12. Quantum groups.

Let R be a Yang-Baxter operator on a finite dimensional vector space V. We saw in Section 10 how R can be used to define a tensor preserving functor

$$\pi : \mathbf{B} \longrightarrow \mathcal{V}ect_{f\mathbb{C}}$$

such that $\pi(1) = V$ and $\pi(c_{1,1}) = R$. If we apply the Tannaka duality machinery to the functor π, we obtain a bialgebra $\mathrm{End}^\vee(\pi)$ that we shall denote by $\mathcal{O}_R(\mathrm{End}(V))$. This notation is borrowed from algebraic geometry where $\mathcal{O}(X)$ usually denotes the ring of regular functions on an affine algebraic variety X. When $X = G$ is a compact Lie group, $O(G)$ is the ring of representative functions on G. If R is the usual symmetry operator $x \otimes y \longmapsto y \otimes x$ then $O_R(\mathrm{End}(V)) = S(\mathrm{End}(V)) = S(\mathrm{End}(V)^*)$ is the symmetric algebra on $\mathrm{End}(V)^*$, or equivalently, the algebra of polynomial functions on End(V). For a general R, the algebra $O_R(\mathrm{End}(V))$ is not commutative, and should be thought of as the algebra of regular functions on a "non-commutative" geometric object. Let us describe the algebra $O_R(\mathrm{End}(V))$ by generators and relations.

Let $e_1, \ldots, e_n$ be a basis for V and let $x_i^j = [e_j^* \otimes e_i]$ be the images of the elements $e_j^* \otimes e_i$ under the canonical map

$$[\ \ ] : \mathrm{End}(V) \longrightarrow O_R(\mathrm{End}(V)).$$

With respect to the basis $\{e_i \otimes e_j\}$ of $V \otimes V$, we have

$$R(e_i \otimes e_j) = \sum_{r,s} e_r \otimes e_s R^{rs}_{ij} .$$

Proposition 1. *A presentation of the algebra* $O_R(\mathrm{End}(V))$ *is provided by the generators* x_i^j *for* $1 \le i, j \le n$ *and the relations*

$$\sum_{k,r} R^{st}_{kr} x^k_i x^r_j = \sum_{k,r} R^{kr}_{ij} x^s_k x^t_r .$$

Before giving a proof, we shall analyse the meaning of the relations in the presentation. For an algebra A, we shall say that a coaction $\alpha : V \longrightarrow V \otimes A$ *respects* the Yang-Baxter operator R when the following square commutes.

$$\begin{array}{ccc} V \otimes V & \xrightarrow{\alpha \underline{\otimes} \alpha} & V \otimes V \otimes A \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R \otimes A} \\ V \otimes V & \xrightarrow[\alpha \underline{\otimes} \alpha]{} & V \otimes V \otimes A \end{array}$$

This condition is expressed by the equations

$$\sum_{k,r} R^{st}_{kr} \alpha^k_i \alpha^r_j = \sum_{k,r} R^{kr}_{ij} \alpha^s_k \alpha^t_r$$

where $\alpha(e_i) = \sum_j e_j \otimes \alpha^j_i$.

The canonical coaction (Section 8 Proposition 3)

$$\gamma_1 : V \longrightarrow V \otimes \mathrm{End}^\vee(\pi)$$

respects the Yang-Baxter operator R since, by the naturality of γ , the square

$$\begin{array}{ccc} V \otimes V & \xrightarrow{\gamma_2} & V \otimes V \otimes \mathrm{End}^\vee(\pi) \\ {\scriptstyle \pi(c_{1,1})}\downarrow & & \downarrow{\scriptstyle \pi(c_{1,1}) \otimes 1} \\ V \otimes V & \xrightarrow[\gamma_2]{} & V \otimes V \otimes \mathrm{End}^\vee(\pi) \end{array}$$

commutes, and we have (Section 10 Proposition 4)

$$\pi(c_{1,1}) = R, \qquad \gamma_2 = \gamma_1 \underline{\otimes} \gamma_1 .$$

This implies that the relations in the purported presentation of $\mathcal{O}_R(\mathrm{End}(V)) = \mathrm{End}^\vee(\pi)$ are satisfied since

$$\gamma_1(e_i) = \sum_j e_i \otimes [e_j^* \otimes e_i] = \sum_j e_j \otimes x_i^j .$$

Proposition 1 is now an immediate consequence of Proposition 2 below (which refers to the triangle).

$$\begin{tikzcd} V \arrow[r, "\gamma_1"] \arrow[dr, "\alpha"'] & V \otimes \mathcal{O}_R(\mathrm{End}(V)) \arrow[d, "V \otimes \alpha^\sim"] \\ & V \otimes A \end{tikzcd}$$

Proposition 2. *For any algebra* A *and any coaction* $\alpha : V \longrightarrow V \otimes A$ *respecting* R*, there is a unique map of algebras* $\alpha^\sim : \mathcal{O}_R(\mathrm{End}(V)) \longrightarrow A$ *such that the above triangle commutes.*

Proof. Let $\alpha : V \longrightarrow V \otimes A$ be a coaction respecting R. Then R is a Yang-Baxter operator on the object (V, α) of the category $\mathcal{C}o_f(A)$ (see Section 10). Using Section 10 Proposition 4, we obtain a tensor functor $\pi^\sim : \mathbf{B} \longrightarrow \mathcal{C}o_f(A)$, or equivalently, a tensor-preserving natural transformation $\pi \longrightarrow \pi \otimes A$. Using Section 8 Proposition 3, we obtain a map of algebras $\mathrm{End}^\vee(\pi) \longrightarrow A$. The rest of the proof is left to the reader. **qed**

The coalgebra structure on $\mathcal{O}_R(\mathrm{End}(V))$ is the usual one. On the generators x_i^j we have

$$\Delta x_i^j = \textstyle\sum_k x_k^j x_i^k , \qquad \varepsilon(x_i^j) = \delta_i^j .$$

If R is the usual symmetry operator $x \otimes y \longmapsto y \otimes x$, the relations reduce to

$$x_i^t x_j^s = x_j^s x_i^t$$

so that $\mathcal{O}_R(\mathrm{End}(V)) = S(\mathrm{End}(V))$.

When R is the Yang-Baxter operator $x \otimes y \longmapsto (-1)^{pq} y \otimes x$ on a $\mathbf{Z}/2$-graded vector space $V = V_0 \oplus V_1$, we have a decomposition

$$\mathrm{End}(V) = \mathrm{End}(V_0) \oplus \mathrm{End}(V_1) \oplus \mathrm{Hom}(V_0, V_1) \oplus \mathrm{Hom}(V_1, V_0)$$

giving rise to a decomposition

$$\mathcal{O}_R(\mathrm{End}(V)) = S(\mathrm{End}(V_0) \oplus \mathrm{End}(V_1)) \otimes \Lambda(\mathrm{Hom}(V_0, V_1) \oplus \mathrm{Hom}(V_1, V_0))$$

where Λ indicates an exterior algebra.

When $R = R_q$ (see just before Proposition 4 in Section 10), we have the following presentation for $\mathcal{O}_R(\mathrm{End}(V)) = \mathcal{O}_q(\mathrm{End}(V))$:

$$x_j^k x_i^r = \begin{cases} x_i^r x_j^k & \text{for } i < j \text{ and } k < r \\ x_i^r x_j^k + (q - q^{-1}) x_j^r x_i^k & \text{for } i < j \text{ and } r < k \\ q\, x_i^k x_j^k & \text{for } i < j \text{ and } k = r \\ q\, x_i^k x_i^r & \text{for } i = j \text{ and } k < r \end{cases} .$$

When R is a dualisable Yang-Baxter operator we shall describe a quantum group $\mathcal{O}_R(GL(V))$. For this we can follow a method similar to the one just used to describe the bialgebra $\mathcal{O}_R(End(V))$. First, let $\mathbf{B'}$ be the free tensor category generated by the quintuple $(D^*, D, \eta, \varepsilon, R)$ where $(\eta, \varepsilon) : D^* \dashv D$ and $R : D\otimes D \longrightarrow D\otimes D$ is a dualisable Yang-Baxter operator on D. It is possible to give a geometric model for $\mathbf{B'}$ like the one described in Section 11 (but, in this case, we use tangles of strings such that the front projection defines an immersion into the yz-plane). However, no explicit description of $\mathbf{B'}$ is necessary. It can be proved that $\mathbf{B'}$ is braided and autonomous (it is in fact the free autonomous braided tensor category on a single object [FY2]). Using the dualisable Yang-Baxter operator R, we can define a tensor functor

$$\pi' : \mathbf{B'} \longrightarrow \mathcal{V}ect_f$$

such that $\pi'((+)) = V$ and $\pi'(c_{+,+}) = R$. We put

$$\mathcal{O}_R(GL(V)) = End^{\vee}(\pi'),$$

which is a Hopf algebra since $\mathbf{B'}$ is autonomous. It is also cobraided since $\mathbf{B'}$ is braided. We shall give a presentation of $\mathcal{O}_R(GL(V))$ in terms of $\mathcal{O}_R(End(V))$. Recall that an action $\alpha : V \longrightarrow V\otimes A$ is (left) dualisable if its matrix $\alpha = (\alpha_{ij})$, with respect to some basis of V, is invertible. The canonical coaction

$$\gamma_+ : V \longrightarrow V \otimes \mathcal{O}_R(GL(V))$$

is left dualisable with

$$\gamma_- : V^* \longrightarrow V^* \otimes \mathcal{O}_R(GL(V))$$

as left dual. We have the following result whose proof is left to the reader.

Proposition 3. *Let* R *be a dualisable Yang-Baxter operator on* V. *For each algebra* A *and each dualisable coaction* $\alpha : V \longrightarrow V\otimes A$ *respecting* R, *there is a unique algebra map* $\alpha^{\sim} : \mathcal{O}_R(GL(V)) \longrightarrow A$ *such that the following triangle commutes.*

$$\begin{tikzcd} V \arrow[r, "\gamma_+"] \arrow[dr, "\alpha"'] & V \otimes \mathcal{O}_R(GL(V)) \arrow[d, "V\otimes \alpha^{\sim}"] \\ & V \otimes A \end{tikzcd}$$

Let us denote by $\mathcal{O}_R(End(V))[x^{-1}]$ the algebra obtained from $\mathcal{O}_R(End(V))$ by adjoining the entries of an inverse x^{-1} of the matrix $x = (x_i{}^j)$ together with the relation $x^{-1} x = x x^{-1} = id$.

Corollary 4. *There is a canonical isomorphism*

$$\mathcal{O}_R(GL(V)) \cong \mathcal{O}_R(End(V))[x^{-1}].$$

When $R = R_q$, the description of $\mathcal{O}_q(GL(V))$ is even simpler. The element

$$d = \sum_{\sigma} (-q)^{|\sigma|} x_1^{\sigma(1)} \dots x_m^{\sigma(m)}$$

turns out to be in the centre of the algebra $\mathcal{O}_q(End(V))$. According to [FRT], it suffices to invert d in order to obtain $\mathcal{O}_q(GL(V))$; that is, we have

$$\mathcal{O}_q(GL(V)) = \mathcal{O}_q(End(V))[d^{-1}].$$

If we force d to be 1, we obtain the quantum group

$$\mathcal{O}_q(SL(V)) \;=\; \mathcal{O}_q(End(V))/d=1\,.$$

A lot of other quantum groups can be obtained as subgroups of $\mathcal{O}_R(GL(V))$. For example, if $\varepsilon : V\otimes V \longrightarrow \mathbf{C}$ is a pairing respected by the Yang-Baxter operator R then there will be a quantum subgroup

$$\mathcal{O}_R(GL(V)) \longrightarrow \mathcal{O}_R(O(V, \varepsilon))$$

of orthogonal transformations preserving ε.

Also, the reader might enjoy reading [W] on the compact quantum group $\mathcal{O}_q(SU(n))$. Finally, we encourage the reader to study quantum Grassmannians and quantum spheres [Pd].

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