Notes by Ross Street 12

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BICATEGORIES

A *bicategory* $\mathcal{W}$ consists of the following data:

1) a set $\mathcal{W}_0$ [elements are called *objects* or 0-*cells*];

2) a category $\mathcal{W}$ [objects are called *arrows* or 1-*cells*, arrows are called 2-*cells*, and composition is denoted by juxtaposition];

3) functors $s, t : \mathcal{W} \longrightarrow \mathcal{W}_0$ [for $U, V \in \mathcal{W}_0$, the subcategory of $\mathcal{W}$ consisting of those $\sigma : f \to g$ with $s(\sigma) = U$, $t(\sigma) = V$ is denoted by $\mathcal{W}(U,V)$, and, such $\sigma : f \to g$ are depicted by

$$\begin{tikzcd} U \arrow[r, bend left, "f"{name=A}] \arrow[r, bend right, "g"'{name=B}] \arrow[from=A, to=B, phantom, "\Downarrow\sigma"] & V \end{tikzcd} \Big];$$

4) functors $c : \mathcal{W}(U,V) \times \mathcal{W}(V,W) \longrightarrow \mathcal{W}(U,W)$ [the value at $(\sigma : f \to g,\ \tau : h \to k)$ is denoted by

$$\begin{tikzcd} U \arrow[r, bend left, "h \circ f"{name=A}] \arrow[r, bend right, "k \circ g"'{name=B}] \arrow[from=A, to=B, phantom, "\Downarrow\tau\circ\sigma"] & W \end{tikzcd} \Big];$$

5) 1-cells $i_U : U \longrightarrow U$;

6) natural isomorphisms

$$\begin{tikzcd} \mathcal{W}(U,V)\times\mathcal{W}(V,W)\times\mathcal{W}(W,X) \arrow[r, "c\times 1"] \arrow[d, "1\times c"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & \mathcal{W}(U,W)\times\mathcal{W}(W,X) \arrow[d, "c"] \\ \mathcal{W}(U,V)\times\mathcal{W}(V,X) \arrow[r, "c"'] & \mathcal{W}(U,X) \end{tikzcd}$$

[whose component at (f,g,h) is $\alpha_{(f,g,h)} : (h\circ g)\circ f \Rightarrow h\circ(g\circ f)$];

7) natural isomorphisms

$$
\begin{tikzcd}
\mathcal{W}(U,V)\times\mathcal{W}(V,V) \arrow[dr, "c"'] & \mathcal{W}(U,V) \arrow[l, "\binom{1}{i_V}"'] \arrow[r, "\binom{i_U}{1}"] \arrow[d, "1"] \arrow[l, phantom, shift right=4, "\overset{\lambda}{\Leftarrow}"] \arrow[r, phantom, shift right=4, "\overset{\rho}{\Rightarrow}"] & \mathcal{W}(U,U)\times\mathcal{W}(U,V) \arrow[dl, "c"] \\
 & \mathcal{W}(U,V) &
\end{tikzcd}
$$

[whose components at f are $\lambda_f : f \Rightarrow i_V \circ f$, $\rho_f : f \Rightarrow f \circ i_U$];

such that the following axioms hold:

$$
\begin{tikzcd}
 & (k\circ h)\circ(g\circ f) \arrow[dr, "\alpha"] & \\
((k\circ h)\circ g)\circ f \arrow[ur, "\alpha"] \arrow[d, "\alpha\circ 1_f"'] & & k\circ(h\circ(g\circ f)) \\
(k\circ(h\circ g))\circ f \arrow[r, "\alpha"'] & k\circ((h\circ g)\circ f) \arrow[ur, "1_k\circ\alpha"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(g\circ i_V)\circ f \arrow[rr, "\alpha"] & & g\circ(i_V\circ f) \\
 & g\circ f \arrow[ul, "\rho\circ 1_f"] \arrow[ur, "1_g\circ\lambda"'] &
\end{tikzcd}
$$

.

MORPHISM OF BICATEGORIES

Suppose $\mathcal{W}$, $\mathcal{W}'$ are bicategories. A *morphism* $T : \mathcal{W} \to \mathcal{W}'$ consists of the following data:

1) a function $T : \mathcal{W}_0 \to \mathcal{W}'_0$ [whose value at U is denoted by TU];

2) a functor $T : \mathcal{W} \to \mathcal{W}'$ with $sT = Ts$, $tT = Tt$ [whose value at
$$\begin{tikzcd} U \arrow[r, bend left=40, "f"{name=A}] \arrow[r, bend right=40, "g"'{name=B}] \arrow[from=A, to=B, phantom, "\Downarrow\sigma"] & V \end{tikzcd}$$
is denoted by
$$\begin{tikzcd} TU \arrow[r, bend left=40, "Tf"{name=A}] \arrow[r, bend right=40, "Tg"'{name=B}] \arrow[from=A, to=B, phantom, "\Downarrow T\sigma"] & TV \end{tikzcd}$$
];

3) 2-cells $\tau_U : i_{TU} \Rightarrow Ti_U$;

4) natural transformations

$$\begin{tikzcd} \mathcal{W}(U,V) \times \mathcal{W}(V,W) \arrow[r, "c"] \arrow[d, "T \times T"'] \arrow[dr, phantom, "\overset{\tau}{\Longrightarrow}"] & \mathcal{W}(U,W) \arrow[d, "T"] \\ \mathcal{W}'(TU,TV) \times \mathcal{W}'(TV,TW) \arrow[r, "c"'] & \mathcal{W}'(TU,TW) \end{tikzcd}$$

[whose component at (f,g) is $\tau_{(f,g)} : Tg \circ Tf \Rightarrow T(g \circ f)$];

such that the following axioms hold:

$$\begin{tikzcd} (Th \circ Tg) \circ Tf \arrow[r, "\alpha"] \arrow[d, "\tau \circ 1"'] & Th \circ (Tg \circ Tf) \arrow[d, "1 \circ \tau"] \\ T(h \circ g) \circ Tf \arrow[d, "\tau"'] & Th \circ T(g \circ f) \arrow[d, "\tau"] \\ T((h \circ g) \circ f) \arrow[r, "T\alpha"'] & T(h \circ (g \circ f)) \end{tikzcd}$$

$$\begin{tikzcd} Tf \arrow[r, "\rho_{Tf}"] \arrow[d, "T\rho_f"'] & Tf \circ i_{TU} \arrow[d, "1 \circ \tau_U"] \\ T(f \circ i_U) & Tf \circ Ti_U \arrow[l, "\tau"] \end{tikzcd}$$

$$\begin{tikzcd} Tf \arrow[r, "\lambda_{Tf}"] \arrow[d, "T\lambda_f"'] & i_{TV} \circ Tf \arrow[d, "\tau_V \circ 1"] \\ T(i_V \circ f) & Ti_V \circ Tf \arrow[l, "\tau"] \end{tikzcd}$$
.

OPTRANSFORMATIONS

Suppose $S, T : \mathcal{W} \to \mathcal{W}'$ are morphisms of bicategories. An <u>optransformation</u> $\theta : S \to T$ consists of the following data:

1) a function $\theta : \mathcal{W}_0 \to \mathcal{W}'$ with $s\,\theta = S$, $t\,\theta = T$ [so that $\theta_U : SU \to TU$ is an arrow of $\mathcal{W}'$];

2) natural transformations

$$\begin{tikzcd}
\mathcal{W}(U,V) \arrow[r, "T"] \arrow[d, "S"'] \arrow[dr, phantom, "\overset{\theta}{\Longrightarrow}"] & \mathcal{W}'(TU,TV) \arrow[d, "-\circ\theta_U"] \\
\mathcal{W}'(SU,SV) \arrow[r, "\theta_V \circ -"'] & \mathcal{W}'(SU,TV)
\end{tikzcd}$$

[with component at f given by
$\begin{tikzcd} SU \arrow[r, bend left, "\theta_V \circ Sf"] \arrow[r, bend right, "Tf \circ \theta_U"'] \arrow[r, phantom, "\Downarrow \theta_f"] & TV \end{tikzcd}$];

such that the following axioms hold:

$$\begin{tikzcd}
(\theta_W \circ Sg) \circ Sf \arrow[r, "\theta_g \circ 1"] \arrow[d, "\alpha"'] & (Tg \circ \theta_V) \circ Sf \arrow[r, "\alpha"] & Tg \circ (\theta_V \circ Sf) \arrow[d, "1 \circ \theta_f"] \\
\theta_W \circ (Sg \circ Sf) \arrow[d, "\theta_W \circ \sigma"'] & & Tg \circ (Tf \circ \theta_U) \arrow[d, "\alpha^{-1}"] \\
\theta_W \circ S(g \circ f) \arrow[r, "\theta_{g \circ f}"'] & T(g \circ f) \circ \theta_U & (Tg \circ Tf) \circ \theta_U \arrow[l, "\tau \circ 1"]
\end{tikzcd}$$

$$\begin{tikzcd}
& \theta_U \arrow[dl, "\rho"'] \arrow[dr, "\lambda"] & \\
\theta_U \circ i_{SU} \arrow[d, "1 \circ \sigma_U"'] & & i_{TU} \circ \theta_U \arrow[d, "\tau_U \circ 1"] \\
\theta_U \circ S i_U \arrow[rr, "\theta_i"'] & & T i_U \circ \theta_U
\end{tikzcd}$$

Note unavoidable use of inverse of α.
See SLN47 p63 (8.4.1)

MODIFICATIONS

Suppose $\theta, \varphi : S \longrightarrow T$ are optransformations. A *modification* $\mu : \theta \longrightarrow \varphi$ is a family of 2-cells

$$\begin{tikzcd} SU \arrow[r, bend left, "\theta_U"] \arrow[r, bend right, "\varphi_U"'] \arrow[r, phantom, "\Downarrow \mu_U"] & TU \end{tikzcd} \quad \text{in } \mathcal{W}'$$

such that

$$\begin{tikzcd} \theta_V \circ Sf \arrow[r, "\theta_f"] \arrow[d, "\mu_V \circ 1"'] & Tf \circ \theta_U \arrow[d, "1 \circ \mu_U"] \\ \varphi_V \circ Sf \arrow[r, "\varphi_f"'] & Tf \circ \varphi_U \end{tikzcd} .$$

A morphism $T : \mathcal{W} \longrightarrow \mathcal{W}'$ is *normal* when each $\tau_U : i_{TU} \Longrightarrow T i_U$ is invertible.

A *homomorphism* $T : \mathcal{W} \to \mathcal{W}'$ is a normal morphism for which each $\tau_{(f,g)} : Tg \circ Tf \Longrightarrow T(g \circ f)$ is invertible.

A *strong transformation* $\theta : S \longrightarrow T$ is an optransformation for which each $\theta_f : \theta_V \circ Sf \Longrightarrow Tf \circ \theta_U$ is invertible.

Exercises. 1. Describe a bicategory $\mathrm{Spn}\, A$ constructed from a category A with pullbacks: 0-cells are the objects of A, 1-cells are spans $a \xleftarrow{\alpha} s \xrightarrow{\beta} b$ in A, and horizontal $\circ$ composition is given by pullback.

0. A morphism which is surjective on objects up to equivalence and locally an equivalence is called a *biequivalence*. Show that biequivalences are homomorphisms.

2. Let $\ell : L \to N$ be a regular function. Describe a bicategory Mod_ℓ whose objects are categories in Set_ℓ, whose 1-cells are modules, and whose horizontal composition is tensor product.

3. To each functor $p : E \to A$ with ℓ-small fibres $E_a = \{ \xi : e \to e' \mid p\xi : pe \to pe' \text{ is } 1_a : a \to a \}$ construct a normal morphism $A^{op} \longrightarrow \mathrm{Mod}_\ell$ which takes $a \in A$ to E_a (as a subcategory of E). Show that $p : E \to A$ is a fibration (as in Bénabou's recent lectures) if and only if this normal morphism is a homomorphism & "factors through" the homomorphism $\mathrm{Cat}_\ell \to \hat{\mathrm{Mod}}_\ell$ which takes a functor $f : X \to Y$ to the module $Yf : X \to Y$. Generalize the "Grothendieck construction" to obtain a functor $p : E \to A$ from a normal morphism $A^{op} \longrightarrow \mathrm{Mod}_\ell$.

4. Describe composition of morphisms $\mathcal{W} \xrightarrow{T} \mathcal{W}' \xrightarrow{T'} \mathcal{W}''$.

5. For bicategories $\mathcal{W}, \mathcal{W}'$, prove that the following data define a bicategory $\mathrm{Bicat}_{op}(\mathcal{W}, \mathcal{W}')$:

— 0-cells are morphisms $\mathcal{W} \to \mathcal{W}'$;

— 1-cells are optransformations and 2-cells are modifications with componentwise composition;

— $s(\theta : S \to T) = S$, $t(\theta : S \to T) = T$;

— for
$$\begin{tikzcd} R \arrow[r, bend left, "\theta"] \arrow[r, bend right, "\theta'"'] \arrow[r, phantom, "\Downarrow\mu"] & S \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\varphi'"'] \arrow[r, phantom, "\Downarrow\nu"] & T \end{tikzcd}, \quad (\varphi \circ \theta)_U = \varphi_U \circ \theta_U,$$
$$(\nu \circ \mu)_U = \nu_U \circ \mu_U$$

$$\begin{tikzcd} (\varphi_V \circ \theta_V) \circ Sf \arrow[rr, "(\varphi \circ \theta)_f"] \arrow[d, "\alpha"'] & & (Tf \circ \varphi_U) \circ \theta_U \\ \varphi_V \circ (\theta_V \circ Sf) \arrow[r, "1 \circ \theta_f"] & \varphi_V \circ (Tf \circ \theta_U) \arrow[r, "\alpha^{-1}"] & (\varphi_V \circ Tf) \circ \theta_U \arrow[u, "\varphi_f \circ 1_U"'] \end{tikzcd}$$

[N.B. strong if θ, φ are]

— $i_S : S \to S$ is the strong (op)transformation with
$(i_S)_U = i_{SU}$, $(i_S)_f = \left(i_{SV}\circ Sf \xrightarrow{\lambda^{-1}} Sf \xrightarrow{\rho} Sf \circ i_{SU}\right)$;

— $\alpha_{(\theta,\varphi,\psi)U} = \alpha_{(\theta_U,\varphi_U,\psi_U)}$;

— $(\lambda_\theta)_U = \lambda_{\theta_U}$, $(\rho_\theta)_U = \rho_{\theta_U}$.

6. Describe the canonical <u>isomorphism</u>:

$$\mathrm{Bicat}_{op}(\mathcal{W}, \mathrm{Bicat}(\mathcal{W}', \mathcal{W}'')) \cong \mathrm{Bicat}(\mathcal{W}', \mathrm{Bicat}_{op}(\mathcal{W}, \mathcal{W}'')).$$

ENRICHED CATEGORIES

Suppose $\mathcal{W}$ is a bicategory. <u>A category</u> X <u>enriched in</u> $\mathcal{W}$, or $\mathcal{W}$-<u>category</u>, consists of the following data:

1) a set X_0 whose elements are called <u>objects</u>;
2) a function $e : X_0 \to \mathcal{W}_0$ [say $x \in X$ <u>is over</u> U when $ex = U$];
3) for each $x, y \in X_0$, an arrow (1-cell)

$$X(y,x) : e(x) \to e(y) \text{ of } \mathcal{W};$$

4) for x, y, z over U, V, W, 2-cells

$$\begin{tikzcd} U \arrow[r, bend left, "i_U"] \arrow[r, bend right, "X(x{,}x)"'] \arrow[r, phantom, "\Downarrow\eta"] & U \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dr, "X(z{,}y)"] \arrow[d, phantom, "\Downarrow\mu"] & \\ U \arrow[ur, "X(y{,}x)"] \arrow[rr, "X(z{,}x)"'] & {} & W \end{tikzcd}$$

of $\mathcal{W}$;
satisfying the following axioms:

$$\begin{tikzcd}
(X(z,y)\circ X(y,x))\circ X(x,w) \arrow[r, "\mu\circ 1"] \arrow[d, "\alpha"'] & X(z,x)\circ X(x,w) \arrow[dd, "\mu"] \\
X(z,y)\circ(X(y,x)\circ X(x,w)) \arrow[d, "1\circ\mu"'] & \\
X(z,y)\circ X(y,w) \arrow[r, "\mu"'] & X(z,w)
\end{tikzcd}$$

$$\begin{tikzcd}
X(y,x)\circ X(x,x) \arrow[r, "\mu"] & X(y,x) & X(y,y)\circ X(y,x) \arrow[l, "\mu"'] \\
X(y,x)\circ i_U \arrow[u, "1\circ\eta"] & X(y,x) \arrow[l, "\rho"] \arrow[r, "\lambda"'] \arrow[u, "1"'] & i_V\circ X(y,x). \arrow[u, "\eta\circ 1"']
\end{tikzcd}$$

Exercises 7. Show that a $\mathcal{W}$-category X amounts precisely to a set X_0 together with a morphism of bicategories $X_{0ch} \longrightarrow \mathcal{W}$ (where X_{0ch} is the category whose set of arrows is $X_0 \times X_0$ and whose composition is $(z,y)(y,x) = (z,x)$).

8. A $\mathcal{W}$-category with one object amounts to a monad in $\mathcal{W}$. If $\mathcal{W}_0 = 1$ and $\mathcal{W}(0,0) = AbGp_\ell$ then such is an ℓ-small ring.

Suppose A, X are $\mathcal{W}$-categories. A $\mathcal{W}$-functor $f : A \longrightarrow X$ consists of the following data:

1) a function $f : A_0 \longrightarrow X_0$ which takes objects over U to objects over U;

2) 2-cells $\begin{tikzcd} e(a) \arrow[r, bend left, "A(b,a)"] \arrow[r, bend right, "X(fb,fa)"'] \arrow[r, phantom, "\Downarrow f"] & e(b) \end{tikzcd}$ of $\mathcal{W}$;

such that

$$\begin{tikzcd} i_{e(a)} \arrow[r, "\eta"] \arrow[dr, "\eta"'] & A(a,a) \arrow[d, "f"] \\ & X(fa,fa) \end{tikzcd} \qquad \begin{tikzcd} A(c,b)\circ A(b,a) \arrow[r, "\mu"] \arrow[d, "f"'] & A(c,a) \arrow[d, "f"] \\ X(fc,fb)\circ X(fb,fa) \arrow[r, "\mu"'] & X(fc,fa) \end{tikzcd}.$$

ENRICHED MODULES

A $\mathcal{W}$-*module* $m : B \to A$ between $\mathcal{W}$-categories A, B consists of:

1) 1-cells $e(b) \xrightarrow{m(a,b)} e(a)$ of $\mathcal{W}$ for $a \in A_0$, $b \in B_0$;

2) 2-cells
$$A(a',a)\circ m(a,b) \xrightarrow{\mu_\ell} m(a',b)$$
$$m(a,b)\circ B(b,b') \xrightarrow{\mu_r} m(a,b') \text{ of } \mathcal{W};$$

satisfying the axioms

$$\begin{tikzcd} m(a,b) \arrow[r, "1"] \arrow[d, "\lambda"'] & m(a,b) \\ i_{e(a)}\circ m(a,b) \arrow[r, "\eta\circ 1"'] & A(a,a)\circ m(a,b) \arrow[u, "\mu_\ell"'] \end{tikzcd}$$

$$\begin{tikzcd} m(a,b) \arrow[r, "1"] \arrow[d, "\rho"'] & m(a,b) \\ m(a,b)\circ i_{e(b)} \arrow[r, "1\circ\eta"'] & m(a,b)\circ B(b,b) \arrow[u, "\mu_r"'] \end{tikzcd}$$

$$\begin{tikzcd}
(A(a'',a')\circ A(a',a))\circ m(a,b) \arrow[r, "\mu\circ 1"] \arrow[d, "\alpha"'] & A(a'',a)\circ m(a,b) \arrow[dd, "\mu_\ell"] \\
A(a'',a')\circ(A(a',a)\circ m(a,b)) \arrow[d, "1\circ\mu_\ell"'] & \\
A(a'',a')\circ m(a',b) \arrow[r, "\mu_\ell"'] & m(a'',b)
\end{tikzcd}$$

$$\begin{tikzcd}
(m(a,b)\circ B(b,b'))\circ B(b',b'') \arrow[r, "\mu_r\circ 1"] \arrow[d, "\alpha"'] & m(a,b')\circ B(b',b'') \arrow[dd, "\mu_r"] \\
m(a,b)\circ(B(b,b')\circ B(b',b'')) \arrow[d, "1\circ\mu"'] & \\
m(a,b)\circ B(b,b'') \arrow[r, "\mu_r"'] & m(a,b'')
\end{tikzcd}$$

$$\begin{tikzcd}
(A(a',a)\circ m(a,b))\circ B(b,b') \arrow[r, "\mu_\ell\circ 1"] \arrow[d, "\alpha"'] & m(a',b)\circ B(b,b') \arrow[dd, "\mu_r"] \\
A(a',a)\circ(m(a,b)\circ B(b,b')) \arrow[d, "1\circ\mu_r"'] & \\
A(a',a)\circ m(a,b') \arrow[r, "\mu_\ell"'] & m(a',b').
\end{tikzcd}$$

A $\mathcal{W}$-<u>module morphism</u> $\sigma : m \to m'$ is a family of 2-cells

$$m(a,b) \xRightarrow{\sigma_{ab}} m'(a,b)$$

such that

$$\begin{tikzcd}
A(a',a)\circ m(a,b) \arrow[r, "\mu_\ell"] \arrow[d, "1\circ\sigma_{ab}"'] & m(a',b) \arrow[d, "\sigma_{a'b}"] \\
A(a',a)\circ m'(a,b) \arrow[r, "\mu_\ell"'] & m'(a',b)
\end{tikzcd}
\qquad
\begin{tikzcd}
m(a,b)\circ B(b,b') \arrow[r, "\mu_r"] \arrow[d, "\sigma_{ab}\circ 1"'] & m(a,b') \arrow[d, "\sigma_{ab'}"] \\
m'(a,b)\circ B(b,b') \arrow[r, "\mu_r"'] & m'(a,b').
\end{tikzcd}$$

$$\begin{tikzcd} & C \arrow[dl, "n"'] \arrow[dr, "p"] \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] & \\ B \arrow[rr, "m"'] & {} & A \end{tikzcd}$$

For modules m, n, p as above, a <u>form</u> $\sigma : (m, n) \Rightarrow p$ is a family of 2-cells

$$m(a,b) \circ n(b,c) \xrightarrow{\sigma^b_{ac}} p(a,c)$$

such that all the 2-cells

$$(((A(a',a) \circ m(a,b)) \circ B(b,b')) \circ n(b',c)) \circ C(c,c') \longrightarrow p(a',c')$$

obtainable using $\alpha, \sigma, \mu_\ell, \mu_r$ are equal.

More generally, there are forms

$$\begin{tikzcd} C = B^k \arrow[r, "m^k"] \arrow[rrrr, bend right, "p"'] & B^{k-1} \arrow[r, "m^{k-1}"] & \cdots \arrow[r] \arrow[d, phantom, "\Downarrow\sigma"] & B^1 \arrow[r, "m^1"] & B^0 = A. \\ & & {} & & \end{tikzcd}$$

There is always the module $A : A \to A$ given by the 1-cells $A(a',a)$ and $\mu_\ell = \mu_r = \mu$.

Each functor $f : B \to A$ and module $p : C \to A$ determine a module $fp : C \to B$ given by $(fp)(b,c) = p(fb,c)$. A functor $f : B \to A$ and module $n : A \to C$ determine a module $nf : B \to C$ given by $(nf)(c,b) = n(c,fb)$. In particular, $fA : A \to B$, $Af : B \to A$ are modules.

<u>Lemma</u>. <u>There are canonical bijections between forms</u>

$$\begin{tikzcd} & C \arrow[dl, "p"'] \arrow[dr, "r"] \arrow[d, phantom, "\Rightarrow"] & \\ A \arrow[rr] & {} & B \end{tikzcd} \longleftrightarrow \begin{tikzcd} C \arrow[d, bend right, "fp"'] \arrow[d, bend left, "r"] \arrow[d, phantom, "\Rightarrow"] \\ B \end{tikzcd}; \quad \begin{tikzcd} & B \arrow[dl, "Af"'] \arrow[dr, "r"] \arrow[d, phantom, "\Rightarrow"] & \\ A \arrow[rr, "m"'] & {} & C \end{tikzcd} \longleftrightarrow \begin{tikzcd} B \arrow[d, bend right, "nf"'] \arrow[d, bend left, "r"] \arrow[d, phantom, "\Rightarrow"] \\ C \end{tikzcd}$$

TENSOR PRODUCT OF MODULES

$$\begin{tikzcd} C \arrow[r, "n"] \arrow[rr, bend right, "m\otimes n"'] & B \arrow[r, "m"] & A \end{tikzcd}$$

A <u>tensor product</u> for m, n is a module $m\otimes n$ together with a form $\pi : (m,n) \Longrightarrow m\otimes n$ such that each form $\sigma : (m,n) \Longrightarrow p$ factors as $\sigma = \tau\pi$ for a unique $\tau : m\otimes n \Longrightarrow p$.

Certain tensor products always exist:

$$fA\otimes p \cong fp \quad , \quad n\otimes Af \cong nf \text{ (by Lemma)}.$$

When either side of the isomorphism

$$m\otimes(n\otimes p) \cong (m\otimes n)\otimes p$$

exists it represents forms out of (m,n,p); hence the isomorphism when both sides exist.

HOM OF MODULES

$$\begin{tikzcd} & C \arrow[dl, "\hom(m,p)"'] \arrow[dr, "p"] \arrow[d, phantom, "\overset{\Longrightarrow}{\varepsilon}"] & \\ B \arrow[rr, "m"'] & {} & A \end{tikzcd}$$

A <u>hom</u> for m, p is a module $\hom(m,p)$ together with a form $\varepsilon : (m, \hom(m,p)) \Longrightarrow p$ (<u>evaluation</u>) such that each form $\sigma : (m,n) \Longrightarrow p$ factors as $\sigma = \varepsilon\,(m,\tau)$ for a unique $\tau : n \Longrightarrow \hom(m,p)$.

<u>Corollary</u>:
$$\begin{array}{c} m\otimes n \Longrightarrow p \\ \hline (m,n) \Longrightarrow p \\ \hline n \Longrightarrow \hom(m,p) \end{array} \quad \square$$

Write m^* for $\hom(m, A)$ when it exists.

Call $m : B \longrightarrow A$ <u>cauchy</u> when $\hom(m, p)$ exists for all modules $p : C \longrightarrow A$ and the canonical form $(m^*, p) \Longrightarrow \hom(m, p)$ is a tensor product for m^*, p.

Suppose $f : B \longrightarrow A$ is an (enriched) functor. Then $\hom(Af, p) \cong fp$ exists and $(Af)^* \cong fA$; also $fA \otimes p \cong fp$. So $Af : B \longrightarrow A$ is cauchy.

Call $m : B \longrightarrow A$ <u>convergent</u> when $m \cong Af$ for some functor $f : B \longrightarrow A$. So convergent implies cauchy.

Call A <u>cauchy complete</u> when every cauchy module $m : B \longrightarrow A$ is convergent. This holds if and only if each cauchy module $m : \mathcal{U} \longrightarrow A$ with $\mathcal{U}$ in $\mathcal{W}$ is convergent.

<u>Colimits.</u> Suppose $m : B \longrightarrow A$ is a module and $g : A \longrightarrow X$ is a functor. A <u>colimit for</u> f <u>weighted by</u> m is a functor $\operatorname{colim}(m, g) : B \longrightarrow X$ together with an isomorphism

$$\operatorname{colim}(m, g)X \cong \hom(m, gX).$$

$$\begin{tikzcd}
 & X \arrow[dl, "\operatorname{colim}(m,g)X"'] \arrow[dr, "gX"] & \\
B \arrow[rr, "m"'] & \overset{\varepsilon}{\Longrightarrow} & A
\end{tikzcd}$$

Exercises 9. X is cauchy complete if and only if it admits all colimits weighted by cauchy modules.

10. If $m \otimes n$ exists and m, n are cauchy then $m \otimes n$ is cauchy.

Construction of tensor products

$$C \xrightarrow{n} B \xrightarrow{m} A$$

For a, c in A, C over $\mathcal{U}, \mathcal{W}$, take the colimit over all b, b' in B of the diagram

$$\begin{tikzcd}[column sep=tiny]
& m(a,b) \circ B(b,b') \circ n(b',c) \arrow[dl, "\mu_r \circ 1"'] \arrow[dr, "1 \circ \mu_l"] & \\
m(a,b') \circ n(b',c) \arrow[dr, dashed] & & m(a,b) \circ n(b,c) \arrow[dl, dashed] \\
& (m \otimes n)(a,c) &
\end{tikzcd}$$

when it exists in $\mathcal{W}(\mathcal{W},\mathcal{U})$ and is preserved by composition $\circ$ in $\mathcal{W}$.

Construction of hom

This needs big enough limits in each $\mathcal{W}(\mathcal{U},\mathcal{V})$ together with adjoints (right) for the functors $\mathcal{W}(\mathcal{U}, w)$ for each 1-cell w of $\mathcal{W}$.