Deligne's conjecture: an interplay between algebra, geometry and higher category theory.

M.A.Batanin

30 November 2006, ANU Canberra

Contents

1 Three principles of noncommutative differential calculus

The commutative algebra of smooth functions on a manifod M is a limit of noncommutative algebras of observables in quantum theory.

Question: can we get this noncommutative algebra from $A = C^\infty(M)$ by a deformation? Deformation means that we consider the algebra of formal series over A and we define a 'deformed' multiplication $f \star g$ by the formula

$$f \star g = fg + \sum_{n \geq 1} \hbar^n B_n(f, g)$$

where

$$B_n(f, g) : A \otimes A \to A$$

are bidifferential operators.

This new product must be associative and this provides some constraints on the operators B_n. In particular, B_1 has to determine the so called Poisson structure on M. This means that it satisfies the Jacobi identity and the Leibniz rule with respect to the usual product of functions.

One of the fundamental questions of the deformation theory was the existence of a $\star$-product on a manifold with a given Poisson structure. A beautiful example of such a deformed product is famous Moyal's product.

In 1997 Kontsevich gave a positive answer to this question. Moreover, he proved that there is a canonical (up to some natural equivalence) $\star$-product on any Poisson manifold.

The main idea is to compare two natural objects from differential calculus: the space of polyvector fields on M and the so called Hochschild complex of $A = C^\infty(M)$, which exists for an arbitrary associative algebra A. It turned out that

- the Hochschild complex of any associative algebra has a very rich algebraic structure (this is the content of Deligne's hypothesis);
- the space of polyvector fields on M has a similar structure;
- these two objects are equivalent in an appropriate sense (up to strong homotopy).

These three observations imply Kontsevich theorem.

Tamarkin and Tsigan have formulated them as three principles of noncommutative differential caculus:

- An object from classical calculus should have its noncommutative analogue;
- If there is an algebraic structure on a classical object, the corresponding noncommutative object should posses a similar structure up to strong homotopy;
- If $A = C^\infty(M)$, one gets two strong homotopy structures, one coming from classical calculus and another from noncommutative one. Those two structures should be equivalent (formality theorems).

2 Operads and strong homotopy structures

To understand these principles we should first talk about abstract algebraic structures and strong homotopy structures. Operads provide one of the solutions.

Classically, the operads were invented to solve the problem of recognition of the n-fold loop spaces and for this purpose it was sufficient to have topological operads.

Let $[n]$ denote the ordinal $0 < 1 < \ldots < n-1$.

Definition 2.1 *A (syymetric) operad A is a sequence of topological spaces*

$$A_{[0]}, \ldots, A_{[n]}, \ldots$$

together with an element $e \in A_{[1]}$ and a multiplication (substitution) map

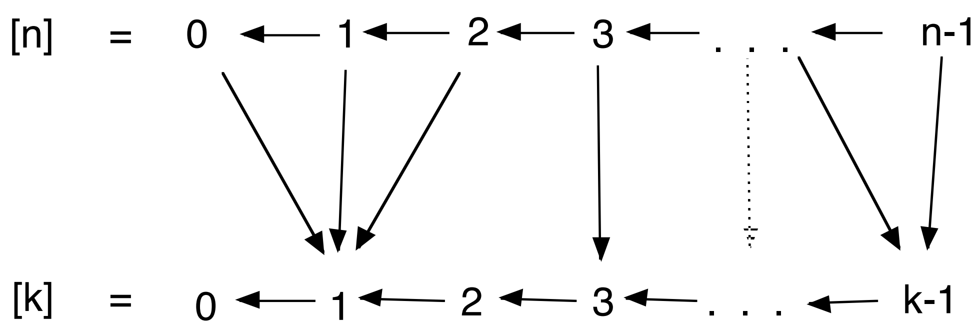
$$m_\sigma : A_{[k]} \times A_{[n_1]} \times \ldots \times A_{[n_{k-1}]} \longrightarrow A_{[n]}$$

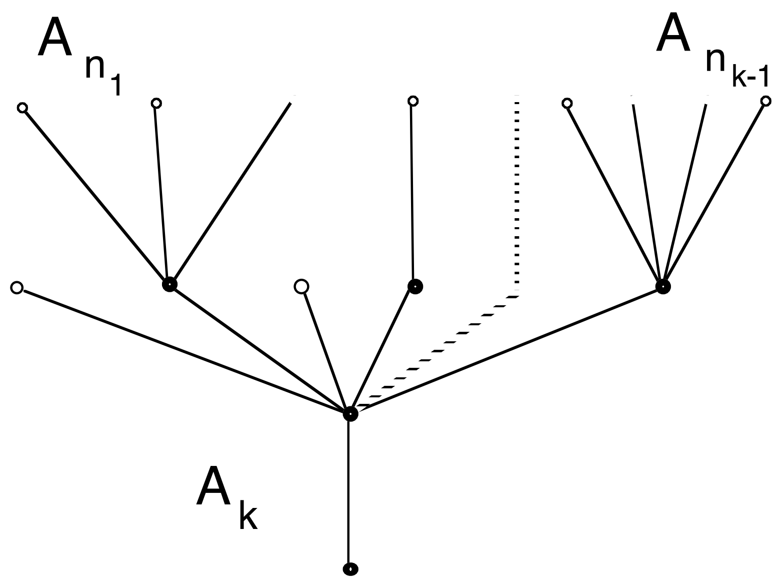
for every map

$$\sigma : [n] \to [k]$$

with the preimages (with their induced orders) $\sigma^{-1}(i) \simeq [n_i]$, satisfying associativity and unitarity conditions with respect to the composition of maps.

A_n called the space of operations of arity n of the operad A.





$$A_k \times A_{n_1} \times \ldots \times A_{n_{k-1}} \longrightarrow A_n$$

A map of operads is a sequence of continuous maps

$$f_{[n]} : A_{[n]} \to B_{[n]}$$

which preserves the operad structures.

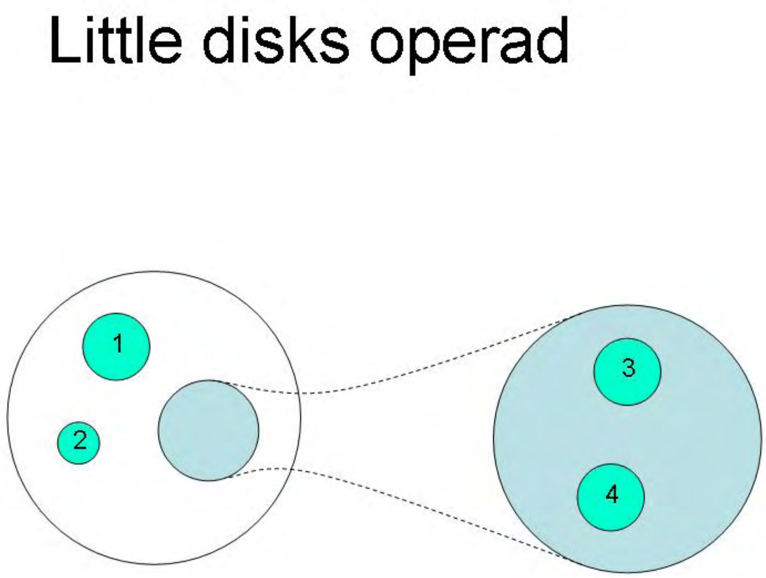
Example 2.1 The endomorphism operad $E(X)$ of a topological space

$$E(X)_{[n]} = Top(X^n, X).$$

The unit element is the identity map $id : X \to X$ and composition is given by substitution of functions.

Example 2.2 The little n-disks operad D^n. (Boardman-Vogt-May)

The space D^n_k is the space of configurations of k nonoverlapping open n-dimensional disks inside closed unit disk in $\Re^n$.



For example, the space D^n_2 (the space of binary operations of the operad D^n) is equivalent to the $(n-1)$-dimensional sphere.

Definition 2.2 *An algebra over an operad A is a topological space X together with a map of operads*

$$k : A \to E(X).$$

We will also say that *the operad A acts on X.*

To construct noncommuative differential calculus we need also algebraic operads. More specifically we need operads with values in vector spaces and chain complexes.

Operads in vector spaces is easy to define. It is sufficient to replace cartesian product of topological spaces by tensor product of vector spaces in the definition of operad. We also need to replace the space of continuous maps by the vector space of linear operators in the definition of endomorphism operad. Then we have a notion of an algebra over an operad.

Example 2.3 An associative algebra (A, m, e) is a vector space A equipped with an associative bilinear operation

$$m : A \otimes A \to A$$

and an element $e \in A$ which is a unit for multiplication m.

There is an operad in vector spaces $Assoc$ whose algebras are exactly associative algebras. The vector space $Assoc_n$ is the vector space freely generataed by the symmetric group on n elements S_n.

Example 2.4 The space of n-ary operations of the operad for commutative algebras is one dimensional vector space.

There are also operads for Lie algebras and many other interesting structures. But there is no operads for Hopf algebras, for example.

Now we extend our definition of operad from vector spaces to chain compexes of vector spaces.

Definition 2.3 *A chain complex* C *is a family of vector spaces indexed by integers and equipped by a sequence of linear operators (differential)*

$$d_n : C^n \rightarrow C^{n+1}$$

such that for any n *the composite*

$$C^{n-1} \stackrel{d_{n-1}}{\longrightarrow} C^n \stackrel{d_n}{\longrightarrow} C^{n+1}$$

is 0.

Since $d_n \cdot d_{n+1} = 0$ we have

$$Im(d_{n-1}) \subset Ker(d_n)$$

and we can form a factor-space

$$H^n(C) = Ker(d_n)/Im(d_{n-1})$$

which is called n-th homology group of C. We will consider the sequence $H^\bullet(C)$ as a chain complex with trivial differential.

Example 2.5 Chain complexes in differential calculus.

- For any smooth manifold M the sequence of spaces of differential forms on M is a chain complex (de Rham complex). The differential is the usual operator of exterior differentiation.
- The n-th homology group is called the n-th de Rham cohomology of M. The famous Poincaré lemma (every closed differential form in $\Re^n$ is exact) can be expressed by the isomorphism $H^p(\Re^n) = 0$.
- Hochschild complex $CH^{\bullet}(A)$ of an associative algebra A will be defined in the next section. If $A = C^{\infty}(M)$ the homology of the Hochschild complex are isomorphic to the graded space of polyvector fields on M (Hochschild-Konstant-Rosenberg theorem .)

Example 2.6 Singular chain complex of a topological space X. In the dimension $n \leq 0$ this is the vector space $S^n(X)$ of linear combinations of all continuous maps from a simplex Δ^{-n} to X. On a singular simplex

$$f : \Delta^{-n} \to X$$

the differential is given by the formula

$$d(f) = \sum_{i=1}^{-n} (-1)^1 d^i(f)$$

where $d^i(f)$ is the restriction of f on the i-th face of Δ^{-n}. The homology of this chain complex are called the singular homology group of X and are denoted $H^\bullet(X)$.

There is also the normalised version $C^\bullet(X)$ of the complex $S^\bullet(X)$

$$C^n(X) = S^n(X)/D$$

where D is the subspace generated by 'degenerate' singular simplices.

We can extend the tensor product of vector spaces to the tensor product of chain complexes. Analogously, we can extend the space of linear operators between vector spaces to chain complex level and construct the chain complex of homomorphisms between two chain complexes. So we can consider the theory of operads with values in chain complexes.

Theorem 2.1 *If A is a topological operad , then $C^{\bullet}(A)$ is an operad in chain complexes and $H^{\bullet}(A)$ is an operad in graded vector spaces. Moreover, if X is an algebra of A then $C^{\bullet}(X)$ and $H^{\bullet}(X)$ has a natural structure of an algebra over $C^{\bullet}(A)$ and $H^{\bullet}(A)$ respectively.*

Finally, we can use operads to define algebraic structures up to strong homotopies in categories like topological spaces or chain complexes. The idea is: an algebraic structure is given by operations and relations. For the corresponding *strong homotopy* structure we leave the same operations but we replace relations by some sort of explicit deformation (*homotopy*). These deformations should satisfy some relations but only up to a new homotopy etc..

For example, if a structure is an associative multiplication on a topological space its strong homotopy counterpart will be a multiplication which can be nonassociative but there should be a path between any points $(ab)c$ and $a(bc)$, these pathes should satisfy the so called pentagon relation up to a homotopy etc. This is called A_∞-space in topology.

One can do a similar thing for chain complexes.

In general it is a hard problem to describe such a structure but, fortunately, it is always possible if our structure is given by an operad.

Theorem 2.2 *For an operad A in topological spaces or chain complexes there exists an operad A' such that the algebras of A' are strong homotopy algebras of A.*

We also want to consider morphisms of algebras up to strong homotopy. There is a way to do this.

3 Gerstenhaber algebras and Hochschild complex

Definition 3.1 *The Hochschild complex* $CH^\bullet(A)$ *of* A *is the sequence of vector spaces*

$$CH^n(A) = Hom(A \otimes \ldots \otimes A, A)$$

equipped with a differential

$$d_n : CH^n(A) \to CH^{n+1}(A)$$

$$d(f)(x_1, \ldots, x_{n+1}) = x_1 f(x_2, \ldots, x_{n+1}) +$$
$$+ \sum_{i=1}^{n} (-1)^i f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) -$$
$$-(-1)^n f(x_1, \ldots, x_{n+1}) x_{n+1}.$$

The homology of $CH^\bullet(A)$ is denoted by $HH^\bullet(A)$ and are called the **Hochschild cohomology** of A.

There are some natural operations defined on Hochschild complex and Hochschild cohomology

- The cup product

$$- \cup - : CH^p(A) \otimes CH^q(A) \to CH^{p+q}(A)$$

  defined by the formula

$$(f \cup g)(x_1, \dots, x_{p+q}) =$$
$$= (-1)^{pq} f(x_1, \dots, x_p) \cdot g(x_{p+1}, \dots, x_{p+q})$$

- The bracket

$$\{-, -\} : CH^p(A) \otimes CH^q(A) \to CH^{p+q-1}(A)$$
$$\{f, g\} = f \circ g - (-1)^{(p-1)(q-1)} g \circ f$$

  where

$$f \circ g(x_1, \dots, x_{p+q-1}) =$$
$$\sum_{i=1}^{p} (-1)^{(q-1)(i-1)} f(x_1, \dots, x_i, g(x_{i+1}, \dots$$
$$\dots, x_{i+q}), x_{i+q+1}, \dots, x_{p+q-1})$$

The cup product is associative but not commutative. However, it induces a multiplication on Hochschild cohomology which is graded commutative. The bracket operation also induces a bracket operation on cohomology.

Definition 3.2 *A Gerstenhaber algebra is a graded vector space C equipped with two binary operations*

$$\cup \quad \textit{and} \quad \{-,-\}$$

satisfying the identities

$$f \cup (g \cup h) = (f \cup g) \cup h$$
$$f \cup g = (-1)^{deg(f)deg(g)} g \cup f$$
$$\{f, g\} = -(-1)^{(deg(f)-1)(deg(g)-1)}\{g, f\}$$

graded Jacobi identity

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(deg(f)-1)(deg(g)-1)}\{g, \{f, h\}\}$$

graded Leibniz rule

$$\{f, g \cup h\} = \{f, g\} \cup h + (-1)^{(deg(f)-1)deg(g)} g \cup \{f, h\}.$$

Theorem 3.1 (Gerstenhaber) *The Hochschild cohomology $HH^\bullet(A)$ has a natural structure of a Gerstenhaber algebra.*

There is an operad for Gerstenhaber algebra structure called the Gerstenhaber operad **G**.

The following result leads to Deligne's conjecture.

Theorem 3.2 (F.Cohen,P.Deligne) *Over a field of characteristic* 0 *the operad of homology of the little* 2*-disks operad* $H^{\bullet}(D^2)$ *is isomorphic to the Gershtenhaber algebra operad* $\boldsymbol{G}$.

One can restate the Gerstenhaber theorem in the following form:

the operad $H^{\bullet}(D^2)$ **acts on the Hochschild cohomology** $HH^{\bullet}(A)$ **of an associative algebra** A.

4 Deligne's hypothesis and higher category theory

At the end of 80s beginning of 90s A whole new family of natural operations on Hochschild complex was discovered by Kadeishvili, Getzler. Gerstenhaber and Voronov.

It turned out that Hochschild complex has a very rich algebraic structure.

Conjecture 4.1 (Deligne 1993)
The $H^\bullet(D^2)$-algebra structure on $HH^\bullet(A)$ is induced by an action of the operad $C^\bullet(D^2)$ on $CH^\bullet(A)$.

Getzler and Jones claimed the proof in 1994. And the result was considered as established and was used in many papers until Tamarkin found a flow in the arguments of Getzler and Jones in 1998. The first real proof is also due to Tamarkin.

After this many people reproved the conjecture and some generalizations of it by different methods (Voronov, Berger-Fresse, McClure-Smith, Kontsevich-Soibelman, R.Kaufman). But all these proofs are highly technical.

Kontsevich : **We need a really short and convincing argument for this very fundamental fact about Hochschild complex.**

Higher category theory provides us with the argument Kontsevich was looking for.

This proof appeared in 2005 in a paper by Tamarkin "What do DG-categories form?". It is based on my theorem about the relation between n-operads (subject of higher category theory) and the little n-disks operad on one side and Tamarkin's observation on the convolution operation in multicategories on the other side.

4.1 Sketch of a proof

A category consists of a class of objects $Ob(C)$ and for every two objects $a, b \in C$ a set of morphisms $Hom(a, b)$. We should have a special morphism $1_a \in Hom(a, a)$ (an identity morphism) and also a composition of the morphisms:

$$Hom(a, b) \times Hom(b, c) \to Hom(a, c).$$

This should satisfy some well known axioms.

We also have a notion of morphism between categories called *functors* . So the categories form a category.

If one replaces the set of morphisms by an object in some other category V (with some extra structure called monoidal structure on V) and composition map by a morphism in this category we get a notion of category *enriched in* V.

Example 4.1 One can consider the categories enriched in the category of abelian groups (additive categories), vector spaces (linear categories or chain complexes (differential graded or DG-categories).

Example 4.2 Categories enriched in the category of categories are called 2-categories. To have a 2-category is the same as to have a class of objects, between any to objects we have a set of 1-arrows and between any two parallel 1-arrows we have a set of 2-arrows. We can compose 1-arrows like in a category and we can compose 2-arrows in two different ways: vertically and horizontally.

One can develop a theory of operads which captures the structure of a category (1-operads) and of 2-category (2-operads). Similar to the usual operad theory we can consider a notion of strong homotopy algebras of a 2-operad. We will call these algebras **strong homotopy 2-categories.**

Now let me recall a classical Eckman-Hilton argument concerning commutativity of the second homotopy group.

Let G be a set with two group structures:

$$- \star - : G \times G \to G$$

$$- \circ - : G \times G \to G$$

with common unit e.

And suppose these two structures are interchangeable in the sense that:

$$(a \star b) \circ (c \star d) = (a \circ c) \star (b \circ d).$$

Theorem 4.1 (Eckman-Hilton argument)

- *The multiplications* $\star$ *and* $\circ$ *coincide;*
- $\star = \circ$ *is commutative.*

The proof is an elementary exercise and can be presented in the following picture

The Eckman-Hilton argument

$$a \bullet b = (a \star e) \bullet (e \star b) = (a \bullet e) \star (e \bullet b) = a \star b = (e \bullet a) \star (b \bullet e) = (e \star b) \bullet (a \star e) = b \bullet a$$

$$a \bullet b = (e \star a) \bullet (b \star e) = (e \bullet b) \star (a \bullet e) = b \star a = (b \bullet e) \star (e \bullet a) = (b \star e) \bullet (e \star a) = b \bullet a$$

We do not use the existence of inverses, so we can generalise the statement a little bit saying that we have *two interchangeable monoid* structures on the same set.

There is an operad for two interchangeable monoid structures on a set. We can express this fact by constructing a symmetric operad which has sets with two interchangeable monoids structures as its algebras. The Eckman-Hilton argument is the statement that this operad is the operad for commutative monoids.

On the other hand we can express the same structure in terms of 2-categorical structures. Indeed, to say that we have two interchangeable monoid structures on a set X is the same as to say that we have a 2-category structure with one object $*$, one arrow 1_* and the set of morphisms

$$Hom(1_*, 1_*) = X.$$

The Eckman-Hilton argument says that **the symmetric operad for one object, one arrow 2-category is the operad for commutative monoids.**

Now we are interested in a strong homotopy version of this statement. That is, we have two operations on the same chain complex C which are interchangeable up to all higher homotopies. Again, this can be reformulated as follows: there is a homotopy 2-category structure with one object $*$, one arrow 1_* and chain complex of morphisms $Ch(1_*, 1_*) = C.$. We have the following *Derived Eckman-Hilton argument*

Theorem 4.2 (Batanin) *The symmetric operad for one object, one arrow strong homotopy* 2*-categories is equivalent to the operad of chains of the little* 2*-disks operad.*

One evidence for this theorem can be seen from the picture above. The picture has a shape of a circle which is the space of binary operations of the little 2-disks operad (up to homotopy). To prove that the corresponding shapes for higher arities are of the right types is more complicated but follows the same idea.

In 2005 Tamarkin answered a question of Drinfeld and Belinson "What do DG-categories form?".

It is well known that one can define a notion of natural transformation between two functors and that categories, functors and natural transformations form a 2-category.

Recall that a DG-category is a category where for every two objects a, b the set of arrows from a to b is a chain complex and composition is bilinear.

Example 4.3 Let A be an associative algebra with unit. Then it can be considered as a DG-category ΣA if we form a category with one object $*$ and with $Chain(*, *) = A$. The composition rule

$$Chain(*, *) \otimes Chain(*, *) \to Chain(*, *)$$

is given by multiplication in A.

For two DG-functors one can talk about natural transformations which are natural in the strong homotopy sense (*homotopy coherent transformations*). So, we can ask what sort of 2-dimensional categorical structure do we have for DG-categories, DG-functors and coherent natural transformations?(Belinson-Drinfeld question.)

Theorem 4.3 (Tamarkin) *DG-categories, DG-functors and homotopy coherent DG-transformations form a strong homotopy* 2*-category.*

Now if we fix a DG-category B one can consider the chain complex of homotopy coherent natural transformations

$$Chain(Id_B, Id_B),$$

where Id_B is an identity functor on B. Since composition of identity with itself is an identity Tamarkin's theorem implies that $Chain(Id_B, Id_B)$ is a Hom chain complex for a strong homotopy 2-category with one object B , and one arrow Id_B. So, by Batanin's theorem $Chain(Id_B, Id_B)$ is an algebra of the chains of the little 2-disks operad.

It is a classical observation that the Hochschild complex of an algebra A is the same as $Chain(Id_{\Sigma A}, Id_{\Sigma A})$ (recall that ΣA means that we consider A as a DG-category with one object). So, we proved the Deligne's conjecture.

5 Higher dimensional Deligne's conjecture and other generalisations

The original version of Deligne's conjecture admits many generalisations:

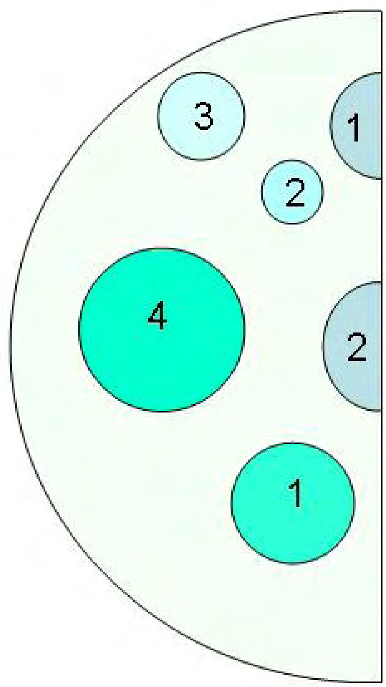
- Higher dimensional generalisation (Kontsevich) There should exist a notion of Hochschild complex for an algebra of the little n-disks operad. It should have a structure of an algebra of the little $(n+1)$-disks operad. There is some partial progress with this hypothesis in the works of Tamarkin and Krizh-Voronov.

- A more sophisticated form of Deligne's hypothesis due to Kontsevich claims that the pair

  (associative algebra, its Hochschild cochain complex)

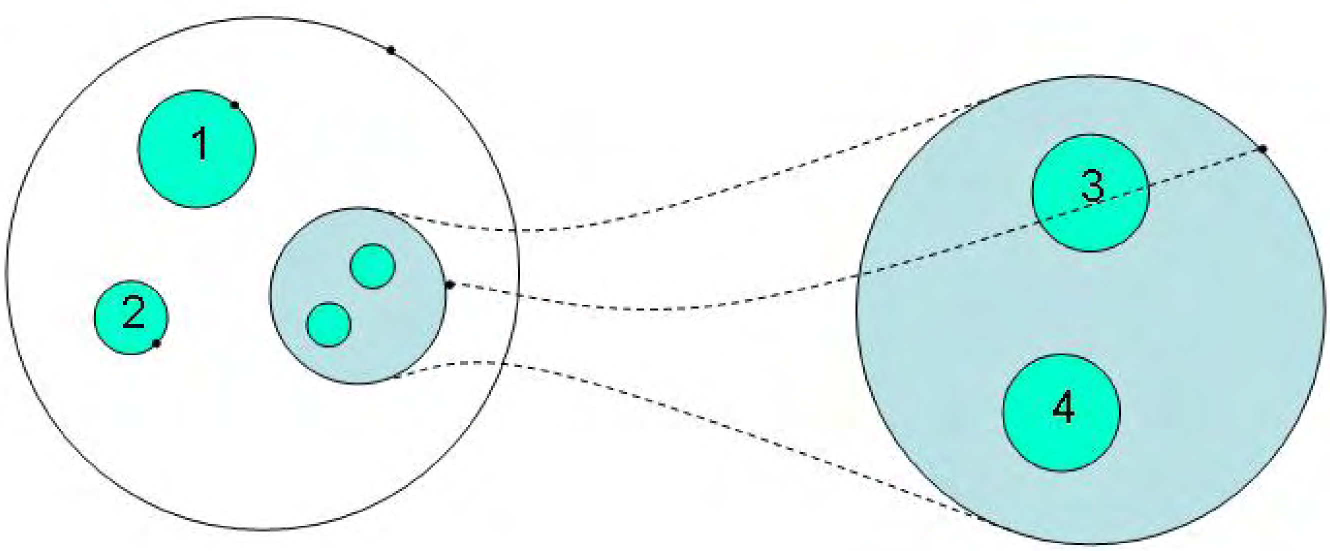
  is an algebra over the Swiss Cheese operad and, moreover, this is a universal pair in some strong homotopy sense.

  The Swiss-Cheese operad is an operad of configurations of disks in a half plane and semidisks on its boundary. It is proved in unpublished notes by Tamarkin by a method similar to what I described for the original Deligne hypothesis. The higher dimensional version is not proved, yet.


3
1
2
4
2
1

- Deligne's conjecture for Frobenious algebras is proved by Kaufman by purely geometric methods (using spineless cacti operad). It involves the action of the *framed* little 2-disks operad. This is an operad of configurations of disks with a marked point on the boundary of each disks. Substitution involves rotation of disks.

  Question: find a higher categorical interpretation and proof of this form of Deligne's conjecture?

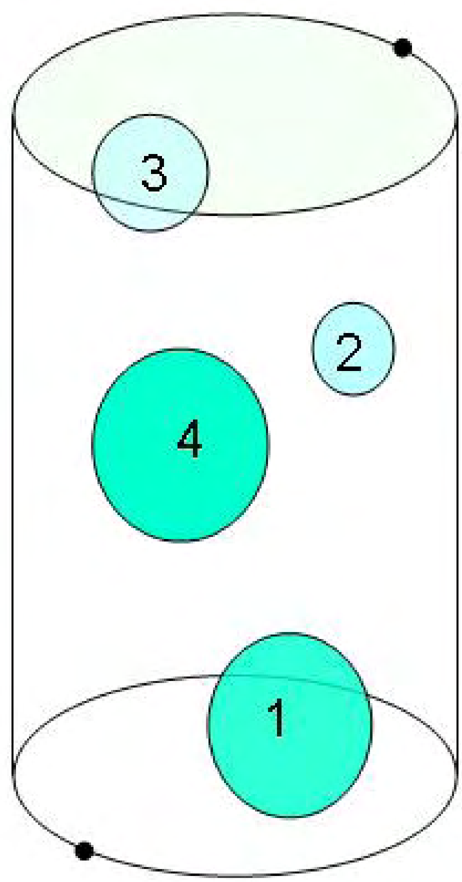

1
2
3
4

- The latest version of the Deligne's conjecture due to Kontsevich-Soibelman and Tamarkin-Tsygan states that on the pair

  (Hochschild chains, Hochschild cochains)

  there is an action of the operad of configurations of little disks on a cylinder with a marked point on each of the connected components of the boundary of the cylinder. This leads to a noncommutative version of Cartan's type calculus which involves both polyvector fields and differential forms. There is a very brief sketch of the proof in the paper of Kontsevich and Soibelman.

  Question: find a higher categorical interpretation and proof of this form of Deligne's conjecture?


3
2
4
1