Riesz transforms and Lie groups of polynomial growth

A.F.M. ter Elst[1], Derek W. Robinson[2] and Adam Sikora[2]

Abstract

Let G be a Lie group of polynomial growth. We prove that the second-order Riesz transforms on $L_2(G\,;dg)$ are bounded if, and only if, the group is a local direct product of a compact group and a nilpotent group, in which case the transforms of all orders are bounded.

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Home institutions:

1. Department of Mathematics and Computing Science
   Eindhoven University of Technology
   P.O. Box 513
   5600 MB Eindhoven
   The Netherlands

2. Centre for Mathematics and its Applications
   School of Mathematical Sciences
   Australian National University
   Canberra, ACT 0200
   Australia

1 Introduction

The Riesz transforms $\partial_i \Delta^{-1/2}$ play an important role in classical harmonic analysis. These operators are bounded on $L_2(\mathbf{R}^d)$ by Fourier theory and on the spaces $L_p(\mathbf{R}^d)$, $p \in \langle 1, \infty \rangle$ by singular integration theory. All higher order transforms are automatically bounded because the partial differential operators commute, e.g., $\partial_i \partial_j \Delta^{-1} = (\partial_i \Delta^{-1/2})(\partial_j \Delta^{-1/2})$. The situation for the analogous transforms on a Lie group G is much more complicated. The transforms of all orders are bounded if G is compact [BER] (see also [Ste2], Chapter I.4) or nilpotent [NRS] [ERS] but it is also known that there are quite simple groups for which the second-order transforms are unbounded [GQS] [Ale1]. Alexopoulos [Ale1] has shown that the second-order transforms are unbounded for the covering group of the group of Euclidean motions in the plane. This example is somewhat surprising as this group only has polynomial growth. Our aim is to analyze this phenomenon in the context of groups with polynomial growth and demonstrate that it always occurs unless the group is the local direct product of a compact group and a nilpotent group.

The unboundedness of the Riesz transforms is directly related to the large time behaviour of the corresponding heat kernel. If the group has polynomial growth then the L_∞-norm of the heat kernel decreases like $V(t)^{-1/2}$ where $V(t)$ is the volume of the ball of radius t measured with respect to a canonical distance. Moreover, Saloff-Coste [Sal] has shown that the derivatives of the heat kernel have a similar asymptotic behaviour with an additional factor $t^{-1/2}$. Higher derivatives can also be bounded with an additional factor $t^{-1/2}$ for each derivative and an overall factor $e^{\omega t}$ with $\omega > 0$. The latter drastically changes the asymptotics. We establish that it is impossible to have $\omega = 0$ for all higher derivatives unless G is the local direct product of a compact and a nilpotent. To be more precise we must introduce some notation. In general we adopt the notation of [Rob] and [ElR2].

Let $a_1, \ldots, a_{d'}$ be an algebraic basis of the Lie algebra $\mathfrak{g}$ of the connected Lie group G and $A_1 = dL(a_1), \ldots, A_{d'} = dL(a_{d'})$ the corresponding representatives of left translations L on the spaces $L_p = L_p(G\,;dg)$. We use a multi-index notation. Let $J(d') = \bigcup_{n=0}^{\infty} \{1, \ldots, d'\}^n$. If $\alpha = (i_1, \ldots, i_n) \in J(d')$ set $A^\alpha = A_{i_1} \ldots A_{i_n}$ and $|\alpha| = n$. The subspace $\bigcap_{|\alpha|=n} D(A^\alpha)$ of L_p formed by the n-times differentiable functions is denoted by $L'_{p;n}$. Furthermore $(g,h) \mapsto d'(g\,;h)$ denotes the right invariant distance associated with the basis and $g \mapsto |g|' = d'(g\,;e)$ the modulus. Then $V(r)$ denotes the volume (Haar measure) of the ball $B'_r = \{g \in G \,:\, |g|' < r\}$. We assume throughout that G has polynomial growth, i.e., one has bounds

$$c^{-1}\, r^D \leq V(r) \leq c\, r^D$$

for some integer $D \geq 1$ and all $r \geq 1$. These bounds automatically imply that G is unimodular. Note that as $D \geq 1$ compact groups are excluded from our considerations.

Next let $H = -\sum_{i=1}^{d'} A_i^2$ denote the sublaplacian associated with the basis. Then H is positive, self-adjoint, on L_2 and since we have excluded compact groups the inverse H^{-1} is a densely defined and self-adjoint operator. It follows readily that

$$\|H^{1/2}\varphi\|_2^2 = \sum_{i=1}^{d'} \|A_i\varphi\|_2^2 \tag{1}$$

for all $\varphi \in D(H^{1/2}) = L'_{2;1}$, i.e., the first-order Riesz transforms $A_i H^{-1/2}$ are bounded for all $i \in \{1, \ldots, d'\}$. It is a much deeper result that $D(H^{n/2}) = L'_{2;n}$ for all $n \in \mathbf{N}$ (see

[ElR1]). The operator H generates a self-adjoint contraction semigroup S with a strictly positive integral kernel K. Moreover, for each $\varepsilon > 0$ there is a $c_\varepsilon > 0$ such that the Gaussian bounds

$$0 < K_t(g) \le c_\varepsilon \, V(t)^{-1/2} e^{-(|g|')^2(4(1+\varepsilon)t)^{-1}} \tag{2}$$

and

$$|(A_i K_t)(g)| \le c_\varepsilon \, t^{-1/2} V(t)^{-1/2} e^{-(|g|')^2(4(1+\varepsilon)t)^{-1}} \tag{3}$$

are valid for all $i \in \{1, \ldots, d'\}$, $g \in G$ and $t > 0$. (See, for example, [Rob], Corollary IV.4.19 and Proposition IV.4.21.) The advantage of these bounds is that they incorporate the behaviour anticipated for large t on groups of polynomial growth. We will show that a similar asymptotic behaviour for all the second derivatives of the kernel is both necessary and sufficient for the boundedness of the Riesz transforms of all orders.

We will establish the following statement.

Theorem 1.1 *Let G be a connected Lie group of polynomial growth. The following conditions are equivalent.*

I. *There is a $c > 0$ such that*

$$\max_{i,j \in \{1,\ldots,d'\}} \|A_i A_j H^{-1}\|_{2\to 2} \le c \quad ,$$

i.e., the second-order Riesz transforms are bounded on L_2.

II. *There is a $c > 0$ such that*

$$\max_{i,j \in \{1,\ldots,d'\}} \|A_i A_j S_t\|_{2\to 2} \le c\, t^{-1}$$

for all $t > 0$.

III. *There are $b, c > 0$ such that*

$$\max_{i,j \in \{1,\ldots,d'\}} |(A_i A_j K_t)(g)| \le c\, t^{-1} V(t)^{-1/2} e^{-b(|g|')^2 t^{-1}}$$

for all $g \in G$ and $t > 0$.

IV. *The group G is the local direct product of a connected compact Lie group K and a connected nilpotent Lie group N, i.e., $G = K \cdot N$ where K and N commute and $K \cap N$ is discrete.*

The equivalence of Conditions I and IV of the theorem states that the second-order Riesz transforms are bounded if, and only if, the group is the local direct product of a compact group and a nilpotent group. The situation is more straightforward if G is simply connected. Then the local direct product becomes a direct product and the groups K, N are also simply connected. In general one has a direct product structure at the Lie algebra level but in some situations there is a possible obstruction which prevents this being lifted to the groups.

Note that the equivalence of Conditions II and III gives the rather surprising conclusion that the pointwise Gaussian bounds on the semigroup kernel hold if, and only if, the derivatives of the semigroup satisfy appropriate L_2-bounds.

The theorem only gives a partial illustration of our results. In fact if G is the local direct product of a connected compact Lie group K and a connected nilpotent Lie group N then all the Riesz transforms $A^\alpha H^{-|\alpha|/2}$ are bounded and all the derivatives $A^\alpha K_t$ of the semigroup kernel satisfy Gaussian bounds with an additional factor $t^{-|\alpha|/2}$ for all $t > 0$. Thus boundedness of the second-order Riesz transforms is equivalent to boundedness of the transforms of all orders and a good asymptotic behaviour of the second derivatives of the kernel K is equivalent to a good asymptotic behaviour of all higher order derivatives. Moreover, Gaussian bounds on a particular derivative $A^\alpha K_t$ of the kernel are equivalent with appropriate L_2-bounds on the corresponding derivative $A^\alpha S_t$ of the semigroup.

If one introduces a notion of fractional derivative then the statements of the theorem can be again strengthened. We establish that 'good' behaviour for the derivatives of some order strictly larger than one implies 'good' behaviour for all derivatives of all orders (see Theorem 4.4).

Although the theorem concentrates on the Riesz transforms on $L_2(G\,;dg)$ its conditions ensure that these transforms are bounded on the spaces $L_p(G\,;dg)$ with $p \in \langle 1, \infty\rangle$. In particular one can combine our results with the standard techniques of singular integration theory to deduce that the Riesz transforms of all orders are bounded on $L_p(G\,;dg)$ with $p \in \langle 1, \infty\rangle$ whenever any of the equivalent conditions I–IV is satisfied.

The theorem has some conceptual interest as it identifies purely analytic properties with an algebraic property. Consequently part of the proof of the theorem is purely analytic and will be described in Section 2 and part is algebraic. The algebraic arguments are developed in Section 3 and the proof of the theorem is completed in Section 4.

2 Analytic structure

In this section we consider various estimates related to the Riesz transforms together with asymptotic estimates on the semigroup S generated by H and on the kernel K of S. The general thrust is to prove that boundedness of the Riesz transforms implies good asymptotic behaviour of S and K. We begin with properties involving monomials of derivatives and subsequently examine properties uniform in the number of derivatives. We also consider Hölder bounds and thereby introduce a continuous scale of derivatives, but this is not directly relevant to the proof of Theorem 1.1 and its multi-derivative extension. The essential features for the latter are Theorem 2.1 and the first statement of Proposition 2.8 together with Proposition 2.9 and Corollary 2.10 with the Hölder parameter $\nu = 1$.

The group G is assumed throughout to have polynomial growth.

First note that $D(H^{n/2}) = D((H+I)^{n/2}) = L'_{2;n} = \bigcap_{|\alpha|\leq n} D(A^\alpha)$ for all $n \in \mathbf{N}$ by [ElR1]. Then for each multi-index α consider the following conditions.

1_α. There is a $c > 0$ such that

$$\|A^\alpha \varphi\|_2 \leq c\,\|H^{|\alpha|/2}\varphi\|_2$$

for all $\varphi \in D(H^{|\alpha|/2})$.

2_α. There are $b, c > 0$ such that

$$|(A^\alpha K_t)(g)| \leq c\,t^{-|\alpha|/2}V(t)^{-1/2}e^{-b(|g|')^2 t^{-1}}$$

for all $g \in G$ and $t > 0$.

3_α. There is a $c > 0$ such that

$$\|A^\alpha S_t\|_{2\to 2} \leq c\, t^{-|\alpha|/2}$$

for all $t > 0$.

4_α. There is a $c > 0$ such that

$$\|A^\alpha K_t\|_\infty \leq c\, t^{-|\alpha|/2} V(t)^{-1/2}$$

for all $t > 0$.

The bounds (1) and (3) establish Conditions 1_α and 2_α for all α with $|\alpha| = 1$. But Condition 4_α follows immediately from Condition 2_α and as G has polynomial growth Condition 3_α also follows from Condition 2_α by a quadrature argument. Therefore all four conditions are fulfilled if $|\alpha| = 1$. The general situation is more complex but one has the following relations.

Theorem 2.1 *The following implications are valid*

$$1_\alpha \Rightarrow 2_\alpha \Leftrightarrow 3_\alpha \Leftrightarrow 4_\alpha$$

for each multi-index α*. Moreover, the exponent* b *in Condition* 2_α *may be chosen arbitrarily close to, but strictly smaller than,* $1/4$.

Remark For compact groups the inequalities of Condition 1_α are established for all α in [BER]. Moreover, if G is nilpotent then Conditions 1_α and 2_α are established for all α in [ERS]. Therefore in both these cases the theorem implies that all the conditions are valid for all multi-indices. Conversely, the example of Alexopoulos [Ale2] is a solvable group with polynomial growth for which Condition 1_α fails for an α with $|\alpha| = 2$.

Proof of Theorem 2.1 The main burden of the proof is to establish that Condition 1_α and Condition 4_α imply Condition 2_α. The other implications are all straightforward and we deal with these first.

As G has polynomial growth a standard quadrature argument establishes $2_\alpha \Rightarrow 3_\alpha$. Next as K satisfies the Gaussian bounds (2) it follows by a second quadrature argument that $\|K_t\|_2 \leq c\, V(t)^{-1/4}$ for some $c > 0$ and all $t > 0$. Therefore

$$\begin{aligned}
\|A^\alpha K_{3t}\|_\infty &\leq \|A^\alpha S_{3t}\|_{1\to\infty} \\
&\leq \|A^\alpha S_{2t}\|_{2\to\infty}\, \|S_t\|_{1\to 2} \\
&= \|A^\alpha K_{2t}\|_2\, \|K_t\|_2 \leq \|A^\alpha S_t\|_{2\to 2}\, \|K_t\|_2^2
\end{aligned}$$

for all $t > 0$. Hence $3_\alpha \Rightarrow 4_\alpha$. Alternatively, if $\delta > 0$ then Condition 3_α implies that

$$\begin{aligned}
\|A^\alpha(\lambda I + H)^{-(\delta+|\alpha|)/2}\|_{2\to 2} &\leq c_\delta \int_0^\infty dt\, t^{-1} e^{-\lambda t} t^{(\delta+|\alpha|)/2} \|A^\alpha S_t\|_{2\to 2} \\
&\leq c\, c_\delta \int_0^\infty dt\, t^{-1} e^{-\lambda t} t^{\delta/2} = c\, c_\delta\, \Gamma(\delta/2)\, \lambda^{-\delta/2}
\end{aligned}$$

for all $\lambda > 0$ with $c_\delta = \Gamma((\delta + |\alpha|)/2)^{-1}$. Thus

$$\|A^\alpha \varphi\|_2 \leq c\, c_\delta\, \Gamma(\delta/2)\, \lambda^{-\delta/2} \|(\lambda I + H)^{(\delta+|\alpha|)/2} \varphi\|_2$$

for all $\varphi \in D(H^{(\delta+|\alpha|)/2})$. Therefore

$$\|A^\alpha \varphi\|_2 \le 2^{(\delta+|\alpha|)/2} c\, c_\delta\, \Gamma(\delta/2) \left(\lambda^{|\alpha|/2}\|\varphi\|_2 + \lambda^{-\delta/2}\|H^{(\delta+|\alpha|)/2}\varphi\|_2\right)$$

for all $\lambda > 0$ and all $\varphi \in D(H^{(\delta+|\alpha|)/2})$. Optimization over λ then establishes the following weak form of Condition 1_α:

$1'_\alpha$. For each $\delta > 0$ there is a $c'_\delta > 0$ such that

$$\|A^\alpha \varphi\|_2 \le c'_\delta \left(\|H^{(\delta+|\alpha|)/2}\varphi\|_2\right)^{|\alpha|/(\delta+|\alpha|)} \left(\|\varphi\|_2\right)^{\delta/(\delta+|\alpha|)}$$

for all $\varphi \in D(H^{(\delta+|\alpha|)/2})$.

Since $1_\alpha \Rightarrow 1'_\alpha$ the implication $1_\alpha \Rightarrow 2_\alpha$ is a consequence of the following result.

Proposition 2.2 *Condition $1'_\alpha$ implies Condition 2_α with an exponent b arbitrarily close to, but smaller than, $1/4$. In particular Conditions $1'_\alpha$ and 2_α are equivalent.*

We establish Condition 2_α as a consequence of an integral bound on $A^\alpha K_t$ which indicates in a precise way that high speed propagation is unlikely. The argument we use is of some independent interest so we separate it into the following lemma.

Lemma 2.3 *Let K denote the kernel of a semigroup generated by a (possibly complex) right invariant operator on a Lie group G of polynomial growth. Fix $b > 0$. Suppose that for each $\varepsilon \in \langle 0, 1\rangle$ there exists an $a > 0$ such that*

$$|K_t(g)| \le a\, V(t)^{-1/2} e^{-b(1-\varepsilon)(|g|')^2 t^{-1}} \tag{4}$$

for all $g \in G$ and $t > 0$.

Then for each multi-index α the following conditions are equivalent.

I. *For each $\varepsilon \in \langle 0, 1\rangle$ there exists an $a > 0$ such that*

$$|(A^\alpha K_t)(g)| \le a\, t^{-|\alpha|/2} V(t)^{-1/2} e^{-b(1-\varepsilon)(|g|')^2 t^{-1}} \tag{5}$$

for all $g \in G$ and $t > 0$.

II. *For each $\varepsilon \in \langle 0, 1\rangle$ there exists an $a > 0$ such that*

$$\int_{\{g \in G : |g|' \ge \rho t^{1/2}\}} dg\, |(A^\alpha K_t)(g)|^2 \le a\, t^{-|\alpha|} V(t)^{-1/2} e^{-2b(1-\varepsilon)\rho^2}$$

for all $\rho, t > 0$.

Proof "I $\Rightarrow$ II". Let $\varepsilon \in \langle 0, 2^{-1}\rangle$ and suppose the bounds (5) are valid. Then by a quadrature estimate there exists an $a' > 0$ such that

$$\begin{aligned}
\int_{\{g \in G : |g|' \ge \rho t^{1/2}\}} dg\, |(A^\alpha K_t)(g)|^2 &\le a^2\, t^{-|\alpha|} \int_{\{g \in G : |g|' \ge \rho t^{1/2}\}} dg\, V(t)^{-1} e^{-2b(1-\varepsilon)(|g|')^2 t^{-1}} \\
&\le a^2\, t^{-|\alpha|} V(t)^{-1/2} e^{-2b(1-2\varepsilon)\rho^2} \int_{\{g \in G : |g|' \ge \rho t^{1/2}\}} dg\, V(t)^{-1/2} e^{-2b\varepsilon(|g|')^2 t^{-1}} \\
&\le a'\, t^{-|\alpha|} V(t)^{-1/2} e^{-2b(1-2\varepsilon)\rho^2}
\end{aligned}$$

for all $\rho, t > 0$.
"II $\Rightarrow$ I". First observe that

$$e^{\rho|g|'}|(A^\alpha K_t)(g)| \leq \Big|\int_G dh\, e^{\rho|h|'}(A^\alpha K_{t/2})(h)\, e^{\rho|h^{-1}g|'}K_{t/2}(h^{-1}g)\Big|$$
$$\leq \Big(\int_G dh\, e^{2\rho|h|'}|(A^\alpha K_{t/2})(h)|^2\Big)^{1/2}\Big(\int_G dh\, e^{2\rho|h|'}|K_{t/2}(h)|^2\Big)^{1/2}$$

for all $\rho > 0$. But

$$e^{2\rho|h|'} = 1 + 2\rho\int_0^{|h|'} dr\, e^{2\rho r}$$

and hence

$$\int_G dh\, e^{2\rho|h|'}|(A^\alpha K_{t/2})(h)|^2 \leq \int_G dh\, |(A^\alpha K_{t/2})(h)|^2$$
$$+ 2\rho\int_0^\infty dr\, e^{2\rho r}\int_{\{h\in G:|h|'\geq r\}} dh\, |(A^\alpha K_{t/2})(h)|^2 \quad .$$

Therefore, using Condition II, one concludes that for each $\varepsilon \in \langle 0, 1\rangle$ there exists an $a > 0$ such that

$$\int_G dh\, e^{2\rho|h|'}|(A^\alpha K_{t/2})(h)|^2 \leq a\, t^{-|\alpha|}V(t)^{-1/2}\Big(1 + 2\rho\int_0^\infty dr\, e^{2\rho r}e^{-4b(1-\varepsilon)r^2t^{-1}}\Big)$$
$$\leq a\, t^{-|\alpha|}V(t)^{-1/2}(1 + \pi^{1/2}b^{-1/2}(1-\varepsilon)^{-1/2}\rho\, t^{1/2}e^{\rho^2t(4b(1-\varepsilon))^{-1}})$$
$$\leq a'\, t^{-|\alpha|}V(t)^{-1/2}e^{\rho^2t(1+\varepsilon)(4b(1-\varepsilon))^{-1}}$$

for all $\rho, t > 0$. Similarly, using the bounds (4) one has

$$\int_G dh\, e^{2\rho|h|'}|K_{t/2}(h)|^2 \leq a'\, V(t)^{-1/2}e^{\rho^2t(1+\varepsilon)(4b(1-\varepsilon))^{-1}}$$

Hence

$$|(A^\alpha K_t)(g)| \leq \inf_{\rho>0} a'\, t^{-|\alpha|/2}V(t)^{-1/2}e^{-\rho|g|'+\rho^2t(1+\varepsilon)(4b(1-\varepsilon))^{-1}}$$
$$= a'\, t^{-|\alpha|/2}V(t)^{-1/2}e^{-b(1-\varepsilon)(1+\varepsilon)^{-1}(|g|')^2t^{-1}}$$

for all $g \in G$ and $t > 0$. □

The principal element in the proof of Proposition 2.2 is the following result on finite propagation speed.

Lemma 2.4 *Let $\psi \in C^\infty(\mathbf{R})$ be an increasing function with $\psi(x) = 0$ if $x \leq -1$ and $\psi(x) = 1$ if $x \geq 0$. Define the family of functions $(F_\rho)_{\rho>2}$ by*

$$F_\rho(x) = \psi(\rho(|x| - \rho))\, e^{-x^2/4}$$

and denote the Fourier transforms by $\widehat{F}_\rho$. Then the kernel $K_{\widehat{F}_\rho((tH)^{1/2})}$ of the self-adjoint operators $\widehat{F}_\rho((tH)^{1/2})$ satisfies

$$K_{\widehat{F}_\rho((tH)^{1/2})}(g) = K_t(g)$$

for all $g \in G$ and all $t > 0$ with $|g|' \geq \rho t^{1/2}$. Moreover, for each $m \in \mathbf{N}$ one has bounds

$$|\widehat{F}_\rho(\lambda)| \leq c_m \frac{\rho^{2m-1}}{(\rho^2+\lambda^2)^m} e^{-\rho^2/4} \tag{6}$$

for all $\rho > 2$ and $\lambda \in \mathbf{R}$.

Proof This follows from (17) and Lemma 3 in [Sik1] but we have used a slightly different convention. □

Proof of Proposition 2.2 The kernel K satisfies the Gaussian bounds (2). Hence to deduce that Condition 2_α is satisfied with an exponent b arbitrarily close to $1/4$ it suffices, by Lemma 2.3, to establish bounds

$$\int_{\{g\in G:|g|'\geq\rho t^{1/2}\}} dg\, |(A^\alpha K_t)(g)|^2 \leq a\, t^{-|\alpha|} V(t)^{-1/2} e^{-(1-\varepsilon)\rho^2/2} \tag{7}$$

for all $\rho, t > 0$. This we achieve by the arguments of [Sik1].

First one has

$$\begin{aligned}\int_{\{g\in G:|g|'\geq\rho t^{1/2}\}} dg\, |(A^\alpha K_t)(g)|^2 &\leq \|A^\alpha K_t\|_2^2 \\ &\leq c_\delta^2\, (\|H^{(\delta+|\alpha|)/2} K_t\|_2)^{2|\alpha|/(\delta+|\alpha|)}\, (\|K_t\|_2)^{2\delta/(\delta+|\alpha|)} \end{aligned}\tag{8}$$

by Condition $1'_\alpha$. But for each $\gamma \geq 0$ one has

$$\|H^\gamma K_t\|_2^2 = \|H^\gamma S_{t/2} K_{t/2}\|_2^2 \leq \|H^\gamma S_{t/2}\|_{2\to2}^2\, \|K_{t/2}\|_2^2 \leq \|H^\gamma S_{t/2}\|_{2\to2}^2\, K_t(e)$$

where the last identity follows from the semigroup property and self-adjointness. Then, however, the Gaussian bounds and spectral theory give

$$\|H^\gamma K_t\|_2^2 \leq a\, t^{-2\gamma} V(t)^{-1/2} \sup_{\lambda\geq0} (\lambda^{2\gamma} e^{-\lambda}) \quad .$$

This estimate, with $\gamma = (\delta+|\alpha|)/2$ and $\gamma = 0$, in combination with (8) establishes (7) for all $\rho \leq 2$. Hence we may now assume $\rho > 2$.

Secondly, let $(F_\rho)_{\rho>2}$ be the family of functions and $(c_m)_{m\in\mathbf{N}}$ the constants as in Lemma 2.4. Then

$$\begin{aligned}\int_{\{g\in G:|g|'\geq\rho t^{1/2}\}} dg\, |(A^\alpha K_t)(g)|^2 &= \int_{\{g\in G:|g|'\geq\rho t^{1/2}\}} dg\, |(A^\alpha K_{\widehat{F}_\rho((tH)^{1/2})})(g)|^2 \\ &\leq \|A^\alpha K_{\widehat{F}_\rho((tH)^{1/2})}\|_2^2 \\ &\leq c_{\alpha,\delta}^2 (\|H^{(\delta+|\alpha|)/2} K_{\widehat{F}_\rho((tH)^{1/2})}\|_2)^{2|\alpha|/(\delta+|\alpha|)} \cdot \\ &\qquad \cdot (\|K_{\widehat{F}_\rho((tH)^{1/2})}\|_2)^{2\delta/(\delta+|\alpha|)} \end{aligned}\tag{9}$$

where we have again used Condition $1'_\alpha$.

Next it follows that for each $\gamma \geq 0$ and $m \geq 1$ with $m+\gamma \in \mathbf{N}$ that

$$\begin{aligned}\|H^\gamma K_{\widehat{F}_\rho((tH)^{1/2})}\|_2 &\leq \|H^\gamma \widehat{F}_\rho((tH)^{1/2})\|_{2\to\infty} \\ &\leq \|H^\gamma(\rho^2 I + tH)^{-\gamma}\|_{2\to 2}\|(\rho^2 I + tH)^{m+\gamma}\widehat{F}_\rho((tH)^{1/2})\|_{2\to 2}\,\cdot \\ &\qquad\qquad \cdot\, \|(\rho^2 I + tH)^{-m}\|_{2\to\infty} \\ &\leq t^{-\gamma}\, c_{m+\gamma}\, \rho^{2(m+\gamma)-1} e^{-\rho^2/4}\, \|(\rho^2 I + tH)^{-m}\|_{2\to\infty} \end{aligned} \tag{10}$$

by (6) and spectral theory. Moreover,

$$\begin{aligned}\|(\rho^2 I + tH)^{-m}\|_{2\to\infty} &\leq \Gamma(m)^{-1}\int_0^\infty ds\, s^{-1} e^{-\rho^2 s} s^m \|S_{st}\|_{2\to\infty} \\ &= \Gamma(m)^{-1}\int_0^\infty ds\, s^{-1} e^{-\rho^2 s} s^m \|K_{st}\|_2 \\ &\leq a_m \int_0^\infty ds\, s^{-1} e^{-\rho^2 s} s^m V(st)^{-1/4}\end{aligned}$$

for all $\rho, t > 0$. But there is a $c > 0$ and an integer N such that

$$V(st)^{-1/4} \leq c\,(1 + s^{-N/4}) V(t)^{-1/4}$$

for all $s, t > 0$ because G has polynomial growth. Hence if $m > N/4$ one has bounds

$$\|(\rho^2 I + tH)^{-m}\|_{2\to\infty} \leq a\, V(t)^{-1/4} \tag{11}$$

uniformly for $\rho \geq 1$. Finally combination of (9), (10) and (11) establishes bounds

$$\int_{\{g\in G: |g|' \geq \rho t^{1/2}\}} dg\, |(A^\alpha K_t)(g)|^2 \leq a\, t^{-|\alpha|} V(t)^{-1/2} \rho^{4m-2+2|\alpha|}\, e^{-\rho^2/2}$$

for all $\rho > 2$. Therefore for each $\varepsilon \in \langle 0, 1\rangle$ there is an $a_\varepsilon > 0$ such that

$$\int_{\{g\in G: |g|' \geq \rho t^{1/2}\}} dg\, |(A^\alpha K_t)(g)|^2 \leq a_\varepsilon\, t^{-|\alpha|} V(t)^{-1/2} e^{-(1-\varepsilon)\rho^2/2}$$

for all $\rho > 2$ and all $t > 0$. This completes the proof of the first statement of Proposition 2.2. The second statement follows because we now have $1'_\alpha \Rightarrow 2_\alpha \Rightarrow 3_\alpha \Rightarrow 1'_\alpha$. □

To complete the proof of Theorem 2.1 it suffices to show that $4_\alpha \Rightarrow 2_\alpha$. The proof is similar to the preceding proof that $1_\alpha \Rightarrow 2_\alpha$ but uses a different functional description of S and K which again incorporates the property of finite propagation speed. We now follow the arguments of [Sik2].

Lemma 2.5 *For each $\mu > -1$ and $r > 0$ introduce F_r^μ as the Fourier transform of the function $x \mapsto \pi^{-1/2}\Big((r^2 - x^2) \vee 0\Big)^\mu$ from $\mathbf{R}$ into $\mathbf{R}_+$.*

Then the kernel $K_{F_r^\mu(H^{1/2})}$ of the self-adjoint operator $F_r^\mu(H^{1/2})$ satisfies

$$\operatorname{supp} K_{F_r^\mu(H^{1/2})} \subseteq B'_r \tag{12}$$

for all $r > 0$. Moreover,

$$e^{-t\lambda^2} = 2^{-1}\Gamma(\mu+1)\,(4t)^{-(\mu+3/2)}\int_0^\infty dr\, re^{-r^2(4t)^{-1}}F_r^\mu(\lambda)$$

for all $\lambda, t > 0$ and there is a $c_\mu > 0$ such that

$$|F_r^\mu(\lambda)| \le c_\mu\, r^{2\mu+1}(1+r^2\lambda^2)^{-(\mu+1)/2}$$

for all $\lambda, r > 0$.

Proof This follows from the proof of Lemma 3 in [Sik2]. □

One immediate consequence of Lemma 2.5 and spectral theory is the representation

$$S_t = 2^{-1}\Gamma(\mu+1)\,(4t)^{-(\mu+3/2)}\int_0^\infty dr\, re^{-r^2(4t)^{-1}}F_r^\mu(H^{1/2})$$

and the corresponding representation

$$K_t = 2^{-1}\Gamma(\mu+1)\,(4t)^{-(\mu+3/2)}\int_0^\infty dr\, re^{-r^2(4t)^{-1}}K_{F_r^\mu(H^{1/2})}$$

for the semigroup kernel. The support property (12) implies that

$$(A^\alpha K_t)(g) = 2^{-1}\Gamma(\mu+1)\,(4t)^{-(\mu+3/2)}\int_{|g|'}^\infty dr\, re^{-r^2(4t)^{-1}}\,(A^\alpha K_{F_r^\mu(H^{1/2})})(g) \tag{13}$$

and hence pointwise bounds on $A^\alpha K_t$ can be inferred from the following result.

Lemma 2.6 *If Condition 4_α is valid then for all large positive μ there is an $a_\mu > 0$ such that*

$$\|A^\alpha K_{F_r^\mu(H^{1/2})}\|_\infty \le a_\mu\, r^{2\mu+1}r^{-|\alpha|}V(r)^{-1}$$

for all $r > 0$.

Proof One has the operator estimate

$$\begin{aligned}\|A^\alpha K_{F_r^\mu(H^{1/2})}\|_\infty &= \|A^\alpha F_r^\mu(H^{1/2})\|_{1\to\infty}\\ &\le \|A^\alpha(I+r^2H)^{-m}\|_{1\to\infty}\|(I+r^2H)^m F_r^\mu(H^{1/2})\|_{\infty\to\infty}\end{aligned} \tag{14}$$

for each positive integer m.

The first term on the right hand side of (14) is bounded by

$$\begin{aligned}\|A^\alpha(I+r^2H)^{-m}\|_{1\to\infty} &\le \Gamma(m)^{-1}\int_0^\infty ds\, s^{-1}e^{-s}s^m\|A^\alpha K_{r^2s}\|_\infty\\ &\le c_m r^{-|\alpha|}\int_0^\infty ds\, s^{-1}e^{-s}s^{m-|\alpha|/2}V(r^2s)^{-1/2}\end{aligned}$$

for all $r > 0$ where the second estimate uses Condition 4_α. Then since G has polynomial growth there is a $c > 0$ and an integer N such that

$$V(r^2s)^{-1/2} \le c\,(1+s^{-N/2})V(r)^{-1}$$

for all $r, s > 0$. Hence if $m > (N + |\alpha|)/2$ one has bounds

$$\|A^\alpha (I + r^2 H)^{-m}\|_{1\to\infty} \le c'_m \, r^{-|\alpha|} V(r)^{-1} \tag{15}$$

for all $r > 0$.

The second term on the right hand side of (14) is, however, bounded by

$$\begin{aligned}\|(I + r^2 H)^m F^\mu_r(H^{1/2})\|_{\infty\to\infty} &= \|(I + r^2 H)^m K_{F^\mu_r(H^{1/2})}\|_1 \\ &\le V(r)^{1/2} \, \|(I + r^2 H)^m K_{F^\mu_r(H^{1/2})}\|_2 \\ &= V(r)^{1/2} \, \|(I + r^2 H)^m F^\mu_r(H^{1/2})\|_{2\to\infty} \end{aligned} \tag{16}$$

where the estimate follows because $\operatorname{supp}(I + r^2 H)^m K_{F^\mu_r(H^{1/2})} \subseteq B'_r$. But

$$\begin{aligned}\|(I + r^2 H)^m F^\mu_r(H^{1/2})\|_{2\to\infty} \le \|(I &+ r^2 H)^{m-(\mu+1)/2}\|_{2\to\infty} \\ &\cdot \|(I + r^2 H)^{(\mu+1)/2} F^\mu_r(H^{1/2})\|_{2\to 2} \quad . \end{aligned} \tag{17}$$

The first term on the right hand side of this last estimate is, however, bounded by (11). Specifically there is an $a > 0$ such that

$$\|(I + r^2 H)^{m-(\mu+1)/2}\|_{2\to\infty} \le a \, V(r)^{-1/2} \tag{18}$$

for all $r > 0$ whenever $(\mu + 1)/2 > m + N/4$. Moreover, the second term on the right hand side of (17) satisfies bounds

$$\|(I + r^2 H)^{(\mu+1)/2} F^\mu_r(H^{1/2})\|_{2\to 2} \le \sup_{\lambda>0} (1 + r^2\lambda^2)^{(\mu+1)/2} |F^\mu_r(\lambda)| \le c_\mu \, r^{2\mu+1} \tag{19}$$

for a suitable $c_\mu > 0$ uniformly for all $r > 0$ by Lemma 2.5. Combination of (16), (17), (18) and (19) then yields bounds

$$\|(I + r^2 H)^m F^\mu_r(H^{1/2})\|_{\infty\to\infty} \le c'_\mu \, r^{2\mu+1} \tag{20}$$

for all $r > 0$ whenever μ is sufficiently large relative to m.

Finally combining (14), (15) and (20) one obtains the desired estimates. □

The proof of the implication $4_\alpha \Rightarrow 2_\alpha$ in Theorem 2.1 is now completed by noting that (13) and Lemma 2.6 give

$$\begin{aligned}|(A^\alpha K_t)(g)| &\le a_{|\alpha|} \, t^{-(\mu+3/2)} \int_{|g|'}^\infty dr \, e^{-r^2(4t)^{-1}} r^{2(\mu+1)-|\alpha|} V(r)^{-1} \\ &\le a_{|\alpha|} \, t^{-(\mu+3/2)} e^{-(1-\varepsilon)(|g|')^2(4t)^{-1}} \int_{|g|'}^\infty dr \, e^{-\varepsilon r^2(4t)^{-1}} r^{2(\mu+1)-|\alpha|} V(r)^{-1}\end{aligned}$$

for all $g \in G$, $t > 0$ and $\varepsilon \in \langle 0, 1\rangle$. Hence by a change of integration variable

$$|(A^\alpha K_t)(g)| \le a_{|\alpha|} \, t^{-|\alpha|/2} e^{-(1-\varepsilon)(|g|')^2(4t)^{-1}} \int_0^\infty ds \, e^{-\varepsilon s^2/4} s^{2(\mu+1)-|\alpha|} V(st^{1/2})^{-1}$$

and then since $V(st^{1/2})^{-1} \le c\,(1 + s^{-N}) V(t)^{-1/2}$ one obtains bounds

$$|(A^\alpha K_t)(g)| \le a_{|\alpha|,\varepsilon} \, t^{-|\alpha|/2} V(t)^{-1/2} e^{-(1-\varepsilon)(|g|')^2(4t)^{-1}}$$

for all $g \in G$, $t > 0$ and $\varepsilon \in \langle 0, 1\rangle$, if μ is large enough.

This completes the proof of Theorem 2.1. □

We next digress to discuss an analogue of Theorem 2.1 for fractional derivatives. To this end we introduce the following conditions for each multi-index α and $\nu \in \langle 0, 1\rangle$.

$1_{\alpha,\nu}$. There is a $c > 0$ such that

$$\|(I - L(h))A^\alpha\varphi\|_2 \leq c\,(|h|')^\nu \|H^{(|\alpha|+\nu)/2}\varphi\|_2$$

for all $h \in G$ and $\varphi \in D(H^{(|\alpha|+\nu)/2})$.

$2_{\alpha,\nu}$. For each $\kappa > 0$ there are $b, c > 0$ such that

$$|((I - L(h))A^\alpha K_t)(g)| \leq c\,(|h|'t^{-1/2})^\nu t^{-|\alpha|/2} V(t)^{-1/2} e^{-b(|g|')^2 t^{-1}}$$

for all $g, h \in G$ and $t > 0$ with $|h|' \leq \kappa\, t^{1/2}$.

$3_{\alpha,\nu}$. There is a $c > 0$ such that

$$\|(I - L(h))A^\alpha S_t\|_{2\to 2} \leq c\,(|h|'t^{-1/2})^\nu t^{-|\alpha|/2}$$

for all $h \in G$ and $t > 0$.

$4_{\alpha,\nu}$. There is a $c > 0$ such that

$$\|(I - L(h))A^\alpha K_t\|_\infty \leq c\,(|h|'t^{-1/2})^\nu t^{-|\alpha|/2} V(t)^{-1/2}$$

for all $h \in G$ and $t > 0$.

One now has the following implications analogous to those of Theorem 2.1.

Proposition 2.7 *Let $\nu \in \langle 0, 1\rangle$. Then $1_{\alpha,\nu} \Rightarrow (2_{\alpha,\nu} + 3_{\alpha,\nu})$ and $(2_{\alpha,\nu} + 2_\alpha) \Rightarrow 3_{\alpha,\nu} \Rightarrow 4_{\alpha,\nu} \Rightarrow 2_{\alpha,\nu}$ for each multi-index α. Moreover, the exponent b in Condition $2_{\alpha,\nu}$ may be chosen arbitrarily close to, but strictly smaller than, $1/4$.*

Proof It follows from spectral theory that $1_{\alpha,\nu} \Rightarrow 3_{\alpha,\nu}$ and by a quadrature estimate that $2_{\alpha,\nu} \Rightarrow 3_{\alpha,\nu}$ under the additional restraint $|h|' \leq \kappa\, t^{1/2}$. But if $|h|' \geq \kappa\, t^{1/2}$ then Condition 2_α implies

$$\|(I - L(h))A^\alpha S_t\|_{2\to 2} \leq 2\,\|A^\alpha S_t\|_{2\to 2} \leq 2c\,t^{-|\alpha|/2} \leq 2c\,\kappa^{-\nu}\,(|h|'t^{-1/2})^\nu t^{-|\alpha|/2} \quad .$$

Hence $2_{\alpha,\nu} + 2_\alpha \Rightarrow 3_{\alpha,\nu}$.

A slight modification of the argument that $3_\alpha \Rightarrow 4_\alpha$ establishes that $3_{\alpha,\nu} \Rightarrow 4_{\alpha,\nu}$

Next Condition $3_{\alpha,\nu}$ implies the following weak form of Condition $1_{\alpha,\nu}$:

$1'_{\alpha,\nu}$. For each $\delta > 0$ there is a $c_\delta > 0$ such that

$$\|(I - L(h))A^\alpha\varphi\|_2 \leq c_\delta\,(|h|')^\nu (\|H^{(\delta+|\alpha|+\nu)/2}\varphi\|_2)^{(|\alpha|+\nu)/(\delta+|\alpha|+\nu)}\,(\|\varphi\|_2)^{\delta/(\delta+|\alpha|+\nu)}$$

for all $h \in G$ and all $\varphi \in D(H^{(\delta+|\alpha|+\nu)/2})$.

The proof is a repetition of the argument used to establish that $3_\alpha \Rightarrow 1'_\alpha$.

To complete the proof of the proposition it suffices to prove that $1'_{\alpha,\nu} \Rightarrow 2_{\alpha,\nu}$ and $4_{\alpha,\nu} \Rightarrow 2_{\alpha,\nu}$. But the proof of these implications is a straightforward variation of the previous reasoning and we omit further details. □

Theorem 2.1 and Proposition 2.7 deal with individual multi-derivatives A^α and next we consider properties uniform in the number $|\alpha|$ of derivatives. For this we need uniform versions of the previous conditions.

Let $s \geq 1$. If $s \in \mathbf{N}$ we define Condition N_s, where $N \in \{1, \ldots, 4\}$, to be valid if Condition N_α holds for all α with $|\alpha| = s$. If, however, $s = n + \nu$ with $n \in \mathbf{N}_0$ and $\nu \in \langle 0, 1\rangle$ we define Condition N_s to be valid if Condition $N_{\alpha,\nu}$ holds for all α with $|\alpha| = n$.

In addition we introduce a fifth family of conditions involving 'cutoff' functions.

5_s. There are $\sigma \in \langle 0,1\rangle$, $c>0$ and a family of C^∞-functions $(\eta_R)_{R>0}$ such that $\operatorname{supp}\eta_R \subset B'_R$, $\eta_R(g)=1$ for all $g\in B'_{\sigma R}$ and $0\le \eta_R\le 1$. In addition, if $s\in\mathbf{N}$ then

$$\|A^\alpha \eta_R\|_\infty \le c\,R^{-|\alpha|}$$

for all multi-indices α with $|\alpha|=s$ uniformly for $R>0$. Alternatively, if $s=n+\nu$ with $n\in\mathbf{N}_0$ and $\nu\in\langle 0,1\rangle$ then

$$\|(I-L(h))A^\alpha\eta_R\|_\infty \le c\,(|h|'R^{-1})^\nu R^{-|\alpha|}$$

for all multi-indices α with $|\alpha|=n$, uniformly for $h\in G$ and $R>0$.

The existence of cutoff functions of this type on a general Lie group, with $s\in\mathbf{N}$, has been established in [ElR3], Lemma 2.3, for all R in a finite subinterval of $\langle 0,\infty\rangle$ and any multi-index α. The crucial feature of Condition 5_s is the requirement that the functions exist with the appropriate bounds on their derivatives uniformly for all $R>0$. If, however, $s=1$ cutoff functions of this type always exist by the following construction.

The kernel K has Gaussian lower bounds with $\omega=0$, by [Rob], Proposition IV.4.21, i.e., there exist $b,c>0$ such that

$$K_t(g) \ge c\,V(t)^{-1/2} e^{-b(|g|')^2 t^{-1}} \tag{21}$$

for all $t>0$ and $g\in G$. Together with the upper bounds (2) it follows that there are $a>1$ and $b_1,b_2>0$ such that

$$a^{-1}e^{-b_1(|g|'/R)^2} \le \frac{K_{R^2}(g)}{K_{R^2}(e)} \le a e^{-b_2(|g|'/R)^2}$$

for all $g\in G$ and $R>0$. Fix an increasing function $\varphi\in C^\infty(\mathbf{R})$ such that $\varphi(x)=0$ if $x\le(4a)^{-1}$ and $\varphi(x)=1$ if $x\ge(2a)^{-1}$. Then define

$$\varphi_R(g) = \varphi\Big(\frac{K_{R^2}(g)}{K_{R^2}(e)}\Big)$$

for all $g\in G$ and $R>0$. Next choose $\tau_1,\tau_2>0$ so that $e^{-b_1\tau_1^2}>2^{-1}$ and $e^{-b_2\tau_2^2}<(4a^2)^{-1}$. Then $\varphi_R(g)=1$ for all $R>0$ and $g\in G$ with $|g|'\le\tau_1 R$ and $\varphi_R(g)=0$ if $|g|'\ge\tau_2 R$. Therefore the functions

$$\eta_R = \varphi_{\tau_2^{-1}R}$$

satisfy the required domain properties.

Next we show that the derivatives have the right decay. It suffices to establish this for the functions φ_R. But

$$(A_i\varphi_R)(g) = \varphi'\Big(\frac{K_{R^2}(g)}{K_{R^2}(e)}\Big)\frac{(A_iK_{R^2})(g)}{K_{R^2}(e)}$$

for all $i\in\{1,\ldots,d'\}$ uniformly for all $g\in G$ and $R>0$. Then

$$|(A_i\varphi_R)(g)| \le c\,R^{-1}$$

by (3) and (21) uniformly for $g\in G$ and $R>0$. Condition 5_1 follows immediately.

Our ultimate aim is to prove that all the Conditions 1_s–5_s are equivalent and if they hold for one $s > 1$ then they hold for all $s > 1$. But the proof of these statements, even restricted to integer s, requires detailed examination of the algebraic structure which we defer to the next section. At this point we have several partial implications summarized in the following proposition. Note that the first statement is the only one essential for the discussion of integer s.

Proposition 2.8 *Let* $\nu \in \langle 0, 1\rangle$.

I. *If* $n \in \mathbf{N}$ *and* $n \geq 2$ *then* $1_n \Rightarrow 2_n \Leftrightarrow 3_n \Leftrightarrow 4_n \Rightarrow 5_n \Rightarrow 5_2$.

II. *If* $n \in \mathbf{N}$ *then* $2_{n+\nu} \Rightarrow 5_{1+\nu}$ *and* $1_{n+\nu} \Rightarrow 3_{n+\nu} \Rightarrow 4_{n+\nu} \Rightarrow 5_{n+\nu} \Rightarrow 5_{1+\nu}$.

III. *If* $\nu' \in \langle 0, \nu\rangle$ *then* $5_2 \Rightarrow 5_{1+\nu} \Rightarrow 5_{1+\nu'}$.

Proof First, it follows from Theorems 2.1 that $1_n \Rightarrow 2_n \Leftrightarrow 3_n \Leftrightarrow 4_n$.

Secondly, as translations on the L_p-spaces are isometric it follows as in [Rob], Lemma III.3.3, that for all $m \in \mathbf{N}$ and $p \in [1,\infty]$ there exists a $c > 0$ such that

$$\|A^\alpha \varphi\|_p \leq \varepsilon^{n-|\alpha|} \max_{|\beta|=n} \|A^\beta \varphi\|_p + c\,\varepsilon^{-|\alpha|}\|\varphi\|_p \tag{22}$$

for all $\varphi \in L'_{p;m}$, $\varepsilon > 0$ and $\alpha \in J(d')$ with $1 \leq |\alpha| < n$. Using these inequalities on L_∞ one immediately deduces that $4_n \Rightarrow 4_m$, and $5_n \Rightarrow 5_m$ for all $m, n \in \mathbf{N}$ with $n > m$.

Thirdly, suppose Condition 4_n is valid, and hence Condition 4_m is valid for all $m < n$. Let $\alpha = (i_1, \ldots, i_n) \in J(d')$. Then

$$(A^\alpha \varphi_R)(g) = \sum \varphi^{(l)}\Big(\frac{K_{R^2}(g)}{K_{R^2}(e)}\Big) \prod_{p=1}^{l} \frac{(A^{\beta_p} K_{R^2})(g)}{K_{R^2}(e)} \tag{23}$$

uniformly for all $g \in G$ and $R > 0$, where the sum is finite and over a subset of all $l \in \{1, \ldots, n\}$ and $\beta_1, \ldots, \beta_l \in J(d')$ with $|\beta_p| \geq 1$ for all $p \in \{1, \ldots, l\}$ and $|\beta_1| + \ldots + |\beta_l| = n$. Then

$$\Big|\prod_{p=1}^{l} \frac{(A^{\beta_p} K_{R^2})(g)}{K_{R^2}(e)}\Big| \leq \prod_{p=1}^{l} c_{\beta_p} c^{-1} R^{-|\beta_p|} = R^{-n} c^{-n} \prod_{p=1}^{l} c_{\beta_p}$$

uniformly for $g \in G$ and $R > 0$. Condition 5_n follows immediately. Hence Condition 5_2 is valid by the previous argument. This completes the proof of Statement I and we now prove Statement II.

First, it follows from Proposition 2.7 that $1_{n+\nu} \Rightarrow 3_{n+\nu} \Rightarrow 4_{n+\nu}$. (We have also sketched the proof that $1_{n+\nu}$ implies $2_{n+\nu}$ but we do not need this implication for the sequel.)

Secondly, if $n \in \mathbf{N}$ and $n \geq 2$ then it follows from the Duhamel formula and some rearrangement that

$$f'(x) = u^{-1}\left(f(x+u) - f(x)\right) - u^{-1}\int_0^u ds\,\Big(f'(x+s) - f'(x)\Big) \quad .$$

Therefore

$$\max_{|\alpha|=n} \|A^\alpha \varphi\|_\infty \leq 2u^{-1} \max_{|\alpha|=n-1} \|A^\alpha \varphi\|_\infty + u^{-1} \max_{|\alpha|=n-1} \max_{i \in \{1,\ldots,d'\}} \int_0^u ds\, \Big\|\Big(I - L(\exp(s a_i))\Big) A_i A^\alpha \varphi\Big\|_\infty$$

$$\leq 2u^{-1}\Big(\varepsilon \max_{|\alpha|=n} \|A^\alpha\varphi\|_\infty + c\,\varepsilon^{-n+1}\|\varphi\|_\infty\Big)$$
$$+ u^{-1} \max_{|\alpha|=n-1} \max_{i\in\{1,\ldots,d'\}} \int_0^u ds\, \|\Big(I - L(\exp(sa_i))\Big)A_iA^\alpha\varphi\|_\infty$$

for all $u > 0$ and $\varepsilon > 0$, by (22) with $p = 2$. Setting $\varepsilon = u/4$ it follows that

$$\max_{|\alpha|=n} \|A^\alpha\varphi\|_\infty \leq 2u^{-1} \max_{|\alpha|=n-1} \max_{i\in\{1,\ldots,d'\}} \int_0^u ds\, \|\Big(I - L(\exp(sa_i))\Big)A_iA^\alpha\varphi\|_\infty + c'\, u^{-n}\|\varphi\|_\infty$$

for a suitable $c' > 0$, uniformly for all $u > 0$ and $\varphi \in L'_{\infty;n}$. Therefore if Condition $2_{n+\nu}$ or $4_{n+\nu}$ is valid then there is a $c'' > 0$ such that

$$\|(I - L(h))A^\alpha K_t\|_\infty \leq c\,(|h|' t^{-1/2})^\nu t^{-|\alpha|/2} V(t)^{-1/2}$$

for all $h \in G$ and $t > 0$ with $|h|' \leq t^{1/2}$. Hence

$$\max_{|\alpha|=n} \|A^\alpha K_t\|_\infty \leq c_1\, u^{-1} \int_0^u ds\, (st^{-1})^\nu t^{-n/2} V(t)^{-1/2} + c_2\, u^{-n} V(t)^{-1/2}$$
$$= c_1(1+\nu)^{-1} u^\nu t^{-(n+\nu)/2} V(t)^{-1/2} + c_2\, u^{-n} V(t)^{-1/2}$$

for all $t > 0$ and $u \in \langle 0, t^{1/2}]$ for suitable $c_1, c_2 > 0$. Choosing $u = t^{1/2}$ implies that Condition 4_n is valid. Thus if $n \geq 2$ then $4_{n+\nu} \Rightarrow 4_n$ and $2_{n+\nu} \Rightarrow 4_n$. The implication $5_{n+\nu} \Rightarrow 5_n$ follows by a similar argument. Moreover, since 2_1 is valid, it follows from Proposition 2.7 that $2_{1+\nu} \Rightarrow (2_{1+\nu} + 2_1) \Rightarrow 3_{1+\nu} \Rightarrow 4_{1+\nu}$. Hence it remains to show that $4_{n+\nu} \Rightarrow 5_{n+\nu}$ for all $n \in \mathbf{N}$. But Condition $4_{n+\nu}$ with $n \in \mathbf{N}$ and $\nu \in \langle 0, 1\rangle$ implies Condition 4_s for all $s \leq n + \nu$ by the foregoing reasoning and a simple interpolation argument. Hence

$$\|(I - L(h))A^\alpha K_t\|_\infty \leq c\,(|h|' t^{-1/2})^\nu t^{-|\alpha|/2} V(t)^{-1/2}$$

and

$$\|A^\alpha K_t\|_\infty \leq c\, t^{-|\alpha|/2} V(t)^{-1/2}$$

for all $h \in G$, $t > 0$ and α with $|\alpha| \leq n$. Since $\|(I-L(h))(\tau\circ\psi)\|_\infty \leq \|\tau'\|_\infty \|(I-L(h))\psi\|_\infty$ and $\|(I - L(h))(\psi_1 \cdot \psi_2)\|_\infty \leq \|\psi_1\|_\infty \|(I - L(h))\psi_2\|_\infty + \|\psi_2\|_\infty \|(I - L(h))\psi_1\|_\infty$ for all $\tau \in C_c^\infty(\mathbf{R})$, $\psi, \psi_1, \psi_2 \in L_\infty$ and $h \in G$ it follows from (23) that there exists a $c > 0$ such that $\|(I - L(h))A^\alpha\varphi_R\|_\infty \leq c\,(|h|' R^{-1})^\nu R^{-n}$ for all $h \in G$ and $R > 0$, i.e., Condition $5_{n+\nu}$ is valid.

Finally Statement III is evident as Condition $5_{1+\nu}$ is a simple consequence of Conditions 5_2 and 5_1 and Condition $5_{1+\nu'}$ follows by combination of $5_{1+\nu}$ and 5_1. □

The cutoff functions introduced by Conditions 5_s play the crucial role in linking the current analytic arguments with the subsequent algebraic reasoning. Their significance lies in the following observation. Note that for the discussion of integer s one only requires the case $\nu = 1$.

Proposition 2.9 *If Condition $5_{1+\nu}$ is valid for some $\nu \in \langle 0, 1]$ then there exist an infinitely differentiable function $\varphi: G \to \mathbf{R}$ and for all $h_1, h_2 \in G$ a $c > 0$ such that*

$$\Big|\Big((I - L(h_1h_2h_1^{-1}h_2^{-1})\varphi\Big)(g)\Big| \leq c\,(|g|')^{-\nu}$$

for all $g \in G$ with $|g|' > 2(|h_1|' + |h_2|')$. Moreover,

$$|g|' - 1 \le \varphi(g)$$

for all $g \in G$.

Proof Let $(\eta_R)_{R>0}$ be the family of functions and $\sigma \in \langle 0,1\rangle$ the parameter in Condition $5_{1+\nu}$. Then $1 - \eta_n(g) = 0$ for all $g \in G$ and $n \ge \sigma^{-1}|g|'$. Therefore we can define $\varphi: G \to \mathbf{R}$ by

$$\varphi(g) = \sum_{n=1}^{\infty} \Big(1 - \eta_n(g)\Big) \quad .$$

Then

$$|g|' - 1 \le \varphi(g) \le \sigma^{-1}|g|' \tag{24}$$

for all $g \in G$. If $g \in G$, $n \in \mathbf{N}$ and $n \notin [|g|', \sigma^{-1}|g|']$, then η_n is constant on a neighbourhood of g and therefore all derivatives of η_n vanish. So

$$(A_i\varphi)(g) = - \sum_{n \in \mathbf{N};\, |g|' \le n \le \sigma^{-1}|g|'} (A_i\eta_n)(g) \tag{25}$$

for all $g \in G$ and $i \in \{1, \ldots, d'\}$. Since $\sup_{n \in \mathbf{N}} n\, \|A_i\eta_n\|_\infty < \infty$ it follows that $A_i\varphi \in L_\infty$ for all $i \in \{1, \ldots, d'\}$.

Now let $g, h \in G$ with $g \ne e$ and suppose that $|h|' \le 2^{-1}|g|'$. Then $2^{-1}|g|' \le |h^{-1}g|' \le 2|g|'$ and therefore

$$\begin{aligned}\Big|\Big((I - L(h))A_i\varphi\Big)(g)\Big| &\le \sum_{n \in \mathbf{N};\, 2^{-1}|g|' \le n \le 2\sigma^{-1}|g|'} \Big|\Big((I - L(h))A_i\eta_n\Big)(g)\Big| \\ &\le \sum_{n \in \mathbf{N};\, 2^{-1}|g|' \le n \le 2\sigma^{-1}|g|'} c\,(|h|' n^{-1})^\nu\, n^{-1} \le 2^{2+\nu} c\, \sigma^{-1} (|h|')^\nu (|g|')^{-\nu}\end{aligned}$$

for all $i \in \{1, \ldots, d'\}$, by Condition $5_{1+\nu}$ if $\nu \in \langle 0,1\rangle$. Alternatively, if $\nu = 1$ then $|\Big((I - L(h))A_i\varphi\Big)(g)| \le c\,|h|' \max_{1 \le j \le d'} |(A_jA_i\varphi)(g)|$ and the same estimate follows. But since $A_i\varphi$ is bounded it follows that there exists a $c > 0$ such that

$$\Big|\Big((I - L(h))A_i\varphi\Big)(g)\Big| \le c\,(|h|')^\nu (|g|')^{-\nu}$$

for all $g, h \in G$ with $g \ne e$.

Next let $g, h_1, h_2 \in G$ with $g \ne e$ and $|h_2|' \le 3^{-1}|g|'$. There exists an absolutely continuous path $\gamma: [0,1] \to G$ such that $\gamma(0) = e$, $\gamma(1) = h_2$,

$$\dot\gamma(t) = \sum_{i=1}^{d'} \gamma_i(t)\, A_i\Big|_{\gamma(t)}$$

for almost every $t \in [0,1]$ and $\int_0^1 dt\, \Big(\sum_{i=1}^{d'} |\gamma_i(t)|^2\Big)^{1/2} \le 2|h_2|'$. Then

$$\begin{aligned}\Big|\Big((I - L(h_1))L(\gamma(t))\, A_i\varphi\Big)(g)\Big| &= \Big|\Big(L(\gamma(t))\,\Big(I - L(\gamma(t)^{-1}h_1\gamma(t))\Big)A_i\varphi\Big)(g)\Big| \\ &\le c\,(|\gamma(t)^{-1}h_1\gamma(t)|')^\nu (|\gamma(t)^{-1}g|')^{-\nu} \\ &\le 3^\nu c\,(|h_1|' + 4|h_2|')^\nu (|g|')^{-\nu}\end{aligned}$$

for all $t \in [0,1]$ and $i \in \{1,\ldots,d'\}$. Therefore

$$\Big|\Big((I-L(h_1))(I-L(h_2))\varphi\Big)(g)\Big| \le \int_0^1 dt\,\sum_{i=1}^{d'}|\gamma_i(t)|\,\Big|\Big((I-L(h_1))L(\gamma(t))\,A_i\varphi\Big)(g)\Big|$$
$$\le 2\cdot 3^\nu c\,d'\,|h_2|'(|h_1|'+4|h_2|')^\nu(|g|')^{-\nu}$$
$$\le 2\cdot 12^\nu c\,d'\,(|h_1|'+|h_2|')^{1+\nu}(|g|')^{-\nu} \quad .$$

Since $\varphi(l) \le \sigma^{-1}|l|'$ for all $l \in G$ it follows that there exists a $c > 0$ such that

$$\Big|\Big((I-L(h_1))(I-L(h_2))\varphi\Big)(g)\Big| \le c\,(|h_1|'+|h_2|')^{1+\nu}(|g|')^{-\nu}$$

for all $g, h_1, h_2 \in G$ with $g \neq e$.

Finally let $h_1, h_2 \in G$ and set $k = h_1h_2h_1^{-1}h_2^{-1}$. Using the identity

$$I-L(h_1h_2h_1^{-1}h_2^{-1}) = -L(h_1)\,(I-L(h_1^{-1}))(I-L(h_2^{-1}))$$
$$- L(h_1h_2^{-1})\,(I-L(h_2))(I-L(h_2h_1^{-1}h_2^{-1}))$$

it follows that there exists a $c' > 0$ such that

$$\Big|\Big((I-L(k))\varphi\Big)(g)\Big| \le c\,(|h_1|'+|h_2|')^{1+\nu}(|h_1^{-1}g|')^{-\nu} + c\,(|h_2|'+|h_2h_1^{-1}h_2^{-1}|')^{1+\nu}(|h_2h_1^{-1}g|')^{-\nu}$$
$$\le c'\,(|g|')^{-\nu}$$

for all $g \in G$ with $|g|' > 2(|h_1|'+|h_2'|)$. □

Corollary 2.10 *If $\nu \in \langle 0,1]$, $h_1, h_2 \in G$ and $c_1, c_2 > 0$ are such that $|k^{-n}|' \ge c_1 n$ for all $n \in \mathbf{N}$ with $n \ge c_2$, where $k = h_1h_2h_1^{-1}h_2^{-1}$ then Condition $5_{1+\nu}$ fails.*

Proof Suppose that h_1, h_2, c_1, c_2 exist with the described properties and Condition $5_{1+\nu}$ is valid. By Proposition 2.9 there exists a $c > 0$ and an infinitely differentiable function $\varphi: G \to \mathbf{R}$ such that $\varphi(g) \ge |g|' - 1$ and $|((I-L(k))\varphi)(g)| \le c\,(|g|')^{-\nu}$ for all $g \in G$ with $|g|' > 2(|h_1|'+|h_2|')$. Apply the last inequality to $g = k^{-n}$. Let $N \in \mathbf{N}$ be such that $N \ge c_2$ and $c_1\,N > 2(|h_1|'+|h_2|')$. Then for all $n \ge N$ one has

$$|\varphi(k^{-n}) - \varphi(k^{-(n+1)})| = \Big|\Big((I-L(k))\varphi\Big)(k^{-n})\Big| \le c\,(|k^{-n}|')^{-\nu} \le c\,(c_1 n)^{-\nu}$$

and hence

$$c_1(N+m) - 1 - \varphi(k^{-N}) \le \varphi(k^{-(N+m)}) - \varphi(k^{-N})$$
$$\le \sum_{l=1}^{m} c\,c_1^{-\nu}(N+l)^{-\nu}$$
$$\le c\,c_1^{-\nu}\,(1-\nu)^{-1}\Big((N+m)^{1-\nu} - N^{1-\nu}\Big)$$

for all $m \in \mathbf{N}$ and $\nu \in \langle 0,1\rangle$, by a quadrature estimate. If $\nu = 1$ the last estimate is replaced by $c\,c_1^{-1}\Big(\log(N+m) - \log N\Big)$. But these bounds are impossible for large m. □

Note that if Condition $5_{1+\nu}$ fails then Conditions 1_s–5_s must also fail for $s \geq 1+\nu$ by Proposition 2.8.

In the next section we demonstrate that Condition $5_{1+\nu}$ has strong implications for the group structure. Our line of argument is most easily illustrated by examining Condition 5_2. If this condition is valid then it follows from (24) and (25) that there exists a $c > 0$ such that

$$|(A^\alpha \varphi)(g)| \leq c \left(\varphi(g)\right)^{-1}$$

for all $g \in G$ with $|g|' \geq 2$ and all multi-indices α with $|\alpha| = 2$. Let $i, j \in \{1, \ldots, d'\}$ and set $b = [a_i, a_j]$. Then

$$\frac{d}{dt}\varphi(\exp tb) = -\Big(dL(b)\,\varphi\Big)(\exp tb) = \Big(A_j A_i - A_i A_j)\varphi\Big)(\exp tb) \leq 2c\left(\varphi(\exp tb)\right)^{-1} \quad .$$

Integrating this differential inequality it follows that there is a $c' > 0$ such that

$$|\exp tb|' - 1 \leq \varphi(\exp tb) \leq c'\, t^{1/2}$$

for all $t \geq 1$. On the covering group of the Euclidean motion group one has, however, lower bounds $|\exp tb|' \geq c''\, t$ for large t, if $b \neq 0$. This then contradicts Condition 5_2. More generally Condition 5_2, and hence Condition 1_2, fails for any group for which one can find an element b which is a commutator and such that $|\exp tb|' \geq c\,t$ for large t. On nilpotent and compact groups this is impossible. On a solvable group which is not nilpotent one can find such a b, but then it is unlikely that it equals a commutator of order 2 in the algebraic basis. Therefore it is appropriate to estimate a group commutator as in Proposition 2.9. Moreover, in Corollary 2.10 the time variable t in the key lower bound $|\exp tb|' \geq c\,t$ has been discretized. The main problem in the next section is to find the candidates for the k in Corollary 2.10.

3 Algebraic structure

In the previous section we demonstrated that boundedness of the second-order Riesz transforms implies that the second derivatives of the semigroup kernel satisfy good Gaussian bounds and hence Condition 5_2 is satisfied. In this section we establish that this is only possible on a group with polynomial growth if the group is the local direct product of a compact group and a nilpotent group. The previous arguments were largely analytic but the proofs of this section are largely algebraic. We rely heavily on the structure theory of Lie groups.

We begin with some geometric observations. First note that two moduli on a Lie group associated with two algebraic bases are equivalent on the complement of any neighbourhood of the identity by [VSC], Proposition III.4.2.

Secondly one has the following simple relationship.

Lemma 3.1 *Let Q, E be Lie groups with moduli $|\cdot|_Q$ and $|\cdot|_E$ and $\Psi: Q \to E$ a Lie group homomorphism. Then there exists a $c > 0$ such that $|\Psi(g)|_E \leq c\,|g|_Q$ for all $g \in G$ with $|\Psi(g)|_E \geq 1$.*

Proof The proof is elementary once one realizes that one can assume that the modulus on E can be taken with respect to a vector space basis. We omit the details. □

Next let $\mathfrak{q}$, $\mathfrak{n}$ and $\mathfrak{m}$ be the radical, the nil-radical and a Levi-subalgebra of $\mathfrak{g}$ and Q, M the connected analytic subgroups of G which have Lie algebras $\mathfrak{q}$ and $\mathfrak{m}$. Then the Killing form on $\mathfrak{m}$ is negative-definite since all eigenvalues of the adjoint representation on a group of polynomial growth are purely imaginary (see [Gui]). Hence M is compact and therefore closed in G by [Hoc], Theorems XIII.1.1 and XIII.1.3. In addition, $G = QM$ and Q is closed in G (see [Var], Theorem 3.18.13).

Since M is compact the moduli on G and Q do not differ much.

Lemma 3.2 *There exist $c_1, c_2 > 0$ such that $c_1|g|_Q \leq |g|'$ for all $g \in Q$ with $|g|_Q \geq c_2$, where $|\cdot|_Q$ is a modulus on Q with respect to some basis.*

Proof Since M is compact in G there exists a $c_1 > 0$ such that $|m|' \leq c_1$ for all $m \in M$. Let $B = \{g \in G : |g|' < 1 + 2c_1\}$. Then $\overline{B}$ is compact in G and Q is closed in G. Therefore $Q \cap \overline{B}$ is compact in G and hence in Q, thus bounded in Q. Let $C > 0$ be such that $|g|_Q \leq C$ for all $g \in Q \cap \overline{B}$.

Now let $g \in Q$ and suppose $|g|_Q > C$. Then $|g|' \geq 1 + 2c_1 \geq 1$. There exists a $n \in \mathbf{N}$ such that $n - 1 \leq |g|' < n$ and a sequence $e = g_0, g_1, \ldots, g_{n-1}, g_n = g$ in G such that $|g_i^{-1} g_{i-1}|' \leq 1$ for all i. Moreover, for all i there exist $q_i \in Q$, $m_i \in M$ such that $g_i = q_i m_i$ where we may assume that $m_0 = m_n = e$. Then $g_i^{-1} g_{i-1} = m_i^{-1} q_i^{-1} q_{i-1} m_{i-1}$ and hence

$$|q_i^{-1} q_{i-1}|' \leq |g_i^{-1} g_{i-1}|' + |m_i^{-1}|' + |m_i|' \leq 1 + 2c_1 \quad .$$

But also $q_i^{-1} q_{i-1} \in Q$. Therefore $q_i^{-1} q_{i-1} \in Q \cap \overline{B}$ and $|q_i^{-1} q_{i-1}|_Q \leq C$. Hence $|g|_Q = |q_n|_Q \leq C\, n \leq C(|g|' + 1) \leq 2C\, |g|'$. □

Proposition 3.3 *If $\nu \in \langle 0, 1]$ and Condition $5_{1+\nu}$ is valid then the radical of $\mathfrak{g}$ is nilpotent, i.e., $\mathfrak{q} = \mathfrak{n}$.*

Proof For all $a \in \mathfrak{q}$ let $S(a)$ and $K(a)$ be the semisimple and nilpotent part of the Jordan decomposition of the derivation $\mathrm{ad}a$. Note that $S(a) = 0$ for all $a \in \mathfrak{n}$. Set $d_Q = \dim \mathfrak{q}$ and $d_0 = \dim \mathfrak{q} - \dim \mathfrak{n}$. Let $\widetilde{Q}$ be the universal covering of Q and $\pi \colon \widetilde{Q} \to Q$ the natural map. Set $\Gamma = \operatorname{Ker} \pi$. We identify the Lie algebras of Q and $\widetilde{Q}$. By [Ale1], Sections 2 and 3, there exist a basis $b_1, \ldots, b_{d_Q}$ for $\mathfrak{q}$, an $r \in \mathbf{N}$, for all $i \in \{1, \ldots, d_Q\}$ there are $R_i \in \{\{0\}, \mathbf{Z}\}$ and $w_i \in \{1, \ldots, r\}$ and, moreover, there are a Lie bracket $[\,\cdot\,,\,\cdot\,]_N$ on $\mathfrak{q}$, ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_{r+1}$ of $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,])$ and vector subspaces $\mathfrak{a}_1, \ldots, \mathfrak{a}_r, \mathfrak{h}_{01}, \ldots, \mathfrak{h}_{0r}, \mathfrak{h}_{11}, \ldots, \mathfrak{h}_{1r}$ of $\mathfrak{q}$ with the following properties.

- **I.** $S(b_i) b_j = 0$ for all $i, j \in \{1, \ldots, d_0\}$ and $\mathfrak{n} = \operatorname{span}\{b_{d_0+1}, \ldots, b_{d_Q}\}$.
- **II.** $[b_i, b_j]_N = [b_i, b_j]$, $[b_i, a]_N = K(b_i) a$ and $[a, b]_N = [a, b]$ for all $i \in \{1, \ldots, d_0\}$ and $a, b \in \mathfrak{n}$.
- **III.** The Lie algebra $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,]_N)$ is nilpotent.
- **IV.** $\mathfrak{q}_1 = \mathfrak{q}$ and $\mathfrak{q}_{i+1} = [\mathfrak{q}, \mathfrak{q}_i]_N$ for all $i \in \{1, \ldots, r\}$. Moreover, $\mathfrak{q}_r \neq \{0\}$ and $\mathfrak{q}_{r+1} = \{0\}$, i.e., r is the rank of the nilpotent Lie algebra $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,]_N)$.

V. $\mathfrak{q}_j = \mathfrak{a}_j \oplus \mathfrak{q}_{j+1}$ and $\mathfrak{a}_j = \mathfrak{h}_{0j} \oplus \mathfrak{h}_{1j}$ for all $j \in \{1, \ldots, r\}$. Also $\mathfrak{h}_{0j} = \{a \in \mathfrak{a}_j : S(b_i)a = 0$ for all $i \in \{1, \ldots, d_0\}$ and $[b, a] = 0$ for all $b \in \mathfrak{m}\}$ and the vector space $\mathfrak{h}_{1j}$ is invariant under the $S(b_i)$ with $i \in \{1, \ldots, d_0\}$ and the $S(a)$ with $a \in \mathfrak{m}$. Moreover, $b_i \in \mathfrak{h}_{0w_i} \cup \mathfrak{h}_{1w_i}$ for all $i \in \{1, \ldots, d_Q\}$ and $1 = w_1 = \ldots = w_{d_0} \leq w_{d_0+1} \leq \ldots \leq w_{d_q}$.

VI. If $i_0 \in \{1, \ldots, d_0\}$, $j \in \{1, \ldots, d_Q\}$ and $S(b_{i_0})b_j \neq 0$ then $R_j = \{0\}$ and there exist $\delta \in \{-1, 1\}$ and $\lambda_1, \ldots, \lambda_{d_0} \in \mathbf{R}$ such that $S(b_i)b_j = \lambda_i \, b_{j+\delta}$ and $S(b_i)b_{j+\delta} = -\lambda_i \, b_j$ for all $i \in \{1, \ldots, d_0\}$.

VII. If $a \in \mathfrak{m}$, $i \in \{1, \ldots, d_Q\}$ and $[a, b_i] \neq 0$ then $R_i = \{0\}$.

VIII. The map $\widetilde{\Phi}: \mathbf{R}^{d_Q} \to \widetilde{Q}$ given by

$$\widetilde{\Phi}(t_1, \ldots, t_{d_Q}) = \exp_{\widetilde{Q}}(t_{d_Q} b_{d_Q}) \ldots \exp_{\widetilde{Q}}(t_1 b_1)$$

is a diffeomorphism and $\Gamma = \widetilde{\Phi}(R_1 \times \ldots \times R_{d_Q})$.

Lemma 3.4 *The Lie algebra* $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,])$ *is the smallest subalgebra of* $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$ *which contains* $\mathfrak{a}_1$.

Proof For the proof we need to introduce one more Lie bracket on $\mathfrak{q}$. For all $t > 0$ define the linear map $\gamma_t: \mathfrak{q} \to \mathfrak{q}$ by

$$\gamma_t(b_i) = t^{w_i} \, b_i$$

for all $i \in \{1, \ldots, d_Q\}$. We define a scale of Lie brackets on the vector space $\mathfrak{q}$. For $t > 0$ define $[\,\cdot\,,\,\cdot\,]_{Nt}: \mathfrak{q} \times \mathfrak{q} \to \mathfrak{q}$ by

$$[a, b]_{Nt} = \gamma_t^{-1}([\gamma_t(a), \gamma_t(b)]_N) \quad .$$

By [NRS], Section 3, $\lim_{t\to\infty}[a, b]_{Nt}$ exists and we set

$$[a, b]_H = \lim_{t\to\infty}[a, b]_{Nt}$$

for all $a, b \in \mathfrak{q}$. Obviously $\gamma_t([a, b]_H) = [\gamma_t(a), \gamma_t(b)]_H$ for all $a, b \in \mathfrak{q}$ and $t > 0$.

The proof now follows by establishing that the elements $b_1, \ldots, b_{d_1}$ form an algebraic basis first for the Lie algebra $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,]_H)$, then for the Lie algebra $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,]_N)$ and finally for the Lie algebra $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,])$, where $d_1 = \dim \mathfrak{a}_1$. If $\alpha = (i_1, \ldots, i_n) \in J(d)$ with $n \in \mathbf{N}$ then set $\|\alpha\| = w_{i+1} + \ldots + w_{i_n}$ and $b_{[\alpha]} = [b_{i_1}, [\ldots [b_{i_{n-1}}, b_{i_n}] \ldots]] \in \mathfrak{q}$. Define similarly $b_{[\alpha]_N}$ and $b_{[\alpha]_H}$. Then

$$b_{[\alpha]_N} = b_{[\alpha]_H} \pmod{\mathfrak{q}_{\|\alpha\|+1}} \tag{26}$$

for all $\alpha \in J(d)$ with $|\alpha| \neq 0$.

We first show that $b_1, \ldots, b_{d_1}$ is an algebraic basis for $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,]_H)$. Let $k \in \{1, \ldots, r\}$ and $a \in \mathfrak{a}_k$. Then for all $\alpha \in J(d)$ with $|\alpha| \geq k$ there exist $c_\alpha \in \mathbf{R}$ such that

$$a = \sum_{\substack{\alpha \in J(d) \\ k \leq |\alpha| \leq r}} c_\alpha \, b_{[\alpha]_N} \quad .$$

By (26) there exists a $b \in \mathfrak{q}_{k+1}$ such that

$$a = b + \sum_{\substack{\alpha \in J(d) \\ k \leq |\alpha| \leq r}} c_\alpha \, b_{[\alpha]_H} \quad .$$

Since $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,]_H)$ is homogeneous one deduces that

$$a = \sum_{\substack{\alpha \in J(d) \\ k \leq |\alpha| \leq r \\ \|\alpha\| = k}} c_\alpha \, b_{[\alpha]_H} \quad .$$

But if $\alpha \in J(d)$ with $\|\alpha\| = k$ and $|\alpha| \geq k$ then $\alpha \in J(d_1)$. So

$$a = \sum_{\substack{\alpha \in J(d_1) \\ |\alpha| = k}} c_\alpha \, b_{[\alpha]_H} \tag{27}$$

and $b_1, \ldots, b_{d_1}$ is an algebraic basis for $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,]_H)$.

Next we prove by induction that $b_1, \ldots, b_{d_1}$ is an algebraic basis for $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,]_N)$. Obviously for all $a \in \mathfrak{q}_r = \mathfrak{a}_r$ there exist $c_\alpha \in \mathbf{R}$ such that

$$a = \sum_{\substack{\alpha \in J(d_1) \\ |\alpha| = r}} c_\alpha \, b_{[\alpha]_H} = \sum_{\substack{\alpha \in J(d_1) \\ |\alpha| = r}} c_\alpha \, b_{[\alpha]_N}$$

by (27) and (26). Let $k \in \{1, \ldots, r-1\}$ and suppose that

$$\mathfrak{q}_{k+1} \subseteq \operatorname{span}\{b_{[\alpha]_N} : \alpha \in J(d_1)\} \quad . \tag{28}$$

Let $a \in \mathfrak{a}_k$. Then there exist $c_\alpha \in \mathbf{R}$ such that (27) is valid. Let $b \in \mathfrak{q}_{k+1}$ be such that

$$a = b + \sum_{\substack{\alpha \in J(d_1) \\ |\alpha| = k}} c_\alpha \, b_{[\alpha]_N} \quad .$$

Then together with (28) it follows that $a \in \operatorname{span}\{b_{[\alpha]_N} : \alpha \in J(d_1)\}$.

Finally we show that $a_1, \ldots, a_{d_1}$ is an algebraic basis for $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,])$. It suffices to prove that $b_{[\alpha]_N} \in \operatorname{span}\{b_{[\beta]} : \beta \in J(d_1)\}$ for all $\alpha \in J(d_1)$. But $[a, b]_N = [a, b] - S(a)b + S(b)a$ for all $a, b \in \{b_1, \ldots, b_{d_0}\} \cup \mathfrak{n}$ and $S(a)$ is a polynomial in $\operatorname{ad} a$ without constant term. Therefore expanding the commutator $b_{[\alpha]_N}$ from inside in terms of the Lie brackets $[\,\cdot\,,\,\cdot\,]$ one deduces that $b_{[\alpha]_N} \in \operatorname{span}\{b_{[\beta]} : \beta \in J(d_1)\}$. □

The next lemma is the main step in the proof of Proposition 3.3. To formulate it we need the Lie algebra $\mathfrak{e}$ of the Euclidean motion group, i.e., the Lie algebra with basis e_1, e_2, e_3 and commutation relations $[e_1, e_2] = 2\pi e_3$, $[e_1, e_3] = -2\pi e_2$ and $[e_2, e_3] = 0$. This algebra provided the counterexample of Alexopoulos [Ale1] on the boundedness of the Riesz transforms. Let E_s be the connected simply connected Lie group with Lie algebra $\mathfrak{e}$ and let $E = E_s / \Gamma_{E_s}$, where $\Gamma_{E_s} = \{\exp_{E_s}(k e_1) : k \in \mathbf{Z}\} = Z(E_s)$, the centre of E_s. It follows from the structure theory of [Ale1], in particular Property VIII, that E is, up to isomorphism, the connected not-simply connected Lie group with Lie algebra $\mathfrak{e}$.

Lemma 3.5 *Let Q be a connected solvable Lie group with Lie algebra $\mathfrak{q}$ and let $\mathfrak{n}$ be the nil-radical of $\mathfrak{q}$. The following are equivalent.*

I. $\mathfrak{q} \neq \mathfrak{n}$.

II. *There is a surjective Lie group homomorphism from Q to the Euclidean motion group E.*

Proof Clearly if the second condition is valid then Q, and hence $\mathfrak{q}$, cannot be nilpotent. Conversely, if $\mathfrak{q} \neq \mathfrak{n}$ then $d_0 \geq 1$. Then $S(b_1) \neq 0$ because otherwise $\mathrm{ad} b_1 = K(b_1)$ would be nilpotent and $b_1 \in \mathfrak{n}$ (see [Var], Corollary 3.8.4). But $(\mathfrak{q}, [\,\cdot\,,\,\cdot\,])$ is spanned as a Lie algebra by $\mathfrak{a}_1$ and $S(b_1)$ is a derivation. Hence there is a $j \in \{1, \ldots, d_1\}$ such that $S(b_1)b_j \neq 0$, where $d_1 = \dim \mathfrak{a}_1$. Then $j > d_0$ by Property I and $b_j \in \mathfrak{n}$. By Property VI there exist $\delta \in \{-1, 1\}$ and $\lambda_1, \ldots, \lambda_{d_0} \in \mathbf{R}$ such that $S(b_i)b_j = \lambda_i\, b_{j+\delta}$ and $S(b_i)b_{j+\delta} = -\lambda_i\, b_j$ for all $i \in \{1, \ldots, d_0\}$. Moreover, $b_{j+\delta} = \lambda_1^{-1} S(b_1)b_j \in \mathfrak{a}_1$ by Property V.

Next define the linear map $\psi: \mathfrak{q} \to \mathfrak{e}$ by

$$\psi(b_j) = e_2 \quad , \quad \psi(b_i) = (2\pi)^{-1}\lambda_i\, e_1 \ \text{ if } i \in \{1, \ldots, d_0\} \quad ,$$

$$\psi(b_{j+\delta}) = e_3 \quad , \quad \psi(b_k) = 0 \ \text{ if } k \notin \{1, \ldots, d_0, j, j+\delta\} \quad .$$

Let $i \in \{1, \ldots, d_0\}$. Then $[b_i, b_j]_N \in \mathfrak{q}_2$ and $\psi([b_i, b_j]_N) = 0$. Hence $\psi([b_i, b_j]) = \psi(S(b_i)b_j) + \psi(K(b_i)b_j) = \psi(\lambda_i\, b_{j+\delta}) + \psi([b_i, b_j]_N) = \lambda_i\, e_3 = [(2\pi)^{-1}\lambda_i\, e_1, e_2] = [\psi(b_i), \psi(b_j)]$. By analogous arguments it follows that ψ is a Lie algebra homomorphism.

We lift ψ to a Lie group homomorphism from $\widetilde{Q}$ to the Euclidean motion group E. There exists a unique Lie group homomorphism $\widetilde{\Psi}: \widetilde{Q} \to E$ such that $\widetilde{\Psi}(\exp_{\widetilde{Q}} a) = \exp_E \psi(a)$ for all $a \in \mathfrak{q}$.

We next show that $\widetilde{\Psi}(\Gamma) = \{e\}$, so that $\widetilde{\Psi}$ factors over Q. Let $i \in \{1, \ldots, d_0\}$ and suppose that $\exp_{\widetilde{Q}} b_i \in \Gamma$. Let Q_2 be the (normal) analytic subgroup of $\widetilde{Q}$ which has Lie algebra $\mathfrak{q}_2$. Then for all $t \in \mathbf{R}$ one has $\exp_{\widetilde{Q}} tb_j = \exp_{\widetilde{Q}} b_i \exp_{\widetilde{Q}} tb_j \exp_{\widetilde{Q}}(-b_i) = \exp_{\widetilde{Q}}(te^{\mathrm{ad} b_i} b_j)$ and hence

$$\exp_{\widetilde{Q}}(tb_j)Q_2 = \exp_{\widetilde{Q}}(te^{\mathrm{ad} b_i} b_j)Q_2 = \exp_{\widetilde{Q}}(te^{S(b_i)} b_j)Q_2 = \exp_{\widetilde{Q}}(t(\cos(\lambda_i)b_j + \sin(\lambda_i)b_{j+\delta}))Q_2 \quad .$$

Therefore $\lambda_i \in 2\pi\mathbf{Z}$. But then $\widetilde{\Psi}(\exp_{\widetilde{Q}} b_i) = \exp_E \psi(b_i) = \exp_E((2\pi)^{-1}\lambda_i\, e_1) = \{e\}$ since $\exp_{E_s}((2\pi)^{-1}\lambda_i\, e_1) \in \Gamma_{E_s}$.

Thus $\widetilde{\Psi}(\Gamma) = \{e\}$ and there exists a unique Lie group homomorphism $\Psi: Q \to E$ such that $\Psi \circ \pi = \widetilde{\Psi}$. Then $\Psi(\exp a) = \Psi(\pi \exp_{\widetilde{Q}} a) = \widetilde{\Psi}(\exp_{\widetilde{Q}} a) = \exp_E \psi(a)$ for all $a \in \mathfrak{q}$.

Finally, since $\lambda_1 \neq 0$ the map $\widetilde{\Psi}$ is surjective. □

Now we are prepared to complete the proof of Proposition 3.3.

Assume Condition $5_{1+\nu}$ is valid and $\mathfrak{q} \neq \mathfrak{n}$. Then the foregoing Lie group homomorphism Ψ from Q to the Euclidean motion group E exists. We use the notation of the proof of Lemma 3.5. Set $h_1 = \exp(\lambda_1^{-1}\pi b_1)$, $h_2 = \exp(b_j)$ and $k = h_1 h_2 h_1^{-1} h_2^{-1}$. Then $\Psi(k) = \exp_E(-2e_2)$ and $\Psi(k^n) = \exp_E(-2ne_2)$ for all $n \in \mathbf{Z}$. Let $|\cdot|_E$ be the modulus on E with respect to the vector basis e_1, e_2, e_3. Obviously $|\exp_E(-2ne_2)|_E \leq 2|n|$ for all $n \in \mathbf{Z}$. We next show that the inequality is actually an equality. There exists a unique $\varphi: E \to \mathbf{R}$ such that

$$\varphi(\exp_E(\xi_3 e_3)\exp_E(\xi_2 e_2)\exp_E(\xi_1 e_1)) = \xi_2$$

for all $(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$. Then

$$\Big(dR(e_1)\varphi\Big)(\exp_E(\xi_3 e_3)\exp_E(\xi_2 e_2)\exp_E(\xi_1 e_1)) = 0 \quad ,$$

$$\Big(dR(e_2)\varphi\Big)(\exp_E(\xi_3 e_3)\exp_E(\xi_2 e_2)\exp_E(\xi_1 e_1)) = \cos 2\pi\xi_1$$

and

$$\Big(dR(e_3)\varphi\Big)(\exp_E(\xi_3 e_3)\exp_E(\xi_2 e_2)\exp_E(\xi_1 e_1)) = -\sin 2\pi\xi_1$$

for all $(\xi_1,\xi_2,\xi_3) \in \mathbf{R}^3$. Now let $\gamma\colon[0,1] \to E$ be an absolutely continuous path with $\gamma(0) = e$ and $\gamma(1) = \exp_E(-2ne_2)$. Then

$$2|n| = -\operatorname{sgn} n\int_0^1 dt\,\dot\gamma(t)\,\varphi = -\operatorname{sgn} n\int_0^1 dt\sum_{i=1}^3 \gamma_i(t)\Big(dR(e_i)\varphi\Big)(\gamma(t)) \le \int_0^1 dt\Big(\sum_{i=1}^3|\gamma_i(t)|^2\Big)^{1/2} .$$

Therefore $2|n| \le |\exp_E(-2ne_2)|_E$ and $|\Psi(k^n)|_E = 2|n|$ for all $n \in \mathbf{Z}$.

By Lemmas 3.1 and 3.2 there exist $c_1, c_2 > 0$ such that $c_1\,|\Psi(g)|_E \le |g|'$ for all $g \in Q$ with $|\Psi(g)|_E \ge c_2$. Hence $|k^n|' \ge 2c_1\,|n|$ for all $n \in \mathbf{Z}$ with $|n| \ge c_2/2$.

By Corollary 2.10 this implies that Condition $5_{1+\nu}$ is not valid. This is a contradiction and hence $\mathfrak{q} = \mathfrak{n}$. □

We are now in a position to establish the principle conclusion of this section.

Theorem 3.6 *If $\nu \in \langle 0,1]$ and Condition $5_{1+\nu}$ is valid then G is the local direct product of a compact and a nilpotent group.*

Proof We use the notation and basis as in the proof of Proposition 3.3. Let $a \in \mathfrak{m}$ and $b \in \mathfrak{q}$. Since $k \mapsto \operatorname{Ad}(k)b$ from the compact K into $\mathfrak{g}$ is bounded and, moreover, all eigenvalues of $S(a)$ are purely imaginary, it follows from the identity $e^{tK(a)}b = e^{-tS(a)}\operatorname{Ad}(\exp(ta))b$ that the function $t \mapsto e^{tK(a)}b$ is bounded from $\mathbf{R}$ into $\mathfrak{g}$. Hence $K(a)b = 0$ and $[a,b] = S(a)b$.

It follows from Proposition 3.3 that the radical $\mathfrak{q}$ of $\mathfrak{g}$ is nilpotent, i.e., $\mathfrak{q} = \mathfrak{n}$. If the semidirect product of $\mathfrak{m}$ and $\mathfrak{q}$ is not direct then by Lemma 3.4 there exists an $a \in \mathfrak{m}$ such that $S(a)\mathfrak{a}_1 \neq \{0\}$. Then $S(a)\mathfrak{h}_{11} \neq \{0\}$. In addition $S(a)\mathfrak{h}_{11} \subseteq \mathfrak{h}_{11}$ by Property VI. If one complexifies the space $\mathfrak{h}_{11}$ and the semisimple operator $S(a)$, also denoted by $S(a)$, then $S(a)$ can be diagonalized. Since G has polynomial growth, each eigenvalue of $\operatorname{ad} a = S(a)$ is purely imaginary. Then the operator $S(a)$ must have a complex eigenvector in $\mathfrak{h}_{11}$ whose eigenvalue is not zero. Passing back to the real vector space this implies that there exist $\lambda \in \mathbf{R}\backslash\{0\}$, $b, c \in \mathfrak{h}_{11}\backslash\{0\}$ such that $S(a)b = \lambda\, c$ and $S(a)c = -\lambda\, b$. Set $h_1 = \exp(\lambda^{-1}\pi\, a)$ and $h_2 = \exp b$. Then $k = h_1 h_2 h_1^{-1} h_2^{-2} = \exp(-2b)$.

Let $d_1' = \dim\mathfrak{h}_{11}$. We may assume that $b_i \in \mathfrak{h}_{11}$ for all $i \in \{1,\ldots,d_1'\}$ and $b_i \in \mathfrak{h}_{01}$ for all $i \in \{d_1'+1,\ldots,\dim\mathfrak{a}_1\}$. Write $b = \sum_{i=1}^{d_1'} t_i\, b_i$ with $t_1,\ldots,t_{d_1'} \in \mathbf{R}$. Then there exists an $i_0 \in \{1,\ldots,d_1'\}$ such that $t_{i_0} \neq 0$ and obviously $b_{i_0} \in \mathfrak{h}_{11}$. But $\mathfrak{h}_{11} = \{\tilde a \in \mathfrak{a}_1 :$ there exists a $\tilde b \in \mathfrak{m}$ such that $[\tilde a, \tilde b] \neq 0\}$ since $\mathfrak{n} = \mathfrak{q}$ and $d_0 = 0$. Therefore $R_{i_0} = \{0\}$ by Property VII. Hence there exists a Lie group homomorphism $\Psi\colon Q \to \mathbf{R}$ such that $\Psi(\exp(tb_{i_0})) = t$ and $\Psi(\exp(tb_j)) = 0$ for all $t \in \mathbf{R}$ and $j \in \{1,\ldots,d_Q\}\backslash\{i_0\}$. Then $\Psi(k^n) = -2nt_{i_0}$ for all $n \in \mathbf{Z}$ and one deduces a contradiction as before.

Thus $\mathfrak{g}$ is the direct product of the Lie algebras $\mathfrak{m}$ and $\mathfrak{n}$. But also $G = QM = NM$. Therefore G is the local direct product of M and N. □

4 Dénouement

In this section we complete the chain of reasoning required to prove Theorem 1.1. It already follows from Proposition 2.8 that $1_n \Rightarrow 2_n \Rightarrow 3_n \Rightarrow 4_n \Rightarrow 5_n \Rightarrow 5_2$. Moreover, Condition 5_2

implies that G is the local direct product of a connected compact Lie group and a connected nilpotent Lie group N by Theorem 3.6. Therefore the proof of Theorem 1.1, and its extension to derivatives of all orders, is completed by the next result.

Proposition 4.1 *Let G be the local direct product of a connected compact Lie group K and a connected nilpotent Lie group N and let $a_1, \ldots, a_{d'}$ be an arbitrary algebraic basis of the Lie algebra of G. If A_i are the left representatives and H the sublaplacian associated with the algebraic basis then for each $n \in \mathbf{N}$ there is a $c_n > 1$ such that*

$$c_n^{-1} \, \|H^{n/2}\varphi\|_2 \leq \sup_{|\alpha|=n} \|A^\alpha \varphi\|_2 \leq c_n \, \|H^{n/2}\varphi\|_2$$

for all $\varphi \in D(H^{n/2})$.

Proof First suppose that G is the direct product of K and N.

Let $g = (k, n)$ with $k \in K$ and $n \in N$ denote a general element of G. Further let dk and dn denote the Haar measures on K and N and $\mathfrak{k}$ and $\mathfrak{n}$ the Lie algebras. We normalize the Haar measure on K by $|K| = 1$. Let L_G, L_K and L_N denote the left regular representations of G, K and N.

Define the projection $P_N \colon L_2(G\,; dg) \to L_2(N\,; dn)$ by

$$(P_N \varphi)(n) = \int_K dk \, \varphi(k^{-1}, n)$$

for almost every $n \in N$ and the isometric lifting $T \colon L_2(N\,; dn) \to L_2(G\,; dg)$ by

$$(T\varphi)(k, n) = \varphi(n)$$

for almost every $(k, n) \in G$. Define the projection $P \colon L_2(G\,; dg) \to L_2(G\,; dg)$ by

$$P = T P_N = \int_K dk \, L_G(k, e) \quad .$$

Then $L_G(k, n)\, P = T L_N(n)\, P_N = P L_G(k, n)$ for all $(k, n) \in G$. Hence the subspace $PL_2(G\,; dg)$ and its orthogonal complement $(I - P)L_2(G\,; dg)$ are both L-invariant. Therefore the restrictions of H to the spaces $PL_2(G\,; dg)$ and $(I - P)L_2(G\,; dg)$ are both self-adjoint. Moreover, H commutes with P.

Each a_i has a unique decomposition $a_i = a_i^{(K)} + a_i^{(N)}$ with $a_i^{(K)} \in \mathfrak{k}$ and $a_i^{(N)} \in \mathfrak{n}$. The $a_1^{(K)}, \ldots, a_{d'}^{(K)}$ are an algebraic basis for $\mathfrak{k}$ and the $a_1^{(N)}, \ldots, a_{d'}^{(N)}$ an algebraic basis for $\mathfrak{n}$. Let $A_i = dL_G(a_i)$, $K_i = dL_K(a_i^{(K)})$ and $N_i = dL_N(a_i^{(N)})$ and set

$$H_K = -\sum_{i=1}^{d'} K_i^2 \quad \text{and} \quad H_N = -\sum_{i=1}^{d'} N_i^2 \quad .$$

If $\varphi \in D(A_i)$ then $P\varphi \in D(A_i)$ and $A_i P\varphi = P A_i \varphi$. Moreover, $A_i P = T N_i P_N$, $A^\alpha P = T N^\alpha P_N$ and $HP = T H_N P_N$ by the various definitions. Therefore one has bounds

$$\|A^\alpha P\varphi\|_2 = \|N^\alpha P_N \varphi\|_2 \leq c_{|\alpha|} \, \|H_N^{|\alpha|/2} P_N \varphi\|_2 = c_{|\alpha|} \, \|H^{|\alpha|/2} P\varphi\|_2 \qquad (29)$$

for all α and all $\varphi \in D(H^{|\alpha|/2})$ because the Riesz transforms on a nilpotent group are bounded by [ERS], Lemma 4.2.

Next we establish similar bounds on $(I-P)L_2(G\,;dg)$. The basic idea is to prove that the restriction $H(I-P)$ of H to $(I-P)L_2(G\,;dg)$ has spectrum in an interval $[\mu,\infty\rangle$ where $\mu>0$.

Fix $n\in N$. Then for each $\varphi\in C_c(G)$ introduce $\varphi_n\in L_2(K\,;dk)$ by setting $\varphi_n(k)=\varphi(k,n)$. The set $\{\varphi_n:\varphi\in C_c^\infty(G)\}$ is dense in $L_2(K\,;dk)$ and $((I-P)\varphi)_n$ is orthogonal to the constant functions on K. Moreover, $(L_G(k,e)\,(I-P)\varphi)_n=L_K(k)\,((I-P)\varphi)_n$ for all $k\in K$, $n\in N$ and $\varphi\in C_c(G)$. Therefore $(dL_G(a_i^{(K)})(I-P)\varphi)_n=K_i((I-P)\varphi)_n$ if $\varphi\in C_c^\infty(G)$. Now H_K acting on $L_2(K\,;dk)$ has a compact resolvent and there is a $\lambda>0$ such that $H_K\geq\lambda I$ on the orthogonal complement of the constant functions. Therefore

$$\begin{aligned}\sum_{i=1}^{d'}\|dL_G(a_i^{(K)})(I-P)\varphi\|_2^2&=\sum_{i=1}^{d'}\int_N dn\,\|K_i((I-P)\varphi)_n\|_2^2\\&=\int_N dn\,(((I-P)\varphi)_n,H_K((I-P)\varphi)_n)\\&\geq\lambda\int_N dn\,\|((I-P)\varphi)_n\|_2^2=\lambda\,\|(I-P)\varphi\|_2^2\end{aligned}\tag{30}$$

for all $\varphi\in C_c^\infty(G)$. Next we derive an upper bound on the sum with the aid of the following asymptotic estimates.

Lemma 4.2 *Let S denote the semigroup generated by H on $L_2(G\,;dg)$. Then for each $n\in\mathbf{N}$ there exist $c_{n,0}>0$ and $c_{n,1}\geq0$ such that*

$$\sup_{|\alpha|=n}\|A^\alpha S_t\|_{2\to2}\leq c_{n,0}\,t^{-n/2}+c_{n,1}t^{-1/4}$$

for all $t>0$. Hence for each $N>n$ there is a $C_N>0$ such that

$$\sup_{|\alpha|=n}\|A^\alpha\varphi\|_2\leq c_N\,\varepsilon^{-2N+1}\,\|H^{N/2}\varphi\|_2+\varepsilon\,\|\varphi\|_2$$

for all $\varphi\in D(H^{N/2})$ and all $\varepsilon\in\langle0,1]$.

Proof Let $\alpha=(\beta,i_n)$ with $|\alpha|=n$. Then

$$\|A^\alpha S_t\varphi\|_2^2=(A_{i_n}S_t\varphi,(-1)^{|\beta|}A^{\beta_*}A^\alpha S_t\varphi)\leq\|A_{i_n}S_t\|_{2\to2}\,\|A^{\beta_*}A^\alpha S_t\|_{2\to2}\,\|\varphi\|_2^2$$

where β_* is the reversal of β. But

$$\|A_{i_n}S_t\|_{2\to2}\leq\|H^{1/2}S_t\|_{2\to2}\leq c\,t^{-1/2}$$

by (1) and spectral theory. Moreover,

$$\|A^{\beta_*}A^\alpha S_t\|_{2\to2}\leq c_n\left(\|S_t\|_{2\to2}+\|H^{(2n-1)/2}S_t\|_{2\to2}\right)$$

for a suitable $c_n>0$ by [ElR1], Theorem 7.2.IV. Then

$$\|A^{\beta_*}A^\alpha S_t\|_{2\to2}\leq c_n'\,(1+t^{-n+1/2})$$

by another application of spectral theory. Combining these estimates gives the first bounds of the lemma.

The second bounds follow from the first using the Laplace transform estimate,

$$\|A^\alpha (H+\varepsilon^4 I)^{-N/2}\psi\|_2 \le \Gamma(N/2)^{-1}\int_0^\infty dt\, t^{-1}\, e^{-\varepsilon^4 t}\, t^{N/2}\|A^\alpha S_t\|_{2\to 2}\,\|\psi\|_2 \quad ,$$

which is valid for all $\psi \in L_2$ and all $\varepsilon > 0$, and rearranging. □

Next since $a_1, \ldots, a_{d'}$ is an algebraic basis each $a_i^{(K)}$ can be expressed as a polynomial in the a_j. The lowest order term in these polynomials is at least one and the highest order term at most r, the rank of the basis. Therefore, by the second estimate of Lemma 4.2, for each $N > 2r$ there is a $c_N > 0$ such that

$$\Big(\sum_{i=1}^{d'} \|dL_G(a_i^{(K)})\varphi\|_2^2\Big)^{1/2} \le c_N\, \varepsilon^{-2N}\|H^{N/2}\varphi\|_2 + \varepsilon\,\|\varphi\|_2$$

for all $\varphi \in D(H^{N/2})$ and all $\varepsilon \in \langle 0,1]$. Replacing φ by $(I-P)\varphi$ and appealing to (30) one then deduces that

$$c_N\, \varepsilon^{-2N}\|H^{N/2}(I-P)\varphi\|_2 \ge (\lambda^{1/2}-\varepsilon)\,\|(I-P)\varphi\|_2$$

for all $\varphi \in C_c^\infty(G)$ and $\varepsilon \in \langle 0,1]$. Therefore choosing ε smaller than $\lambda^{1/2}$ one readily concludes that there is a $\mu > 0$ such that

$$\|H^{N/2}(I-P)\varphi\|_2 \ge \mu^{N/2}\,\|(I-P)\varphi\|_2 \tag{31}$$

for all $\varphi \in C_c^\infty(G)$ and, since $C_c^\infty(G)$ is dense in $D(H^{N/2})$, for all $\varphi \in D(H^{N/2})$. Hence the spectrum of H restricted to $(I-P)L_2(G\,;dg)$ must lie in $[\mu,\infty\rangle$ and the bounds (31) are valid for all $N \in \mathbf{N}$.

Now consider the unitary representation $g \mapsto L(g)(I-P)$ of G on $(I-P)L_2(G\,;dg)$. It follows from [ElR1], Theorem 7.2.IV, that one has bounds

$$\|A^\alpha(I-P)\varphi\|_2 \le c_{|\alpha|}\,(\|H^{|\alpha|/2}(I-P)\varphi\|_2 + \|(I-P)\varphi\|_2)$$

for some $c_{|\alpha|} > 0$ and all $\varphi \in (I-P)D(H^{|\alpha|/2})$. Then using (31) with $N = |\alpha|$ one obtains bounds

$$\|A^\alpha(I-P)\varphi\|_2 \le c'_{|\alpha|}\,\|H^{|\alpha|/2}(I-P)\varphi\|_2 \tag{32}$$

for all $\varphi \in (I-P)D(H^{|\alpha|/2})$.

Finally combination of (29) and (32) yields

$$\begin{aligned}\|A^\alpha\varphi\|_2 &\le \|A^\alpha P\varphi\|_2 + \|A^\alpha(I-P)\varphi\|_2\\ &\le c_{|\alpha|}\,\|H^{|\alpha|/2}P\varphi\|_2 + c'_{|\alpha|}\,\|H^{|\alpha|/2}(I-P)\varphi\|_2 \le C_{|\alpha|}\,\|H^{|\alpha|/2}\varphi\|_2\end{aligned}$$

for a suitable $C_{|\alpha|} > 0$ and all $\varphi \in D(H^{|\alpha|/2})$. This completes the proof Proposition 4.1 if G is the direct product of K and N.

Secondly, we drop the condition that G is the direct product, but merely assume that G is a local direct product of K and N. Let $\widetilde{G} = K\cdot N$ be the direct product of K and N and let $D = K \cap N$. Then D is a discrete central subgroup of G and $D \subseteq K$. Therefore D is finite. Moreover, G is isomorphic with $\widetilde{G}/D$. Hence it suffices to show that the Riesz transforms on $\widetilde{G}/D$ are bounded. Let $\pi: \widetilde{G} \to \widetilde{G}/D$ be the quotient map. We normalize the Haar measure on D by $|D| = 1$. Next normalize the Haar measure on $\widetilde{G}/D$ such that

$$\int_{\widetilde{G}} d\tilde{g}\, \varphi(\tilde{g}) = \int_{\widetilde{G}/D} d\dot{g} \int_D dh\, \varphi(gh)$$

for all $\varphi \in C_c(\widetilde{G})$, where $\dot{g} = \pi(g)$. For all functions $\varphi: \widetilde{G}/D \to \mathbf{C}$ define $\pi^*\varphi: \widetilde{G} \to \mathbf{C}$ by $\pi^*\varphi = \varphi \circ \pi$. Then $\int_{\widetilde{G}} \pi^*\varphi = \int_{\widetilde{G}/D} \varphi$ and hence $\|\pi^*\varphi\|_{\tilde{2}} = \|\varphi\|_2$ for all $\varphi \in C_c(\widetilde{G}/D)$, where $\|\cdot\|_{\tilde{2}}$ and $\|\cdot\|_2$ denote the L_2-norms on $\widetilde{G}$ and $\widetilde{G}/D$. Since D is zero-dimensional we can and do identify the Lie algebras of $\widetilde{G}$ and G. Let $\widetilde{A}_i$ and A_i denote the infinitesimal generators on $\widetilde{G}$ and $\widetilde{G}/D$. Then $\widetilde{A}_i\pi^*\varphi = \pi^* A_i\varphi$ for all $\varphi \in C_c^\infty(\widetilde{G}/D)$.

Let $\alpha \in J(d')$. By the above there exists a $c > 0$ such that $\|\widetilde{A}^\alpha\psi\|_{\tilde{2}} \le c\,\|\widetilde{H}^{|\alpha|/2}\psi\|_{\tilde{2}}$ for all $\psi \in C_c^\infty(\widetilde{G})$. Hence

$$\|A^\alpha\varphi\|_2 = \|\pi^* A^\alpha\varphi\|_{\tilde{2}} = \|\widetilde{A}^\alpha\pi^*\varphi\|_{\tilde{2}} \le c\,\|\widetilde{H}^{|\alpha|/2}\pi^*\varphi\|_{\tilde{2}} = c\,\|H^{|\alpha|/2}\pi^*\varphi\|_2$$

for all $\varphi \in C_c^\infty(\widetilde{G}/D)$ and the proposition follows by a density argument.

Finally, the lower bounds of the proposition are easy. For even n they are obvious and the case $n = 1$ follows from (1). But then the case $n = 2k + 1$ with $k \in \mathbf{N}$ is also elementary. □

Boundedness of the fractional analogues of the Riesz transforms can now be established by interpolation theory but one needs to exercise care since the spaces $L'_{2;n}$ equipped with the norms $\varphi \mapsto N'_n(\varphi) = \max_{|\alpha|=n} \|A^\alpha\varphi\|_2$ and the spaces $D(H^\gamma)$ equipped with the norms $\varphi \mapsto \|\varphi\|_{D(H^\gamma)} = \|H^\gamma\varphi\|_2$ are not complete. This gives some difficulty with the application of standard complex interpolation theory.

Proposition 4.3 *If G is the local direct product of a connected compact group and a connected nilpotent group, $n \in \mathbf{N}$ and $\nu \in \langle 0, 1\rangle$ then there exists a $c > 0$ such that*

$$\sup_{h \in G\setminus\{e\}} \max_{|\alpha|=n} (|h|')^{-\nu} \|(I - L(h))A^\alpha\varphi\|_2 \le c\,\|H^{(n+\nu)/2}\varphi\|_2$$

for all $\varphi \in D(H^{(n+\nu)/2})$.

Proof Let $\alpha \in J(d')$ with $|\alpha| = n$. Since H is self-adjoint it has a bounded H_∞-functional calculus and hence

$$M_0 = \sup_{\varepsilon \in [0,1]} \|A^\alpha (H + \varepsilon I)^{-n/2}\|_{2\to 2} \le \sup_{\varepsilon \in [0,1]} \|A^\alpha H^{-n/2}\|_{2\to 2} \|H^{n/2}(H + \varepsilon I)^{-n/2}\|_{2\to 2} < \infty \quad .$$

Similarly,

$$M_1 = \sup_{\varepsilon \in [0,1]} \max_{i \in \{1,\ldots,d'\}} \|A_i\, A^\alpha (H + \varepsilon I)^{-(n+1)/2}\|_{2\to 2} < \infty \quad .$$

Next for all $\varepsilon \in \langle 0, 1]$ and $\gamma > 0$ equip the spaces $D((H + \varepsilon I)^\gamma)$ with the norm $\varphi \mapsto \|\varphi\|_{D((H+\varepsilon I)^\gamma)} = \|(H + \varepsilon I)^\gamma\varphi\|_2$. Note that these spaces are complete.

Let $\varepsilon > 0$ and $h \in G$. Then the operator $(I - L(h))A^\alpha$ is a bounded operator from $D((H + \varepsilon I)^{n/2})$ into L_2, with norm less than or equal to $2M_0$. Moreover, the operator

$(I - L(h))A^\alpha$ is a bounded operator from $D((H+\varepsilon I)^{(n+1)/2})$ into L_2 with norm less than or equal to $M_1 \, |h|'$. Then complex interpolation gives

$$\|(I - L(h))A^\alpha \varphi\|_2 \leq (2M_0)^{1-\nu} (M_1 \, |h|')^\nu \|\varphi\|_{[D((H+\varepsilon I)^{n/2}), D((H+\varepsilon I)^{(n+1)/2})]_\nu}$$

uniformly for all $\varphi \in [D((H+\varepsilon I)^{n/2}), D((H+\varepsilon I)^{(n+1)/2})]_\nu$. Since the operators $H + \varepsilon I$ have bounded imaginary powers, uniformly for $\varepsilon > 0$, it follows from the proof of Step 3 of Theorem 1.15.3 in [Tri78] that there exists a $c > 0$, independent of $\varepsilon \in \langle 0, 1]$ and h, such that

$$\|\varphi\|_{[D((H+\varepsilon I)^{n/2}), D((H+\varepsilon I)^{(n+1)/2})]_\nu} \leq c \, \|(H+\varepsilon I)^{(n+\nu)/2} \varphi\|_2$$

uniformly for all $\varphi \in D((H+\varepsilon I)^{(n+\nu)/2})$. Combining the two estimates it follows that

$$\|(I - L(h))A^\alpha \varphi\|_2 \leq c_1 \, (|h|')^\nu \|(H+\varepsilon I)^{(n+\nu)/2} \varphi\|_2$$

uniformly for all $\varphi \in D((H+\varepsilon I)^{(n+\nu)/2})$, where $c_1 = (2M_0)^{1-\nu} M_1^\nu c$ is independent of ε and h.

The estimates of the proposition now follow by taking the limit $\varepsilon \to 0$. □

Note that on any group with polynomial growth Condition 1_1 is valid and $\|(I - L(h))\varphi\|_2 \leq 2 \, \|\varphi\|_2$. Hence $\|(I - L(h))\varphi\|_2 \leq c \, (|h|')^\nu \|H^{\nu/2} \varphi\|_2$ for all $\varphi \in D(H^{\nu/2})$ and all $\nu \in \langle 0, 1 \rangle$ by the last argument.

Finally it follows by combination of the statements of Proposition 2.8, Theorem 3.6, Proposition 4.1 and Proposition 4.3 that one has the following conclusion.

Theorem 4.4 *Conditions 1_s–5_s are equivalent for all $s > 1$ and are valid if, and only if, G is the local direct product of a connected compact group and a connected nilpotent group.*

This theorem incorporates Theorem 1.1. As Conditions 1_1–5_1 are always valid the theorem states that the corresponding bounds are the best possible for a general group with polynomial growth.

5 Concluding remarks

The foregoing discussion focussed on the Riesz transforms associated with the sublaplacian H acting on $L_2(G\,; dg)$. But one can also deduce boundedness properties etc. on the L_p-spaces with $p \in \langle 1, \infty \rangle$. If G is the local direct product of a connected compact group and a connected nilpotent group, then one has boundedness of the Riesz transforms on the L_p-spaces and, in addition, optimal kernel bounds of any order.

Proposition 5.1 *If G is the local direct product of a connected compact group and a connected nilpotent group, $p \in \langle 1, \infty \rangle$ and $n \in \mathbf{N}$ then there exists a $c_n > 1$ such that*

$$c_n^{-1} \, \|H^{n/2} \varphi\|_p \leq \sup_{|\alpha| = n} \|A^\alpha \varphi\|_p \leq c_n \, \|H^{n/2} \varphi\|_p$$

for all $\varphi \in D(H^{n/2})$.

Proof It follows by an argument similar to the proof of Proposition 4.7 in [ERS] that the operator $A^\alpha H^{-|\alpha|/2}$ is of weak type $(1,1)$. Hence by interpolation the Riesz transforms are bounded on L_p for all $p \in \langle 1,2]$. But the dual operators of the Riesz transforms are bounded on L_2 and one has similar kernel estimates for these operators. So the same argument applies and the Riesz transforms are bounded on L_p for all $p \in [2,\infty\rangle$. This proves the upper bounds of the proposition.

The lower bounds are again easy, except for the case $n = 1$. Let $\varphi \in D(H) \subset L_p$ and $\psi \in D(H^{-1/2}) \subset L_q$, where q is the dual exponent. Then

$$(\psi, H^{1/2}\varphi) = (\psi, H^{-1/2}\, H\varphi) = (H^{-1/2}\psi, H\varphi) = -\sum_{i=1}^{d'}(H^{-1/2}\psi, A_i^2\varphi) = \sum_{i=1}^{d'}(A_i H^{-1/2}\psi, A_i\varphi)$$

since the range of $H^{-1/2}$ is contained in the domain of the operator A_i in L_q. But the Riesz transforms are bounded on L_q and therefore there exists a $c > 0$ such that

$$|(\psi, H^{1/2}\varphi)| \leq c \sum_{i=1}^{d'} \|\psi\|_q \, \|A_i\varphi\|_p$$

uniformly for all $\varphi \in D(H)$ and $\psi \in D(H^{-1/2})$. Since $D(H^{-1/2})$ is dense in L_q it follows that $\|H^{1/2}\varphi\|_p \leq c \sum_{i=1}^{d'} \|A_i\varphi\|_p$ for all $\varphi \in D(H)$ and then, by density, for all $\varphi \in L'_{p;1}$. □

Finally, since the operator H on L_p has a bounded H_∞-functional calculus (see, for example, [DuR], Theorem 3.4) the proof of Proposition 4.3 can be carried over line by line and one deduces boundedness of the fractional Riesz transforms on the L_p-spaces.

Proposition 5.2 *If G is the local direct product of a connected compact group and a connected nilpotent group, $n \in \mathbf{N}_0$, $\nu \in \langle 0,1\rangle$ and $p \in \langle 1,\infty\rangle$ then there exists a $c > 0$ such that*

$$\sup_{h\in G} \max_{|\alpha|=n} (|h|')^{-\nu} \|(I - L(h))A^\alpha\varphi\|_p \leq c \, \|H^{(n+\nu)/2}\varphi\|_p$$

for all $\varphi \in D(H^{(n+\nu)/2})$.

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