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A smooth subadditive homogeneous norm on a homogeneous group

by

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Abstract. We prove that on every homogeneous group there exists a smooth, subadditive and homogeneous norm.

Introduction. Around 1970 E. M. Stein introduced the notion of a homogeneous group. Such a group G admits a homogeneous norm $\|\cdot\|$, which for a $\gamma \geqslant 1$ satisfies

$$\|xy\| \leqslant \gamma(\|x\| + \|y\|) \quad \text{for all } x, y \in G.$$

The group equipped with $\|\cdot\|$ and the Haar (Lebesgue) measure is a space of homogeneous type in the sense of [1]. A number of estimates become easier if $\gamma = 1$, i.e. if the homogeneous norm is subadditive, so that it gives rise to a left-invariant metric. It is known that for some homogeneous groups such a norm exists, e.g. for Heisenberg groups and the like [2]. Also for stratified groups the optimal control metric is homogeneous.

The aim of this note is to show that a homogeneous and subadditive norm exists for every homogeneous group and in fact the construction is quite simple. More information about such norms is supplied by Theorem 2.

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A smooth subadditive homogeneous norm on a homogeneous group. A family of *dilations* on a nilpotent Lie algebra G is a one-parameter group $\{\delta_t\}_{t>0}$ $(\delta_t \circ \delta_s = \delta_{ts})$ of automorphisms of G determined by

$$\delta_t e_j = t^{d_j} e_j,$$

where $e_1, \ldots, e_n$ is a linear basis for G, the d_j are real numbers and $d_n \geqslant \ldots \geqslant d_1 \geqslant 1$. If we put $(x_1, \ldots, x_n) = \sum x_i e_i$, then

$$\delta_t(x_1, \ldots, x_n) = (t^{d_1} x_1, \ldots, t^{d_n} x_n).$$

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If we regard G as a Lie group with multiplication given by the Campbell–Hausdorff formula, then the dilations δ_t are also automorphisms of the group structure on G, and the nilpotent group G equipped with these dilations is called a *homogeneous group* (cf. [3]).

We are going to show that on every homogeneous group G there exists a subadditive and homogeneous norm, i.e. a function $\|\cdot\|$: $G \to \mathbf{R}^+ \cup \{0\}$ such that

(a) $\|xy\| \leqslant \|x\| + \|y\|$, (b) $\|\delta_t x\| = t\|x\|$,
(c) $\|x\| = 0 \Leftrightarrow x = 0$, (d) $\|x\| = \|x^{-1}\|$,
(e) $\|\cdot\|$ is continuous, (f) $\|\cdot\|$ is smooth on $G - \{0\}$.

The existence of $\|\cdot\|$ which satisfies (a)–(e) is equivalent to the existence of a set $A \subset G$ which satisfies the following conditions:

(α) A is open and $\bar{A}$ is compact,
(β) A is convex, i.e. if $x \in A$ and $y \in A$, $1 \geqslant t \geqslant 0$, then $\delta_t x \cdot \delta_{1-t} y \in A$,
(γ) A is symmetric, i.e. if $x \in A$, then $x^{-1} \in A$.

In fact, given a set A satisfying (α)–(γ), we put

$$\|x\| = \inf\{t\colon\ \delta_{1/t} x \in A\}.$$

Now, if $\|x\| < \varepsilon$ and $\|y\| < \varepsilon'$, then $\delta_{1/\varepsilon} x \in A$, $\delta_{1/\varepsilon'} y \in A$ and by (β)

$$\delta_{1/(\varepsilon+\varepsilon')} xy = \delta_{\varepsilon/(\varepsilon+\varepsilon')} \delta_{1/\varepsilon} x \cdot \delta_{\varepsilon'/(\varepsilon+\varepsilon')} \delta_{1/\varepsilon'} y \in A,$$

so $\|xy\| < \varepsilon + \varepsilon'$. This proves (a). The rest is easy.

The converse is obtained by putting $A = \{x \in G\colon\ \|x\| < 1\}$.

Moreover, we see that the condition

(ε) (i) the boundary ∂A of A is a smooth manifold,
(ii) $(d/dt)\delta_t x|_{t=1} \notin T_x \partial A$ for every $x \in \partial A$,

is equivalent to (f).

THEOREM 1. *For every homogeneous group G there exists a set A which satisfies (α)–(ε), hence G admits a norm which satisfies (a)–(f).*

Proof. If G is abelian we put $A = \{x = (x_1, \ldots, x_n)\colon \sum x_i^2 < 1\}$. To see that A satisfies (β) note that $d_i \geqslant 1$, so

$$\begin{aligned}\Big(\sum (t^{d_i} x_i + (1-t)^{d_i} y_i)^2\Big)^{1/2} &\leqslant \Big(\sum (t^{d_i} x_i)^2\Big)^{1/2} + \sum\big((1-t)^{d_i} y_i)^2\big)^{1/2} \\ &\leqslant t\Big(\sum x_i^2\Big)^{1/2} + (1-t)\Big(\sum y_i^2\Big)^{1/2}.\end{aligned}$$

(α), (γ) and (ε) are obvious.

We notice that if G is not abelian, then $d_n \geqslant 2$ and e_n is in the center of G, for $\delta_t[e_i, e_j] = [\delta_t e_i, \delta_t e_j] = t^{d_i+d_j}[e_i, e_j]$ and we assume that $1 \leqslant d_1 \leqslant \ldots \leqslant d_n$. By the Campbell–Hausdorff formula we have

$$\begin{aligned}(x_1, \ldots, x_n)(y_1, \ldots, y_n) = (x_1 + y_1 + P_1(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}), \\ \ldots, x_n + y_n + P_n(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})),\end{aligned}$$

where the P_i are polynomials and since e_n is in the center of G ($[e_n, e_i] = 0$ for $1 \leqslant i \leqslant n$), neither x_n nor y_n appears in any of the P_i.

Now we proceed by induction on dim G. Let A' be a subset of the quotient group $G' = G/\mathrm{lin}\{e_n\} = \{\bar{x} = (x_1, \ldots, x_{n-1})\colon x_i \in \mathbf{R}\}$ which satisfies (α)–(ε) and $\|\cdot\|'$ the corresponding norm. There exists a constant C such that

$$(*) \qquad |P_n(\delta_t x, \delta_{1-t} y)| \leqslant 2Ct(1-t) \quad \text{for all } x, y \in A',\ 0 \leqslant t \leqslant 1.$$

Indeed, since $P_n(x, 0) = P_n(0, y) = 0$, we see that every monomial in P_n depends both on x and y; hence, since A' is bounded, (*) holds for some C. If $x = (x_1, \ldots, x_n)$, then put $\bar{x} = (x_1, \ldots, x_{n-1})$. We prove that the set

$$A = \{x \in G\colon\ \bar{x} \in A' \text{ and } |x_n| < C + f(\|\bar{x}\|')\}$$

satisfies (α)–(ε) too, where C is the constant from (*), $f \in C^\infty(0, 1)$, $f' \leqslant 0$, $f'' \leqslant 0$, $f^{(k)}(0) = 0$, $f(0) = 1$, $f^{(k)}(1) = -\infty$, $f(1) = 0$ for $k = 1, 2, \ldots$

Remark. With $f = 0$ the construction yields a set A which satisfies (α)–(γ) but of course not (ε).

Proof of (α)–(ε) for A. (α) and (γ) are obvious. To show (β) notice that if $x \in A$ and $y \in A$, then $\overline{\delta_t x \cdot \delta_{1-t} y} = \delta_t \bar{x} \cdot \delta_{1-t} \bar{y} \in A'$. So, it is sufficient to prove the following inequality:

$$|t^{d_n} x_n + (1-t)^{d_n} y_n + P_n(\delta_t \bar{x}, \delta_{1-t} \bar{y})| < C + f(\|\delta_t \bar{x} \cdot \delta_{1-t} \bar{y}\|').$$

But $d_n \geqslant 2$, $0 \leqslant t \leqslant 1$, $f' \leqslant 0$, $f'' \leqslant 0$ and hence, by the definition of A

$$\begin{aligned}|t^{d_n} x_n + (1-t)^{d_n} y_n &+ P_n(\delta_t \bar{x}, \delta_{1-t} \bar{y})| \\ &< t^2(C + f(\|\bar{x}\|')) + (1-t)^2(C + f(\|\bar{y}\|')) + 2Ct(1-t) \\ &\leqslant C(t^2 + 2t(1-t) + (1-t)^2) + tf(\|\bar{x}\|') + (1-t)f(\|\bar{y}\|') \\ &\leqslant C + f(t\|\bar{x}\|' + (1-t)\|\bar{y}\|') \leqslant C + f(\|\delta_t \bar{x} \cdot \delta_{1-t} \bar{y}\|').\end{aligned}$$

(ε)(i) is obvious. We first prove (ε)(ii) for $x = (x_1, \ldots, x_n) \in \partial A$ such that $|x_n| \leqslant C$. Then $\bar{x} \in \partial A'$ and $T_x \partial A = T_{\bar{x}} \partial A' \oplus \mathbf{R} e_n$. So if $(d/dt)\delta_t x|_{t=1} \in T_x \partial A$, then $(d/dt)\delta_t x|_{t=1} = (d/dt)\delta_t \bar{x}|_{t=1} \in T_{\bar{x}} \partial A'$. But this contradicts the induction hypothesis. Now, we observe that the set $\partial A \cap \{x \in \mathbf{R}^n\colon x_n > C\}$ is the graph of the function $g(\bar{x}) = C + f(\|x\|')$, $g\colon A' \to \mathbf{R}$, and that if $v = (v_1, \ldots, v_n) \in T_{(\bar{x}, g(\bar{x}))} M$, where M is the graph of a function $g\colon X \to \mathbf{R}$, $\bar{x} \in X \subset \mathbf{R}^{n-1}$, then $v_n = (d/dt) g(\bar{x} + t\bar{v})|_{t=0} = \bar{v} g(\bar{x})$. Hence if $(d/dt)\delta_t x|_{t=1} \in T_x \partial A$, where $x = (\bar{x}, C + f(\|\bar{x}\|'))$, then by the definition of f ($f' \leqslant 0$),

$$\begin{aligned}0 < d_n x_n &= \big((d/dt)\delta_t \bar{x}|_{t=1}\big)\big(f(\|\bar{x}\|') + C\big) \\ &= (d/dt) f(\|\delta_t \bar{x}\|') = (d/dt) f(t\|\bar{x}\|') = f'(\|\bar{x}\|')\|\bar{x}\|' \leqslant 0.\end{aligned}$$

This contradiction proves (ε)(ii) for $\partial A \cap \{x \in \mathbf{R}^n\colon x_n > C\}$. For $\partial A \cap \{x \in \mathbf{R}^n\colon x_n < -C\}$, (ε)(ii) follows by symmetry.

Theorem 2 below exhibits a very simple "convex body", i.e. a set satisfying (α)–(ε), which yields a homogeneous subadditive norm. The proof, however, is more complicated.

THEOREM 2. *Let G be a homogeneous group and $x = (x_1, \ldots, x_n)$ homogeneous coordinates ($\delta_t x = (t^{d_1} x_1, \ldots, t^{d_n} x_n)$). There exists $\varepsilon > 0$ such that for $r < \varepsilon$ the set*

$$A = \{x\colon \textstyle\sum x_i^2 < r^2\}$$

satisfies the conditions (α)–(ε). Consequently there is a homogeneous subadditive norm on G

$$\|x\|' = \inf\{t\colon \|\delta_{1/t} x\| < r\}$$

such that the unit ball $\{x\colon \|x\|' < 1\}$ coincides with the Euclidean ball $\{x\colon \|x\| < r\}$ ($\|x\| = (\sum x_i^2)^{1/2}$).

Proof. We verify only the condition (β) because the others are satisfied trivially. Put

$$V_1 = \operatorname{lin}\{e_i\colon d_i < 2\}, \qquad V_2 = \operatorname{lin}\{e_i\colon d_i \geqslant 2\};$$

then $G = V_1 \oplus V_2$ as a linear space. Define $(x_1, x_2) = x_1 + x_2$, where $x_1 \in V_1$, $x_2 \in V_2$. Since $\delta_t [e_i, e_j] = t^{d_i + d_j} [e_i, e_j]$ and $d_k \geqslant 1$, it follows that $[x, y] \in V_2$ for all $x, y \in G$, so for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we have

$$x \cdot y = (x_1 + y_1,\ x_2 + y_2 + R(x, y)).$$

Let $R_1(x, y) = R((x_1, 0), (y_1, 0))$ and $R_2 = R - R_1$. In virtue of the Campbell–Hausdorff formula there is a constant C_1' such that for all $\|x\|, \|y\| < 1$

$$\|R_1(x, y)\| \leqslant C_1' \|[x_1, y_1]\|.$$

Hence, by the inequality

$$\|[x, y]\| \leqslant C_1'' \|x\| \, \|y\| \, \big\| x/\|x\| - y/\|y\| \big\|,$$

which is an easy consequence of the bilinearity and antisymmetry of [,], we have for some constant C_1

(1) $$\|R_1(x, y)\| \leqslant C_1 \|x_1\| \, \|y_1\| \, \big\| x_1/\|x_1\| - y_1/\|y_1\| \big\|$$

for all $\|x\|, \|y\| < 1$. Also by the Campbell–Hausdorff formula there is a constant C' such that for $\|x\|, \|y\| < 1$

(*) $$\|R_2(x, y)\| \leqslant C'(\|x_1\| \, \|y_2\| + \|x_2\| \, \|y_1\| + \|x_2\| \, \|y_2\|).$$

Let $v = \delta_t x_2 + \delta_{1-t} y_2 + R_2(\delta_t x, \delta_{1-t} y)$. By the definition $d_i \geqslant 2$ for $e_i \in V_2$, so in virtue of (*)

$$\|v\| \leqslant t^2 \|x_2\| + (1-t)^2 \|y_2\| + C' t(1-t)(\|x_1\| \, \|y_2\| + \|x_2\| \, \|y_1\| + \|x_2\| \, \|y_2\|).$$

Now, if we assume that $C'(\|x_1\| + \|x_2\| + \|y_1\|) \leqslant 1/2$ and $0 \leqslant t \leqslant 1$, then

$$\|v\| \leqslant t^2 \|x_2\| + (1-t)^2 \|y_2\| + \tfrac{1}{2} t(1-t)(\|x_2\| + \|y_2\|) \leqslant \|x_2\| + \|y_2\|$$

and

$$\begin{aligned}\|v\| &\leqslant t^2 \|x_2\| + (1-t)^2 \|y_2\| + \tfrac{1}{2} t(1-t)(\|x_2\| + \|y_2\|) \\ &= t\|x_2\| + (1-t)\|y_2\| - \tfrac{1}{2} t(1-t)(\|x_2\| + \|y_2\|).\end{aligned}$$

Therefore $\|v\| + \frac{1}{2} t(1-t)(\|x_2\| + \|y_2\|) \leqslant t\|x_2\| + (1-t)\|y_2\|$ and

(2) $$\begin{aligned}\|v\|^2 (1 + t(1-t)) &\leqslant \|v\|^2 + t(1-t)\|v\|(\|x_2\| + \|y_2\|) \\ &\leqslant (\|v\| + \tfrac{1}{2} t(1-t)(\|x_2\| + \|y_2\|))^2 \leqslant (t\|x_2\| + (1-t)\|y_2\|)^2.\end{aligned}$$

Note that $2(v_1, v_2) \leqslant t(1-t)\|v_1\|^2 + 4\|v_2\|^2/(t(1-t))$, where $(x, y) = \sum x_i y_i$ is the scalar product. Hence

(3) $$\|v + R_1(\delta_t x, \delta_{1-t} y)\|^2 \leqslant \|v\|^2 (1 + t(1-t)) + \|R_1\|^2 [1 + 4/(t(1-t))].$$

Observe also that

(4) $$(\|x\| + \|y\|)^2 = \|x + y\|^2 + \|x\| \, \|y\| \, \big\| x/\|x\| - y/\|y\| \big\|^2.$$

Finally, by (1)–(4) we have

$$\begin{aligned}\|\delta_t x \cdot \delta_{1-t} y\|^2 &= \|\delta_t x_1 + \delta_{1-t} y_1\|^2 + \|v + R_1(\delta_t x, \delta_{1-t} y)\|^2 \\ &\leqslant (\|\delta_t x_1\| + \|\delta_{1-t} y_1\|)^2 - \|\delta_t x_1\| \, \|\delta_{1-t} y_1\| \\ &\quad \times \big\| \delta_t x_1/\|\delta_t x_1\| - \delta_{1-t} y_1/\|\delta_{1-t} y_1\| \big\|^2 \\ &\quad + \|v\|^2 (1 + t(1-t)) + \|R_1\|^2 [1 + 4/(t(1-t))] \\ &\leqslant (t\|x_1\| + (1-t)\|y_1\|)^2 + (t\|x_2\| + (1-t)\|y_2\|)^2 \\ &\quad + [1 + 4/(t(1-t))] C_1^2 t(1-t) \|x_1\| \, \|y_1\| \, \|\delta_t x_1\| \, \|\delta_{1-t} y_1\| \\ &\quad \times \big\| \delta_t x_1/\|\delta_t x_1\| - \delta_{1-t} y_1/\|\delta_{1-t} y_1\| \big\|^2 \\ &\quad - \|\delta_t x_1\| \, \|\delta_{1-t} y_1\| \, \big\| \delta_t x_1/\|\delta_t x_1\| - \delta_{1-t} y_1/\|\delta_{1-t} y_1\| \big\|^2.\end{aligned}$$

However, if $5C_1^2 \|x_1\| \, \|y_1\| < 1$, then the sum of the last two expressions will be nonpositive, so

$$\begin{aligned}\|\delta_t x \cdot \delta_{1-t} y\|^2 &\leqslant (t\|x_1\| + (1-t)\|y_1\|)^2 + (t\|x_2\| + (1-t)\|y_2\|)^2 \\ &\leqslant (t\|x\| + (1-t)\|y\|)^2.\end{aligned}$$

This proves Theorem 2.