

BICATEGORIES

A bicategory W consists of the following data:

- 1) a set W_0 [elements are called objects or 0-cells];
- 2) a category W [objects are called arrows or 1-cells, arrows are called 2-cells, and composition is denoted by juxtaposition];
- 3) functors $s, t: W \rightarrow W_0$ [for $u, v \in W_0$, the subcategory of W consisting of those $\sigma: f \rightarrow g$ with $s(\sigma) = u, t(\sigma) = v$ is denoted by $W(u, v)$, and, such $\sigma: f \rightarrow g$ are depicted by

$$u \begin{array}{c} \xrightarrow{f} \\ \downarrow \sigma \\ \xrightarrow{g} \end{array} v];$$

- 4) functors $c: W(u, v) \times W(v, w) \rightarrow W(u, w)$ [the value at $(\sigma: f \rightarrow g, \tau: h \rightarrow k)$ is denoted by

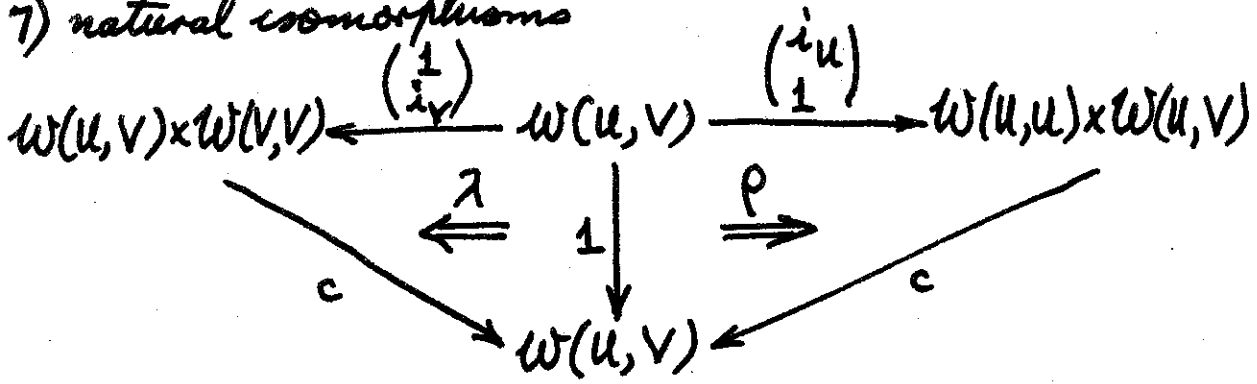
$$u \begin{array}{c} \xrightarrow{h \circ f} \\ \downarrow \tau \circ \sigma \\ \xrightarrow{k \circ g} \end{array} w];$$

- 5) 1-cells $i_u: u \rightarrow u$;
- 6) natural isomorphisms

$$\begin{array}{ccc} W(u, v) \times W(v, w) \times W(w, x) & \xrightarrow{c \times 1} & W(u, w) \times W(w, x) \\ \downarrow 1 \times c & \Downarrow \alpha & \downarrow c \\ W(u, v) \times W(v, x) & \xrightarrow{c} & W(u, x) \end{array}$$

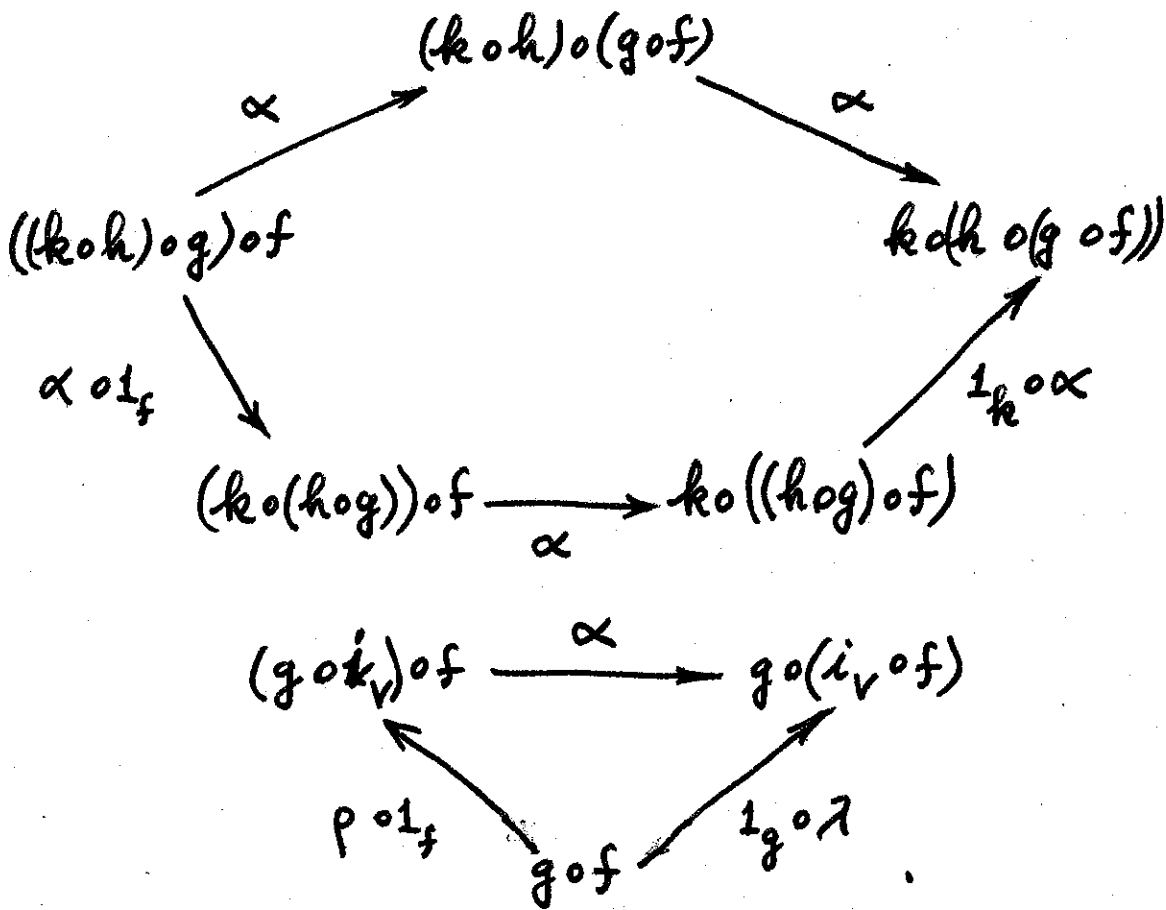
[whose component at (f, g, h) is $\alpha_{(f, g, h)}: (h \circ g) \circ f \Rightarrow h \circ (g \circ f)$];

7) natural isomorphisms



[whose components at f are $\lambda_f: f \Rightarrow i_V \circ f$, $\rho_f: f \Rightarrow f \circ i_U$];

such that the following axioms hold:



MORPHISM OF BICATEGORIES

Suppose W, W' are bicategories. A morphism $T: W \rightarrow W'$ consists of the following data:

- 1) a function $T: W_0 \rightarrow W'_0$ [whose value at U is denoted by TU];
- 2) a functor $T: W \rightarrow W'$ with $sT = Ts, tT = Tt$ [whose value at $U \xrightarrow[f]{\sigma} V$ is denoted by $TU \xrightarrow[Tg]{Tf} TV$];
- 3) 2-cells $\tau_U: i_{TU} \Rightarrow Ti_U$;
- 4) natural transformations

$$\begin{array}{ccc}
 W(U, V) \times W(V, W) & \xrightarrow{c} & W(U, W) \\
 T \times T \downarrow & \xRightarrow{\tau} & \downarrow T \\
 W'(TU, TV) \times W'(TV, TW) & \xrightarrow{c} & W'(TU, TW)
 \end{array}$$

[whose component at (f, g) is $\tau_{(f, g)}: Tg \circ Tf \Rightarrow T(g \circ f)$]

such that the following axioms hold:

$$\begin{array}{ccc}
 (Th \circ Tg) \circ Tf & \xrightarrow{\alpha} & Th \circ (Tg \circ Tf) \\
 \tau_{01} \downarrow & & \downarrow 1 \circ \tau \\
 T(h \circ g) \circ Tf & & Th \circ T(g \circ f) \\
 \tau \downarrow & & \downarrow \tau \\
 T(T(h \circ g) \circ Tf) & \xrightarrow{T\alpha} & T(h \circ (g \circ f))
 \end{array}$$

$$\begin{array}{ccc}
 Tf & \xrightarrow{\rho_{Tf}} & Tf \circ i_{TU} \\
 T\rho_f \downarrow & & \downarrow 1 \circ \tau_U \\
 T(f \circ i_U) & \xleftarrow{\tau} & Tf \circ Ti_U \\
 Tf & \xrightarrow{\lambda_{Tf}} & i_{TV} \circ Tf \\
 T\lambda_f \downarrow & & \downarrow \tau_V \circ 1 \\
 T(i_V \circ f) & \xleftarrow{\tau} & Ti_V \circ Tf
 \end{array}$$

OPTRANSFORMATIONS

Suppose $S, T: W \rightarrow W'$ are morphisms of bicategories. An optransformation $\theta: S \rightarrow T$ consists of the following data:

1) a function $\theta: W_0 \rightarrow W'_0$ with $\lambda \theta = S, \tau \theta = T$ [so that $\theta_u: Su \rightarrow Tu$ is an arrow of W'];

2) natural transformations

$$\begin{array}{ccc} W(u, v) & \xrightarrow{T} & W'(Tu, Tv) \\ S \downarrow & \theta \Rightarrow & \downarrow - \circ \theta_u \\ W'(Su, Sv) & \xrightarrow{\theta_v \circ -} & W'(Su, Tv) \end{array}$$

[with component at f given by $Su \begin{array}{c} \xrightarrow{\theta_v \circ Sf} \\ \Downarrow \theta_f \\ \xrightarrow{Tf \circ \theta_u} \end{array} Tv$];

such that the following axioms hold:

$$\begin{array}{ccc} (\theta_w \circ Sg) \circ Sf \xrightarrow{\theta_g \circ 1} (Tg \circ \theta_v) \circ Sf \xrightarrow{\alpha} Tg \circ (\theta_v \circ Sf) \\ \alpha \downarrow & & \downarrow 1 \circ \theta_f \\ \theta_w \circ (Sg \circ Sf) & & Tg \circ (Tf \circ \theta_u) \\ \theta_w \circ \sigma \downarrow & & \downarrow \alpha^{-1} \\ \theta_w \circ S(g \circ f) \xrightarrow{\theta_{g \circ f}} T(g \circ f) \circ \theta_u \xleftarrow{\tau \circ 1} (Tg \circ Tf) \circ \theta_u \end{array}$$

$$\begin{array}{ccc} & \theta_u & \\ \rho \swarrow & & \searrow \lambda \\ \theta_u \circ i_{Su} & & i_{Tu} \circ \theta_u \\ \downarrow 1 \circ \sigma_u & & \downarrow \tau_u \circ 1 \\ \theta_u \circ S i_u & \xrightarrow{\theta_i} & T i_u \circ \theta_u \end{array}$$

Note unavoidable use of inverse of α .
See SLN 47 p63 (8.4.1)

MODIFICATIONS

Suppose $\theta, \phi : S \rightarrow T$ are optransformations.
 A modification $\mu : \theta \rightarrow \phi$ is a family of 2-cells

$$S U \begin{array}{c} \xrightarrow{\theta_U} \\ \Downarrow \mu_U \\ \xrightarrow{\phi_U} \end{array} T U \quad \text{in } W'$$

such that

$$\begin{array}{ccc} \theta_V \circ S f & \xrightarrow{\theta_f} & T f \circ \theta_U \\ \mu_V \circ 1 \downarrow & & \downarrow 1 \circ \mu_U \\ \phi_V \circ S f & \xrightarrow{\phi_f} & T f \circ \phi_U \end{array}$$

A morphism $T : W \rightarrow W'$ is normal when each $\tau_U : i_{TU} \Rightarrow T i_U$ is invertible.

A homomorphism $T : W \rightarrow W'$ is a normal morphism for which each $\tau_{(f,g)} : T g \circ T f \Rightarrow T(g \circ f)$ is invertible.

A strong transformation $\theta : S \rightarrow T$ is an optransformations for which each $\theta_f : \theta_V \circ S f \Rightarrow T f \circ \theta_U$ is invertible.

Exercises. 1. Describe a bicategory $\text{Spn } A$ constructed from a category A with pullbacks: 0-cells are the objects of A , 1-cells are spans $a \xleftarrow{\alpha} r \xrightarrow{\beta} b$ in A , and horizontal \circ composition is given by pullback.

2. A morphism which is surjective on objects up to equivalence and locally ~~is~~ is called a biequivalence. Show that biequivalences are invertible.

2. Let $l: L \rightarrow N$ be a regular function. Describe a bicategory Mod_l whose objects are categories in Set_l , whose 1-cells are modules, and whose horizontal composition is tensor product.

3. To each functor $p: E \rightarrow A$ with l -small fibres $E_a = \{ \xi: e \rightarrow e' \mid p\xi: pe \rightarrow pe' \text{ is } 1_a: a \rightarrow a \}$ construct a normal morphism $A^{op} \rightarrow \text{Mod}_l$ which takes $a \in A$ to E_a (as a subcategory of E). Show that $p: E \rightarrow A$ is a fibration (as in Bénabou's recent lectures) if and only if this normal morphism ^{is a homomorphism} & "factors through" the homomorphism $\text{Cat}_l \rightarrow \text{Mod}_l$ which takes a functor $f: X \rightarrow Y$ to the module $Yf: X \rightarrow Y$. Generalize the "Grothendieck construction" to obtain a functor $p: E \rightarrow A$ from a normal morphism $A^{op} \rightarrow \text{Mod}_l$.

4. Describe composition of morphisms $W \xrightarrow{T} W' \xrightarrow{T'} W''$.

5. For bicategories W, W' , prove that the following data define a bicategory $\text{Bicat}_{op}(W, W')$:

- 0-cells are morphisms $W \rightarrow W'$;
- 1-cells are optransformations and 2-cells are modifications with componentwise composition;
- $s(\theta: S \rightarrow T) = S, t(\theta: S \rightarrow T) = T$;

- for $R \xrightarrow{\theta} S \xrightarrow{\varphi} T$, $(\varphi \circ \theta)_u = \varphi_u \circ \theta_u$,
 $(\nu \circ \mu)_u = \nu_u \circ \mu_u$
 $(\varphi_\nu \circ \theta_\nu) \circ S \xrightarrow{(\varphi \circ \theta)_\nu} (T \circ \varphi_u) \circ \theta_u$ N.B. strong if θ, φ are
 $\alpha \downarrow \quad \uparrow \varphi_\nu \circ 1_u$
 $\varphi_\nu \circ (\theta \circ S) \xrightarrow{1 \circ \theta_\nu} \varphi_\nu \circ (T \circ \theta_u) \xrightarrow{\alpha'} (\varphi_\nu \circ T) \circ \theta_u$

— $i_S: S \rightarrow S$ is the strong (op)transformation with

$$(i_S)_u = i_{Su}, \quad (i_S)_f = (i_{Sv} \circ Sf \xrightarrow{\lambda^{-1}} Sf \xrightarrow{\rho} Sf \circ i_{Su});$$

$$\alpha_{(\theta, \varphi, \psi)} u = \alpha_{(\theta_u, \varphi_u, \psi_u)};$$

$$(\lambda_\theta)_u = \lambda_{\theta_u}, \quad (\rho_\theta)_u = \rho_{\theta_u}.$$

6. Describe the canonical isomorphism:

$$\text{Bicat}_{op}(W, \text{Bicat}(W', W'')) \cong \text{Bicat}(W', \text{Bicat}_{op}(W, W'')).$$

ENRICHED CATEGORIES

Suppose W is a bicategory. A category X enriched in W , or W -category, consists of the following data:

- 1) a set X_0 whose elements are called objects;
- 2) a function $e: X_0 \rightarrow W_0$ [say $x \in X_0$ is over U when $e x = U$];
- 3) for each $x, y \in X_0$, an arrow (1-cell)

$$X(y, x): e(x) \rightarrow e(y) \text{ of } W;$$

- 4) for x, y, z over U, V, W , 2-cells

$$\begin{array}{ccc} & & V \\ & X(y, x) \nearrow & \searrow X(z, y) \\ U & & W \\ & \downarrow \mu & \\ & X(z, x) & \end{array}$$

of W ;

satisfying the following axioms:

$$\begin{array}{ccc}
 (X(z, y) \circ X(y, x)) \circ X(x, w) & \xrightarrow{\mu \circ 1} & X(z, x) \circ X(x, w) \\
 \alpha \downarrow & & \downarrow \mu \\
 X(z, y) \circ (X(y, x) \circ X(x, w)) & & \\
 \downarrow 1 \circ \mu & & \\
 X(z, y) \circ X(y, w) & \xrightarrow{\mu} & X(z, w)
 \end{array}$$

$$\begin{array}{ccccc}
 X(y, x) \circ X(x, x) & \xrightarrow{\mu} & X(y, x) & \xleftarrow{\mu} & X(y, y) \circ X(y, x) \\
 \uparrow 1 \circ \eta & & \uparrow 1 & & \uparrow \eta \circ 1 \\
 X(y, x) \circ i_u & \xleftarrow{\rho} & X(y, x) & \xrightarrow{\lambda} & i_v \circ X(y, x)
 \end{array}$$

Exercises 7. Show that a \mathcal{W} -category X amounts precisely to a set X_0 together with a morphism of bicategories $X_{och} \rightarrow \mathcal{W}$ (where X_{och} is the category whose set of arrows is $X_0 \times X_0$ and whose composition is $(z, y)(y, x) = (z, x)$).

8. A \mathcal{W} -category with one object amounts to a monad in \mathcal{W} . If $\mathcal{W}_0 = \mathbb{1}$ and $\mathcal{W}(0, 0) = \text{AbGrp}$, then such is an \mathbb{k} -small ring.

Suppose A, X are \mathcal{W} -categories. A \mathcal{W} -functor $f: A \rightarrow X$ consists of the following data:

1) a function $f: A_0 \rightarrow X_0$ which takes objects over U to objects over U ;

2) 2-cells

$$\begin{array}{ccc}
 & A(b, a) & \\
 e(a) & \begin{array}{c} \curvearrowright \\ \downarrow f \\ \curvearrowleft \end{array} & e(b) \\
 & \forall (r, l, f, a) &
 \end{array}$$

such that

$$\begin{array}{ccc}
 i_{e(a)} \xrightarrow{\eta} A(a, a) & & A(c, b) \circ A(b, a) \xrightarrow{\mu} A(c, a) \\
 \searrow \eta & \downarrow f & \downarrow f \\
 & X(fa, fa) & X(fc, fb) \circ X(fb, fa) \xrightarrow{\mu} X(fc, fa)
 \end{array}$$

ENRICHED MODULES

A W -module $m: B \rightarrow A$ between W -categories A, B consists of:

- 1) 1-cells $e(b) \xrightarrow{m(a,b)} e(a)$ of W for $a \in A_0, b \in B_0$;
- 2) 2-cells

$$A(a', a) \circ m(a, b) \xrightarrow{\mu_l} m(a', b)$$

$$m(a, b) \circ B(b, b') \xrightarrow{\mu_r} m(a, b')$$

satisfying the axioms

$$\begin{array}{ccc}
 m(a, b) & \xrightarrow{1} & m(a, b) \\
 \lambda \downarrow & & \uparrow \mu_l \\
 i_{e(a)} \circ m(a, b) & \xrightarrow{\eta \circ 1} & A(a, a) \circ m(a, b) \\
 \rho \downarrow & & \uparrow \mu_r \\
 m(a, b) \circ i_{e(b)} & \xrightarrow{1 \circ \eta} & m(a, b) \circ B(b, b)
 \end{array}$$

$$\begin{array}{ccc}
 (A(a'', a') \circ A(a', a)) \circ m(a, b) & \xrightarrow{\mu \circ 1} & A(a'', a) \circ m(a, b) \\
 \alpha \downarrow & & \downarrow \mu_L \\
 A(a'', a') \circ (A(a', a) \circ m(a, b)) & & \\
 1 \circ \mu_L \downarrow & & \downarrow \mu_L \\
 A(a'', a') \circ m(a', b) & \xrightarrow{\mu_L} & m(a'', b)
 \end{array}$$

$$\begin{array}{ccc}
 (m(a, b) \circ B(b, b')) \circ B(b', b'') & \xrightarrow{\mu_r \circ 1} & m(a, b) \circ B(b', b'') \\
 \alpha \downarrow & & \downarrow \mu_r \\
 m(a, b) \circ (B(b, b') \circ B(b', b'')) & & \\
 1 \circ \mu_r \downarrow & & \downarrow \mu_r \\
 m(a, b) \circ B(b, b'') & \xrightarrow{\mu_r} & m(a, b'')
 \end{array}$$

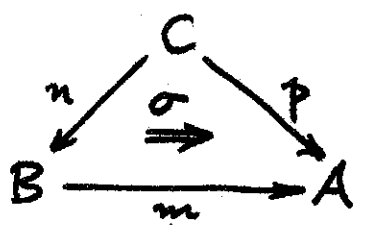
$$\begin{array}{ccc}
 (A(a', a) \circ m(a, b)) \circ B(b, b') & \xrightarrow{\mu_L \circ 1} & m(a', b) \circ B(b, b') \\
 \alpha \downarrow & & \downarrow \mu_r \\
 A(a', a) \circ (m(a, b) \circ B(b, b')) & & \\
 1 \circ \mu_r \downarrow & & \downarrow \mu_r \\
 A(a', a) \circ m(a, b') & \xrightarrow{\mu_L} & m(a', b')
 \end{array}$$

A W-module morphism $\sigma: m \rightarrow m'$ is a family of 2-cells

$$m(a, b) \xrightarrow{\sigma_{a,b}} m'(a, b)$$

such that

$$\begin{array}{ccc}
 A(a', a) \circ m(a, b) & \xrightarrow{\mu_L} & m(a', b) & m(a, b) \circ B(b, b') & \xrightarrow{\mu_r} & m(a, b') \\
 1 \circ \sigma_{a,b} \downarrow & & \downarrow \sigma_{a',b} & \sigma_{a,b} \circ 1 \downarrow & & \downarrow \sigma_{a,b'} \\
 A(a', a) \circ m'(a, b) & \xrightarrow{\mu_L} & m'(a', b) & m'(a, b) \circ B(b, b') & \xrightarrow{\mu_r} & m'(a, b')
 \end{array}$$



For modules m, n, p as above, a form $\sigma: (m, n) \Rightarrow p$ is a family of 2-cells

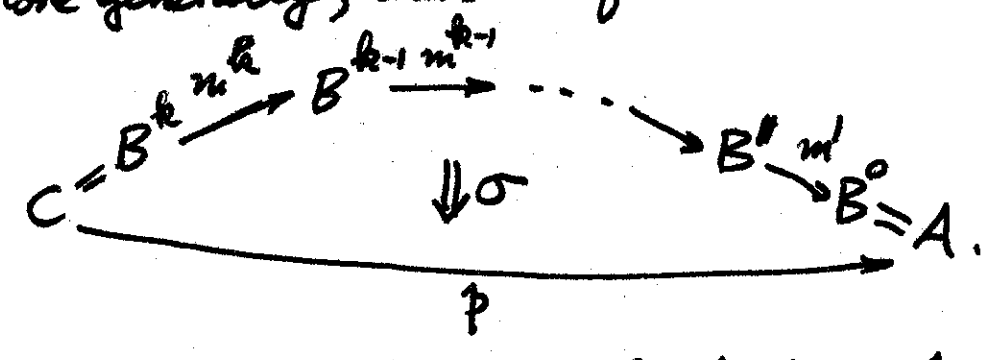
$$m(a, b) \circ n(b, c) \xrightarrow{\sigma_{ac}^b} p(a, c)$$

such that all the 2-cells

$$(((A(a', a) \circ m(a, b)) \circ B(b, b')) \circ n(b', c)) \circ C(c, c') \rightarrow p(a', c')$$

obtainable using $\alpha, \sigma, \mu_l, \mu_r$ are equal.

More generally, there are forms

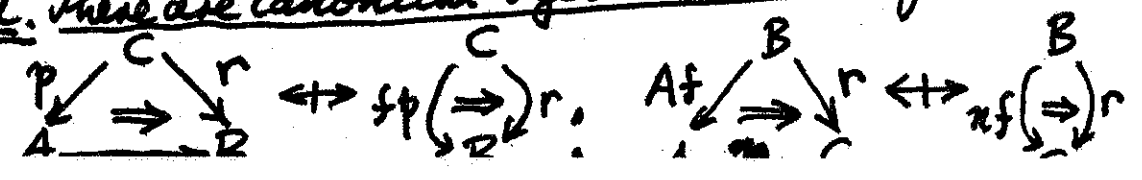


There is always the module $A: A \rightarrow A$ given by the 1-cells $A(a', a)$ and $\mu_l = \mu_r = \mu$.

Each functor $f: B \rightarrow A$ and module $p: C \rightarrow A$ determine a module $fp: C \rightarrow B$ given by $(fp)(b, c) = p(fb, c)$. A functor $f: B \rightarrow A$ and module $n: A \rightarrow B$ determine a module $nf: B \rightarrow C$ given by $(nf)(c, b) = n(c, fb)$.

In particular, $fA: A \rightarrow B, Af: B \rightarrow A$ are modules.

Lemma. There are canonical bijections between forms



TENSOR PRODUCT OF MODULES

$$\begin{array}{ccc}
 C & \xrightarrow{n} & B \xrightarrow{m} A \\
 & \searrow & \nearrow \\
 & & m \otimes n
 \end{array}$$

A tensor product for m, n is a module $m \otimes n$ together with a form $\pi: (m, n) \Rightarrow m \otimes n$ such that each form $\sigma: (m, n) \Rightarrow p$ factors as $\sigma = \tau \pi$ for a unique $\tau: m \otimes n \Rightarrow p$.

Certain tensor products always exist:

$$fA \otimes p \cong fp, \quad n \otimes Af \cong nf \quad (\text{by lemma}).$$

When either side of the isomorphism

$$m \otimes (n \otimes p) \cong (m \otimes n) \otimes p$$

exists it represents forms out of (m, n, p) ; hence the isomorphism when both sides exist.

HOM OF MODULES

$$\begin{array}{ccc}
 & C & \\
 \text{hom}(m, p) \swarrow & & \searrow p \\
 B & \xrightarrow[\quad m \quad]{} & A \\
 & \xrightarrow{\varepsilon} &
 \end{array}$$

A hom for m, p is a module $\text{hom}(m, p)$ together with a form $\varepsilon: (m, \text{hom}(m, p)) \Rightarrow p$ (evaluation) such that each form $\sigma: (m, n) \Rightarrow p$ factors as $\sigma = \varepsilon(m, \tau)$ for a unique $\tau: n \Rightarrow \text{hom}(m, p)$.

Corollary:
$$\frac{m \otimes n \Rightarrow p}{(m, n) \Rightarrow p} = \frac{m \otimes n \Rightarrow p}{n \Rightarrow \text{hom}(m, p)} \quad \square$$

Write m^* for $\text{hom}(m, A)$ when it exists.

Call $m: B \rightarrow A$ cauchy when $\text{hom}(m, p)$ exists for all modules $p: C \rightarrow A$ and the canonical form $(m^*, p) \Rightarrow \text{hom}(m, p)$ is a tensor product for m^*, p .

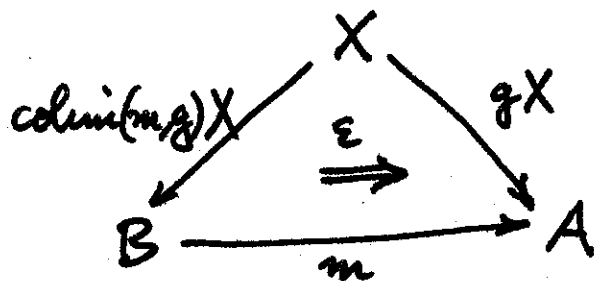
Suppose $f: B \rightarrow A$ is an (enriched) functor. Then $\text{hom}(Af, p) \cong fp$ exists and $(Af)^* \cong fA$; also $fA \otimes p \cong fp$. So $Af: B \rightarrow A$ is cauchy.

Call $m: B \rightarrow A$ convergent when $m \cong Af$ for some functor $f: B \rightarrow A$. So convergent implies cauchy.

Call A cauchy complete when every cauchy module $m: B \rightarrow A$ is convergent. This holds if and only if each cauchy module $m: U \rightarrow A$ with U in \mathcal{W} is convergent.

Colimits. Suppose $m: B \rightarrow A$ is a module and $f: A \rightarrow X$ is a functor. A colimit for f weighted by m is a functor $\text{colim}(m, f): B \rightarrow X$ together with an isomorphism

$$\text{colim}(m, f)X \cong \text{hom}(m, fX).$$



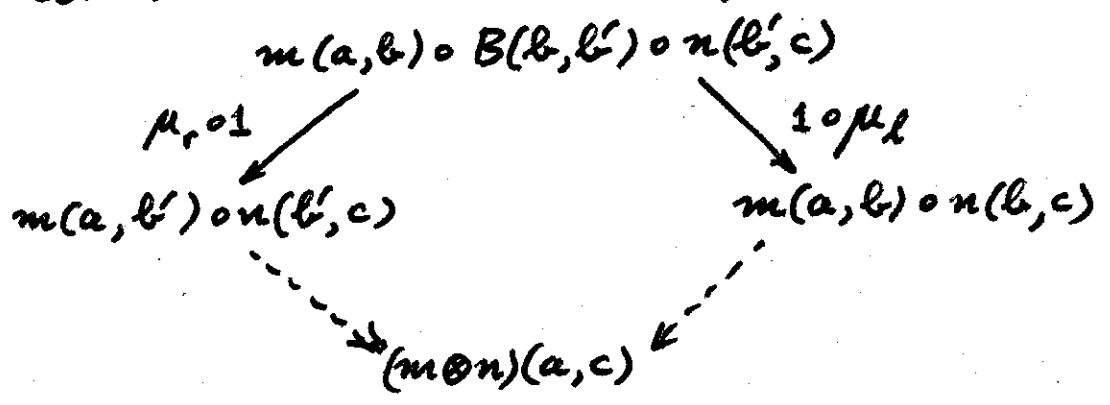
Exercises 9. X is cauchy complete if and only if it admits all colimits weighted by cauchy modules.

10. If $m \otimes n$ exists and m, n are cauchy then $m \otimes n$ is cauchy.

Construction of tensor products

$$C \xrightarrow{n} B \xrightarrow{m} A$$

For a, c in A, C over U, V , take the colimit over all b, b' in B of the diagram



when it exists in $\mathcal{W}(W, U)$ and is preserved by composition \circ in \mathcal{W} .

Construction of hom

This needs big enough limits in each $\mathcal{W}(U, V)$ together with adjoints (right) for the functors $\mathcal{W}(U, w)$ for each 1-cell w of \mathcal{W} .