SHARP POINTWISE ESTIMATES ON HEAT KERNELS

by
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1. Introduction. Let M be a connected and complete Riemannian manifold. By ρ we denote the Riemannian distance on M and by $-L$ the closure of the Laplace-Beltrami operator on $C_c^\infty(M)$ in $L^2(dx)$, where dx is the riemanian measure on M.

For every bounded Borel function $F:\ [0.\infty) \mapsto \mathbb{C}$, we define an operator $F(L):\ L^2(dx) \mapsto L^2(dx)$ by the formula

$$F(L) = \int_0^\infty F(\lambda)\, dE(\lambda),$$

where $E(\lambda)$ is the spectral decomposition of the operator L. We denote by $K_{F(L)}$ the kernel of the operator $F(L)$ i.e. the distribution

$$[F(L)(\psi)](x) = \int_M \psi(y) K_{F(L)}(x,y)\, dy,$$

where $\psi \in C_c^\infty(M)$ and $x, y \in M$. Next for $H_t(\lambda) = \exp(-t\lambda^2)$, we put

$$p_t(x,y) = K_{H_t(\sqrt{L})}(x,y). \tag{1}$$

The function $p_t(x,y)$ is called the heat kernel. Since L is symmetric with respect to dx, for any bounded Borel function F we have

$$K_{F(L)}(x,y) = K_{F(L)}(y,x) \tag{2}$$

and

$$p_{2t}(x,x) = \|p_t(x,\cdot)\|^2_{L^2(dx)}.$$

On the present paper we consider the manifolds which satisfy the following assamption, which seems to appear first in Davies and Pang [3].

$$\sup_{x\in M} p_t(x,x) \le \begin{cases} Ct^{-d/2} & \text{if } t \le 1 \\ Ct^{-D/2} & \text{if } t > 1. \end{cases} \tag{3}$$

Under this assumption Davies and Pang proved (cf. also [1]),that if $d \le D$ then

$$p_t(x,y) \le C' \min\Big\{t^{-d/2}(1+\rho(x,y)/\sqrt{t})^d, \tag{4}$$
$$t^{-D/2}(1+\rho(x,y)/\sqrt{t})^D\Big\} \exp\left(\frac{-\rho(x,y)^2}{4t}\right)$$

and if $d > D$

(5) $$p_t(x,y) \leq C' \max\left\{t^{-d/2}(1+\rho(x,y)/\sqrt{t})^d,\right.$$
$$\left. t^{-D/2}(1+\rho(x,y)/\sqrt{t})^D\right\} \exp\left(\frac{-\rho(x,y)^2}{4t}\right).$$

A slightly weaker result of this type had been earlier obtained by Varopoulos in [11]. The aim of this paper is to improve the esitmates above. In view of [7], they seem to be the best possible. We use the connection of the heat and the vave equation which has long history, notably [6].

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2. Main Theorem. We are going to prove the following theorem

Theorem 1 *Let p_t be the heat kernel defined by (1). Assume that p_t satisfies condition (3). Then, there exists a constant $C_{d,D}$ depending only on d, D such that: if $d \leq D$, then*

(6) $$p_t(x,y) \leq CC_{d,D}\min\left\{t^{-d/2}(1+\rho(x,y)/\sqrt{t})^{d-1},\right.$$
$$\left. t^{-D/2}(1+\rho(x,y)/\sqrt{t})^{D-1}\right\} \exp\left(\frac{-\rho(x,y)^2}{4t}\right)$$

and if $d > D$, then

(7) $$p_t(x,y) \leq CC_{d,D}\max\left\{t^{-d/2}(1+\rho(x,y)/\sqrt{t})^{d-1},\right.$$
$$\left. t^{-D/2}(1+\rho(x,y)/\sqrt{t})^{D-1}\right\} \exp\left(\frac{-\rho(x,y)^2}{4t}\right).$$

Proof. First we note that the proof of Theorem 1 can be reduced to the following lemma

Lemma 1 *Let $d \geq D \geq 0$. Suppose there exist constants α_d and α_D such that*

(8) $$\sup_{x\in M} p_t(x,x) \leq \alpha_d t^{-d/2} + \alpha_D t^{-D/2}.$$

Then

$$p_1(x,y) \le C_{d,D}(\alpha_d(1+\rho(x,y))^{d-1} + \alpha_D(1+\rho(x,y))^{D-1})\exp(\frac{-\rho(x,y)^2}{4}). \tag{9}$$

The constant $C_{d,D}$ depends only on d and D, and does not depend on α_d and α_D.

First we are going to show that in fact Theorem 1 follows from Lemma 1.

Case $d \ge D$. For $s > 0$ we put $L_s = sL$ and let $p_{s,t}$ be the heat kernel corresponding to L_s. Then

$$p_{s,1} = p_s \quad \text{and} \quad \rho_s = \frac{\rho}{\sqrt{s}}, \tag{10}$$

where ρ_s is a Riemannian distance corresponding to the operator L_s. On the other hand, estimate (8) is true for $p_{s,t}$ with constants $\alpha_{s,d} = \alpha_d s^{-d/2}$ and $\alpha_{s,D} = \alpha_D s^{-D/2}$. Now, in virtue of Lemma 1,

$$\begin{aligned} p_s(x,y) = p_{s,1}(x,y) &\le C_{d,D}(\alpha_{s,d}(1+\rho_s(x,y))^{d-1} \\ &+\alpha_{s,D}(1+\rho_s(x,y))^{D-1})\exp(\frac{-\rho_s(x,y)^2}{4}) \\ &\le C_{d,D}(\alpha_d s^{-d/2}(1+\frac{\rho(x,y)}{\sqrt{s}})^{d-1} \\ &+\alpha_D s^{-D/2}(1+\frac{\rho(x,y)}{\sqrt{s}})^{D-1})\exp(\frac{-\rho(x,y)^2}{4s}) \\ \le 2C_{d,D}\max\{\alpha_d s^{-d/2}(1+\frac{\rho(x,y)}{\sqrt{s}})^{d-1}, &\,\alpha_D s^{-D/2}(1+\frac{\rho(x,y)}{\sqrt{s}})^{D-1}\} \\ &\exp(\frac{-\rho(x,y)^2}{4s}). \end{aligned}$$

Case $d \le D$. If $d \le D$ then, by (8), $\sup p_t(x,x) \le Ct^{-\gamma/2}$ for all $d \le \gamma \le D$ so, in virtue of what we have shown above we obtain

$$p_t(x,y) \le CC_{\gamma,\gamma}t^{-\gamma/2}(1+\frac{\rho(x,y)}{\sqrt{t}})^{d-1}e^{-\frac{x^2}{4}} \tag{11}$$

for all $d \leq \gamma \leq D$, so

$$p_t(x,y) \leq \min\{CC_{d,d}t^{-d/2}(1+\frac{\rho(x,y)}{\sqrt{t}})^{d-1}e^{-\frac{x^2}{4}},$$
$$CC_{D,D}t^{-D/2}(1+\frac{\rho(x,y)}{\sqrt{t}})^{d-1}e^{-\frac{x^2}{4}}\}.$$

This proves Theorem 1 in case $d \leq D$.

The proof of Lemma 1 is divided into 5 lemmas.

Lemma 2 *Assume that $\hat{F} \in C_c^\infty(I\!R)$ is a Fourier transform of an even function F and that* supp $\hat{F} \subset [-r,r]$, *then*

$$\text{supp } K_F(\sqrt{L}) \subset \{(x,y) \in M^2 : \rho(x,y) \leq r\}.$$

Proof. If we put $C_t = \cos(t\lambda)$, then we can state the property of the finite speed of propagation of the solution of the wave equation in the following way

Theorem 2 *If B_t is a ball of radius t in the metric corresponding to the operator L, then*

(12) $$\text{supp } K_{C_t(\sqrt{L})}(x,\cdot) \subset B_t(x).$$

For a proof see [6], or[10] ch. IV, or remark to Theorem 3. Next, if F is an even function, then by the Fourier inversion formula,

(13) $$F(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{F}(t)\cos(t\lambda)\, dt,$$

and in virtue of (13)

(14) $$F(\sqrt{L}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{F}(t)C_t(\sqrt{L})\, dt.$$

But since supp $\hat{F} \subset [-r,r]$, so

(15) $$F(\sqrt{L}) = \frac{1}{2\pi}\int_{-r}^{r} \hat{F}(t)C_t(\sqrt{L})\, dt$$

and Lemma 2 follows by (15) and Theorem 2.

Lemma 3 *For $s > 2$, we define the family of functions ϕ_s by the formula*

$$\phi_s(x) = \psi(s(|x| - s)),$$

where $\psi \in C_c^\infty$ and

$$\psi(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ 1 & \text{if } x \geq 0\ . \end{cases}$$

If we define the function F_s by

$$F_s(x) = \phi_s(x) \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right),$$

then for any natural number N and $s > 2$

$$|\hat{F}_s(\lambda)| \leq C_N \frac{s^{N-1}}{(s + |\lambda|)^N} e^{-\frac{s^2}{2}},$$

where C_N is a constant depending only on N.

Proof. Integration by parts N times yields

$$\begin{aligned} \int \phi_s(x) e^{-\frac{x^2}{2}} e^{-ix\lambda} dx &= \int \left(\frac{-1}{x + i\lambda} \phi_s(x)\right)\left(e^{-\frac{x^2}{2} - i\lambda x}\right)' \, dx \\ &= \int \left(\frac{1}{x + i\lambda} \phi_s(x)\right)' e^{-\frac{x^2}{2} - i\lambda x} dx \\ = \int & \underbrace{\left(\frac{1}{x + i\lambda}\left(\dots\left(\frac{1}{x + i\lambda} \phi_s(x)'\dots\right)'\right.\right.}_{N} e^{-\frac{x^2}{2} - i\lambda x} dx. \end{aligned}$$

$\underbrace{\left(\frac{1}{x + i\lambda}\left(\dots\left(\frac{1}{x + i\lambda} \phi_s(x)'\dots\right)'\right.\right.}_{N}$ is a linear combination of the following terms:

$$\begin{aligned} \text{(16)} \quad & \frac{1}{(x + i\lambda)^N} \phi_s(x)^{(N)} e^{-\frac{x^2}{2} - i\lambda x}, \frac{1}{(x + i\lambda)^{N+1}} \phi_s(x)^{(N-1)} e^{-\frac{x^2}{2} - i\lambda x}, \\ & \dots, \frac{1}{(x + i\lambda)^{2N-1}} \phi_s(x)' e^{-\frac{x^2}{2} - i\lambda x}, \frac{1}{(x + i\lambda)^{2N}} \phi_s(x) e^{-\frac{x^2}{2} - i\lambda x}. \end{aligned}$$

But

$$|\int \frac{1}{(x+i\lambda)^N}\phi_s(x)^{(N)}e^{-\frac{x^2}{2}-i\lambda x}| \tag{17}$$

$$\leq \frac{1}{(s-\frac{1}{s}+|\lambda|)^N}e^{-\frac{(s-\frac{1}{s})^2}{2}}\int_{s-\frac{1}{s}}^{s}|\phi_s(x)^{(N)}|$$

$$\leq \frac{1}{(s-\frac{1}{s}+|\lambda|)^N}e^{-\frac{(s-\frac{1}{s})^2}{2}}s^{N-1}\beta \leq c_1\frac{s^{N-1}}{(s+|\lambda|)^N}e^{-\frac{s^2}{2}},$$

where $\beta = \sup|\psi^{(N)}|$. Similarly

$$|\int \frac{1}{(x+i\lambda)^{N+k}}\phi_s(x)^{(N-k)}e^{-\frac{x^2}{2}-i\lambda x}| \leq c_k\frac{s^{N-1-k}}{(s+|\lambda|)^{N+k}}e^{-\frac{s^2}{2}} \tag{18}$$

and finally

$$|\int \frac{1}{(x+i\lambda)^{2N}}\phi_s(x)e^{-\frac{x^2}{2}-i\lambda x}| \tag{19}$$

$$\leq \frac{1}{(s-\frac{1}{s}+|\lambda|)^{2N}}2\int_{s-\frac{1}{s}}^{\infty}e^{-\frac{x^2}{2}} \leq c_N\frac{1}{(s+|\lambda|)^{2N}}e^{-\frac{s^2}{2}}. \tag{20}$$

Now in virtue of (17),(18) and (19), Lemma 3 follows.

Lemma 4 *If* $\sup_x p_t(x,x) \leq \alpha_d t^{-d/2} + \alpha_D t^{-D/2}$, *then*

$$\int_{\rho(x,y)\geq s} p_{\frac{1}{2}}(x,y)^2 dy \leq C'_{d,D}(\alpha_d(1+s)^{d-2} + \alpha_D(1+s)^{D-2})e^{-s^2}.$$

Proof. If $s < 2$, then we have

$$\int_{\rho(x,y)\geq s} p_{\frac{1}{2}}(x,y)^2 dy \leq \int_M p_{\frac{1}{2}}(x,y)^2 dy = p_1(x,x) \leq \alpha_d + \alpha_D, \tag{21}$$

so for $s < 2$, Lemma 4 is obvious. If $s \geq 2$, note that

$$p_{\frac{1}{2}}(x,y) = K_{\hat{F}_s(\sqrt{L})}(x,y) \text{ for } \rho(x,y) \geq s, \tag{22}$$

where $\hat{F}_s$ is the function defined in Lemma 3. Indeed,

$$(\hat{F}_s(\lambda) - e^{-\lambda^2/2})\hat{} = \sqrt{2\pi}(\phi_s(x)-1)e^{-\frac{x^2}{2}}$$

and in virtue of Lemma 2

$$\text{supp}\ (K_{\hat{F}_s(\sqrt{L})}(x,y) - p_{\frac{1}{2}}(x,y)) \subset \{(x,y) \in M^2 : \rho(x,y) \le s\}.$$

This proves (22). By (22) we obtain

$$\int_{\rho(x,y)\ge s} p_{\frac{1}{2}}(x,y)^2 dy = \int_{\rho(x,y)\ge s} |K_{\hat{F}_s(\sqrt{L})}(x,y)|^2 dy \le \int_M |K_{\hat{F}_s(\sqrt{L})}(x,y)|^2 dy.$$

So to prove Lemma 4 it is enough to show that

$$(23) \int_M |K_{\hat{F}_s(\sqrt{L})}(x,y)|^2 dy \le C'_{d,D}(\alpha_d(1+s)^{d-2}e^{-s^2} + \alpha_D(1+s)^{D-2}e^{-s^2}).$$

It is easy to check that for the positive measure μ_x defined by the formula

$$\int_0^\infty F(\lambda)\mu_x(\lambda) = \int_0^\infty F(\lambda)(e^{-\lambda^2})^{-2} 2\lambda\, d(E(\lambda^2)p_1(x,\cdot), p_1(x,\cdot)),$$

we have

$$\|F(\sqrt{L})(x,\cdot)\|^2_{L^2(dx)} = \int_0^\infty |F(\lambda)|^2\, d\mu_x(\lambda).$$

On the other hand, in virtue of our main assumptions,

$$\|p_t(x,\cdot)\|^2_{L^2(dx)} = p_{2t}(x,x) \le \alpha_d(2t)^{-d/2} + \alpha_D(2t)^{-D/2},$$

hence

$$(24) \qquad \mu_x([0,r)) \le e \int_0^r e^{\lambda^2 r^{-2}} d\mu_x(\lambda)$$
$$\le e \int_0^\infty e^{\lambda^2 r^{-2}} d\mu_x(\lambda) = e\|p_{\frac{1}{r^2}}(x,\cdot)\|_{L^2} \le e(\alpha_d 2^{-\frac{d}{2}} r^d + \alpha_D 2^{-\frac{D}{2}} r^D).$$

Using (24) it is not difficlult to show that there exist positive measures $\mu_{x,d}$ and $\mu_{x,D}$ and a constant c' such that $\mu_x = \mu_{x,d} + \mu_{x,D}$ and

$$\mu_{x,d}([0,t)) \le c'\alpha_d t^d, \qquad \mu_{x,D}([0,t)) \le c'\alpha_D t^D.$$

Now

$$(25) \qquad \int_M |K_{\hat{F}_s(\sqrt{L})}(x,y)|^2 dy = \int_0^\infty |\hat{F}_s(\lambda)|^2\, d\mu_x(\lambda)$$
$$= \int_0^\infty |\hat{F}_s(\lambda)|^2\, d\mu_{x,d}(\lambda) + \int_0^\infty |\hat{F}_s(\lambda)|^2\, d\mu_{x,D}(\lambda).$$

Next in virtue of Lemma 3

$$\int_0^\infty |\hat{F}_s(\lambda)|^2 \mu_{x,d}(\lambda) \le C_d \int_0^\infty \frac{s^{2(d-1)}}{(s+\lambda)^{2d}} e^{-s^2} \mu_{x,d}(\lambda) \tag{26}$$

$$= C_d \int_0^\infty \int_\lambda^\infty -(\frac{d}{dt}(\frac{s^{2(d-1)}}{(s+t)^{2d}}))e^{-s^2} \mu_{x,d}(\lambda)$$

$$= C_d \int_0^\infty -(\frac{d}{dt}(\frac{s^{2(d-1)}}{(s+t)^{2d}}))e^{-s^2} \int_0^t \mu_{x,d}(\lambda)\, dt$$

$$\le c' C_d \int_0^\infty -(\frac{d}{dt}(\frac{s^{2(d-1)}}{(s+t)^{2d}}))e^{-s^2} \alpha_d t^d\, dt$$

$$= 2dc' C_d \int_0^\infty \frac{s^{2(d-1)}}{(s+t)^{2d+1}} e^{-s^2} \alpha_d C_d t^d\, dt$$

$$= 2dc' C_d e^{-s^2} s^{d-2} \int_0^\infty \frac{1}{(1+\frac{t}{s})^{2d+1}} \alpha_d (\frac{t}{s})^d \frac{dt}{s} = C'_d \alpha_d s^{d-2} e^{-s^2}.$$

In the same way we prove that

$$\int_0^\infty |\hat{F}_s(\lambda)|^2 \mu_{x,D}(\lambda) \le C'_D \alpha_D s^{D-2} e^{-s^2}. \tag{27}$$

Thus in virtue of (25),(26) and (27), (23) is proved.

Lemma 5 *Under the assumptions of Lemma 1 there exists constant $c_{d,D}$ such that for $\tau \ge 0$ the following estimates are true*

$$\sup_x \int_M p_{\frac{1}{2}}(x,y)^2 e^{\tau\rho(x,y)}\, dy \le c_{d,D}(\alpha_d(1+\tau)^{d-1} + \alpha_D(1+\tau)^{D-1})e^{\frac{\tau^2}{4}}.$$

Proof. First we note that

$$\int_M p_{\frac{1}{2}}(x,y)^2 e^{\tau\rho(x,y)}\, dy = \int_M p_{\frac{1}{2}}(x,y)^2 (1 + \int_0^{\rho(x,y)} \tau e^{\tau s}\, ds)\, dy \tag{28}$$

$$= p_1(x,y) + \int_M p_{\frac{1}{2}}(x,y)^2 \int_0^{\rho(x,y)} \tau e^{\tau s}\, ds\, dy$$

$$\le \alpha_d + \alpha_D + \int_M p_{\frac{1}{2}}(x,y)^2 \int_0^{\rho(x,y)} \tau e^{\tau s}\, ds\, dy$$

By the Fubini Theorem,

$$\int_M p_{\frac{1}{2}}(x,y)^2 \int_0^{\rho(x,y)} \tau e^{\tau s}\, ds\, dy = \int_0^\infty \tau e^{\tau s} \int_{\rho(x,y)\ge s} p_{\frac{1}{2}}(x,y)^2\, dy\, ds. \tag{29}$$

In virtue of Lemma 4,

$$\int_0^\infty \tau e^{\tau s} \int_{\rho(x,y)\geq s} p_{\frac{1}{2}}(x,y)^2 \, dy \, ds \tag{30}$$
$$\leq C'_{d,D} \int_0^\infty \tau e^{\tau s}(\alpha_d(1+s)^{d-2}e^{-s^2} + \alpha_D(1+s)^{D-2}e^{-s^2}) \, ds.$$

But it is not difficult to check that there exists a constant c'_d such that

$$\int_0^\infty \tau e^{\tau s}(1+s)^{d-2}e^{-s^2} \, ds \leq c'_d(1+\tau)^{d-1}e^{\frac{\tau^2}{4}}.$$

By (30), this inequality proves Lemma 5.

Lemma 6 *(9) in Lemma 1 follows from Lemma 5.*

Proof. We note that

$$p_1(x,y)e^{\tau\rho(x,y)} = \int_M p_{\frac{1}{2}}(x,z)p_{\frac{1}{2}}(y,z)e^{\tau\rho(x,y)} \, dz \tag{31}$$
$$\leq \int_M p_{\frac{1}{2}}(x,z)p_{\frac{1}{2}}(y,z)e^{\tau\rho(x,z)+\tau\rho(z,y)} \, dz. \tag{32}$$

By the Schwarz inequality

$$\int_M p_{\frac{1}{2}}(x,z)p_{\frac{1}{2}}(y,z)e^{\tau\rho(x,z)+\rho(z,y)} \, dz \tag{33}$$
$$\leq (\int_M p_{\frac{1}{2}}(x,z)^2 e^{2\tau\rho(x,z)} \, dz)^{\frac{1}{2}} (\int_M p_{\frac{1}{2}}(y,z)^2 e^{2\tau\rho(y,z)} \, dz)^{\frac{1}{2}}.$$

So in virtue of Lemma 5

$$p_1(x,y) \leq C'_{d,D} e^{-\tau\rho(x,y)}(\alpha_d(1+2\tau)^{d-1} + \alpha_D(1+2\tau)^{D-1})e^{\tau^2}.$$

If we put $\tau = \rho(x,y)/2$ we obtain Lemma 6 and Lemma 1.

Remarks. 1. In [7] p.23 Molchanov has proved that if N is the north and S is the south pole of the d-dimensional sphere of radius R so that $\rho(N,S) = \pi R$, then

$$p_t(N,S) \sim t^{-d/2}(1+\rho(S,N)/\sqrt{t})^{d-1}\exp\left(-\frac{\rho(N,S)^2}{4t}\right) \quad \text{as } t \downarrow 0.$$

This shows that our estimates are sharp.

2. Theorem 2 is true also for sublaplacians on Lie groups (see [6], [8] or the remark to the Theorem 3). Hence Theorem 1 is also valid for such operators.

3. Finite speed propagation of the solution of the wave equation. We conclude the paper by showing that a fairly weak Gaussian estimates on the heat kernel implies finite speed propagation of the solution of the wave equation.

Theorem 3 *Let M be a space with a measure dx and a metric $\rho(x,y)$ and let L be a self adjoint, positive definite operator on $L^2(dx)$. Assume that*

$$\|H_t(\sqrt{L})\|_{L^2(dx)\mapsto L^\infty(dx)} \le \begin{cases} Ct^{-d/4} & \text{if } t \le 1 \\ Ct^{-D/4} & \text{if } t > 1, \end{cases}$$

where $H_t(\lambda) = e^{-t\lambda^2}$. Next assume that the following Gaussian estimates hold for kernels $p_t(x,y)$ of semigroup $H_t(\sqrt{L})$

$$|p_t(x,y)| \le \begin{cases} Ct^{-d/2}\exp(\frac{-b^2\rho(x,y)^2}{4t}) & \text{if } t \le 1 \\ Ct^{-D/2}\exp(\frac{-b^2\rho(x,y)^2}{4t}) & \text{if } t > 1. \end{cases} \tag{34}$$

Now, if $\phi, \psi \in L^2(dx)$ and $\phi(y) = 0$ for $y \notin B(\xi_1, x_1)$, $\psi(y) = 0$ for $y \notin B(\xi_2, x_2)$, then

$$< C_t(\sqrt{L})\phi, \psi >= 0, \tag{35}$$

for $|t| < b(\rho(x_1,x_2) - \xi_1 - \xi_2)$.

Proof. We prove Theorem 3 in the case when $d = D$. The proof in the other cases is similar. First we note that we can replace the metric ρ by the metric $\rho' = b\rho$ hence we can assume that $b = 1$. Next, by assumptions the operator L is self adjoint and positive definite so that $H_z(\sqrt{L})$ can be extended to an analytic semigroup for all z such that $\Re e\ z \le 0$. In [3] Davies proved the following Theorem

Theorem 4 *(Davies Theorem 3.4.8 [3] p.103) If $p_z(x_1,x_2)$ is the kernel of an analytic semigroup $H_z(L)$, which satisfies the assumptions of Theorem 3($b = 1$), then for any $\delta > 0$ there exists a constant c_δ such that*

$$|p_z(x,y)| \le c_\delta(\mathrm{Re}\ z)^{-d/2}\exp(-\mathrm{Re}\ \frac{\rho(x,y)^2}{(4+\delta)z}). \tag{36}$$

On the other hand note that by (14)

$$< H_s(\sqrt{L})\phi, \psi >= \int_0^\infty < C_r(\sqrt{L})\phi, \psi > \frac{1}{\sqrt{\pi s}} e^{-\frac{r^2}{4s}} dr,$$

for $\phi, \psi \in L^2(dx)$. Hence

$$(37) \sqrt{1/s} < H_{(4s)^{-1}}(\sqrt{L})\phi, \psi >= \int_0^\infty (\sqrt{\pi r})^{-1} < C_{\sqrt{r}}(\sqrt{L})\phi, \psi > e^{-sr} dr,$$

so the function $v(s) = \sqrt{1/s} < H_{(4s)^{-1}}(L)\phi, \psi >$ is a Fourier-Laplace transform of the function $w(r) = (\sqrt{\pi r})^{-1} < C_{\sqrt{r}}(\sqrt{L})\phi, \psi >$. In virtue of Theorem 4, for $\eta = \rho(x, y) - \xi_1 - \xi_2$ we obtain

$$| < H_{(4s)^{-1}}(\sqrt{L})\phi, \psi > | \leq C_\delta(\Re e s^{-1})^{-d/2} \exp(\frac{4}{4+\delta}\eta^2 \Re e s)$$

for $\Re e s^{-1} > 0$. By the Paley-Wiener Theorem (Theorem 7.4.3 [5])

$$(38) \qquad \text{supp } w(r) \subset [\eta^2 \frac{4}{4+\delta}, \infty).$$

for any $\delta > 0$, hence it is also true for $\delta = 0$. This proves Theorem 3.

Remark. Note that in virtue of Theorem 3 we can obtain alternative proof of Theorem 2 using the estimates (4) and (5). Since these estimates are also true for the heat kernels corresponding to sublaplacians on Lie groups (see [1]) we can use Theorem 3 to prove the finite speed propagation of the solution of the wave equation for such operators.

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