On Groupoids and Stuff

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Introduction

The theory of combinatorial species developed by André Joyal [3] is a method for analysis of finite structures using generating functions. A combinatorial species can be thought of as a map assigning to each finite set the set of structures of the species which exist on that set.

A structure type works in the opposite direction, assigning to a structure its underlying set. However the collection of structures is not a set, but a groupoid. This mapping "forgets" the structure, and so its fibre (or inverse) will rebuild it, becoming a species. A stuff type is a generalisation of a structure type, and in the same context can be thought of as mapping a structure to some subset of its underlying set. This mapping forgets not only structure, but extra "stuff".

The advantage of such an approach is two-fold. Firstly it provides a categorical underpinning to the concept of a generating function, giving an explanation as to the form of the power series. Also it allows for a different approach to the familiar operations (such as addition, multiplication, and composition) on species.

The theory of stuff types was introduced by John Baez and James Dolan in a paper [2] and a series of talks [1], in which they use stuff types to 'categorify' the concept of Feynman diagrams and the foundations of quantum mechanics.

Combinatorial species provide an environment for the theory of operads, which were introduced to study the homotopy of iterated loop spaces [6]. Stuff types provide the environment for the theory of clubs which were introduced to study categories with structure [4],[5].

The purpose of this essay is to describe the concept of structure and stuff types, and show how operations on species can be viewed as operations on stuff types.

I would like to thank Professor Ross Street for his advice and support.

Chapter 1

Groupoids

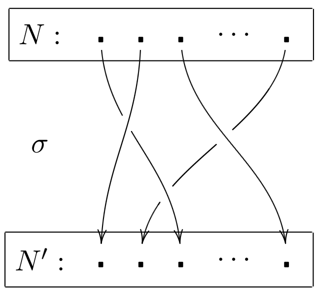
1.1 An introduction

A groupoid is a category in which every morphism is an isomorphism. This paper only considers groupoids which are 'small', that is the objects of the groupoid form a set.

Example 1.1.1. A groupoid which we will use often is $\mathbb{E}$, a subcategory of **FinSet** in which the only morphisms are bijections. An object is a finite set N, which we can depict as:

$$N : \; \cdot \; \cdot \; \cdot \; \cdots \; \cdot$$

where the **.**'s represent the elements of the set. A morphism is a bijective function $\sigma : N \to N'$, which we can depict as:



A useful property of groupoids is the ability to split them into their isomorphism classes, which themselves form groupoids. These are called connected groupoids, as every object in the groupoid has a morphism to every other object. The automorphisms of any element form a group under composition (hence the origins of the term groupoid).

Lemma 1.1.2. *If X, X', Y, Y' are objects in the same isomorphism class of a groupoid $\mathscr{X}$, then $\mathscr{X}(X, X') \cong \mathscr{X}(Y, Y')$*

Proof. Since $X, X', Y'Y'$ are in the same isomorpism class, we know there exists morphisms $u : X \to Y$ and $v : X' \to Y'$. We can construct a function $\theta : \mathscr{X}(X, X') \to \mathscr{X}(Y, Y')$ where $\theta(f) = v \circ f \circ u^{-1}$. Furthermore, θ is invertible, with $\theta^{-1}(g) = v^{-1} \circ g \circ u : X \to X'$. Therefore we have $\mathscr{X}(X, X') \cong \mathscr{X}(Y, Y')$. □

Note that this isomorphism is not natural, as it depended on the choice of u and v.

From this we can build a *skeleton* of the groupoid, in which the only objects are representative objects of each isomorphism class. The only morphisms in a skeleton will be the automorphisms on the representative objects.

Example 1.1.3. A skeleton of $\mathbb{E}$ is $\mathbb{P}$ the groupoid whose objects are $\langle n \rangle = \{1, 2, \ldots, n\}$ and the morphisms are permutations of the elements.

We will denote the set of isomorphism classes of a groupoid $\mathscr{X}$ as $\pi_0(\mathscr{X})$ and the isomorphism class of an object X in $\mathscr{X}$ by $[X]$. That is X is a representative object of the isomorphism class $[X]$.

This inspires the definition of groupoid cardinality (as given by [2]):

Definition 1.1.4. The cardinality of a groupoid $\mathscr{X}$ is:

$$|\mathscr{X}| = \sum_{[X] \in \pi_0(\mathscr{X})} \frac{1}{|\operatorname{Aut}(X)|}$$

where $\operatorname{Aut}(X) = \mathscr{X}(X, X)$, the set of automorphisms on X.

Remark 1.1.5. Obviously for this to be of any use, it is required that the sum converge.

This definition has several useful properties, the most obvious of which is the cardinality of a groupoid is equal to that of its skeleton. Also, it corresponds with the notion of set cardinality, since a set can be thought of as a category whose objects are elements, with only the identity morphisms.

A group $\mathcal{G}$ can be thought of as a category $\mathbf{B}\mathcal{G}$ with one object I, in which the morphisms (which will be automorphisms on I) are the elements of $\mathcal{G}$. The identity element of the group e corresponds to the identity morphism morphism 1_I, and so every morphism has an inverse, hence $\mathbf{B}\mathcal{G}$ is a groupoid.

However, this means that $|\mathbf{B}\mathcal{G}|$ will be the reciprocal of the order of the group, $\#\mathcal{G}$. While this may at first seem odd, we will later show that it allows for a type of quotient on a groupoid.

Example 1.1.6. The isomorphism classes of $\mathbb{E}$ consists of sets with the same number of elements. An automorphism is a permutation on the set. On a set with n elements, there are $n!$ permutations, so we have:

$$|\mathbb{E}| = \sum_{n=0}^{\infty} \frac{1}{n!} = e$$

Proposition 1.1.7. *If two groupoids $\mathscr{X}$ and $\mathscr{Y}$ are equivalent, then $|\mathscr{X}| = |\mathscr{Y}|$.*

Proof. Let $S : \mathscr{X} \longrightarrow \mathscr{Y}$ be an equivalence. A functor is an equivalence if and only if it is full, faithful and essentially surjective. S being full implies that each isomorphism class $[X]$ of $\mathscr{X}$ correspondonds uniquely to the isomorphism class $[S(X)]$ in $\mathscr{Y}$. Furthermore since S is essentially surjective, for every isomorpism class $[Y]$ of $\mathscr{Y}$, there exists an isomorphism class $[X]$ in $\mathscr{X}$ such that $[X] = [S(X)]$.

Also, since S is full and faithful we have $|\operatorname{Aut}(X)| = |\operatorname{Aut}(S(X))|$, and so we have:

$$\begin{aligned} |\mathscr{X}| &= \sum_{[X]\in\pi_0(\mathscr{X})} \frac{1}{|\operatorname{Aut}(X)|} = \sum_{[X]\in\pi_0(\mathscr{X})} \frac{1}{|\operatorname{Aut}(S(X))|} \\ &= \sum_{[Y]\in\pi_0(\mathscr{Y})} \frac{1}{|\operatorname{Aut}(Y)|} = |\mathscr{Y}| \end{aligned}$$

□

1.2 Category of groupoids

The category **Gpd** of groupoids is the categiry whose objects are grouopids and morphisms are functors between groupoids, and so is a subcategory of the category **Cat** of small categories.

The initial object is the empty category **0**, which has no objects and no morphisms, and the terminal object is **1**, with one object and only the identity morphism on that object.

The operations of coproduct and product are easily defined on **Gpd**.

Addition of groupoids

The sum (or coproduct) of a family of groupoids $(\mathscr{X}_i)_{i\in\Lambda}$ is the groupoid $\sum_{i\in\Lambda} \mathscr{X}_i$, in which the set of objects is the disjoint union of the set of objects of each $\mathscr{X}_i$, and the included morphisms between those objects. The coprojections are the inclusion functors:

$$I_j : \mathscr{X}_j \longrightarrow \sum_{i\in\Lambda} \mathscr{X}_i$$

Any isomorphism class of $\sum \mathscr{X}_i$ must correpond uniquely to an isomorphism class of some groupoid $\mathscr{X}$, so we have:

$$\begin{aligned} \left|\sum_{i\in\Lambda} \mathscr{X}_i\right| &= \sum_{[X]\in\pi_0(\sum \mathscr{X}_i)} \frac{1}{|\operatorname{Aut}(X)|} \\ &= \sum_{i\in\Lambda} \left(\sum_{[X_i]\in\pi_0(\mathscr{X})} \frac{1}{|\operatorname{Aut}(X_i)|} \right) = \sum_{i\in\Lambda} |\mathscr{X}_i| \end{aligned}$$

Multiplication of groupoids

The product can be interpreted as the cartesian product, and so for a family of groupoids $(\mathscr{X}_i)_{i\in\Lambda}$, we define the category $\prod_{i\in\Lambda}\mathscr{X}_i$ which has objects $(X_i)_{i\in\Lambda}$ where X_i is an object in $\mathscr{X}_i$, and morphisms $(f_i)_{i\in\Lambda} : (X_i)_{i\in\Lambda} \to (X_i')_{i\in\Lambda}$, where each $f_i : X_i \to X_i'$ is a morphism in $\mathscr{X}_i$. Therefore we have an obvious choice for projections:

$$P_j : \sum_{i\in\Lambda}\mathscr{X}_i \longrightarrow \mathscr{X}_j$$

Any isomorphism class of $\prod \mathscr{X}_i$ corresponds to an isomorphism class of each $\mathscr{X}_i$, and an automorphism on $(X_i)_{i\in\Lambda}$ will correspond to an automorphism on each X_i. Therefore we have:

$$\left|\prod_{i\in\Lambda}\mathscr{X}_i\right| = \sum_{[(X_i)_{i\in\Lambda}]\in\pi_0(\prod\mathscr{X}_i)} \frac{1}{|\operatorname{Aut}(X_i)_{i\in\Lambda}|}$$
$$= \prod_{i\in\Lambda}\sum_{[X_i]\in\pi_0(\mathscr{X})}\frac{1}{|\operatorname{Aut}(X_i)|} = \prod_{i\in\Lambda}|\mathscr{X}_i|$$

1.3 Weak quotient

One of the main reasons the groupoid cardinality is defined as such is that it fits in nicely with the idea of a weak quotient. Informally, a weak quotient is a group action on the set of objects in a groupoid.

Definition 1.3.1. A *group action* of a finite group $\mathcal{G}$ on a groupoid $\mathscr{X}$ is a functor $A : \mathbf{B}\mathcal{G} \longrightarrow \mathbf{Gpd}$ such that $A(I) = \mathscr{X}$ where I is the object in $\mathbf{B}\mathcal{G}$.

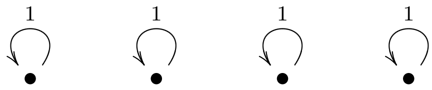
This means that each element $g \in \mathcal{G}$ induces a functor $A(g) : \mathscr{X} \longrightarrow \mathscr{X}$. For an object X of $\mathscr{X}$ we write $A(g)X = gX$, and similarly for a morphism $f : X \to X'$, we write $A(g)(f) = gf : gX \to gX'$. Also these functors preserve composition, so for $g, g' \in \mathcal{G}$ we have $A(g)A(g') = A(gg')$.

Definition 1.3.2. The *weak quotient* of a group action on a groupoid is the category $\mathscr{X}//\mathcal{G}$ whose objects are those of $\mathscr{X}$, and morphisms $(f, g) : X \to X'$, with $g \in \mathcal{G}$ and a morphism $f : gX \to X'$ in $\mathscr{X}$.

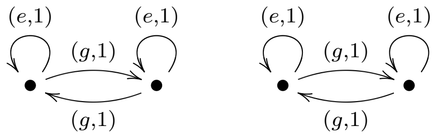
The identity morphism on an object X is $(e, 1_X)$, and composition of morphisms $(g, f) : X \to X'$ and $(g', f') : X' \to X''$ is defined as $(g', f') \circ (g, f) = (g'g, f' \circ g'f)$. Hence morphism $(g, f) : X \to X'$ has an inverse $(g^{-1}, g^{-1}f^{-1})$, and so $\mathscr{X}//\mathcal{G}$ is a groupoid.

So how may this be considered a quotient? Essentially a group action can be thought of as introducing new morphisms to a groupoid, and these will combine distinct isomorphism classes or introduce new automorphisms, both of which will reduce the cardinality of such a groupoid.

Example 1.3.3. Consider a set with 4 elements as a groupoid **4**. Then clearly $|\mathbf{4}| = 4$, and such a groupoid may be depicted as:



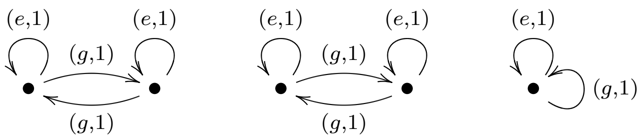
A group action of $\mathbb{Z}_2 = \{e, g\}$ can be defined to act on **4**. This action can be chosen such that the groupoid $\mathbf{4}//\mathbb{Z}_2$ can be depicted as:



There are two isomorphism classes, and each object has one automorphism, so $|\mathbf{4}//\mathbb{Z}_2| = 2$.

The above group action combined isomorphism classes. However the action may not always be so 'nice':

Example 1.3.4. Consider a set with 5 elements as a groupoid **5**. We have a group action of $\mathbb{Z}_2 = \{e, g\}$ on **5** such that the groupoid $\mathbf{5}//\mathbb{Z}_2$ can be depicted as:



Now there are three isomorphism classes, but one has two automorphisms, so $|\mathbf{5}//\mathbb{Z}_2| = 2\frac{1}{2}$.

Theorem 1.3.5. *If $\mathcal{G}$ is a group acting on a groupoid $\mathscr{X}$, the cardinality of $\mathscr{X}//G$ is given by:*

$$|\mathscr{X}//\mathcal{G}| = \frac{|\mathscr{X}|}{\#\mathcal{G}}$$

Proof. Firstly, note that if X and Y are objects in $\mathscr{X}$, and $g \in \mathcal{G}$ such that there exists an isomorphism $f : gX \to Y$ in $\mathscr{X}$ then there also exists an isomorphism $g^{-1}f : X \to g^{-1}Y$ in $\mathscr{X}$.

For an object X of $\mathscr{X}$, define its orbit as:

$$\mathrm{orb}(X) = \{[Y] \in \pi_0(\mathscr{X}) \mid Y \cong gX \text{ for some } g \in \mathcal{G}\}$$

These are the isomorphism classes of $\mathscr{X}$ which will 'merge' to become the isomorphism class of X in $\mathscr{X}//\mathcal{G}$. Note that if $[Y] \in \mathrm{orb}(X)$, then $\mathrm{orb}(Y) = \mathrm{orb}(X)$, and also that:

$$\pi_0(\mathscr{X}) = \sum_{[X] \in \pi_0(\mathscr{X}//\mathcal{G})} \mathrm{orb}(X)$$

For any objects X and Y in $\mathscr{X}$, such that $[Y] \in \mathrm{orb}(X)$, define:

$$\mathcal{G}_{X,Y} = \{g \in \mathcal{G} \mid gX \cong Y \text{ in } \mathscr{X}\}$$

Then there exists a $k \in \mathcal{G}$ such that $kX \cong Y$, and so:

$$k^{-1}\mathcal{G}_{X,Y} = \{k^{-1}g \in \mathcal{G} \mid gX \cong Y\} = \{h \in \mathcal{G} \mid hX \cong X\} = \mathcal{G}_{X,X}$$

For any object X in $\mathscr{X}$ we have:

$$\mathcal{G} = \sum_{[Y]\in\mathrm{orb}(X)} \mathcal{G}_{X,Y}$$

and so we can write:

$$\#\mathcal{G} = \sum_{[Y]\in\mathrm{orb}(X)} |\mathcal{G}_{X,Y}| = \sum_{[Y]\in\mathrm{orb}(X)} |\mathcal{G}_{X,X}| = |\,\mathrm{orb}(X)| \cdot |\mathcal{G}_{X,X}|$$

Finally, note that for any automorphism $(f, g) : X \to X$ in $\mathscr{X}//\mathcal{G}$ we require that $g \in \mathcal{G}_{X,X}$, and so we have:

$$|(\mathscr{X}//\mathcal{G})(X, X)| = \sum_{g\in\mathcal{G}_{X,X}} |\mathscr{X}(gX, X)| = |\mathcal{G}_{X,X}| \cdot |\mathscr{X}(X, X)|$$

Combining all of this we have:

$$\begin{aligned}
|\mathscr{X}//\mathcal{G}| &= \sum_{[X]\in\pi_0(\mathscr{X}//\mathcal{G})} \frac{1}{|(\mathscr{X}//\mathcal{G})(X, X)|} \\
&= \sum_{[X]\in\pi_0(\mathscr{X}//\mathcal{G})} \sum_{[Y]\in\mathrm{orb}(X)} \frac{1}{|\,\mathrm{orb}(Y)|} \times \frac{1}{|(\mathscr{X}//\mathcal{G})(Y, Y)|} \\
&= \sum_{[Y]\in\pi_0(\mathscr{X})} \frac{1}{|\,\mathrm{orb}(Y)| \times |\mathcal{G}_{Y,Y}| \times |\mathscr{X}(Y, Y)|} \\
&= \sum_{[Y]\in\pi_0(\mathscr{X})} \frac{1}{\#\mathcal{G} \times |\mathscr{X}(Y, Y)|} = \frac{|\mathscr{X}|}{\#\mathcal{G}}
\end{aligned}$$

□

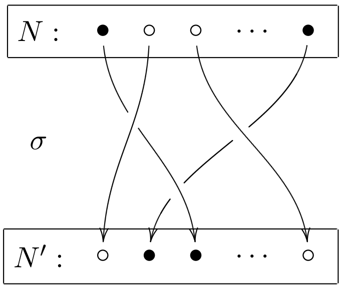
1.4 Colouring

We will often work with groupoids of coloured finite sets. That is, objects of the groupoid are finite sets in which each element has some further property called a "colour". A morphism is a bijective map (that is a morphism in $\mathbb{E}$) which preserves the "colour" of the elements.

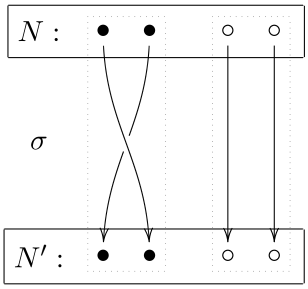
Example 1.4.1. Consider $\mathbb{E}^2$, the groupoid of 2-coloured finite sets. An object of the groupoid is a finite set N in which each element has a label "black" or "white":

$$\boxed{N : \ \bullet \quad \circ \quad \circ \quad \cdots \quad \bullet}$$

A morphism is a bijection $\sigma : N \to N'$ on the underlying finite set with the property that $\sigma(x) \in N'$ has the same colour as $x \in N$:



We can split the objects and consider the elements with a distinct colouring separately:



The isomorphism classes of $\mathbb{E}^2$ will consist of sets which have the same number of black coloured elements and white coloured elements. An automorphism on an object in $\mathbb{E}^2$ will consist of permutations within the colourings, and hence:

$$|\mathbb{E}^2| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!\,m!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{\infty} \frac{1}{m!} = e^2$$

This makes sense, as we can think of this groupoid as the product of a white $\mathbb{E}$ with a black $\mathbb{E}$.

So what exactly is a colouring? In the above example it was a 2-element set $\{\text{black}, \text{white}\}$, which experienced no change under a morphism. This can be thought of as a category with 2 objects, with the only morphisms being the identity (a groupoid!). So continuing in the spirit of replacing sets with groupoids, we obtain the definition:

Definition 1.4.2. For any groupoid $\mathscr{Z}$, let $\mathbb{E}^{\mathscr{Z}}$ be the groupoid of $\mathscr{Z}$-coloured finite sets, which has:

- *Objects*: $(N; \mathbf{Z})$ where N is an object in $\mathbb{E}$ and $\mathbf{Z} = (Z_i)_{i \in N}$ is a family of objects in $\mathscr{Z}$;
- *Morphisms*: $(\sigma; \mathbf{u}) : (N; \mathbf{Z}) \to (N'; \mathbf{Z}')$ where $\sigma : N \to N'$ is a morphism in $\mathbb{E}$ and $\mathbf{u} = (u_i)_{i \in N}$ is a family of morphisms $u_i : Z_i \to Z'_{\sigma(i)}$.

The identity morphism for an object $(N; \mathbf{Z})$ is $(1_N; \mathbf{1_Z})$, where $\mathbf{1_Z} = (1_{Z_i})_{i \in N}$. The composite of $(\sigma; \mathbf{u}) : (N; \mathbf{Z}) \to (N'; \mathbf{Z}')$ and $(\rho; \mathbf{v}) : (N'; \mathbf{Z}') \to (N''; \mathbf{Z}'')$ is $(\rho \circ \sigma; \mathbf{v} \circ \mathbf{u}) : (N; \mathbf{Z}) \to (N''; \mathbf{Z}'')$, where $\mathbf{v} \circ \mathbf{u} = (v_{\sigma(i)} \circ u_i)_{i \in N}$.

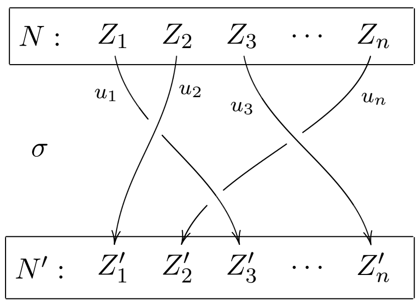
This definition can be considered as the wreath product of $\mathbb{E}$ with $\mathscr{Z}$.

Remark 1.4.3. Technically one should say that the morphisms preserve colouring up to isomorphism. Also our previous example should be referred to as the groupoid of **2**-coloured finite sets.

Using a similar representation as before, an object of $\mathbb{E}^{\mathscr{Z}}$ would look like:

$$\boxed{N: \quad Z_1 \quad Z_2 \quad Z_3 \quad \cdots \quad Z_n}$$

A morphism would look like:



Looking at Example 1.4.1 inspires the following:

Proposition 1.4.4. *For a groupoid $\mathscr{Z}$, the groupoid of $\mathscr{Z}$-coloured finite sets has cardinality*

$$\left| \mathbb{E}^{\mathscr{Z}} \right| = e^{|\mathscr{Z}|}$$

Proof. An isomorphism class of $\mathbb{E}^{\mathscr{Z}}$ consists of sets where each isomorphism class $[Z]$ of $\mathscr{Z}$ has some fixed number $n_{[Z]}$ of elements with colouring in $[Z]$.

An automorphism $(\sigma; \mathbf{u}) : (N; \mathbf{Z}) \to (N; \mathbf{Z})$ will consist of a permutation $\sigma : N \to N$ and a family of morphisms $u_i : Z_i \to Z_{\sigma(i)}$. However since u_i exists only if Z_i and $Z_{\sigma(i)}$ are in the same isomorphism class, we require that σ be restricted to permuting elements which have isomorphic colouring, of which there are:

$$\prod_{[Z]} n_{[Z]}!$$

For each $i \in N$, there are $|\operatorname{Hom}(Z_i, Z_{\sigma(i)})|$ possible choices for u_i, and so the number of possible choices for $\mathbf{u}$ are:

$$\prod_{i \in N} \left| \operatorname{Hom}(Z_i, Z_{\sigma(i)}) \right| = \prod_{i \in N} \left| \operatorname{Aut}(Z_i) \right| = \prod_{[Z]} \left| \operatorname{Aut}(Z) \right|^{n_{[Z]}}$$

by Lemma 1.1.2.

Therefore the cardinality of $\mathbb{E}^{\mathscr{Z}}$.

$$|\mathbb{E}^{\mathscr{Z}}| = \prod_{[Z]} \sum_{n=0}^{\infty} \frac{1}{|\operatorname{Aut}(Z)|^n . n!} = \prod_{[Z]} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{|\operatorname{Aut}(Z)|}\right)^n}{n!}$$
$$= \prod_{[Z]} e^{1/|\operatorname{Aut}(Z)|} = e^{\sum_{[Z]} 1/|\operatorname{Aut}(Z)|} = e^{|\mathscr{Z}|} \qquad \square$$

This definition does allow for some non-intuitive cases, such as $\frac{1}{2}$-colouring:

Example 1.4.5. Let $\mathscr{I}$ be a groupoid with one object I with two automorphisms $1, p$. Then we have $|\mathscr{I}| = \frac{1}{2}$. We can think of n-element $\mathscr{I}$-coloured finite sets as n-dimensional "cubes", whose axes are labelled by elements of the set. Then morphisms on $\mathbb{E}^{\mathscr{I}}$ will correspond to transformations of the cubes. We can think of a morphism $(\sigma; \mathbf{u})$, where σ is a rearrangement of the axes, and each $u_i = p$ takes the mirror image about the ith axis.

These morphisms do not correspond to rotational operations. For example in the 3-dimensional case, we can turn the cube inside out.

Chapter 2

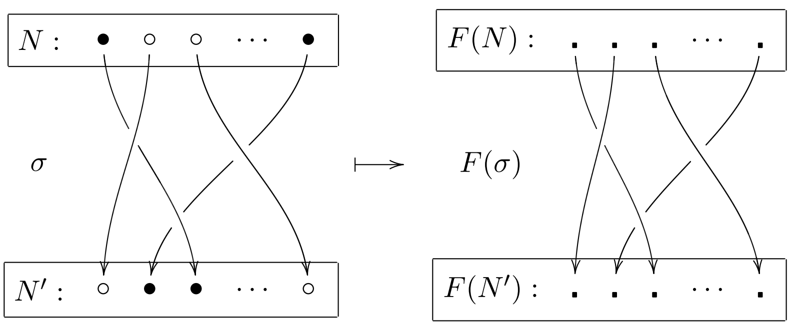
Stuff types

2.1 Stuff types

A species of structures (also called a combinatorial species) maps each finite set to the set of possible structures which may be placed upon it. In this context, the groupoid of $\mathscr{X}$-coloured sets may be considered as the image of some combinatorial species.

A stuff type works the opposite way, by mapping a structured set to its underlying set.

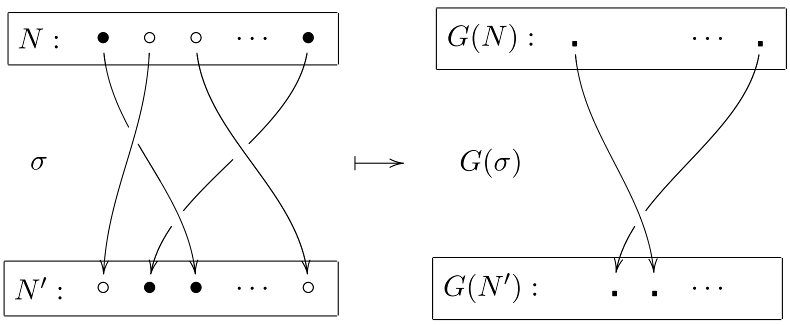
Example 2.1.1. Consider $\mathbb{E}^2$, the groupoid of 2-coloured finite sets, with a functor $F : \mathbb{E}^2 \longrightarrow \mathbb{E}$ which forgets all colouring:



Then the fibre of F would be equivalent to the species which applies a 2-colouring structure to a finite set.

However not all such functors may produce such a species:

Example 2.1.2. Consider $\mathbb{E}^2$, the groupoid of 2-coloured finite sets, with a functor $G : \mathbb{E} \longrightarrow \mathbb{E}$ which forgets all elements of white colouring



The fibre of G will be a messy construction (certainly not a finite set), and definitely not a species.

Definition 2.1.3. A *stuff type* is a groupoid $\mathscr{X}$ together with a functor

$$F : \mathscr{X} \longrightarrow \mathbb{E}$$

Objects of $\mathscr{X}$ are called 'F-stuffed sets', and so $\mathscr{X}$ may be called the groupoid of F-stuffed sets. Note that since a morphism in $\mathscr{X}$ will map to a morphism in $\mathbb{E}$, objects in the same isomorphism class of $\mathscr{X}$ will all map to sets with the same number of elements.

For any $n \in \mathbb{N}$, define $(\mathscr{X}, F)_n$ as the subgroupoid of $\mathscr{X}$ in which all the objects map to n-element sets, and the morphisms between these objects. We may call $(\mathscr{X}, F)_n$ the groupoid of F-stuffed n-element sets. Also we have:

$$\mathscr{X} = \sum_{n=0}^{\infty} (\mathscr{X}, F)_n$$

We can also construct a functor $F_n : (\mathscr{X}, F)_n \longrightarrow \mathbb{E}$ which acts on $(\mathscr{X}, F)_n$ in the same way that F acts on $\mathscr{X}$.

2.1.1 Structure types

As mentioned before, the concept of stuff is a generalisation of structure, and so we seek to reconcile the concept of a stuff type with that of species of structure.

Definition 2.1.4. A species of structure (as given by [7]) is a functor

$$S : \mathbb{E} \longrightarrow \mathbf{FinSet}$$

We can easily create a stuff type from a species S by defining a groupoid $\mathscr{X}$ with:

- *Objects*: (N, t) where N is an object in $\mathbb{E}$ and t is an element of the set $S(N)$;

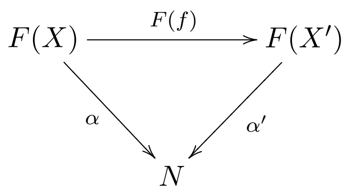
- *Morphisms*: $(\sigma) : (N, t) \to (N', t')$ where $\sigma : N \to N'$ is a morphism in $\mathbb{E}$ and $S(\sigma)(t) = t'$.

Then the associated stuff type is the functor $F : \mathscr{X} \longrightarrow \mathbb{E}$ such that $F(N, t) = N$ and $F(\sigma) = \sigma$. Working the other way is considerably more complex, as we have shown that a stuff type may not always induce a species.

If a stuff type is thought of as a mapping of a structure to its underlying set, then the fibre (or inverse image) of a stuff type will map a finite set to the set of all corresponding structures.

Definition 2.1.5. The *fibre functor* of a stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$ is the functor $F^{-1} : \mathbb{E} \longrightarrow \mathbf{Gpd}$ such that for any finite set N, the groupoid $F^{-1}(N)$ is defined by:

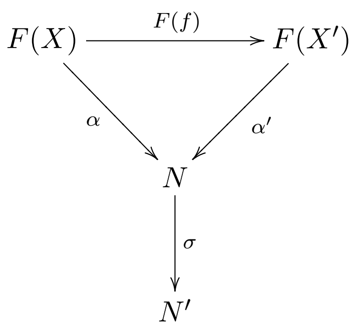
- *Objects*: (X, α), where X is an object in $\mathscr{X}$ and α is an isomorphism $\alpha : F(X) \to N$;
- *Morphisms*: $f : (X, \alpha) \to (X', \alpha')$, where $f : X \to X'$ is a morphism in $\mathscr{X}$ such that the following commutes:



Any bijection $\sigma : N \to N'$ in $\mathbb{E}$ will induce a map $F^{-1}(\sigma) : F^{-1}(N) \longrightarrow F^{-1}(N')$ which maps:

- *Objects*: $(X, \alpha) \longmapsto (X, \sigma \circ \alpha)$;
- *Morphisms*: $f \longmapsto f$.

This is clearly a functor, since the following commutes:



Technically this should be called a pseudofibre (or weak inverse image), as it maps sets up to isomorphism. For this to correspond to a species we require that the fibre functor of any finite set be equivalent to a set.

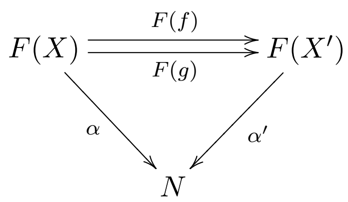
Proposition 2.1.6. *For any stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$, $F^{-1}(N)$ is equivalent to a set for all finite sets N, if and only if F is faithful.*

Proof. The set $\pi_0(F^{-1}(N))$ is a groupoid whose objects are the isomorphism classes of $F^{-1}(N)$, and morphisms are the identity morphisms on each object. Then we have a functor:

$$[\,\cdot\,] : F^{-1}(N) \longrightarrow \pi_0(F^{-1}(N))$$

which maps all objects to their isomorphism class and all morphisms to the identity morphism on that class.

$[\,\cdot\,]$ is obviously essentially surjective and full, so we are concerned with the conditions under which $[\,\cdot\,]$ is faithful. For any morphisms $f, g : (X, \alpha) \to (X', \alpha')$ in $F^{-1}(N)$, then $[(X, \alpha)] = [(X', \alpha')]$, and so $[f] = [g] = 1_{[(X,\alpha)]}$. However the following also commutes:



and so we have $F(f) = F(g) = (\alpha')^{-1} \circ \alpha$. Therefore $[\,\cdot\,]$ is faithful, and hence an equivalence, if and only if F is faithful. $\square$

Therefore when a stuff type is faithful, it corresponds to some species. In such a case the stuff type maps a structure to its underlying set, and so we can say it 'forgets structure'.

One other specific case is also worth mentioning. If a stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$ is both full and faithful, then $\mathscr{X}$ is equivalent to some subgroupoid of $\mathbb{E}$, that is to finite sets which have some particular property. In such a case the stuff type 'forgets properties'.

Definition 2.1.7. A stuff type $F : \mathscr{X} \to \mathbb{E}$ is:

- a *structure type* if F is faithful.
- a *property type* if F is full and faithful.

Example 2.1.8. Consider the groupoid $\mathscr{X}$, whose objects are sets with an even number of elements, and morphisms being the bijections. Then if $F : \mathscr{X} \to \mathbb{E}$ is the inclusion functor, then F is a property type associated with the property of "being an even set".

Example 2.1.9. Let $I : \mathbb{E} \longrightarrow \mathbb{E}$ be the identity functor. Then I is the property type of "being a finite set".

2.2 Generating functions

A generating function of a combinatorial species is a formal power series from which we can extract certain information, namely the number of structures which may be placed on a set of a given size. We can create a similar definition for stuff types:

Definition 2.2.1. The *generating function* of a stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$ is the formal power series:

$$|F|(z) = \sum_{n=0}^{\infty} |(\mathscr{X}, F)_n|\, z^n$$

Firstly we aim to show that this corresponds to the definition of a generating function of a species. The generating function of a species $S : \mathbb{E} \longrightarrow$ **FinSet** is the formal power series:

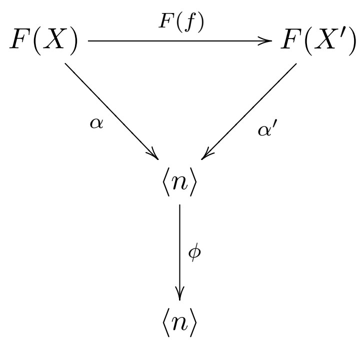
$$|S|(z) = \sum_{n=0}^{\infty} s_n \frac{z^n}{n!}$$

where s_n is the number of structures which can be placed on an n-element set.

Proposition 2.2.2. *For any stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$ we have:*

$$|(\mathscr{X}, F)_n| = |F^{-1}\langle n\rangle|/n!$$

Proof. Let $\mathcal{S}_n$ be the group of permutations on $\langle n\rangle$. Then we can construct a group action A of $\mathcal{S}_n$ on $F^{-1}\langle n\rangle$ such that for each $\phi \in \mathcal{S}_n$ we have $A(\phi)(X, \alpha) = (X, \phi \circ \alpha)$ and $A(\phi)f = f$, since the following commutes:



We now want to show that the weak quotient $F^{-1}\langle n\rangle//\mathcal{S}_n$ is equivalent to $(\mathscr{X}, F)_n$. Construct the functor $T : \left(F^{-1}\langle n\rangle//\mathcal{S}_n\right) \longrightarrow (\mathscr{X}, F)_n$ which maps:

- *Objects*: $(X, \alpha) \longmapsto X$;

- *Morphisms*: $(\phi, f) \longmapsto f$.

This is obviously essentially surjective. Given objects (X, α), (X', α') in $F^{-1}\langle n\rangle//\mathcal{S}_n$, and a morphism $f : X \to X'$, there exists a unique $\phi = \alpha' \circ F(f) \circ \alpha^{-1}$ such that the following commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(X') \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ \langle n\rangle & \xrightarrow[\phi]{} & \langle n\rangle \end{array}$$

Hence there is a unique morphism $(\phi, f) : (X, \alpha) \to (X', \alpha')$ in $F^{-1}\langle n\rangle//\mathcal{S}_n$ such that $T(\phi, f) = f : X \to X'$, and so T is fully faithful, and hence an equivalence. It follows that:

$$|(\mathscr{X}, F)_n| = |F^{-1}\langle n\rangle//\mathcal{S}_n| = \frac{|F^{-1}\langle n\rangle|}{|\mathcal{S}_n|} = \frac{|F^{-1}\langle n\rangle|}{n!} \qquad \square$$

It follows from this that in the case when F is a structure type, the correseponding species will have the same generating function.

Example 2.2.3. Consider the stuff type from example 2.1.1, being $F : \mathbb{E}^2 \longrightarrow \mathbb{E}$ which forgets 2-colouring. For any $n \in \mathbb{N}$ the groupoid $(\mathbb{E}^2, F)_n$ will be the groupoid of n-element 2-coloured sets.

The isomorphism classes will correspond to each $k = 0, 1, 2, \ldots, n$ where k is the number of white-coloured elements. The number of automorphisms on a typical element in such an isomorphism class will be $k!(n-k)!$, since there are $k!$ permutations on the white elements, and $(n-k)!$ on the black-coloured elements, so we have:

$$\begin{aligned} |F|(z) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^n}{k!(n-k)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!}(1+1)^n = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = e^{2z} \end{aligned}$$

Example 2.2.4. Consider the stuff type from example 2.1.2, being $G : \mathbb{E}^2 \longrightarrow \mathbb{E}$ which forgets the white-coloured elements. For any $n \in \mathbb{N}$ the groupoid $(\mathbb{E}^2, G)_n$ will be the groupoid of 2-coloured finite sets in which n of the elements are black-coloured.

The isomorphism classes will correspond to each $k = 0, 1, 2, \ldots$ where k is the number of white-coloured elements. The number of automorphisms on a typical element in such an isomorphism class will be $k!n!$, since there are $k!$ permutations on the white elements, and $n!$ on the black-coloured elements, so we have:

$$|F|(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{z^n}{n!k!} = \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{\infty}\frac{1}{k!} = e^z.e = e^{z+1}$$

2.2.1 A categorical view

Our aim is to determine if the generating function can be interpreted as a category, or ideally a groupoid. This is achieved by colouring the underlying set.

Definition 2.2.5. For a stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$, and some groupoid $\mathscr{Z}$, define the groupoid $\hat{F}(\mathscr{Z})$ of F-stuffed $\mathscr{Z}$-coloured finite sets, as:

- *Objects*: $(X, \alpha, (N; \mathbf{Z}))$ where X is an object in $\mathscr{X}$, $(N; \mathbf{Z})$ is an object in $\mathbb{E}^{\mathscr{Z}}$ and α is an isomorphism $\alpha : F(X) \to N$;
- *Morphisms*: $(f, (\sigma; \mathbf{u})) : (X, \alpha, (N; \mathbf{Z})) \to (X', \alpha', (N'; \mathbf{Z}'))$ where $f : X \to X'$ and $(\sigma; \mathbf{u}) : (N; \mathbf{Z}) \to (N'; \mathbf{Z}')$ such that the following diagram commutes:

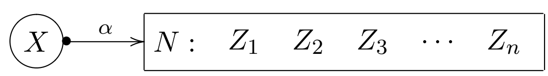
$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha} & N \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle \sigma} \\ F(X') & \xrightarrow{\alpha'} & N' \end{array}$$

Composition of morphisms $(f, (\sigma; \mathbf{u})) : (X, \alpha, (N; \mathbf{Z})) \to (X', \alpha', (N'; \mathbf{Z}'))$ and $(f', (\sigma'; \mathbf{u}')) : (X', \alpha', (N'; \mathbf{Z}')) \to (X'', \alpha'', (N''; \mathbf{Z}''))$ may be defined by:

$$(f', (\sigma'; \mathbf{u}')) \circ (f, (\sigma; \mathbf{u})) = (f' \circ f, (\sigma' \circ \sigma; \mathbf{u}' \circ \mathbf{u}))$$

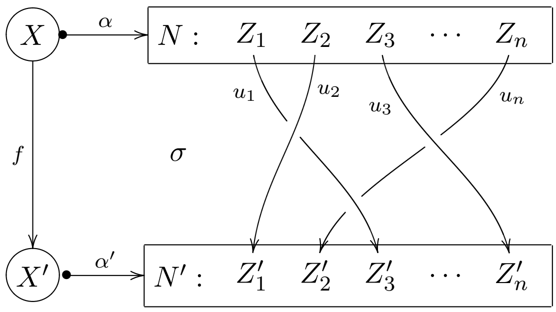
where $\mathbf{u}' \circ \mathbf{u} = (u_{\sigma(i)} \circ u_i)_{i \in N}$. The identity morphism is obvious, and clearly every morphism is invertible, so $\hat{F}(\mathscr{Z})$ is a groupoid.

Using notation similar to before, we can depict objects of $\hat{F}(\mathscr{Z})$ as:



Here X can be thought of as a tag attached to $(N; \mathbf{Z})$ by α.

Similarly, a morphism in $\hat{F}(\mathscr{Z})$ can be depicted as:



Proposition 2.2.6. *For any stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$ and any groupoid $\mathscr{Z}$ we have:*

$$\left|\hat{F}(\mathscr{Z})\right| = \left|F\right|\left(|\mathscr{Z}|\right)$$

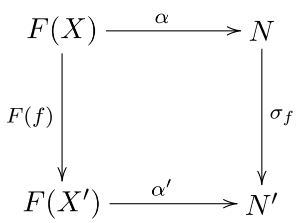
Proof. Our aim is to show:

$$\left|\hat{F}(\mathscr{Z})\right| = \sum_{n=0}^{\infty} \left|(\mathscr{X}, F)_n\right| . |\mathscr{Z}|^n$$

Given an object $(X, \alpha, (N; \mathbf{Z}))$ of $F(\mathscr{Z})$, then X will be an object in $(\mathscr{X}, F)_n$ where $n = |N|$, and so $(X, \alpha, (N; \mathbf{Z}))$ is an object in $\hat{F}_n(\mathscr{Z})$. Furthermore $(f, (\sigma; \mathbf{u})) : (X, \alpha, (N; \mathbf{Z})) \to (X', \alpha', (N'; \mathbf{Z}'))$ will also be a morphism in $\hat{F}_n(\mathscr{Z})$, so we can write:

$$\hat{F}(\mathscr{Z}) \cong \sum_{n=0}^{\infty} \hat{F}_n(\mathscr{Z}) \tag{2.1}$$

Consider objects $(X, \alpha, (N; \mathbf{Z}))$ and $(X', \alpha', (N'; \mathbf{Z}'))$ in the groupoid $F_n(\mathscr{Z})$. Then $|N| = |N'| = n$, and for any $f : X \to X'$ there exists a unique morphism $\sigma_f = \alpha' \circ F(f) \circ \alpha^{-1} : N \to N'$ such that the following commutes:



This implies that $(X, \alpha, (N; \mathbf{Z}))$ and $(X', \alpha', (N'; \mathbf{Z}'))$ are isomorphic if and only if there exists a morphism $f : X \to X'$ and a family of morphisms $\mathbf{u} = (u_i)_{i \in N}$ such that $u_i : Z_i \to Z'_{\sigma_f(i)}$. Therefore the number of possible automorphisms on $(X, \alpha, (N; \mathbf{Z}))$ is:

$$\left|\operatorname{Aut}\left(X, \alpha, (N; \mathbf{Z})\right)\right| = \left|\operatorname{Aut}(X)\right| \prod_{i \in N} \left|\operatorname{Hom}(Z_i, Z_{\sigma_f(i)})\right|$$

and so we have:

$$\begin{aligned}
\big|F_n(\mathscr{Z})\big| &= \sum_{[X]\in\pi_0(\mathscr{X})} \frac{1}{|\operatorname{Aut}(X)|} \prod_{i\in N} \left(\sum_{[Z_i]\in\pi_0(\mathscr{Z})} \frac{1}{|\operatorname{Hom}(Z_i, Z_{\sigma_f(i)})|} \right) \\
&= \sum_{[X]\in\pi_0(\mathscr{X})} \frac{1}{|\operatorname{Aut}(X)|} \left(\sum_{[Z]\in\pi_0(\mathscr{Z})} \frac{1}{|\operatorname{Aut}(Z)|} \right)^n \\
&= \big|(X,F)_n\big| \times |\mathscr{Z}|^n
\end{aligned}$$

Combining this with (2.1) gives the desired result. □

This gives a basis for the following definition:

Proposition 2.2.7. *For any stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$, we have:*

$$\hat{F}(\mathbf{1}) \simeq \mathscr{X} \simeq \hat{E}(\mathscr{Z})$$

where $\mathbf{1}$ *is the groupoid with one object and only the identity morphism on that object, and $E : \mathscr{X} \longrightarrow \mathbb{E}$ is the stuff type which maps every object to the empty set.*

Proof. Define the functor $T : \hat{F}(\mathbf{1}) \longrightarrow \mathscr{X}$ as the projetion:

- for an object: $T(X, \alpha, (N; \mathbf{Z})) = X$;
- for a morphism: $T(f, (\sigma; \mathbf{u})) = f$.

Then T is clearly essentially surjective and full. To show it is faithful, one may see that for a morphism $(f, (\sigma; \mathbf{u})) : (X, \alpha, (N; \mathbf{Z})) \to (X', \alpha', (N'; \mathbf{Z}'))$ in $\hat{F}(\mathbf{1})$ we require that $\sigma = \alpha' \circ F(f) \circ \alpha^{-1}$, and since there is exactly one choice for $\mathbf{u}$, then T must be faithful, and hence an equivalence.

An object of the groupoid $\hat{E}(\mathscr{Z})$ is of the form $(X, \alpha, (N; \mathbf{Z}))$; however, N must be the empty set, α the morphism on the empty set, and $\mathbf{Z}$ is indexed by the empty set (and hence is empty).

Similarly, for a morphism $(f, (\sigma; \mathbf{u}))$, σ is the morphism on the empty set, and $\mathbf{u}$ is empty. It becomes obvious that we can create an equivalence between $\hat{E}(\mathscr{Z})$ and $\mathscr{X}$ (in fact this is an isomorphism). □

Remark 2.2.8. We call the stuff type $E : \mathscr{X} \longrightarrow \mathbb{E}$ in Theorem 2.2.7 a *null type*.

Example 2.2.9. If I is the stuff type of being a finite set (Example 2.1.9), then the groupoid of I-stuffed $\mathscr{Z}$-coloured finite sets is just the groupoid of $\mathscr{Z}$-coloured finite sets, so we can write:

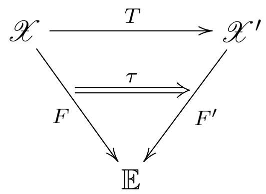
$$\hat{I}(\mathscr{Z}) \simeq \mathbb{E}^{\mathscr{Z}}$$

2.3 The 2-category of stuff types

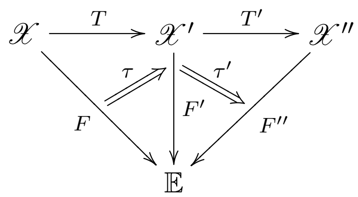
Definition 2.3.1. The 2-category **Stuff** of stuff types has the following:

- *0-cells*: Stuff types $(\mathscr{X}, F)$ where $F : \mathscr{X} \longrightarrow \mathbb{E}$ is a stuff type (may be simply written as F).

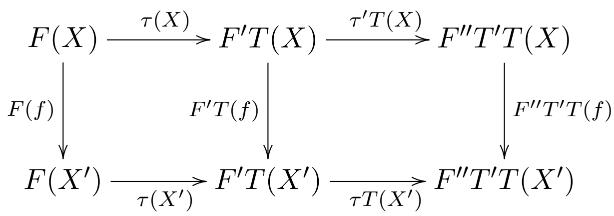
- *1-cells*: Stuff type morphisms $(T, \tau) : (\mathscr{X}, F) \to (\mathscr{X}', F')$ where $T : \mathscr{X} \longrightarrow \mathscr{X}'$ is a functor and τ is a natural transformation (actually a natural isomorphism):

  Composition of morphisms between stuff types $(T, \tau) : F \to F'$ and $(T', \tau') : F' \to F''$ can be defined by "pasting", thus:

  For any morphism $f : X \to X'$ in $\mathscr{X}$ the following commutes:

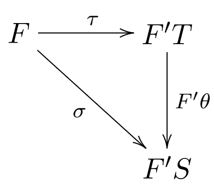
  

  Therefore we have that the pasting composite above amounts to

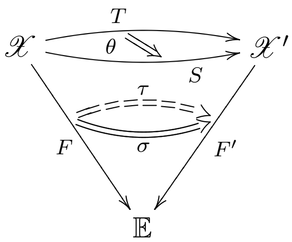
  $$(T', \tau') \circ (T, \tau) = (T'T, \tau'T \circ \tau)$$

  The identity morphism on $F : \mathscr{X} \longrightarrow \mathbb{E}$ is $(1_{\mathscr{X}}, 1_F)$, where 1_F is the identity natural transformation on F.

- *2-cells*: Stuff type transforms $\theta : (T, \tau) \to (S, \sigma)$, where $(T, \tau), (S, \sigma) : F \to F'$ are stuff type morphisms and $\theta : T \to S$ is a natural transformation such that the following commutes:

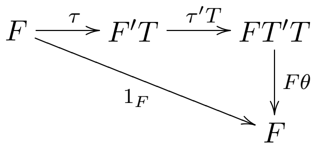
This can be represented as a 3-dimensional cone:

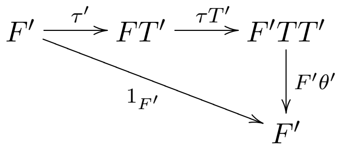


> Vertical and horizontal compositions of stuff type transforms are defined by a similar pasting process.

Defining **Stuff** as a 2-category (as opposed to a category) allows for a concept of equivalence between stuff types, similar to that which exists between categories.

Two stuff types $F : \mathscr{X} \longrightarrow \mathbb{E}$ and $F' : \mathscr{X}' \longrightarrow \mathbb{E}$ are equivalent if there exists stuff type morphisms $(T, \tau) : F \to F'$ and $(T', \tau') : F' \to F$ such that $(T', \tau') \circ (T, \tau) \cong (1_{\mathscr{X}}, 1_F)$ and $(T, \tau) \circ (T', \tau') \cong (1_{\mathscr{X}'}, 1_{F'})$. That is, there exists natural isomorphisms $\theta : T'T \to 1_F$ and $\theta' : TT' \to 1_{F'}$ such that the following diagrams commute:

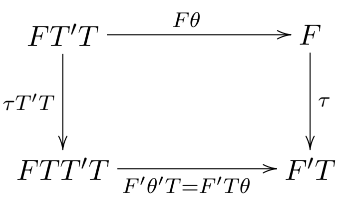




Proposition 2.3.2. *Two stuff types $F : \mathscr{X} \longrightarrow \mathbb{E}$ and $F' : \mathscr{X}' \longrightarrow \mathbb{E}$ are equivalent if and only if there exists a stuff type morphism $(T, \tau) : F \to F'$ such that T is an equivalence.*

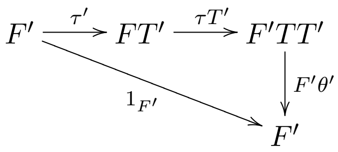
Proof. If F and F' are equivalent, then there exist stuff type morphisms $(T, \tau) : F \to F'$ and $(T', \tau') : F' \to F$ and natural isomorphisms $\theta : T'T \to 1_F$ and $\theta' : TT' \to 1_{F'}$. Therefore T is an equivalence.

To prove the converse, suppose there exist natural isomorphisms $\theta : T'T \to 1_F$ and $\theta^* : TT' \to 1_{F'}$. Since T is fully faithful, there exists a unique $\theta' : TT' \to 1_{F'}$ such that $T\theta = \theta' T$, and so the following commutes:

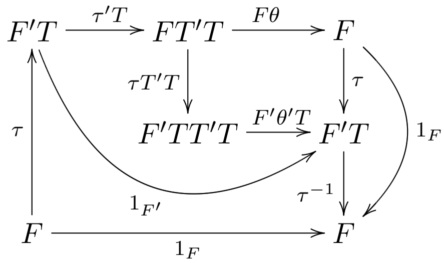


(2.2)

Define τ' such that the following commutes:



Therefore the following commutes:

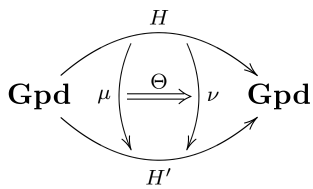


Therefore (T, τ) is an equivalence. □

We can now think of **Gpd** as a 2-category, whose 2-cells are natural transformations between functors. There is also a 2-category [**Gpd**, **Gpd**] of endo-2-functors on **Gpd** with:

- *0-cells*: 2-functors $H : \mathbf{Gpd} \longrightarrow \mathbf{Gpd}$;
- *1-cells*: 2-natural transformations on 2-functors $\mu : H \to H'$ where H, H' are endo-2-functors on **Gpd**;
- *2-cells*: modifications on 2-natural transformations $\Theta : \mu \to \nu$ where $\mu, \nu : H \to H'$ are 2-natural transformations on endo-2-functors.

We can depict these as:



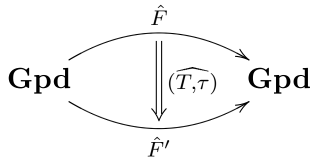
Proposition 2.3.3. *The operation* $(\hat{\ })$ *is a 2-functor from* **Stuff** *to the 2-category* [**Gpd**, **Gpd**] *of endo-2-functors on* **Gpd**.

Proof. Firstly it is necessary to show that for any stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$, $\hat{F}$ is an endo-2-functor on **Gpd**. We have already shown that a groupoid $\mathscr{X}$ maps to a groupoid $\hat{F}(\mathscr{X})$.

For any functor $K : \mathscr{X} \longrightarrow \mathscr{X}'$, we can construct a functor $\hat{F}(K) : \hat{F}(\mathscr{X}) \longrightarrow \hat{F}(\mathscr{X}')$, such that $(X, \alpha, (N; \mathbf{Z})) \longmapsto (X, \alpha, (N; K\mathbf{Z}))$, where $K\mathbf{Z} = (KZ_i)_{i \in N}$, and $(f, (\sigma, \mathbf{u})) \longmapsto (f, (\sigma, K\mathbf{u}))$, where $K\mathbf{u} = (Ku_i)_{i \in N}$.

For any natural transformation $\kappa : K \to L$ where $K, L : \mathscr{X} \longrightarrow \mathscr{X}'$, there is a natural transformation $\hat{F}(\kappa) : \hat{F}(K) \to \hat{F}(L)$ such that $\hat{F}(\kappa)(X, \alpha, (N; \mathbf{Z})) = (1_X, (1_N; \kappa\mathbf{Z})) : (X, \alpha, (N; K\mathbf{Z})) \to (X, \alpha, (N; L\mathbf{Z}))$ where $\kappa\mathbf{Z} = (\kappa Z_i)_{i \in N}$.

We have shown that $(\hat{\cdot})$ does indeed map stuff types to groupoid endofunctors, now we need to show that a morphism between stuff types $(T, \tau) : F \to F'$ will map to a 2-natural transformation:



That is, we want to show that for any groupoids $\mathscr{X}, \mathscr{X}'$ and any functor $K : \mathscr{X} \longrightarrow \mathscr{X}'$, the following diagram commutes:

$$\begin{array}{ccc} \hat{F}(\mathscr{X}) & \xrightarrow{\widehat{(T,\tau)}\mathscr{X}} & \hat{F}'(\mathscr{X}) \\ {\scriptstyle \hat{F}(K)}\downarrow & & \downarrow{\scriptstyle \hat{F}'(K)} \\ \hat{F}(\mathscr{X}') & \xrightarrow[\widehat{(T,\tau)}\mathscr{X}']{} & \hat{F}'(\mathscr{X}') \end{array}$$

and the same with K replaced by a natural transformation κ.

We need to show $\widehat{(T, \tau)}\mathscr{X}$ is a functor:

- For an object $(X, \alpha, (N; \mathbf{Z}))$ in $\hat{F}(\mathscr{X})$ we have:

$$F'T(X) \xleftarrow{\tau(X)} F(X) \xrightarrow{\alpha} N$$

There is an isomorphism $\alpha \circ (\tau(X))^{-1} : F'T(X) \to N$, and so we have:

$$\big(\widehat{(T, \tau)}\mathscr{X}\big)(X, \alpha, (N; \mathbf{Z})) = (T(X), \alpha \circ (\tau(X))^{-1}, (N; \mathbf{Z}))$$

- For any morphism $(f, (\sigma, \mathbf{u})) : (X, \alpha, (N; \mathbf{Z})) \to (X', \alpha', (N'; \mathbf{Z}'))$ in $\hat{F}(\mathscr{X})$ the following commutes:

$$\begin{array}{ccccc} F'T(X) & \xleftarrow{\tau(X)} & F(X) & \xrightarrow{\alpha} & N \\ {\scriptstyle F'T(f)}\downarrow & & {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle \sigma} \\ F'T(X') & \xleftarrow{\tau(X')} & F(X') & \xrightarrow{\alpha'} & N' \end{array}$$

So we have:

$$\big(\widehat{(T, \tau)}\mathscr{X}\big)(f, (\sigma; \mathbf{u})) = (T(f), (N; \mathbf{Z}))$$

Since T is a functor, $\widehat{(T, \tau)}\mathscr{X}$ will preserve the composition and the identity morphism.

Then for any object $(X, \alpha, (N; \mathbf{Z}))$ in $\hat{F}(\mathscr{Z})$ we have:

$$\begin{array}{ccc}
(X, \alpha, (N; \mathbf{Z})) & \xmapsto{\widehat{(T,\tau)}\mathscr{Z}} & (T(X), \alpha \circ (\tau(X))^{-1}, (N; \mathbf{Z})) \\
\Big\downarrow{\scriptstyle \hat{F}(K)} & & \Big\downarrow{\scriptstyle \hat{F}'(K)} \\
(X, \alpha, (N; K\mathbf{Z})) & \xmapsto[\widehat{(T,\tau)}\mathscr{Z}']{} & (T(X), \alpha \circ (\tau(X))^{-1}, (N; K\mathbf{Z}))
\end{array}$$

and for any morphism $(f, (\sigma; \mathbf{u}))$:

$$\begin{array}{ccc}
(f, (\sigma; \mathbf{u})) & \xmapsto{\widehat{(T,\tau)}\mathscr{Z}} & (T(f), (\sigma; \mathbf{u})) \\
\Big\downarrow{\scriptstyle \hat{F}(K)} & & \Big\downarrow{\scriptstyle \hat{F}'(K)} \\
(f, (\sigma; K\mathbf{u})) & \xmapsto[\widehat{(T,\tau)}\mathscr{Z}']{} & (T(f), (\sigma; K\mathbf{u}))
\end{array}$$

Therefore $\widehat{(T, \tau)}$ is a natural transformation, in fact a 2-natural transformation by a similar argument.

Clearly the identity morphism on a stuff type maps to the identity transformation, and we now show that composition of morphisms is preserved.

Given morphisms of stuff types $(T, \tau) : F \to F'$ and $(T', \tau') : F' \to F''$, for any morphism $(f, (\sigma, \mathbf{u})) : (X, \alpha, (N; \mathbf{Z})) \to (X', \alpha', (N'; \mathbf{Z}'))$ in $\hat{F}(\mathscr{Z})$ the following commutes:

$$\begin{array}{ccccccc}
F''T'T(X) & \xleftarrow{\tau' T(X)} & F'T(X) & \xleftarrow{\tau(X)} & F(X) & \xrightarrow{\alpha} & N \\
\Big\downarrow{\scriptstyle F''T'T(f)} & & \Big\downarrow{\scriptstyle F'T(f)} & & \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle \sigma} \\
F''T'T(X') & \xleftarrow{\tau' T(X)} & F'T(X') & \xleftarrow{\tau(X')} & F(X') & \xrightarrow{\alpha'} & N'
\end{array}$$

For an object $(X, \alpha, (N; \mathbf{Z}))$ in $\hat{F}(\mathscr{Z})$:

$$\begin{aligned}
&\big(\widehat{(T', \tau')} \circ \widehat{(T, \tau)}\mathscr{Z}\big)(X, \alpha, (N; \mathbf{Z})) \\
&= \big(\widehat{(T', \tau')}\mathscr{Z}\big)(T(X), \alpha \circ (\tau(X))^{-1}, (N; \mathbf{Z})) \\
&= (T'T(X), \alpha \circ (\tau(X))^{-1} \circ (\tau' T(X))^{-1}, (N; \mathbf{Z})) \\
&= (T'T(X), \alpha \circ (\tau' T \circ \tau)^{-1}(X), (N; \mathbf{Z})) \\
&= \big((T'T, \tau T \circ \tau)\mathscr{Z}\big)(X, \alpha, (N; \mathbf{Z}))
\end{aligned}$$

and a morphism $(f, (\sigma; \mathbf{u}))$ in $\hat{F}(\mathscr{Z})$

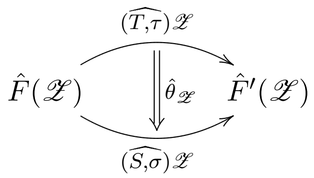
$$\begin{aligned}
\big(\widehat{(T', \tau')} \circ \widehat{(T, \tau)}\mathscr{Z}\big)(f, (\sigma; \mathbf{u})) &= \big(\widehat{(T', \tau')}\mathscr{Z}\big)(T(f), (\sigma; \mathbf{u})) = (T'T(f), (\sigma; \mathbf{u})) \\
&= \big(\widehat{(T'T, \tau T \circ \tau)}\mathscr{Z}\big)(f, (\sigma; \mathbf{u}))
\end{aligned}$$

Hence we have:

$$\widehat{(T',\tau')} \circ \widehat{(T,\tau)} = \widehat{(T',\tau') \circ (T,\tau)}$$

Therefore $(\hat{\cdot})$ is a functor. To show it is a 2-functor, we need to define it on 2-cells.

Given a stuff type transform $\theta : (T,\tau) \to (S,\sigma)$ where $(T,\tau),(S,\sigma) : F \to F'$, define $\hat{\theta} : \widehat{(T,\tau)} \to \widehat{(S,\sigma)}$ such that for any groupoid $\mathscr{X}$, we have a natural transformation $\hat{\theta}_{\mathscr{X}}$:



where $\hat{\theta}_{\mathscr{X}}(X,\alpha,(N;\mathbf{Z})) = (\theta(X),(1_N;\mathbf{1_Z})) : (T(X),\alpha\circ(\tau(X))^{-1},(N;\mathbf{Z})) \to (S(X),\alpha\circ(\sigma(X))^{-1},(N;\mathbf{Z}))$. This is easily shown to preserve vertical and horizontal compositions. □

As a consequence of Proposition 2.3.3, if two stuff types F and F' are equivalent in **Stuff**, then $\hat{F}(\mathscr{X})$ and $\hat{F}'(\mathscr{X})$ are equivalent in **Gpd**.

Chapter 3

Operations on stuff types

For any operation on species of structure, we create an analagous operation on stuff types.

3.1 Addition

Addition of species is defined as the disjoint union of the image. Since stuff types can be interpreted as the fibre of a species, the logical definition for addition of stuff types is the disjoint union of the domains.

Definition 3.1.1. Given a family of stuff types $(F_i)_{i\in\Lambda}$, where $F_i : \mathscr{X}_i \longrightarrow \mathbb{E}$, define the stuff type:

$$\sum_{i\in\Lambda} F_i : \sum_{i\in\Lambda} \mathscr{X}_i \longrightarrow \mathbb{E}$$

such that

$$\sum_{i\in\Lambda} F_i(X) = F_j(X) \quad \text{if } X \text{ is an object of } \mathscr{X}_j$$

and the obvious definition for morphisms.

Using our previous notation, for any stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$, we can write:

$$F = \sum_{n=0}^{\infty} F_n : \sum_{n=0}^{\infty} (\mathscr{X}, F)_n \longrightarrow \mathbb{E}$$

The additive identity is the initial object in **Stuff**; the stuff type $0 : \mathbf{0} \longrightarrow \mathbb{E}$, where $\mathbf{0}$ is the empty groupoid.

Theorem 3.1.2. *Given a family of stuff types $(F_i)_{i\in\Lambda}$ we have:*

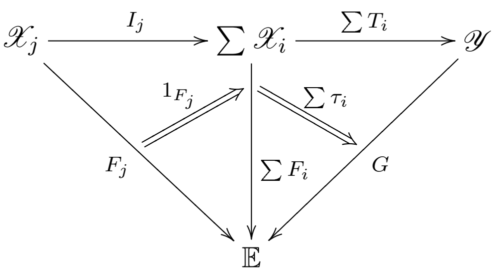
$$\left(\widehat{\sum_{i\in\Lambda} F_i}\right)(\mathscr{Z}) \cong \sum_{i\in\Lambda} \hat{F}(\mathscr{Z})$$

Proof. The result is obvious when one notes that for any object $(X, \alpha, (N; \mathbf{Z}))$ of $\left(\widehat{\sum F_i}\right)(\mathscr{Z})$, there is a unique object X in some unique $\mathscr{X}_j$, and so $(X, \alpha, (N; \mathbf{Z}))$ corresponds to a unique object in $\hat{F}_j(\mathscr{Z})$. □

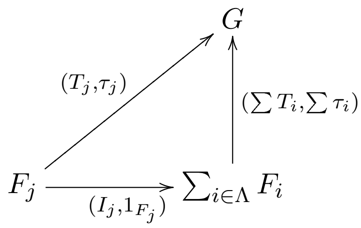
Proposition 3.1.3. *The addition operation corresponds to the coproduct on* **Stuff**.

Proof. For any $j \in \Lambda$, let $I_j : \mathscr{X}_i \longrightarrow \sum \mathscr{X}_j$ be the inclusion functor, so we have $F_j = I_j \circ \sum F_i$. If 1_{F_j} is the identity natural transformation on F_j, then there is a stuff type morphism $(I_j, 1_{F_j}) : F_j \to F$. We can show that this is the coprojection for the coproduct on **Stuff**.

Given a stuff type $G : \mathscr{Y} \longrightarrow \mathbb{E}$, and a collection of stuff type morphisms $(T_j, \tau_j) : F_j \to G$ for each $j \in \Lambda$, we can construct the functor $\sum T_i : \sum \mathscr{X}_i \longrightarrow (Y)$ and the natural transformation $\sum \tau_i : \sum F_i \to G \circ \sum T_i$, where $\sum \tau_i(X) = \tau_j(X)$ when X is an object in $\mathscr{X}_j$. Then we have a stuff type morphism $(\sum T_i, \sum \tau_i) : \sum F_i \to G$ such that for any $j \in \Lambda$:



This composite is equal to (T_j, τ_j), and so the following commutes:



To see this is unique, for any $(T', \tau') : \sum F_i \to G$, we can write $T' = \sum T' \circ I_i$ and $\tau' = \sum \tau' I_i$. □

Example 3.1.4. Consider the property types F being the empty set and G being a 1 element set. Then we have $|F| = 1$ and $|G| = z$. Then $F + G$ corresponds to the property type of being an empty or 1 element set, with generating function $|F + G|(z) = 1 + z$.

3.2 Multiplication

Multiplication of species is achieved by partitioning the set, and applying the constituent species to the individual partitions. The corresponding operation on stuff types is achieved by taking the disjoint union of the image.

Definition 3.2.1. Given a finite family of stuff types $(F_i)_{i\in\Lambda}$, where $F_i : \mathscr{X}_i \longrightarrow \mathbb{E}$, define the stuff type:

$$\prod_{i\in\Lambda} F_i : \prod_{i\in\Lambda} \mathscr{X}_i \longrightarrow \mathbb{E}$$

such that:

- for objects $(X_i)_{i\in\Lambda}$ in $\prod \mathscr{X}_i$, we define

$$\left(\prod_{i\in\Lambda} F_i\right)\big((X_i)_{i\in\Lambda}\big) = \sum_{i\in\Lambda} F_i(X_i)$$

- for morphisms $(f_i)_{i\in\Lambda} : (X_i)_{i\in\Lambda} \to (X'_i)_{i\in\Lambda}$ in $\prod \mathscr{X}_i$, we define

$$\left(\prod_{i\in\Lambda} F_i\right)\big((f_i)_{i\in\Lambda}\big) = \sum_{i\in\Lambda} F_i(f_i) : \sum_{i\in\Lambda} F_i(X_i) \to \sum_{i\in\Lambda} F_i(X'_i)$$

Remark 3.2.2. Given finite sets N and M, we denote the disjoint union of these sets as $N + M$. Also given morphisms $\sigma : N \to N'$ and $\rho : M \to M'$, we define the morphism $(\sigma + \rho) : (N + M) \to (N' + M')$ such that:

$$(\sigma+\rho)(k) = \begin{cases} \sigma(k) & \text{if } k \in N \\ \rho(k) & \text{if } k \in M \end{cases}$$

Such a morphism can be split into its constituent morphisms.

The requirement that the family be finite is to ensure that the disjoint union is still a finite set. The multiplicitive identity is the stuff type $E : \mathbf{1} \longrightarrow \mathbb{E}$, where $\mathbf{1}$ is defined as usual and E maps the object to the empty set.

Interestingly, this definition does not correspond to the product operation on **Stuff**. Given a stuff type morphism $(T, \tau) : F \to G$, we require that the F-stuffed set map to a G-stuffed set of the same size. Since a $\prod F_i$-stuffed set will usually be larger than an F_j-stuffed set, it is impossible to create the necessary projections.

Theorem 3.2.3. *Given a finite family of stuff types* $(F_i)_{i\in\Lambda}$ *we have:*

$$\left(\widehat{\prod_{i\in\Lambda} F_i}\right)(\mathscr{Z}) \cong \prod_{i\in\Lambda} \hat{F}_i(\mathscr{Z})$$

Proof. $(\widehat{\prod F})(\mathscr{X})$ is a groupoid with:

- *Objects*: $\big((X_i)_{i\in\Lambda}, \alpha, (N;\mathbf{Z})\big)$ where $\alpha : \sum F_i(X_i) \to N$;
- *Morphisms*:$\big((f_i)_{i\in\Lambda}, (\sigma;\mathbf{u})\big) : \big((X_i)_{i\in\Lambda}, \alpha, (N;\mathbf{Z})\big) \to \big((X_i')_{i\in\Lambda}, \alpha', (N';\mathbf{Z}')\big)$ such that the following commutes:

$$\begin{array}{ccc} \sum F_i(X_i) & \xrightarrow{\alpha} & N \\ {\scriptstyle \sum F_i(f_i)}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \sum F_i(X_i') & \xrightarrow{\alpha'} & N' \end{array}$$

For each object $\big((X_i), \alpha, (N;\mathbf{Z})\big)$, and for each $i \in \Lambda$ define $N_i = \alpha(F_i(X_i))$. Clearly $N = \sum N_i$, and there exists isomorphisms $\alpha_i : F_i(X_i) \to N_i$ such that $\alpha = \sum \alpha_i$.

For any morphism $\big((f_i)_{i\in\Lambda}, (\sigma;\mathbf{u})\big) : \big((X_i)_{i\in\Lambda}, \alpha, (N;\mathbf{Z})\big) \to \big((X_i')_{i\in\Lambda}, \alpha', (N';\mathbf{Z}')\big)$, and for each $i \in \Lambda$ we can create morphisms $\sigma_i : N_i \to N_i'$ such that $\sigma = \sum \sigma_i$ and the following diagram commutes:

$$\begin{array}{ccc} F_i(X_i) & \xrightarrow{\alpha_i} & N_i \\ {\scriptstyle F_i(f_i)}\downarrow & & \downarrow{\scriptstyle \sigma_i} \\ F_i(X_i') & \xrightarrow{\alpha_i'} & N_i' \end{array}$$

We can construct a functor $T : (\widehat{\prod F_i})(\mathscr{X}) \longrightarrow \prod \hat{F}_i(\mathscr{X})$ where:

- *Objects*: $T\big((X_i)_{i\in\Lambda}, \alpha, (N;\mathbf{Z})\big) = \big(X_i, \alpha_i, (N_i;\mathbf{Z}_i)\big)_{i\in\Lambda}$ where $\mathbf{Z}_i = (Z_j)_{j\in N_i}$;
- *Morphisms*: $T\big((f_i)_{i\in\Lambda}, (\sigma;\mathbf{u})\big) = \big(f_i, (\sigma_i;\mathbf{u}_i)\big)_{i\in\Lambda}$ where $\mathbf{u}_i = (u_j)_{j\in N_i}$.

We also have a functor $S : \prod \hat{F}_i(\mathscr{X}) \longrightarrow (\widehat{\prod F_i})(\mathscr{X})$:

- *Objects*: $S\big(X_i, \alpha_i, (N_i;\mathbf{Z}_i)\big)_{i\in\Lambda} = \big((X_i)_{i\in\Lambda}, \sum \alpha_i, (\sum N_i; \sum \mathbf{Z}_i)\big)$;
- *Morphisms*: $S\big(f_i, (\sigma_i;\mathbf{u}_i)\big)_{i\in\Lambda} = \big((f_i)_{i\in\Lambda}, (\sum \sigma_i; \sum \mathbf{u}_i)\big)$.

Clearly $T \circ S = 1_{\prod \hat{F}_i(\mathscr{X})}$ and $S \circ T = 1_{(\widehat{\prod F_i})(\mathscr{X})}$, so the groupoids are isomorphic, and hence equivalent. □

Example 3.2.4. The product of stuff types provides another way of thinking of examples 2.1.1 and 2.1.2.

Since a 2-coloured set can be thought of as the disjoint union of a 'black' and 'white' set, the stuff type F which forgets 2-colouring can be thought of as the product of two copies of the property type $I : \mathbb{E} \longrightarrow \mathbb{E}$, associated with "being a finite set".

The stuff type G which forgets white-coloured elements can be thought of as the product of the property type $I : \mathbb{E} \longrightarrow \mathbb{E}$ with the stuff type $E : \mathbb{E} \longrightarrow \mathbb{E}$ which maps everything to the empty set.

Multiplication by a scalar can be thought of as a special case of multiplication of stuff types. However, yet again, our scalar is not a number, but a groupoid.

Given a groupoid $\mathscr{U}$ and a stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$, we can construct the scalar multiple stuff type:

$$(\mathscr{U}F) : \mathscr{U} \times \mathscr{X} \longrightarrow \mathbb{E}$$

by creating the stuff type $E : \mathscr{U} \longrightarrow \mathbb{E}$ which maps everything to the empty set, and letting $\mathscr{U}F = E \times F$.

By Theorem 2.2.7, we have:

$$(\widehat{\mathscr{U}F})(\mathscr{Z}) \simeq \mathscr{U} \times \hat{F}(\mathscr{Z})$$

Example 3.2.5. Let $\mathscr{X}$ be the groupoid of one-element sets, and $J : \mathscr{X} \longrightarrow \mathbb{E}$ be the inclusion functor. Then J is a property type of "being a one-element set". Then we have:

$$\hat{F}(\mathscr{Z}) = \mathscr{Z}$$

If we let $\mathbf{2}$ be a groupoid with two objects (which we will call 'black' and 'white') and only the identity morphisms on those objects. Then $\mathbf{2} \times \mathscr{X}$ becomes the groupoid of 2-colored 1-element sets, and $\mathbf{2}F$ becomes the stuff type which forgets the colouring of a 2-colored 1-element set, with:

$$(\widehat{\mathbf{2}F})(\mathscr{Z}) = \mathbf{2}\mathscr{Z}$$

3.3 Composition

When working with species, composition is a somewhat complicated operation involving partitioning, applying a species to each partition and applying another species to the set of partitions.

The composition of two stuff types F and G can be thought of as the disjoint union of a family of G-stuffed sets indexed by an F-stuffed set.

Definition 3.3.1. Given two stuff types $F : \mathscr{X} \longrightarrow \mathbb{E}$ and $G : \mathscr{Y} \longrightarrow \mathbb{E}$ define the stuff type $F \circ G : \hat{F}(\mathscr{Y}) \longrightarrow \mathbb{E}$ such that:

- for an object $(X, \alpha, (N; \mathbf{Y}))$ in $F(\mathscr{Y})$, we have:

$$(F \circ G)(X, \alpha, (N; \mathbf{Y})) = \sum_{i \in N} G(Y_i)$$

- for a morphism $(f,(\sigma;\mathbf{g})) : (X,\alpha,(N;\mathbf{Y})) \to (X',\alpha',(N';\mathbf{Y}'))$ in $F(\mathscr{Y})$ we define:

$$(F\circ G)(f,(\sigma;\mathbf{g})) = \sum_{i\in N} G(g_i) : \sum_{i\in N} G(Y_i) \to \sum_{i\in N} G(Y'_{\sigma(i)})$$

In the definition of composition of species, it is necessary to place additional restrictions on G. When composing stuff types, we have no such restrictions.

Theorem 3.3.2. *For stuff types $F : \mathscr{X} \to \mathbb{E}$ and $G : \mathscr{Y} \to \mathbb{E}$, we have:*

$$\widehat{(F\circ G)}(\mathscr{Z}) \cong \hat{F}\big(\hat{G}(\mathscr{Z})\big)$$

$\hat{F}\big(\hat{G}(\mathscr{Z})\big)$ is the groupoid of F-stuffed finite sets, whose elements are labelled by objects of $\hat{G}(\mathscr{Z})$. That is, the groupoid of F-stuffed finite sets, with elements labelled by G-stuffed $\mathscr{Z}$-coloured finite sets. On the other hand $\widehat{(F\circ G)}(\mathscr{Z})$ is the groupoid of families of G-stuffed $\mathscr{Z}$-coloured finite sets, indexed by F-stuffed finite sets.

Proof. The groupoid $\widehat{(F\circ G)}(\mathscr{Z})$ has:

- *Objects*: $\big((X,\alpha,(N;\mathbf{Y})),\beta,(M;\mathbf{Z})\big)$ where $\alpha : F(X) \to N$ and $\beta : \sum_{i\in N} G(Y_i) \to M$;
- *Morphisms*: $\big((f,(\sigma;\mathbf{g})),(\rho;\mathbf{u})\big) : \big((X,\alpha,(N;\mathbf{Y})),\beta,(M;\mathbf{Z})\big) \to \big((X',\alpha',(N';\mathbf{Y}')),\beta',(M';\mathbf{Z}')\big)$ such that the following commute:

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha} & N \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle \sigma} \\ F(X') & \xrightarrow{\alpha'} & N' \end{array} \qquad \begin{array}{ccc} \sum_{i\in N} G(Y_i) & \xrightarrow{\beta} & M \\ {\scriptstyle \sum_{i\in N} G(g_i)}\downarrow & & \downarrow{\scriptstyle \rho} \\ \sum_{i\in N} G(Y'_{\sigma(i)}) & \xrightarrow{\beta'} & M' \end{array}$$

We can apply a similar process to that used in the proof of Theorem 3.2.3. For the above objects and morphisms, we can decompose β, β', M, M' and ρ into their constituents such that for each $i \in N$ the following commutes:

$$\begin{array}{ccc} G(Y_i) & \xrightarrow{\beta_i} & M_i \\ {\scriptstyle G(g_i)}\downarrow & & \downarrow{\scriptstyle \rho_i} \\ G(Y'_{\sigma(i)}) & \xrightarrow{\beta'_{\sigma(i)}} & M'_{\sigma(i)} \end{array}$$

Therefore we have a functor $T : \widehat{(F\circ G)}(\mathscr{Z}) \longrightarrow \hat{F}\big(\hat{G}(\mathscr{Z})\big)$ defined as:

- For an object $\big((X, \alpha, (N; \mathbf{Y})), \beta, (M; \mathbf{Z})\big)$ we have:

$$T\big((X, \alpha, (N; \mathbf{Y})), \beta, (M; \mathbf{Z})\big) = (X, \alpha, (N; \mathbf{U}))$$

  where $U_i = (Y_i, \beta_i, (M_i; \mathbf{Z}_i))$ for each $i \in N$.

- For a morphism $\big((f, (\sigma; \mathbf{g})), (\rho; \mathbf{u})\big)$ we have:

$$T\big((f, (\sigma; \mathbf{g})), (\rho; \mathbf{u})\big) = (f, (\sigma; \mathbf{v}))$$

  where $v_i = (g_i, (\rho_i, \mathbf{u}_i))$ for each $i \in N$.

Then T is invertible. □

Example 3.3.3. This gives yet another way of thinking of examples 2.1.1. Let $I : \mathbb{E} \longrightarrow \mathbb{E}$ be the identity functor, that is the property type of "being a finite set", and compose it with the stuff type $\mathbf{2}J$ from example 3.2.5 (the structure type which forgets 2-colouring on 1-element sets). The resultant stuff type maps finite sets in which each element has a 2-colouring, that is a 2-coloured finite set, to the underlying set. In other words $F \cong I \circ \mathbf{2}J$, and so we have:

$$|F|(z) = |I|\big(|\mathbf{2}J|(z)\big) = e^{2z}$$

Let $\mathscr{Y}$ be the groupoid of empty and 1-element sets, and $K : \mathscr{Y} \longrightarrow \mathbb{E}$ be the inclusion functor. Then the generating function of K is clearly $|K|(z) = 1 + z$. The groupoid $\hat{I}(\mathscr{Y})$ has objects which are finite sets in which each element has been coloured either by a singleton or the empty set. The composition stuff type $I \circ K : \hat{I}(\mathscr{Y}) \longrightarrow \mathbb{E}$ will map these objects to set of elements which were coloured by singletons, that is forgetting the elements coloured by the empty set. This is essentially the same as G in example 2.1.2, and in fact we they share generating functions:

$$|G|(z) = |I|\big(|K|(z)\big) = e^{1+z}$$

This also shows that the composition of structure types may not necessarily be a structure type.

3.4 Pointing

The pointing operation in a species selects a distinguished element of the set. The analagous operation on a stuff type modifies the groupoid, so that each object has an associated element of the output set.

Definition 3.4.1. For a stuff type $F : \mathscr{X} \to \mathbb{E}$, define $\mathscr{X}^*$ with:

- *Objects*: (X, k) where X is an object of $\mathscr{X}$ and k is an element of the set $F(X)$;

- *Morphisms*: $f : (X, k) \to (X', k')$ where $f : X \to X'$ is a morphism in $\mathscr{X}$ such that $F(f)(k) = k'$.

Define the stuff type $\bullet F : \mathscr{X}^* \longrightarrow \mathbb{E}$ such that $\bullet F(X, k) = F(X)$ and $\bullet F(f) = F(f)$.

Theorem 3.4.2. *If $F : \mathscr{X} \to \mathbb{E}$ is a stuff type, then:*

$$|\bullet F|(z) = z\frac{\mathrm{d}}{\mathrm{d}z}|F|(z)$$

Proof. There is a group action A of $\mathcal{S}_{n-1}$, the group of permutations on $\langle n-1\rangle$, on $F^{-1}\langle n\rangle$, such that for any $\phi \in \mathcal{S}_{n-1}$, we have $A(\phi)(X, \alpha) = (X, \phi \circ \alpha)$ and $A(\phi)f = f$.

Construct the functor $T : (F^{-1}\langle n\rangle//\mathcal{S}_n) \longrightarrow (\mathscr{X}^*, \bullet F)_n$ which maps:

- *Objects*: $(X, \alpha) \longmapsto (X, \alpha^{-1}(n))$;
- *Morphisms*: $(\phi, f) \longmapsto f$.

This is obviously essentially surjective, since α can be chosen such that $\alpha^{-1}(n) = k$ for any $k \in F(X)$. Given objects $(X, \alpha), (X', \alpha')$ in $F^{-1}\langle n\rangle//\mathcal{S}_n$ and a morphism $f : (X, \alpha^{-1}(n)) \to (X', \alpha'^{-1}(n))$ in $\mathscr{X}^*$, there exists a unique permutation ϕ on $\langle n\rangle$ where $\phi = \alpha' \circ F(f) \circ \alpha^{-1}$ such that the following commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(X') \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha'} \\ \langle n\rangle & \xrightarrow[\phi]{} & \langle n\rangle \end{array}$$

Since f is a morphism in $\mathscr{X}^*$, then $f(\alpha^{-1}(n)) = \alpha'^{-1}(n)$, and so ϕ is actually a permutation on $\langle n-1\rangle$. Therefore T is fully faithful, and hence an equivalence. Proposition 2.2.2 gives the followng:

$$\big|(\mathscr{X}^*, \bullet F)_n\big| = \frac{|F^{-1}\langle n\rangle|}{(n-1)!} = n\frac{|F^{-1}\langle n\rangle|}{n!} = n\big|(\mathscr{X}, F)_n\big|$$

The result follows. □

3.5 Derivative

The derivative operation on a species is the structure of a set which is one element larger. The derivative on a stuff type will map to a set which is one element smaller.

Definition 3.5.1. For a stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$, define the groupoid $\mathscr{X}^*$ as for the pointing operation. Define the functor $\partial F : \mathscr{X}^* \longrightarrow \mathbb{E}$ as $\partial F(X,k) = F(X) \setminus \{k\}$ (the set $F(X)$ excluding k) and $\partial F(f) = F(f)$, where the domain is restricted to $F(X) \setminus \{k\}$.

Theorem 3.5.2. *If $F : \mathscr{X} \to \mathbb{E}$ is a stuff type, then:*

$$|\partial F|(z) = \frac{\mathrm{d}}{\mathrm{d}z}|F|(z)$$

Proof. This result follows from Theorem 3.4.2, since:

$$\big|(\mathscr{X}^*, \partial F)_n\big| = \big|(\mathscr{X}^*, \bullet F)_{n+1}\big| \qquad \square$$

The groupoid $\hat{F}'(\mathscr{X})$ has:

- *Objects*: $\big((X,n), \alpha, (N;\mathbf{Z})\big)$ where $\alpha : F(X) \setminus \{n\} \to N$;
- *Morphisms*: $(f, (\sigma, \mathbf{u})) : \big((X,n), \alpha, (N;\mathbf{Z})\big) \to \big((X',n'), \alpha', (N';\mathbf{Z}')\big)$ such that the following commutes:

$$\begin{array}{ccc} F(X) \setminus \{n\} & \xrightarrow{\alpha} & N \\ {\scriptstyle F'(f)}\downarrow & & \downarrow{\scriptstyle \sigma} \\ F(X') \setminus \{n'\} & \xrightarrow{\alpha'} & N' \end{array}$$

Conceptually a ∂F-stuffed set can be thought of as an F-stuffed set with one element dropped.

A common feature for any concept called a derivative is that it satisfies the chain rule. If an $F \circ G$-stuffed set is the disjoint union of a family of G-stuffed sets indexed by an F-stuffed set, then an $\partial(F \circ G)$-stuffed set is the disjoint union in which one of the elements have been dropped. This is equivalent to dropping the entire G-stuffed set containing the dropped element (that is dropping an element of the index set) which is an $\partial F \circ G$-stuffed set, and the disjoint union with a G-stuffed set with a dropped element (a ∂G-stuffed set).

Theorem 3.5.3 (Chain rule)**.** *For stuff types $F : \mathscr{X} \to \mathbb{E}$ and $G : \mathscr{Y} \to \mathbb{E}$, we have:*

$$\partial(F \circ G) \simeq (\partial F \circ G)\partial G$$

Proof. We need to show that there exists an equivalence $(T, \tau) : \partial(F \circ G) \to (\partial F \circ G)\partial G$. Define the functor $T : \hat{F}(\mathscr{Y})^* \longrightarrow \widehat{\partial F}(\mathscr{Y})\mathscr{Y}^*$ as:

- For any object $\big((X, \beta, (M; \mathbf{Y})), k\big)$ in $\hat{F}(\mathscr{Y})^*$ define T such that:

  $$T\big((X, \beta, (M; \mathbf{Y})), k\big) = \big(((X, j), \beta^*, (M^*; \mathbf{Y})), (Y_{\beta(j)}, k)\big)$$

  where j is the element of $F(X)$ such that k is an element of $G(Y_{\beta(j)})$, and β^* is β over the domain $F(X) \setminus \{j\}$, and M^* is the range of β^* (that is $M \setminus \{\beta(j)\}$).

- For any morphism $(f, (\rho; \mathbf{g})) : \big((X, \beta, (M; \mathbf{Y})), k\big) \to \big((X', \beta', (M'; \mathbf{Y}')), k'\big)$ in $\hat{F}(\mathscr{Y})^*$:

  $$T(f, (\rho; \mathbf{g})) = \big((f, (\rho^*; \mathbf{g})), (g_{\beta(j)})\big)$$

  where j is defined as above and σ^* is the function σ with domain restricted to M^*. Therefore the following diagram commutes:

  $$\begin{array}{ccc} \partial F(X) & \xrightarrow{\beta^*} & M^* \\ {\scriptstyle \partial F(f)} \downarrow & & \downarrow {\scriptstyle \sigma^*} \\ \partial F(X') & \xrightarrow{\beta'^*} & M'^* \end{array}$$

  Given $(f, (\rho; \mathbf{g})) : \big((X, \beta, (M; \mathbf{Y})), k\big) \to \big((X', \beta', (M'; \mathbf{Y}')), k'\big)$ and $(f', (\rho'; \mathbf{g}')) : \big((X', \beta', (M'; \mathbf{Y}')), k'\big) \to \big((X'', \beta'', (M''; \mathbf{Y}'')), k''\big)$, with $j \in F(X)$ and $j' \in F(X')$ such that $k \in G(Y_{\beta(j)})$ and $k' \in G(Y'_{\beta'(j')})$, then we can show T preserves composition:

  $$\begin{aligned} & T(f, (\rho; \mathbf{g})) \circ T(f', (\rho'; \mathbf{g}')) \\ & = \big((f, (\rho^*; \mathbf{g})), (g_{\beta(j)})\big) \circ \big((f', (\rho'^*; \mathbf{g}')), (g'_{\beta'(j')})\big) \\ & = \big((f, (\rho^*; \mathbf{g})) \circ (f', (\rho'^*; \mathbf{g}')), (g_{\beta(j)}) \circ (g'_{\beta'(j')})\big) \\ & = \big((f \circ f', (\rho^* \circ \rho'^*; \mathbf{g} \circ \mathbf{g}')), (g_{\beta(j)} \circ g'_{\sigma(\beta(j))})\big) \\ & = T\big((f, (\rho; \mathbf{g})) \circ (f', (\rho'; \mathbf{g}'))\big) \end{aligned}$$

  since $\beta'(j') = \sigma(\beta(j))$ and hence $\rho^* \circ \rho'^* = (\rho \circ \rho')^*$.

T is clearly essentially surjective, full and faithful, and hence is an equivalence. □

Similar results can be shown, such as distribution over addition:

$$\partial(F + G) \simeq \partial F + \partial G$$

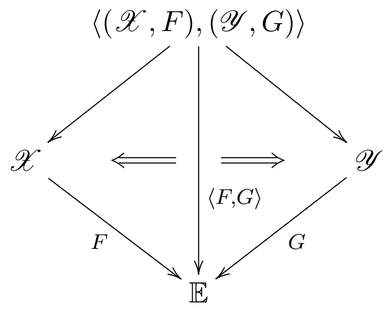
and the product rule:

$$\partial(FG) \simeq (\partial F)G + F(\partial G)$$

3.6 Cartesian product

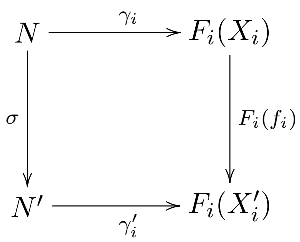
When working with species, the cartesian product involves applying several structures to the same set. The analogue for stuff types requires the construction of a type of product groupoid such that objects correspond to sets of the same size from the component stuff types.

For stuff types $F : \mathscr{X} \longrightarrow \mathbb{E}$ and $G : \mathscr{Y} \longrightarrow \mathbb{E}$ we can construct a pseudo-pullback $\langle(\mathscr{X}, F), (\mathscr{Y}, G)\rangle$ and the functor $\langle F, G\rangle : \langle(\mathscr{X}, F), (\mathscr{Y}, G)\rangle \longrightarrow \mathbb{E}$:



Definition 3.6.1. Given a finite family of stuff types $(F_i)_{i\in\Lambda}$, where $F_i : \mathscr{X}_i \longrightarrow \mathbb{E}$, define the groupoid $\langle(\mathscr{X}_i, F_i)\rangle_{i\in\Lambda}$ with:

- *Objects*: $\big(N, (X_i, \gamma_i)_{i\in\Lambda}\big)$ where N is a finite set, X_i is an object in $\mathscr{X}_i$ and an isomorphism $\gamma_i : N \to F_i(X_i)$;
- *Morphisms*: $\big(\sigma, (f_i)_{i\in\Lambda}\big) : \big(N, (X_i, \gamma_i)_{i\in\Lambda}\big) \to \big(N', (X_i', \gamma_i')_{i\in\Lambda}\big)$, where $\sigma : N \to N'$ in $\mathbb{E}$ and $f_i : X_i \to X_i$ in $\mathscr{X}_i$ such that:



commutes for each $i \in \Lambda$.

The *cartesian product* is the stuff type $\langle F_i\rangle_{i\in\Lambda} : \langle(\mathscr{X}_i, F_i)\rangle_{i\in\Lambda} \longrightarrow \mathbb{E}$ which maps:

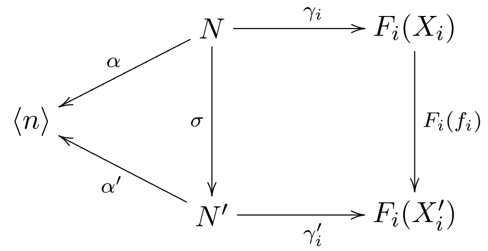
- *Objects*: $\big(N, (X_i, \gamma_i)_{i\in\Lambda}\big) \longmapsto N$;
- *Morphisms*: $\big(\sigma, (f_i)_{i\in\Lambda}\big) \longmapsto \sigma$.

This definition is somewhat complex, but we can show it corresponds to the definition of the cartesian product of species.

Theorem 3.6.2. *For any a family of stuff types*$(F_i)_{i\in\Lambda}$*, where* $F_i : \mathscr{X}_i \longrightarrow \mathbb{E}$*, for any* $n \in \mathbb{N}$ *we have:*

$$\langle F_i\rangle^{-1}_{i\in\Lambda}\langle n\rangle \simeq \prod_{i\in\Lambda} F_i^{-1}\langle n\rangle$$

Proof. A typical morphism in $\langle F_i\rangle^{-1}_{i\in\Lambda}\langle n\rangle$ is $\big(\sigma, (f_i)_{i\in\Lambda}\big) : \big(N, (X_i, \gamma_i)_{i\in\Lambda}, \alpha\big) \to \big(N', (X'_i, \gamma'_i)_{i\in\Lambda}, \alpha'\big)$ such that for each $i \in \Lambda$ the following commutes:



Construct the functor $T : \langle F_i\rangle^{-1}_{i\in\Lambda}\langle n\rangle \longrightarrow \prod_{i\in\Lambda} F_i^{-1}\langle n\rangle$ which maps:

- *Objects*: $\big(N, (X_i, \gamma_i)_{i\in\Lambda}, \alpha\big) \longmapsto (X_i, \alpha\circ\gamma_i^{-1})_{i\in\Lambda}$;
- *Morphisms*: $\big(\sigma, (f_i)_{i\in\Lambda}\big) \longmapsto (f_i)_{i\in\Lambda}$.

This is clearly essentially surjective, and is easily shown to be fully faithful.

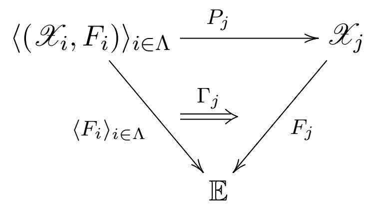
Given objects $\big(N, (X_i, \gamma_i)_{i\in\Lambda}, \alpha\big)$ and $\big(N', (X'_i, \gamma'_i)_{i\in\Lambda}, \alpha'\big)$ in $\langle F_i\rangle^{-1}_{i\in\Lambda}\langle n\rangle$ and a morphism $(f_i)_{i\in\Lambda} : (X_i, \alpha\circ\gamma_i^{-1})_{i\in\Lambda} \to (X'_i, \alpha'\circ\gamma_i'^{-1})_{i\in\Lambda}$ in $\prod_{i\in\Lambda} F_i^{-1}\langle n\rangle$, there is a unique $\sigma = \alpha'^{-1}\circ\alpha$ such that $T(\sigma, (f_i)_{i\in\Lambda}) = (f_i)_{i\in\Lambda}$. $\square$

We showed previously that the multiplication operation did not correspond to the product operation on **Stuff**. The cartesian product somewhat fulfills this role, but not quite.

For any $j \in \Lambda$ we can construct the functor $P_j : \langle(\mathscr{X}_i, F_i)\rangle_{i\in\Lambda} \longrightarrow \mathscr{X}_j$ with:

- *Objects*: $\big(N, (X_i, \gamma_i)_{i\in\Lambda}\big) \longmapsto X_j$;
- *Morphisms*: $\big(\sigma, (f_i)_{i\in\Lambda}\big) \longmapsto f_j$.

Then there exists a natural isomorphism $\Gamma_j : \langle F_i\rangle_{i\in\Lambda} \to F_j\circ P_j$ where $\Gamma_j\big(N, (X_i, \gamma_i)_{i\in\Lambda}\big) = \gamma_j$ so we have the following:



The stuff type morphism $(P_j, \Gamma_j) : \langle F_i \rangle_{i\in\Lambda} \to F_j$ becomes the projection onto F_j.

Given a stuff type $G : \mathscr{Y} \longrightarrow \mathbb{E}$ and a collection of stuff type morphisms $(T_i, \tau_i)_{i\in\Lambda}$, where $(T_j, \tau_j) : G \to F_j$ for each $j \in \Lambda$, we can construct the functor $[T_i, \tau_i]_{i\in\Lambda} : \mathscr{Y} \longrightarrow \langle(\mathscr{X}_i, F_i)\rangle_{i\in\Lambda}$ defined thus:

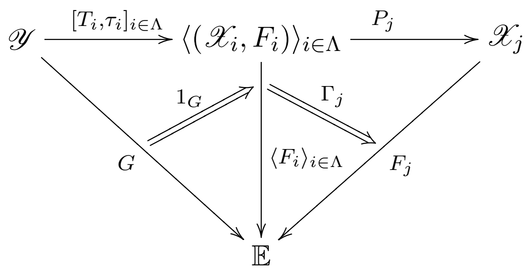
- for an object Y in $\mathscr{Y}$ we have:

$$[T_i, \tau_i]_{i\in\Lambda}(Y) = \big(G(Y), (T_i(Y), \tau_i(Y))_{i\in\Lambda}\big)$$

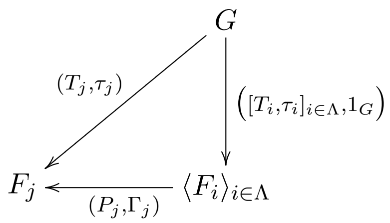
- for a morphism $g : Y \to Y'$ in $\mathscr{Y}$ we have:

$$[T_i, \tau_i]_{i\in\Lambda}(g) = \big(G(g), (T_i(g))_{i\in\Lambda}\big)$$

Then clearly $\langle F_i \rangle_{i\in\Lambda} \circ [T_i, \tau_i]_{i\in\Lambda} = G$, and so we have a stuff type morphism $\big([T_i, \tau_i]_{i\in\Lambda}, 1_G\big) : G \to \langle F_i \rangle_{i\in\Lambda}$ such that the composite with (P_j, Γ_j) gives the following:



This is equal to (T_j, τ_j), and so the following commutes:

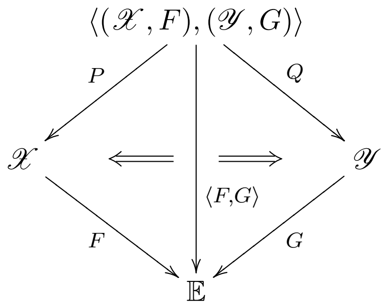


However, the stuff type morphism $([T_i, \tau_i]_{i\in\Lambda}, 1_G)$ is not unique in having this property, and so this may be called a "bicategorical" product in **Stuff**.

3.7 Inner product

An inner product on a real vector space maps two vectors to a real number. Since a stuff type can be represented as a series, it is possible to think of it as a vector, whose coefficients are not real numbers, but groupoids. Therefore the inner product of stuff types will map two stuff types to a groupoid.

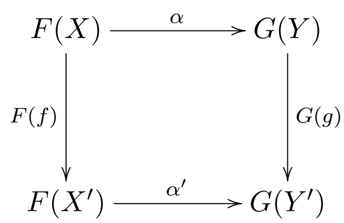
There already exists such a object: the groupoid constructed for the cartesian product. Given two stuff types $F : \mathscr{X} \longrightarrow \mathbb{E}$ and $G : \mathscr{Y} \longrightarrow \mathbb{E}$, we have constructed the groupoid $\langle(\mathscr{X}, F), (\mathscr{Y}, G)\rangle$ so that we have the following:



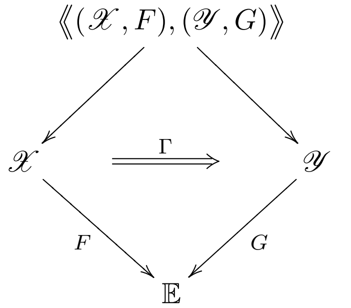
We call this construction the *strict inner product.* However since we are only concerned with the groupoid itself, it is possible to discard some information, such as the projection to $\mathbb{E}$.

Definition 3.7.1. The *inner product* of two stuff types $F : \mathscr{X} \longrightarrow \mathbb{E}$ and $G : \mathscr{Y} \longrightarrow \mathbb{E}$ is the groupoid $\langle\!\langle(\mathscr{X}, F), (\mathscr{Y}, G)\rangle\!\rangle$ with:

- *Objects*: (X, Y, γ), where X and Y are objects in $\mathscr{X}$ and $\mathscr{Y}$ respectively, and $\gamma : F(X) \to G(Y)$ is an isomorphism;
- *Morphisms*: $(f, g) : (X, Y, \gamma) \to (X', Y', \gamma')$, where $f : X \to X'$ and $g : Y \to Y'$ such that the following commutes:



Essentially, we are creating the pseudo pullback without the projection to $\mathbb{E}$:



where Γ is a natural isomorphism. One may easily see that:

$$\langle\!\langle(\mathscr{X}, F), (\mathscr{Y}, G)\rangle\!\rangle \simeq \langle(\mathscr{X}, F), (\mathscr{Y}, G)\rangle$$

This construction has properties analagous to that of an inner product on real vector spaces:

Proposition 3.7.2. *(i) Let* $\mathbf{0}$ *denote the groupoid with no objects, then for any stuff type* $F : \mathscr{X} \longrightarrow \mathbb{E}$*:*

$$\langle\!\langle (\mathscr{X}, F), (\mathscr{X}, F) \rangle\!\rangle \cong \mathbf{0} \quad \text{if and only if} \quad \mathscr{X} \cong \mathbf{0}$$

(ii) For any stuff types $F : \mathscr{X} \longrightarrow \mathbb{E}$*,* $F' : \mathscr{X}' \longrightarrow \mathbb{E}$ *and* $G : \mathscr{Y} \longrightarrow \mathbb{E}$ *we have:*

$$\langle\!\langle (\mathscr{X}+\mathscr{X}', F+F'), (\mathscr{Y}, G) \rangle\!\rangle \cong \langle\!\langle (\mathscr{X}, F), (\mathscr{Y}, G) \rangle\!\rangle + \langle\!\langle (\mathscr{X}', F'), (\mathscr{Y}, G) \rangle\!\rangle$$

(iii) For any stuff types $F : \mathscr{X} \longrightarrow \mathbb{E}$ *and* $G : \mathscr{Y} \longrightarrow \mathbb{E}$ *we have:*

$$\langle\!\langle (\mathscr{X}, F), (\mathscr{Y}, G) \rangle\!\rangle \cong \langle\!\langle (\mathscr{Y}, G), (\mathscr{X}, F) \rangle\!\rangle$$

(iv) For any stuff types $F : \mathscr{X} \longrightarrow \mathbb{E}$ *and* $G : \mathscr{Y} \longrightarrow \mathbb{E}$*, and any groupoid* $\mathscr{U}$ *we have:*

$$\langle\!\langle (\mathscr{U} \times \mathscr{X}, \mathscr{U}F), (\mathscr{Y}, G) \rangle\!\rangle \cong \mathscr{U} \times \langle\!\langle (\mathscr{X}, F), (\mathscr{Y}, G) \rangle\!\rangle$$

Proof. (i) The 'if' part is obvious, and the 'only if' part can be seen by noticing that for any object X in the groupoid $\mathscr{X}$, there is always an object $(X, X, 1_{F(X)})$ in the groupoid $\langle\!\langle (\mathscr{X}, F), (\mathscr{X}, F) \rangle\!\rangle$.

(ii) Any object in $\langle\!\langle (\mathscr{X} + \mathscr{X}', F + F'), (\mathscr{Y}, G) \rangle\!\rangle$ corresponds to a unique object in $\langle\!\langle (\mathscr{X}, F), (\mathscr{Y}, G) \rangle\!\rangle$ or $\langle\!\langle (\mathscr{X}', F'), (\mathscr{Y}, G) \rangle\!\rangle$.

(iii) For an object (X, Y, γ) in $\langle\!\langle (\mathscr{X}, F), (\mathscr{Y}, G) \rangle\!\rangle$, the morphism $\gamma : F(X) \to G(Y)$ is invertible, and so there exists an isomorphism

$$T : \langle\!\langle (\mathscr{X}, F), (\mathscr{Y}, G) \rangle\!\rangle \longrightarrow \langle\!\langle (\mathscr{Y}, G), (\mathscr{X}, F) \rangle\!\rangle$$

which maps:

- *Objects*: $(X, Y, \gamma) \longmapsto (Y, X, \gamma^{-1})$;
- *Morphisms*: $(f, g) \longmapsto (g, f)$.

(iv) For any objects U and X in $\mathscr{U}$ and $\mathscr{X}$ respectively, we have $\mathscr{U}F(U, X) = F(X)$, and so we can construct an isomorphism

$$S : \langle (\mathscr{U} \times \mathscr{X}, \mathscr{U}F), (\mathscr{Y}, G) \rangle \longrightarrow \mathscr{U} \times \langle (\mathscr{X}, F), (\mathscr{Y}, G) \rangle$$

which maps:

- *Objects*: $\big((U, X), Y, \gamma\big) \longmapsto \big(U, (X, , Y, \gamma)\big)$;

- *Morphisms*: $\big((h,f),g\big) \longmapsto \big(h,(f,g)\big)$. □

Remark 3.7.3. All these results are also true for the strict inner product.

Theorem 3.7.4. *For any stuff types $F : \mathscr{X} \longrightarrow \mathbb{E}$ and $G : \mathscr{Y} \longrightarrow \mathbb{E}$ we have:*

$$\big|\langle\!\langle(\mathscr{X},F),(\mathscr{Y},G)\rangle\!\rangle\big| = \sum_{n=0}^{\infty} \big|(\mathscr{X},F)_n\big| \cdot \big|(\mathscr{Y},G)_n\big| \cdot n!$$

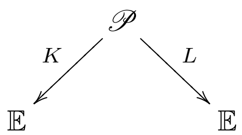
Proof. By Proposition 2.2.2 and Theorem 3.6.2 we have:

$$\begin{aligned}
\big|\langle\!\langle(\mathscr{X},F),(\mathscr{Y},G)\rangle\!\rangle\big| &= \big|\langle(\mathscr{X},F),(\mathscr{Y},G)\rangle\big| \\
&= \sum_{n=0}^{\infty} \Big|\big(\langle(\mathscr{X},F),(\mathscr{Y},G)\rangle, \langle F,G\rangle\big)_n\Big| \\
&= \sum_{n=0}^{\infty} \frac{|\langle F,G\rangle^{-1}\langle n\rangle|}{n!} = \sum_{n=0}^{\infty} \frac{|F^{-1}\langle n\rangle|}{n!} \cdot \frac{|G^{-1}\langle n\rangle|}{n!} \cdot n! \\
&= \sum_{n=0}^{\infty} \big|(\mathscr{X},F)_n\big| \cdot \big|(\mathscr{Y},G)_n\big| \cdot n!
\end{aligned}$$ □

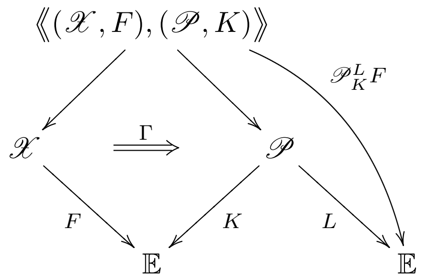
3.8 Stuff operator

Continuing in the spirit of thinking of stuff types as vectors, we seek to create an analogous construct on stuff types. In fact we have already seen two such examples, being the pointing (•) and the derivative (∂) operations.

Definition 3.8.1. An operator on a stuff type $\mathscr{P}_K^L$ is a groupoid $\mathscr{P}$ and two functors $K : \mathscr{P} \longrightarrow \mathbb{E}$ and $L : \mathscr{P} \longrightarrow \mathbb{E}$:



An operator acts on a stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$ by constructing the pseudo pullback:



The resultant stuff type is the composite: $\mathscr{P}_K^L F : \langle\!\langle(\mathscr{X},F),(\mathscr{P},K)\rangle\!\rangle \longrightarrow \mathbb{E}$ which maps:

- *Objects*: $(X, P, \gamma) \longmapsto L(P)$;
- *Morphisms*: $(f, h) \longmapsto L(h)$.

Linear operators on vector spaces preserve vector addition and scalar multiplication. We can show stuff operators have analagous properties:

Proposition 3.8.2. *(i) For any stuff operator $\mathscr{P}^L_K$ and stuff types $F : \mathscr{X} \longrightarrow \mathbb{E}$ and $F' : \mathscr{X}' \longrightarrow \mathbb{E}$ we have:*

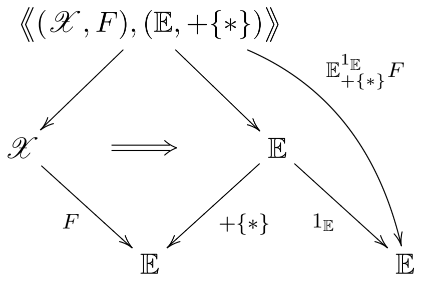
$$\mathscr{P}^L_K(F + F') \cong \mathscr{P}^L_K F + \mathscr{P}^L_K F'$$

(ii) For any stuff operator $\mathscr{P}^L_K$, stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$ and groupoid $\mathscr{U}$ we have:

$$\mathscr{P}^L_K(\mathscr{U} F) \cong \mathscr{U}(\mathscr{P}^L_K F)$$

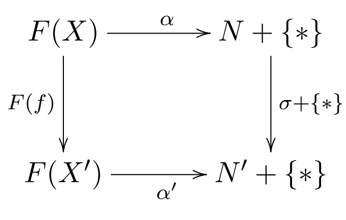
Proof. Both of these results follow from Proposition 3.7.2, which allow the creation of the relevant isomorphisms in **Stuff**. □

Example 3.8.3. Consider the operator $\mathbb{E}^{1_\mathbb{E}}_{+\{*\}}$, where $+\{*\} : \mathbb{E} \longrightarrow \mathbb{E}$ is the functor which adds the element $*$ to the set, with the obvious mapping on morphisms.



The groupoid $\langle\!\langle(\mathscr{X}, F), (\mathbb{E}, +\{*\})\rangle\!\rangle$

- *Objects*: (X, N, α) such that $\alpha : F(X) \to N + \{*\}$ which maps to N under $\mathbb{E}^{1_\mathbb{E}}_{+\{*\}}F$;
- *Morphisms*: $(f, \sigma) : (X, N, \alpha) \to (X', N', \alpha')$ such that the following commutes:



which maps to σ under $\mathbb{E}^{1_\mathbb{E}}_{+\{*\}}F$.

This acts in essentially the same manner to the derivative operator (∂), by dropping a distinguished element. In fact for any stuff type $F : \mathscr{X} \longrightarrow \mathbb{E}$ it is possible to show that $\partial F \simeq \mathbb{E}^{1_\mathbb{E}}_{+\{*\}}F$.

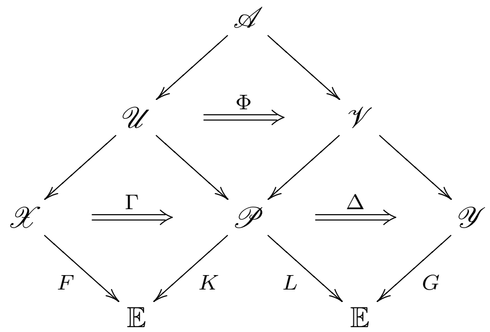
The obvious question is what happens if we switch the functors of the operator. We can show that this is becomes the adjoint on the inner product.

Theorem 3.8.4. *Given two stuff types* $F : \mathscr{X} \longrightarrow \mathbb{E}$ *and* $G : \mathscr{Y} \longrightarrow \mathbb{E}$ *and a stuff type operator* $\mathscr{P}_K^L$*, we have:*

$$\langle\!\langle (\mathscr{U}, \mathscr{P}_K^L F), (\mathscr{Y}, G) \rangle\!\rangle \cong \langle\!\langle (\mathscr{X}, F), (\mathscr{V}, \mathscr{P}_L^K G) \rangle\!\rangle$$

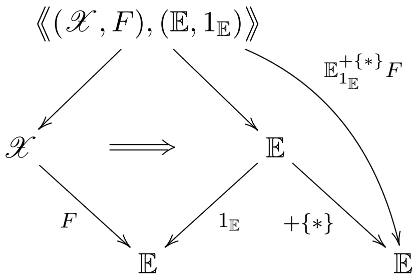
where $\mathscr{U} = \langle\!\langle (\mathscr{X}, F), (\mathscr{P}, K) \rangle\!\rangle$ *and* $\mathscr{V} = \langle\!\langle (\mathscr{Y}, G), (\mathscr{P}, L) \rangle\!\rangle$.

Proof. This proof can be obtained rather easily by constructing the following:



where Γ, Δ, Φ are all natural isomorphisms. □

Example 3.8.5. Considering the adjoint of the previous example, we obtain:



So the operator $\mathbb{E}_{1_{\mathbb{E}}}^{+\{*\}}$ adds a new element to the set. Let $I : \mathbb{E}_1 \longrightarrow \mathbb{E}$ be the property type of "being a one-element set". Then it is possible to show that:

$$\mathbb{E}_{1_{\mathbb{E}}}^{+\{*\}} F \simeq I \times F$$

where $I \times F$ is the multiplication of stuff types.

The operators from Examples 3.8.3 and 3.8.5 are used in [1] to correspond to the annihilation (A) and creation (A^*) operators in the quantum harmonic oscillator, as they have the nice property that:

$$AA^*F = A^*AF + F$$

for any stuff type F.

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