IMAGINARY POWERS OF LAPLACE OPERATORS

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ABSTRACT. We show that if L is a second-order uniformly elliptic operator in divergence form on $\mathbf{R}^d$, then $C_1(1+|\alpha|)^{d/2} \le \|L^{i\alpha}\|_{L^1\to L^{1,\infty}} \le C_2(1+|\alpha|)^{d/2}$. We also prove that the upper bounds remain true for any operator with the finite speed propagation property.

1. Introduction. Assume that $a_{ij} \in C^\infty(\mathbf{R}^d)$, $a_{ij} = a_{ji}$ for $1 \le i,j \le d$ and that $\kappa I \le (a_{ij}) \le \tau I$ for some positive constants κ and τ. We define a positive self-adjoint operator L on $L^2(\mathbf{R}^d)$ by the formula

$$L = -\sum \partial_i a_{ij} \partial_j. \tag{1}$$

We refer readers to [8] for the precise definition and basic properties of L. In particular, L admits a spectral resolution $E(t)$ and we can define the operator $L^{i\alpha}$ by the formula

$$L^{i\alpha} = \int_0^\infty t^{i\alpha} dE(t).$$

By spectral theory $\|L^{i\alpha}\|_{L^2\to L^2} = 1$. It is well known that $L^{i\alpha}$ falls within the scope of classical Calderón-Zygmund theory (as described in [3] or [22]) and so it extends to a bounded operator on $L^p, 1 < p < \infty$, and is also weak type (1,1). The main aim of this paper is to obtain the sharp estimate for the weak type $(1,1)$ norm of $L^{i\alpha}$ in terms of α.

The study of imaginary powers of operators is an important part of the theory of operators of type ω with H^∞ functional calculus, see e.g., [6], [9] and [17]. What is perhaps more interesting and relevant from the point of view of this paper is that the weak type $(1,1)$ norm of imaginary powers of self-adjoint operators can play a central role in the theory of spectral multipliers. See [5] and [15]. Imaginary powers of Laplace operators on compact Lie groups were also investigated in [20]. Theorem 2 below applied to Laplace operators on compact Lie groups gives the sharp endpoint result of Theorem 3 in [20], pp. 58. See also Corollary 4 of [20], pp. 121.

However, the starting point for this paper is the following observation from [2]. If we denote the weak type (1,1) norm of an operator T on a measure space (X,μ) by $\|T\|_{L^1\to L^{1,\infty}} = \sup \lambda\ \mu(\{x \in X : |Tf(x)| > \lambda\})$ where the supremum is taken over $\lambda > 0$ and functions f with $L^1(X)$ norm less than one, then for the standard Laplace operator on $\mathbf{R}^d$,

$$C_1(1+|\alpha|)^{d/2} \le \|(-\Delta_d)^{i\alpha}\|_{L^1\to L^{1,\infty}} \le C_2(1+|\alpha|)^{d/2}\log(1+|\alpha|). \tag{2}$$

The classical Hörmander multiplier theorem (see [13]) states that a multiplier operator T_m on $\mathbf{R}^d$ with multiplier m satisfies

$$\|T_m\|_{L^1\to L^{1,\infty}} \ \le\ C_s \sup_{t>0} \|\eta(\cdot)m(t\cdot)\|_{H_s} \ \le\ A \tag{3}$$

for any $s > d/2$ and any $\eta \in C_c^\infty(\mathbf{R}_+)$ not identically zero. Here H_s is the Sobolev space of order s on $\mathbf{R}^d$. Since the Sobolev norm in (3) behaves like $(1+|\alpha|)^s$ for the multiplier

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$m(x) = |x|^{i\alpha}$ of $(-\Delta)^{i\alpha}$, (2) shows that the exponent $d/2$ in Hörmander's theorem is sharp. Furthermore, if (3) is satisfied with $A < \infty$, then the distribution $K = \hat{m}$ agrees with a locally integrable function away from the origin which satisfies

$$I(B) \;=\; \sup_{y\neq 0} \int_{|x|\geq B|y|} |K(x-y) - K(x)|\, dx \;\leq\; A \tag{4}$$

for $B \geq 2$ and Hörmander's theorem actually shows that the weak type (1,1) norm of T_m is bounded by $I(B) + \|m\|_{L^\infty}^2 + B^d$. One can easily compute that for the convolution kernel K of $(-\Delta)^{i\alpha}$, the integral $I(B)$ is bounded above and below by $(1+|\alpha|)^{d/2}\log(1+|\alpha|/B)$. Hence Hörmander's theorem gives the upper bound in (2). The lower bound is a simple consequence of the explicit formula for the kernel K of $(-\Delta)^{i\alpha}$. See for example, [21] pp. 51-52.

The main observation of this paper is to note that there is a slight improvement of the bound $I(B) + \|m\|_{L^\infty}^2 + B^d$ to $I(B) + (\|m\|_{L^\infty}^2 B^d)^{1/2}$. This can be achieved either by using C. Fefferman's ideas in [11] of exploiting more information of L^2 bounds or by varying the level of the Calderón - Zygmund decomposition and optimising. Hence we will be able to remove the *log* term in (2). We will show that this more precise estimate holds for a general class of operators.

Theorem 1. *Suppose that L is defined by (1). Then*

$$C_1(1+|\alpha|)^{d/2} \leq \|L^{i\alpha}\|_{L^1\to L^{1,\infty}} \leq C_2(1+|\alpha|)^{d/2} \tag{5}$$

for all $\alpha \in \mathbf{R}$.

Proof of the lower bound. We begin with some known estimates for the kernel $p_t(x,y)$ of the heat operator e^{-tL} associated to L. Firstly, this kernel satisfies Gaussian bounds

$$C_1\,\frac{1}{t^{d/2}}\,\mathrm{e}^{-b_1\rho^2(x,y)/t} \;\leq\; p_t(x,y) \;\leq\; C_2\,\frac{1}{t^{d/2}}\,\mathrm{e}^{-b_2\rho^2(x,y)/t} \tag{6}$$

(see [8]) for some positive constants C_1, C_2, b_1 and b_2 and where $\rho(x,y)$ denotes the geodesic distance between x and y given by the Riemannian metric $(a_{i,j})$. In this setting of uniform ellipticity, $\kappa|x-y| \leq \rho(x,y) \leq \tau|x-y|$. Secondly, from the construction of a parametrix for the heat equation with respect to L (either via Hadamard's construction, see §17.4 of [14] , or using pseudodifferential operator techniques, see chapter 7, §13 of [23]), we have for each $y \in \mathbf{R}^d$, a ball $B(y,r)$ such that for $x \in B(y,r)$ and $0 < t < 1$,

$$|p_t(x,y) - (\det a_{ij}(y))^{-1/2}(4\pi t)^{-d/2}\mathrm{e}^{-\rho^2(x,y)/4t}| \leq Ct^{1/2}t^{-d/2}. \tag{7}$$

Here we are using the fact that p_t is symmetric, $p_t(x,y) = p_t(y,x)$. ¿From (6) and (7), we have for $x \in B(y,r)$ the bound

$$|p_t(x,y) - (\det a_{ij}(y))^{-1/2}(4\pi t)^{-d/2}\mathrm{e}^{-\rho^2(x,y)/4t}| \leq Ct^{1/4}t^{-d/2}\exp\left(-b'\rho(x,y)^2/t\right)$$

which translates into a bound for the kernel $K_{L^{i\alpha}}$ of $L^{i\alpha}$ since the functional calculus for L gives us the relationship

$$L^{i\alpha} = \Gamma(-i\alpha)^{-1}\int_0^\infty t^{-i\alpha-1}e^{-tL}dt$$

for $\alpha \neq 0$. Thus for $x \in B(y,r)$,

$$|K_{L^{i\alpha}}(x,y) - (\det a_{ij}(y))^{-1/2}4^{i\alpha}\pi^{-d/2}\gamma(\alpha)\rho(x,y)^{-d-i2\alpha}| \leq C|\Gamma(-i\alpha)|^{-1}\rho(x,y)^{-d+1/2} \tag{8}$$

where $\gamma(\alpha)=\Gamma(i\alpha+d/2)/\Gamma(-i\alpha)$. Using (8) with $y=0$ we obtain for λ large enough

$$\mu(\{|K_{L^{i\alpha}}(x,0)|\geq\lambda\})\geq\mu(\{C_1|\gamma(\alpha)|\rho^{-d}(x,0)\geq 2\lambda\})-\mu(\{C_2|\Gamma(-i\alpha)|\rho^{-d+\frac{1}{2}}(x,0)\geq\lambda\})$$
$$=\mu(B(0,(2C_1|\gamma(\alpha)|/\lambda)^{1/d}))-\mu(B(0,(C_2|\Gamma(-i\alpha)|/\lambda)^{1/(d-1/2)}))\geq C'|\gamma(\alpha)|/\lambda.$$

Here μ is Lebesgue measure and the sets above have the further restriction that $x\in B(0,r)$. Since $K_{L^{i\alpha}}$ is smooth away from the diagonal, we see that $L^{i\alpha}\phi_\delta(x)$ tends to $K_{L^{i\alpha}}(x,0)$ as $\delta\to 0$ for any $x\neq 0$ and any approximation of the identity $\{\phi_\delta\}$. Hence the above estimate shows that the weak type (1,1) norm of $L^{i\alpha}$ is bounded below by $|\gamma(\alpha)|=|\Gamma(i\alpha+d/2)/\Gamma(-i\alpha)|\sim(1+|\alpha|)^{\frac{d}{2}}$ (see [10]).

The upper bound in Theorem 1 holds in a much more general setting which we describe now. Assume that (X,μ,ρ) is a space with measure μ and metric ρ. If $\|P\|_{L^2\to L^\infty}<\infty$ then we can define the kernel K_P of the operator P by the formula

$$\langle P(\psi),\phi\rangle=\int P(\psi)\overline{\phi}d\mu=\int K_P(x,y)\psi(x)\overline{\phi(y)}d\mu(x)d\mu(y).$$

Note that $\sup_x\|K_P(x,\cdot)\|_{L^2}=\|P\|_{L^2\to L^\infty}$. Next, we say that

$$\text{supp } K_P\subset\{(x,y)\in X^2:\rho(x,y)\leq r\} \tag{9}$$

if $\langle P(\psi),\phi\rangle=0$ for every $\phi,\psi\in L^2$ and every $r_1+r_2+r<\rho(x',y')$ such that $\psi(x)=0$ for $\rho(x,x')>r_1$ and $\phi(x)=0$ for $\rho(x,y')>r_2$. This definition (9) makes sense even if $\|P\|_{L^2\to L^\infty}=\infty$. Now if L is a self-adjoint positive definite operator acting on $L^2(\mu)$ then we say that it satisfies the finite speed propagation property of the corresponding wave equation if

$$\text{supp } K_{C_t(\sqrt{L})}\subset\{(x,y)\in X^2:\rho(x,y)\leq t\}, \tag{10}$$

where $C_t(\sqrt{L})=\int\cos(t\sqrt{\lambda})\,dE(\lambda)$.

Theorem 2. *Suppose that L satisfies (10). Next assume that*

$$\|\exp(-tL)\|^2_{L^2\to L^\infty}\leq C_1V_{d,D}(t^{1/2})^{-1}\leq C\mu(B(x,t^{1/2}))^{-1}\leq C_2V_{d,D}(t^{1/2})^{-1} \tag{11}$$

for all $t>0$ and $x\in X$, where $B(x,t)$ is a ball with radius t centred at x and

$$V_{d,D}(t)=\begin{cases} t^d & \text{for } t\leq 1\\ t^D & \text{for } t>1\end{cases}$$

for $d,D\geq 0$. Then

$$\|L^{i\alpha}\|_{L^1\to L^{1,\infty}}\leq C_2(1+|\alpha|)^{\max(d,D)/2}$$

for all $\alpha\in\mathbf{R}$.

We remark that (10) and (11) are equivalent to having Gaussian upper bounds on the heat kernel and the associated volume growth on balls. See [18]. Furthermore, the upper bound in Theorem 1 follows from Theorem 2. Indeed, if $X=\mathbf{R}^d$, $\rho(x,y)=\tau|x-y|$ and μ is Lebesgue measure then it is well known (see e.g. [8] and [19]) that (11) and (10) hold. We are going to prove Theorem 2 only in the case $d=D$. The argument for the other cases is similar.

2. Preliminaries. The following lemma is a very simple but useful consequence of (10).

Lemma 1. *Assume that L satisfies (10) and that $\hat{F}$ is a Fourier transform of an even bounded Borel function F with* supp $\hat{F}\subset[-r,r]$. *Then*

$$\text{supp } K_{F(\sqrt{L})}\subset\{(x,y)\in X^2:\rho(x,y)\leq r\}.$$

Proof. If F is an even function, then by the Fourier inversion formula,

$$F(\sqrt{L}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{F}(t)C_t(\sqrt{L})\, dt.$$

But since supp $\hat{F} \subset [-r, r]$,

$$F(\sqrt{L}) = \frac{1}{2\pi}\int_{-r}^{r} \hat{F}(t)C_t(\sqrt{L})\, dt$$

and Lemma 1 follows from (10).

Lemma 2. *Let $\phi \in C_c^\infty(\mathbf{R})$ be even, $\phi \geq 0$, $\|\phi\|_{L^1} = 1$, $\mathrm{supp}(\phi) \subset [-1, 1]$, and set $\phi_r(x) = 1/r\, \phi(x/r)$ for $r > 0$. Let Φ denote the Fourier transform of ϕ and Φ^r denote the Fourier transform of ϕ_r. If (11) and (10) hold, then the kernel $K_{\Phi^r(\sqrt{L})}$ of the self-adjoint operator $\Phi^r(\sqrt{L})$ satisfies*

$$\text{supp } K_{\Phi^r(\sqrt{L})} \subset \{(x, y) \in X^2; \rho(x, y) \leq r\} \tag{12}$$

and

$$|K_{\Phi^r(\sqrt{L})}(x, y)| \leq C\, r^{-d} \tag{13}$$

for all $r > 0$ and $x, y \in X$.

Proof. (12) follows from Lemma 1. For any $m \in \mathbf{N}$ and $r > 0$, we have the relationship

$$(I + rL)^{-m} \;=\; \frac{1}{m!}\int_0^\infty \mathrm{e}^{-rtL}\mathrm{e}^{-t}t^{m-1}\, dt$$

and so when $m > d/4$, (11) implies

$$\|(I + rL)^{-m}\|_{L^2\to L^\infty} \;\leq\; \frac{1}{m!}\int_0^\infty \|\exp(-rtL)\|_{L^2\to L^\infty} e^{-t}t^{m-1}dt \;\leq\; C_1\, r^{-d/4} \tag{14}$$

for all $r > 0$. Now $\|(I + r^2L)^{-m}\|_{L^1\to L^2} = \|(I + r^2L)^{-m}\|_{L^2\to L^\infty}$ and so

$$\|\Phi^r(\sqrt{L})\|_{L^1\to L^\infty} \;\leq\; \|(I + r^2L)^{2m}\Phi^r(\sqrt{L})\|_{L^2\to L^2}\, \|(I + r^2L)^{-m}\|^2_{L^2\to L^\infty}.$$

The L^2 operator norm of the first term is equal to the L^∞ norm of the function $(1 + r^2|t|)^{2m}\Phi(r\sqrt{|t|})$ which is uniformly bounded in $r > 0$ and so (13) follows by (14).

Next we recall the Calderón-Zygmund decomposition in the general setting of spaces of homogeneous type (see e.g. [3] or [22]) .

Lemma 3. *There exists C such that, given $f \in L^1(X, \mu)$ and $\lambda > 0$, one can decompose f as*

$$f = g + b = g + \sum b_i$$

so that

1. *$|g(x)| \leq C\lambda$, a.e. x and $\|g\|_{L^1} \leq C\|f\|_{L^1}$.*
2. *There exists a sequence of balls $B_i = B(x_i, r_i)$ such that the support of each b_i is contained in B_i and*

$$\int |b_i(x)|d\mu(x) \;\leq\; C\lambda\mu(B_i).$$

3. *$\sum \mu(B_i) \leq C\,\frac{1}{\lambda}\int |f(x)|\, d\mu(x)$.*

(4) *There exists $k \in \mathbf{N}$ such that each point of X is contained in at most k of the balls $B(x_i, 2r_i)$.*

We are now in a position to prove Theorem 2.

3. Proof of Theorem 2. The proof follows closely the line of argument in [1] (which of course generalises to this setting). We are out to prove

$$\lambda\,\mu(\{x \in X\,:\,|L^{i\alpha} f(x)| \geq \lambda\,\}) \;\leq\; C(1+|\alpha|)^{\frac{d}{2}}\,\|f\|_{L^1}.$$

As usual we start by decomposing f into $g + \sum b_i$ at the level of λ according to Lemma 3. We will follow the idea of C. Fefferman [11] of using more information of the L^2 operator norm (in our case, $\|L^{i\alpha}\|_{L^2\to L^2} = 1$) by smoothing out the bad functions b_i at a scale smaller than the size of it's support and considering this part of the good function where L^2 estimates can be used (see also [4]). In our case for each b_i, consider $\Phi^{s_i}(\sqrt{L})b_i$ where $s_i = \theta r_i$, $\theta = (1+|\alpha|)^{-\frac{1}{2}}$, and let $G = g + \sum \Phi^{s_i}(\sqrt{L})b_i$ be the modified good function. Hence $f = G + B$ where $B = \sum (I - \Phi^{s_i}(\sqrt{L}))b_i$ and we write

$$\lambda\,\mu(\{|L^{i\alpha} f(x)| \geq \lambda\}) \;\leq\; \lambda\,\mu(\{|L^{i\alpha} G(x)| \geq \lambda/2\}) \;+\; \lambda\,\mu(\{|L^{i\alpha} B(x)| \geq \lambda/2\}). \tag{15}$$

The first term is less than $4/\lambda\,\|L^{i\alpha} G\|_{L^2}^2 \;\leq\; 4/\lambda\,\|G\|_{L^2}^2$. However, according to Lemma 2,

$$|\Phi^{s_i}(\sqrt{L})b_i(x)| \;\leq\; \int |K_{\Phi^{s_i}(\sqrt{L})}(x,y) b_i(y)|\,d\mu(y) \;\leq\; C\,(\theta r_i)^{-d}\|b_i\|_{L^1}\mathbb{1}_{B(x_i,2r_i)}$$

and therefore by Lemma 3, $|G(x)| \;\leq\; C\theta^{-d}\lambda$ for *a.e.*, x. Using Lemma 2 again which shows that the $L^p \to L^p$ operator norms of $\Phi^r(\sqrt{L})$ are uniformily bounded in $r > 0$, we also have that $\|G\|_{L^1} \leq \|g\|_{L^1} + C\sum \|\Phi^{s_i}(\sqrt{L})b_i\|_{L^1} \leq \|g\|_{L^1} + C\sum\|b_i\|_{L^1} \leq C\|f\|_{L^1}$. Therefore the first term in (15) is bounded by $(1+|\alpha|)^{\frac{d}{2}}\,\|f\|_{L^1}$.

Since $\mu(\cup B(x_i, \theta^{-1}r_i)) \leq C\theta^{-d}\sum \mu(B_i) \leq C(1+|\alpha|)^{\frac{d}{2}}\|f\|_{L^1}/\lambda$, then to bound the second term in (15), it suffices to show

$$\int\limits_{x\notin \cup B_i^*} |L^{i\alpha} B(x)|\,d\mu(x) \;\leq\; C(1+|\alpha|)^{\frac{d}{2}}\|f\|_{L^1} \tag{16}$$

where $B_i^* = B(x_i, \theta^{-1}r_i)$. Let $H^\alpha(t) = |t|^{2i\alpha}$ so that $L^{i\alpha}B(x) = \sum H^\alpha(1-\Phi^{s_i})(\sqrt{L})b_i(x)$ and therefore the left side of (16) is less than

$$\sum_i \int\limits_{x\notin\cup_j B_j^*} \Big|\int K_{H^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)b_i(y)\,d\mu(y)\Big|\,d\mu(x)$$

$$\leq\; \sum_i \int |b_i(y)| \int\limits_{x\notin B_i^*} |K_{H^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\,d\mu(x)\,d\mu(y).$$

Since $F(L)^* = \overline{F}(L)$, we may interchange the roles of x and y, and so (16) will follow from Lemma 3 once we establish

$$\sup_{x,i} \int\limits_{\rho(x,y)\geq\theta^{-1}r_i} |K_{H^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\,d\mu(y) \;\leq\; C\,(1+|\alpha|)^{\frac{d}{2}}. \tag{17}$$

We now fix $x \in X$ and i. Let $\eta \in C_c^\infty(\mathbf{R})$ be an even function supported in $\{t \in \mathbf{R} : 1 \leq |t| \leq 4\}$ such that

$$\sum_{n=-\infty}^{\infty} \eta(2^{-n}t) = 1 \text{ for all } t \neq 0.$$

We put $H_n^\alpha(t) = \eta(2^{-n}t)H^\alpha(t)$ so that

$$H^\alpha(1-\Phi^{s_i})(\sqrt{L}) = \sum_n H_n^\alpha(1-\Phi^{s_i})(\sqrt{L}).$$

Thus

$$\int_{y\notin B_i^*} |K_{H^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\, d\mu(y) \ \leq\ \sum_n \int_{y\notin B_i^*} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L}))}(x,y)|\, d\mu(y) \tag{18}$$

and we will estimate each term in the sum on the right side in terms of n and i, uniformly in $x \in X$.

Let $k_o = [d/2]+1$ so that

$$\int_{y\notin B_i^*} (1+2^n\rho(x,y))^{-2k_o} d\mu(y) \leq C \int_{\theta^{-1}r_i}^{\infty} (1+2^n r)^{-2k_0} r^{d-1} dr \leq C 2^{-2nk_o}(\theta^{-1}r_i)^{d-2k_o}$$

and therefore by the Cauchy-Schwarz inequality,

$$\int_{y\notin B_i^*} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L}))}(x,y)|\, d\mu(y) \tag{19}$$
$$\leq\ C\, 2^{-nk_o}(\theta^{-1}r_i)^{\frac{d}{2}-k_o}\Big(\int_{\rho(x,y)\geq\theta^{-1}r_i} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|^2(1+2^n\rho(x,y))^{2k_o} d\mu(y)\Big)^{1/2}.$$

We break up the integral on the right side of (19) where $2^n\rho(x,y)$ is roughly constant and consider

$$\sum_{2^j\geq 2^n r_i\theta^{-1}} 2^{2jk_o} \int_{2^{j-1-n}<\rho(x,y)\leq 2^{j-n}} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|^2\, d\mu(y). \tag{20}$$

Fix a nonnegative even $\varphi \in C_c^\infty(\mathbf{R})$ such that $\varphi = 1$ on $[-1/4, 1/4]$ and $\varphi = 0$ on $\mathbf{R}\setminus[-1/2,1/2]$. Then the Fourier transforms of $H_n^\alpha(1-\Phi^{s_i})$ and $H_n^\alpha(1-\Phi^{s_i})*(\delta-\hat{\varphi}_{2^{n-j}})$ agree on $\{\xi : |\xi| \geq 2^{j-1-n}\}$ and so by Lemma 1, the kernels of $H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})$ and $H_n^\alpha(1-\Phi^{s_i})*(\delta-\hat{\varphi}_{2^{n-j}})(\sqrt{L})$ agree on the set $\{(x,y)\in X^2 : \rho(x,y) \geq 2^{j-1-n}\}$. Here δ denotes the Dirac mass at 0. For each j, the integrals in (20) satisfy the bound

$$\int_{2^{j-1-n}<\rho(x,y)\leq 2^{j-n}} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|^2\, d\mu(y) \ \leq\ \|K_{F_{n,j}^\alpha(\sqrt{L})}\|^2_{L^2\to L^\infty}$$

where we are defining $F_{n,j}^\alpha(t) = H_n^\alpha(1-\Phi^{s_i})*(\delta-\hat{\varphi}_{2^{n-j}})(t)$. So by (14), the right side of this inequality is bounded by $\|(I+2^{-2n}L)^m F_{n,j}^\alpha(\sqrt{L})\|^2_{L^2\to L^2}\, 2^{nd}$ as long as $m > d/4$. Everything then comes down to estimating the L^∞ norm of $(1+2^{-2n}t^2)^m F_{n,j}^\alpha(t)$. We make the following claim.

Claim: For each j, n and $m > d/4$,

$$(1+2^{-2n}t^2)^m|F_{n,j}^\alpha(t)| \ \leq\ C_m|\alpha|^{k_o}2^{-jk_o}\min(1,(2^n r_i\theta)^2)\,\min(1,|\alpha|2^{-j})$$

uniformly in $t \in \mathbf{R}$.

The claim shows that

$$\|K_{F^\alpha_{n,j}}(\sqrt{L})\|_{L^2\to L^\infty} \;\le\; C\,|\alpha|^{k_o}2^{-jk_o}2^{\frac{nd}{2}}\;\min(1,(2^n r_i\theta)^2)\;\min(1,|\alpha|2^{-j})$$

and hence the sum in (20) is bounded by

$$|\alpha|^{2k_o}2^{nd}\min{}^2(1,(2^n r_i\theta)^2)\sum_{2^j\ge 2^n r_i\theta^{-1}}\min{}^2(1,|\alpha|2^{-j}) \le |\alpha|^{2k_o}2^{nd}\min{}^2(1,(2^n r_i\theta)^2)\log(2+\frac{|\alpha|}{2^n r_i\theta^{-1}}).$$

Recall that θ and α are related so that $\theta|\alpha| = |\alpha|/(1+|\alpha|)^{\frac12}\le\theta^{-1}$. Plugging this into (19) gives

$$\int_{y\notin B_i^*}|K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)d\mu(y)\;\le\;\theta^{-d}(2^n r_i\theta)^{\frac d2-k_o}\;\min(1,(2^n r_i\theta)^2)\log(2+\frac{1}{2^n r_i\theta})$$

and this makes the sum in (18) bounded by $\theta^{-d}=(1+|\alpha|)^{\frac d2}$, proving (17) and hence Theorem 2.

Proof of the Claim. If $G_n(t)=H_n^\alpha(t)(1-\Phi^{s_i}(t))$, then $F^\alpha_{n,j}(t)=2^{(n-j)k_o}G_n^{(k_o)}*\hat\psi_{2^{n-j}}(t)$ where $\psi(\xi)=\xi^{-k_o}(1-\varphi(\xi))$ (and so $\hat\psi$ is continuous, rapidly decreasing and has vanishing moments, $\int t^\ell\hat\psi(t)dt \;=\; 0,\;\; \ell=0,1,2,...$). Hence

$$\begin{aligned} F^\alpha_{n,j}(t) &= 2^{(n-j)k_o}\int_{\mathbf{R}}\big[G_n^{(k_o)}(t-s)-G_n^{(k_o)}(t)\big]\hat\psi_{2^{n-j}}(s)\,ds\\ &=\;2^{(n-j)k_o}\int_{\mathbf{R}}\big[G_n^{(k_o)}(t-2^{n-j}s)-G_n^{(k_o)}(t)\big]\hat\psi(s)\,ds.\end{aligned}$$

However $G_n(t)=\eta(2^{-n}t)|t|^{2i\alpha}(1-\Phi(s_it))$ and thereby each time we take a derivative, we gain a factor of 2^{-n}. $G_n^{(k_o)}(t)$ is thus a finite sum of terms of the form $\alpha^p2^{-nk_o}\tilde\eta(2^{-n}t)|t|^{2i\alpha}\Psi(s_it)$ where $\tilde\eta\in C_c^\infty(\mathbf{R})$, $\mathrm{supp}(\tilde\eta)\subset\mathrm{supp}(\eta)$ and Ψ is a Schwartz function which is $0(t^2)$ as $t\to0$ (note that $\Phi'(0)=\int x\phi(x)dx=0$ since ϕ is even). The worst power p is k_o which occurs when all derivatives land on the factor $|t|^{2i\alpha}$.

Without loss of generality, let us suppose that $G^{(k_o)}(t)\;=\;\alpha^{k_o}2^{-nk_o}\eta(2^{-n}t)|t|^{2i\alpha}\Psi(s_it)$. ¿From the above integral representation of $F^\alpha_{n,j}(t)$, we see that the main contribution to $(1+2^{-2n}t^2)^m|F^\alpha_{n,j}(t)|$ occurs when $|t|\sim2^n$ and in this case,

$$|F^\alpha_{n,j}(t)|\le C|\alpha|^{k_o}2^{(n-j)k_o}2^{-nk_o}\min(1,(s_i2^n)^2)\le C|\alpha|^{k_o}2^{-jk_o}\min(1,(2^nr_i\theta)^2).$$

However we may write

$$F^\alpha_{n,j}(t)=-2^{(n-j)k_o}2^{n-j}\int_0^1\int_{\mathbf{R}}G_n^{(k_o+1)}(t-\sigma2^{j-n}s)s\hat\psi(s)\,ds\,d\sigma$$

and therefore we also have

$$|F^\alpha_{n,j}(t)|\le C|\alpha|^{k_o+1}2^{(n-j)k_o}2^{n-j}2^{-n(k_o+1)}\min(1,(s_i2^n)^2)\le C|\alpha|^{k_o}2^{-jk_o}|\alpha|2^{-j}\min(1,(2^nr_i\theta)^2),$$

establishing the claim.

Remarks. Theorem 1 holds also for Laplace-Beltrami operators on compact manifolds of dimension d. The proof is essentially the same as the proof of Theorem 1.

The hypotheses of Theorem 2 are satisfied for Laplace operators on Lie groups of polynomial growth. However, if L is a sub-Laplacian on the three dimensional Heisenberg group, then $d = 4$ but

$$C_1(1+|\alpha|)^{3/2} \le \|L^{i\alpha}\|_{L^1\to L^{1,\infty}} \le C_\epsilon(1+|\alpha|)^{3/2+\epsilon}.$$

See [16]. (See also [12]). The same estimates hold for a sub-Laplacian on SU(2) for which $d = 4$ and $D = 0$ (see [7]). Thus there are situations where the upper bound is better than the one given by Theorem 2 and where the lower-bound in Theorem 1 is false. For general groups of polynomial growth Theorem 2 gives the best known estimates, however as the above examples show, these bounds are not always best possible.

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