PLANCHEREL TYPE ESTIMATES AND SHARP SPECTRAL MULTIPLIERS

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Abstract. We study general spectral multiplier theorems for self-adjoint positive definite operators on $L^2(X,\mu)$, where X is any open subset of a space of homogeneous type. We show that the sharp Hörmander-type spectral multiplier theorems follow from the appropriate estimates of the L^2 norm of the kernel of spectral multipliers and the Gaussian bounds for the corresponding heat kernel. The sharp Hörmander-type spectral multiplier theorems are motivated and connected with sharp estimates for the critical exponent for the Riesz means summability, which we also study here. We discuss several examples, which include sharp spectral multiplier theorems for a class of scattering operators on $\mathbf{R}^3$ and new spectral multiplier theorems for the Laguerre and Hermite expansions.

1. Introduction

Suppose that A is a positive definite self-adjoint operator acting on $L^2(X)$, where X is a measure space. Such an operator admits a spectral resolution $E_A(\lambda)$ and for any bounded Borel function $F\colon [0,\infty) \to \mathbf{C}$, we define the operator $F(A)$ by the formula

$$F(A) = \int_0^\infty F(\lambda)\, \mathrm{d}E_A(\lambda). \tag{1.1}$$

By the spectral theorem the operator $F(A)$ is continuous on $L^2(X)$. Spectral multiplier theorems investigate sufficient conditions on function F which ensure that the operator $F(A)$ extends to a bounded operator on L^q for some q, $1 \le q \le \infty$.

The theory of spectral multipliers is related to and motivated by study of convergence of the Riesz means or convergence of other eigenfunction expansions of self-adjoint operators. To define the Riesz means of the operator A we put

$$\sigma_R^\alpha(\lambda) = \begin{cases} (1-\lambda/R)^\alpha & \text{for } \lambda \le R \\ 0 & \text{for } \lambda > R. \end{cases} \tag{1.2}$$

We then define the operator $\sigma_R^\alpha(A)$ using (1.1). We call $\sigma_R^\alpha(A)$ the Riesz or the Bochner-Riesz means of order α. The basic question in the theory of the Riesz means is to establish the critical exponent for the continuity and convergence of the Riesz means. More precisely we want to study the optimal range of α for which the Riesz means $\sigma_R^\alpha(A)$ are uniformly bounded on $L^1(X)$ (or other $L^q(X)$ spaces). Since the publication of Riesz's paper [44] the summability of the Riesz means has been one of the most fundamental problems in Harmonic Analysis (see e.g. [56, IX.2 and §IX.6B]). Despite the fact that the Riesz means have been extensively studied we do not have the full description of the optimal range of α even if we study only the space $L^1(X)$. On one hand we know that for the Laplace operator $\Delta_d = -\sum_{k=1}^d \partial_k^2$ acting on $\mathbf{R}^d$ and the Laplace-Beltrami operator acting on

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Date: 23/03/2001.

1991 *Mathematics Subject Classification.* Primary 42B15; Secondary 35P99.

Key words and phrases. Spectral multipliers.

compact d-dimensional Riemannian manifolds the critical exponent is equal $(d-1)/2$ (see [53]). This means that the Riesz means are uniformly continuous on $L^1(X)$ if and only if $\alpha > (d-1)/2$ (see also [8, 59]). On the other hand, if we consider more general operators like e.g. uniformly elliptic operators on R^d it is only known that the Riesz means are uniformly continuous on $L^1(X)$ if $\alpha > d/2$ (see [25]). One of the main points of our paper is to investigate the summability of the Riesz means for $d/2 \geq \alpha > (d-1)/2$.

Now we discuss two fairly specific but important examples of spectral multiplier theorems concerning group invariant Laplace operators acting on Lie groups of polynomial growth. As we will see this discussion is closely related to the summability of the Riesz means for $d/2 \geq \alpha > (d-1)/2$.

Let $\mathbf{G}$ be a Lie group of polynomial growth and let $X_1, \dots, X_k$ be a system of left-invariant vector fields on $\mathbf{G}$ satisfying the Hörmander condition. We define the Laplace operator L acting on $L^2(\mathbf{G})$ by the formula

$$L = -\sum_{i=1}^{k} X_i^2. \tag{1.3}$$

If $B(x,r)$ is the ball defined by the distance associated with system $X_1, \dots, X_k$ (see e.g. [62, §III.4]), then there exist natural numbers $d_0, d_\infty \geq 0$ such that $\mu(B(x,r)) \sim r^{d_0}$ for $r \leq 1$ and $\mu(B(x,r)) \sim r^{d_\infty}$ for $r > 1$ (see e.g. [62, §VIII.2]). We call $\mathbf{G}$ a homogeneous group if there exists a family of dilations on $\mathbf{G}$. A family of dilations on a Lie group $\mathbf{G}$ is a one-parameter group $(\tilde{\delta}_t)_{t>0}$ $(\tilde{\delta}_t \circ \tilde{\delta}_s = \tilde{\delta}_{ts})$ of automorphisms of $\mathbf{G}$ determined by

$$\tilde{\delta}_t Y_j = t^{d_j} Y_j, \tag{1.4}$$

where $Y_1, \dots, Y_l$ is a linear basis of Lie algebra of $\mathbf{G}$ and $d_j \geq 1$ for $1 \leq j \leq l$ (see [23]). We say that an operator L defined by (1.3) is homogeneous if $\tilde{\delta}_t X_i = t X_i$ for $1 \leq i \leq k$. For the homogeneous Laplace operator $d_0 = d_\infty = \sum_{j=1}^{l} d_j$ (see [23]).

Spectral multiplier theorems for the homogeneous Laplace operators acting on homogeneous groups were investigated by Hulanicki and Stein [33] (see also [23, Theorem 6.25]) and De Michele and Mauceri [16]. The following theorem was obtained independently by Christ [10] and Mauceri and Meda [37].

Theorem 1.1. *Let L be the homogeneous operator defined by the formula (1.3) acting on a homogeneous group $\mathbf{G}$. Denote by $d = d_0 = d_\infty$ the homogeneous dimension of the underlying group $\mathbf{G}$. Next suppose that $s > d/2$ and that $F\colon [0,\infty) \to \mathbf{C}$ is a bounded Borel function such that*

$$\sup_{t>0} \|\eta\, \delta_t F\|_{W^2_s} < \infty, \tag{1.5}$$

where $\delta_t F(\lambda) = F(t\lambda)$, $\|F\|_{W^p_s} = \|(I - \mathrm{d}^2/\,\mathrm{d}x^2)^{s/2} F\|_{L^p}$ and $\eta \in C_c^\infty(\mathbf{R}_+)$ is a fixed function, not identically zero. Then $F(L)$ is of weak type $(1,1)$ and bounded on L^q when $1 < q < \infty$.

Condition (1.5) is actually independent of the choice of η. Once and for all we fix a nonzero cut-off function $\eta \in C_c^\infty(\mathbf{R}_+)$.

The Hörmander multiplier theorem gives a sufficient condition for a Fourier multiplier to extend to an operator bounded on $L^p(\mathbf{R}^d)$ for $p \in (1,\infty)$ (see [29] and [32, Theorem 7.9.5, pp. 243]). If we apply Theorem 1.1 to $\mathbf{R}^d$ we obtain a result equivalent to the Hörmander multiplier theorem restricted to radial Fourier multipliers. Therefore we call Theorem 1.1 the Hörmander-type multiplier theorem and condition (1.5) the Hörmander-type condition.

In the setting of general Lie groups of polynomial growth spectral multipliers were investigated by Alexopoulos. The following theorem is equivalent to the spectral multiplier theorem obtained by Alexopoulos (see [2], see also Section 8.3).

Theorem 1.2. *Let L be a group invariant operator acting on a Lie group of polynomial growth defined by (1.3). Suppose that $s > d/2 = \max(d_0, d_\infty)/2$ and that $F\colon [0,\infty) \to \mathbf{C}$ is a bounded Borel function such that*

$$\sup_{t>0} \|\eta\, \delta_t F\|_{W^\infty_s} < \infty, \tag{1.6}$$

where $\delta_t F(\lambda) = F(t\lambda)$ and $\|F\|_{W^p_s} = \|(I - \mathrm{d}^2/\,\mathrm{d}x^2)^{s/2} F\|_{L^p}$. Then $F(L)$ is of weak type $(1,1)$ and bounded on L^q when $1 < q < \infty$.

Condition (1.6) is also independent of the choice of η. In [28] Hebisch extended Theorem 1.2 to a class of abstract operators acting on spaces satisfying the doubling condition (see also [3]). The order of differentiability in the Alexopoulos-Hebisch multiplier theorem is optimal. This means that for any $s < d/2$ we can find a function F such that F satisfies condition (1.6) but $F(A)$ is not of weak type $(1,1)$. Indeed, let A be a uniformly elliptic, self-adjoint second-order differential operator on $\mathbf{R}^d$, e.g. $A = \Delta_d$, where Δ_d is the standard Laplace operator. One can prove that

$$C_1(1+|\alpha|)^{d/2} \le \|A^{i\alpha}\|_{L^1 \to L^{1,\infty}} \le C_2(1+|\alpha|)^{d/2} \tag{1.7}$$

(see [50]). (See also [55, pp. 52] and Christ [10]). However, if we put $F_\alpha(\lambda) = |\lambda|^{i\alpha}$, then

$$C_1'(1+|\alpha|)^{s/2} \le \sup_{t>0} \|\eta\, \delta_t F_\alpha\|_{W^\infty_s} \le C_2'(1+|\alpha|)^{s/2}. \tag{1.8}$$

Therefore for any $s < d/2$ Theorem 1.2 does not hold.[1] Although the exponent $d/2$ is optimal, the Alexopoulos-Hebisch multiplier theorem is not sharp, as it does not give the optimal range of the exponent α for the Riesz summability. Indeed, if $\|\sigma_1^\alpha\|_{W^\infty_s} < \infty$, then $\alpha \ge s$. However, $\|\sigma_1^\alpha\|_{W^2_s} < \infty$ if and only if $\alpha > s - 1/2$. This means that in virtue of Theorem 1.2 one obtains uniform continuity of the Riesz means on L^q for any $\alpha > d/2$ and for all $q \in (1,\infty)$, whereas Theorem 1.1 shows the Riesz summability for $\alpha > (d-1)/2$ (see also [10, pp. 74]). As we mentioned earlier $(d-1)/2$ is the critical index for the Riesz summability for the standard Laplace operator on R^d and for the Laplace-Beltrami operator on compact manifolds.

The main aims of this paper is to investigate when it is possible to replace condition (1.6) in the Alexopoulos-Hebisch multiplier theorem by condition (1.5) from Theorem 1.1. However we investigate spectral multipliers in a general setting of abstract operators rather than in a specific setting of group invariant operators acting on Lie groups.

If we consider the harmonic oscillator i.e. the operator $A = -\,\mathrm{d}^2/\,\mathrm{d}x^2 + x^2 = \Delta + x^2$ on $\mathbf{R}$, then $d = 1$ and $(d-1)/2 = 0$. However, in [60, Theorem 2.1] Thangavelu proved that

Theorem 1.3. *If $A = \Delta + x^2$ and the operators $\sigma_R^\alpha(A)$ are uniformly bounded on L^q for $q \le 4$, then we necessarily have $q \ge 4/(6\alpha+3)$. In particular $\sigma_R^\alpha(A)f$ cannot converge in the norm for all $f \in L^1(\mathbf{R})$ unless $\alpha > 1/6$.*

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[1] See however [27, 41, 13].

Hence the analogue of Theorem 1.1 does not hold for the harmonic oscillator. See Section 7.5 for further discussion of the multiplier theorems for the harmonic oscillator. Thus if we want to generalise Theorem 1.1 we have to introduce some additional conditions. The additional conditions which we study here describes the L^2 norm of the kernels of spectral multipliers. We call such estimates the Plancherel estimates. If $\mu(X) < \infty$, then these Plancherel estimates are related to the sharp Weyl formula (see Section 7.3).

To provide rationale for the additional assumptions which we introduce here we discuss several examples in Section 7 including elliptic differential operators on compact manifolds, the Hermit and Laguerre expansions, scattering type operators on $\mathbf{R}^3$. Analysis of these examples seems to be of interest in its own right.

One striking feature of our results is their simplicity. Even though the proofs of Theorems 3.1 and 3.2 are quite easy, most of known multiplier results follow from Theorems 3.1 and 3.2. Examples of multiplier theorems which follow from Theorem 3.1 are Theorem 1.2 and Theorem 1.1.

The subject of the Bochner-Riesz means and spectral multipliers is so broad that it is impossible to provide comprehensive bibliography of it here. Hence we quote only papers directly related to our investigation and refer reader to [2, 10, 9, 8, 13, 16, 19, 18, 25, 28, 29, 33, 37, 39, 46, 53, 54, 48, 55, 56, 59] and their references.

2. Preliminaries

In this section we introduce some notation and describe the hypotheses under which we work. We also prove a few lemmas which will be useful in stating our main results.

Assumption 2.1. *Let X be an open subset of $\widetilde{X}$, where $\widetilde{X}$ is a topological space equipped with a Borel measure μ and a distance ρ. Let $B(x,r) = \{y \in \widetilde{X}, \rho(x,y) < r\}$ be the open ball (of $\widetilde{X}$) with centre at x and radius r. We suppose throughout that $\widetilde{X}$ satisfies the doubling property, i.e., there exists a constant C such that*

$$\mu(B(x,2r)) \le C\mu(B(x,r)) \ \forall x \in \widetilde{X}, \forall r > 0. \tag{2.1}$$

Note that (2.1) implies that there exist positive constants C and d such that

$$\mu(B(x,\gamma r)) \le C(1+\gamma)^d \mu(B(x,r)) \quad \forall \gamma > 0, x \in \widetilde{X}, r > 0. \tag{2.2}$$

In a sequel we always assume that (2.2) holds.

We state our results in terms of the value d in (2.2). Of course for any $d' \ge d$ (2.2) also holds. However, the smaller d the stronger multiplier theorem we will be able to obtain. Therefore we want to take d as small as possible. Note that in the case of the group of polynomial growth the smallest possible d in (2.2) is equal to $\max(d_0, d_\infty)$. Hence our notation is consistent with statements of Theorems 1.1 and 1.2.

Note that we do not assume that X satisfies the doubling property. This enables us to investigate singular integrals on the spaces without the doubling property (see Section 7.3).

Suppose that T is a bounded operator on $L^2(X,\mu)$. We say that a measurable function $K_T \colon X^2 \to \mathbf{C}$ is the (singular) kernel of T if

$$\langle Tf_1, f_2 \rangle = \int_X Tf_1 \overline{f_2} \, \mathrm{d}\mu = \int_X K_T(x,y) f_1(y) \overline{f_2(x)} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y). \tag{2.3}$$

for all $f_1, f_2 \in C_c(X)$ (for all $f_1, f_2 \in C_c(X)$ such that $\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \emptyset$ respectively).

Next we denote the weak type $(1,1)$ norm of an operator T on a measure space (X,μ) by $\|T\|_{L^1(X,\mu)\to L^{1,\infty}(X,\mu)} = \sup \lambda\, \mu(\{x \in X : |Tf(x)| > \lambda\})$, where the supremum is taken over $\lambda > 0$ and functions f with $L^1(X,\mu)$ norm less than one; this is often called the "operator norm", though in fact it is not a norm.

Assumption 2.2. *Let A be a self-adjoint positive definite operator. We suppose that the semigroup generated by $-A$ on L^2 has the kernel $p_t(x,y) = K_{\exp(-tA)}(x,y)$ defined by (2.3) which satisfies following Gaussian upper bound*

$$|p_t(x,y)| \le C\mu(B(y,t^{1/m}))^{-1} \exp\Big(- b\frac{\rho(x,y)^{m/(m-1)}}{t^{1/(m-1)}}\Big), \quad \forall t > 0, x,y \in X \tag{2.4}$$

where C, b and m are positive constants and $m \ge 2$.

Such estimates are typical for elliptic or sub-elliptic differential operators of order m (see e.g. [14, 45, 62]). We will call $p_t(x,y)$ the heat kernel associated with A.

In a sequel we always suppose that Assumptions 2.1 and 2.2 hold. To avoid repetition we often skip these assumptions in the statements of our results but **Assumption 2.1 and Assumption 2.2 should always be added to the hypothesis of all our results.** All examples of operators and spaces which we discuss here satisfy Assumptions 2.1 and 2.2. The values d and m always refer to (2.2) and (2.4).

Now we describe some simple but useful consequences of Assumptions 2.1 and 2.2.

Lemma 2.1. *Suppose that (2.4) and (2.2) hold. Then*

$$\int_{X-B(y,r)} |p_t(x,y)|^2 \, d\mu(x) \le C\mu(B(y,t^{1/m}))^{-1} \exp\big(- b \sqrt[m-1]{r^m/t}\big). \tag{2.5}$$

In particular

$$\|p_t(x,\cdot)\|^2_{L^2(X,\mu)} = \|p_t(\cdot,x)\|^2_{L^2(X,\mu)} \le C\mu(B(x,t^{1/m}))^{-1}.$$

Proof. By (2.4) and (2.2) (see also [11, Lemma 2.1])

$$\begin{aligned}\int_{X-B(y,r)} |p_t(x,y)|^2 \, d\mu(x) &\le C\mu(B(y,t^{1/m}))^{-2} \int_{X-B(y,r)} \exp\big(- 2b \sqrt[m-1]{\rho(x,y)^m/t}\big) \, d\mu(x) \\ &\le C \exp(-b \sqrt[m-1]{r^m/t}) \mu(B(y,t^{1/m}))^{-2} \int_X \exp\big(- b \sqrt[m-1]{\rho(x,y)^m/t}\big) \, d\mu(x) \\ &\le C\mu(B(y,t^{1/m}))^{-1} \exp(-b \sqrt[m-1]{r^m/t}).\end{aligned}$$

□

The following lemma is important for our further study and it motivates the Plancherel type condition which we introduce in Theorem 3.1.

Lemma 2.2. *Suppose that $\|p_t(\cdot,y)\|^2_{L^2(X,\mu)} \le C\mu(B(y,t^{1/m})^{-1}$. Then*

$$\|K_{F(\sqrt[m]{A})}(\cdot,y)\|^2_{L^2(X,\mu)} = \|K_{\overline{F}(\sqrt[m]{A})}(y,\cdot)\|^2_{L^2(X,\mu)} \le C\mu(B(y,R^{-1}))^{-1}\|F\|^2_{L^\infty} \tag{2.6}$$

for any Borel function F such that $\operatorname{supp} F \subset [0,R]$.

Proof. Put $G_1(\lambda) = F(\sqrt[m]{\lambda})e^{\lambda/R^m}$ and $G_2(\lambda) = e^{-\lambda/R^m}$ so that $F(\sqrt[m]{\lambda}) = G_1(\lambda)G_2(\lambda)$. Then $\|G_1(A)\|^2_{L^2(X,\mu)\to L^2(X,\mu)} \le \|G_1\|_{L^\infty} \le e\|F\|_{L^\infty}$. Next note that for any $f \in C_c(X)$ and any $t > 0$ we have $\int \|p_t(\,\cdot\,,y)f(y)\|_{L^2(X,\mu)}\,\mathrm{d}\mu(y) < \infty$ so

$$\begin{aligned} F(\sqrt[m]{A})f = G_1(A)(G_2(A)f) &= G_1(A)\Big(\int_X p_{R^{-m}}(\,\cdot\,,y)f(y)\,\mathrm{d}\mu(y)\Big) \\ &= \int_X G_1(A)\Big(p_{R^{-m}}(\,\cdot\,,y))(y)\Big)f(y)\,\mathrm{d}\mu(y) \end{aligned}$$

for every $f \in C_c(X)$. Hence if $\operatorname{supp} F \subset [0,R]$ and $\|F\|_{L^\infty} < \infty$ Then operator $F(\sqrt[m]{A})$ has the kernel given by the formula

$$K_{F(\sqrt[m]{A})}(x,y) = (G_1(A)K_{G_2(A)}(\,\cdot\,,y))(x).$$

In addition

$$\begin{aligned} \int_X |K_{F(\sqrt[m]{A})}(x,y)|^2\,\mathrm{d}\mu(x) &\le \|G_1(A)\|^2_{L^2(X,\mu)\to L^2(X,\mu)}\|p_{R^{-m}}(\,\cdot\,,y)\|^2_{L^2(X,\mu)} \\ &\le C\mu(B(y,R^{-1}))^{-1}\|G_1\|^2_{L^\infty} \le C\mu(B(y,R^{-1}))^{-1}\|F\|^2_{L^\infty}. \end{aligned} \tag{2.7}$$

□

3. Main results

Our main results are Theorem 3.1 and Theorem 3.2 below.

Theorem 3.1. *Suppose that $s > d/2$ and assume that for any $R > 0$ and all Borel functions F such that* $\operatorname{supp} F \subseteq [0,R]$

$$\int_X |K_{F(\sqrt[m]{A})}(x,y)|^2\,\mathrm{d}\mu(x) \le C\mu(B(y,R^{-1}))^{-1}\|\delta_R F\|^2_{L^p} \tag{3.1}$$

for some $p \in [2,\infty]$. Then for any Borel bounded function F such that $\sup_{t>0}\|\eta\,\delta_t F\|_{W^p_s} < \infty$ the operator $F(A)$ is of weak type $(1,1)$ and is bounded on $L^q(X)$ for all $1 < q < \infty$. In addition

$$\|F(A)\|_{L^1(X,\mu)\to L^{1,\infty}(X,\mu)} \le C_s\Big(\sup_{t>0}\|\eta\,\delta_t F\|_{W^p_s} + |F(0)|\Big). \tag{3.2}$$

Note that if (3.1) holds for some $p < \infty$, then the pointwise spectrum of A is empty. Indeed, for all $p < \infty$ and all $y \in X$

$$0 = C\|\delta_R \chi_{\{a\}}\|_{L^p} \ge \mu(B(y,1/R)^{1/2}\|K_{\chi_{\{a\}}(\sqrt[m]{A})}(\,\cdot\,,y)\|_{L^2(X,\mu)} \tag{3.3}$$

so $\chi_{\{a\}}(\sqrt[m]{A}) = 0$. Hence for elliptic operators on compact manifolds or for the harmonic oscillator, (3.1) cannot be true for any $p < \infty$. To be able to study these operators as well we introduce some variation of condition (3.1). Following [13] for a Borel function F such that $\operatorname{supp} F \subseteq [-1,2]$ we define the norm $\|F\|_{N,p}$ by the formula

$$\|F\|_{N,p} = \Big(\frac{1}{N}\sum_{l=1-N}^{2N}\sup_{\lambda\in[\frac{l-1}{N},\frac{l}{N})}|F(\lambda)|^p\Big)^{1/p},$$

where $p \in [1,\infty)$ and $N \in \mathbf{Z}_+$. For $p = \infty$ we put $\|F\|_{N,\infty} = \|F\|_{L^\infty}$. It is obvious that $\|F\|_{N,p}$ increases monotonically in p. The next theorem is a variation of Theorem 3.1. This variation can be used in the case of operators with nonempty pointwise spectrum (compare [13, Theorem 3.6]).

Theorem 3.2. *Suppose that* κ *is a fixed natural number,* $s > d/2$ *and that for any* $N \in \mathbf{Z}_+$ *and for all Borel functions* F *such that* $\operatorname{supp} F \subseteq [-1, N+1]$

$$\int_X |K_{F(\sqrt[m]{A})}(\,\cdot\,, y)|^2 \,\mathrm{d}\mu(x) \le C\mu(B(y, 1/N))^{-1} \|\delta_N F\|^2_{N^\kappa, p} \tag{3.4}$$

for some $p \ge 2$*. In addition we assume that for any* $\varepsilon > 0$ *there exists a constant* C_ε *such that for all* $N \in Z_+$ *and all Borel functions* F *such that* $\operatorname{supp} F \subseteq [-1, N+1]$

$$\|F(\sqrt[m]{A})\|^2_{L^1(X,\mu)\to L^1(X,\mu)} \le C_\varepsilon N^{\kappa d+\varepsilon} \|\delta_N F\|^2_{N^\kappa, p}. \tag{3.5}$$

Then for any Borel bounded function F *such that* $\sup_{t>1} \|\eta\, \delta_t F\|_{W^p_s} < \infty$ *the operator* $F(A)$ *is of weak type* $(1,1)$ *and is bounded on* $L^q(X)$ *for all* $q \in (1, \infty)$*. In addition*

$$\|F(A)\|_{L^1(X,\mu)\to L^{1,\infty}(X,\mu)} \le C_s \Big(\sup_{t>1} \|\eta\, \delta_t F\|_{W^p_s} + \|F\|_{L^\infty} \Big). \tag{3.6}$$

Remarks 1. Note that in virtue of Lemma 2.2 (3.1) always holds with $p = \infty$. This means that Theorem 1.2 follows from Theorem 3.1. Theorem 1.1 also follows from Theorem 3.1. Indeed, it is easy to check that for homogeneous operators (3.1) holds for $p = 2$ (see Section 7.1 or [10, Proposition 3]).

2. The harmonic oscillator satisfies Assumption 2.2 (see e.g. (7.8) below). However, the Hörmander-type multiplier theorem (i.e. (3.2) for $p = 2$) does not hold for the harmonic oscillator (see Theorem 1.3 and Section 7.5). Hence Theorems 3.1 and 3.2 do not hold without conditions (3.1) or (3.4).

3. The main point of this paper is that if one can obtain (3.1) or (3.4) then one can prove stronger multiplier results. If one shows (3.1) or (3.4) for $p = 2$, then this implies the sharp Hörmander-type multiplier result. Actually we believe that to obtain any sharp spectral multiplier theorem one has to investigate conditions of the same type as (3.1) or (3.4). This means conditions which allow us to estimate the norm $\|K_{F(\sqrt[m]{A})}(\,\cdot\,, y)\|_{L^2(X,\mu)}$ in terms of some kind of L^p norm of the function F. We hope that examples which we analyse would convince readers that our supposition has a sound rationale.

4. We call hypotheses (3.1) or (3.4) the Plancherel estimates or the Plancherel conditions. In the proof of Theorems 3.1 and 3.2 one does not have to assume that $p \ge 2$ in estimates (3.1) or (3.4). However (3.1) or (3.4) for $p < 2$ would imply the Riesz summability for $\alpha < (d-1)/2$ and we do not expect such a situation.

Note that (3.4) is weaker than (3.1) and we need additional hypothesis (3.5) in Theorem 3.2. However, once (3.4) is proved, (3.5) is usually easy to check. Often we can put $\varepsilon = 0$. For example (see also Lemma 7.9)

Lemma 3.3. *Suppose that* $X \subset B(z, \gamma)$ *and (3.4) holds for* $\kappa = 1$*. Then*

$$\|F(\sqrt[m]{A})\|^2_{L^1(X,\mu)\to L^1(X,\mu)} \le C N^d \|\delta_N F\|^2_{N,p}$$

for all $N \in Z_+$ *and all Borel functions* F *such that* $\operatorname{supp} F \subseteq [-1, N+1]$.

Proof. Indeed

$$\begin{aligned}\|F(\sqrt[m]{A})\|^2_{L^1(X,\mu)\to L^1(X,\mu)} &= \sup_{y\in X}\|K_{F(\sqrt[m]{A})}(\,\cdot\,,y)\|^2_{L^1(X,\mu)}\\ &\le \mu(B(z,\gamma))\sup_{y\in X}\|K_{F(\sqrt[m]{A})}(\,\cdot\,,y)\|^2_{L^2(X,\mu)}\\ &\le C\mu(B(z,\gamma))\sup_{y\in X}\mu(B(y,1/N))^{-1}\|\delta_N F\|^2_{N,p}.\end{aligned}$$

But by Assumption 2.1 for any $y\in X$

$$\sup_{y\in X}\mu(B(y,1/N))^{-1}\mu(B(z,\gamma))\le C\sup_{y\in X}\mu(B(y,1/N))^{-1}\mu(B(y,2\gamma))\le C'N^d.$$

□

4. Proofs of Theorem 3.1 and Theorem 3.2

We split the proofs of Theorem 3.1 and 3.2 into a few lemmas. First we note that (compare [6, 42])

Lemma 4.1. *For any $s\ge 0$ there exists a constant C such that*

$$\int_X |p_{(1+i\tau)R^{-m}}(x,y)|^2\rho(x,y)^s\,\mathrm{d}\mu(x)\le C\mu(B(y,1/R))^{-1}R^{-s}(1+|\tau|)^s, \tag{4.1}$$

where $p_{(1+i\tau)R^{-m}}=K_{\exp(-(1+i\tau)R^{-m}A)}$.

Proof. Assume that $\|f\|_{L^2(X,\mu)}=1$ and that $\operatorname{supp} f\subset X-B(y,r)$. We define the holomorphic function $F_y\colon\{z\in\mathbf{C}\colon\ \Re e\, z>0\}\to\mathbf{C}$ by the formula

$$F_y(z)=e^{-zR^m}\mu(B(y,1/R))\Big(\int_X p_z(x,y)f(x)\,\mathrm{d}\mu(x)\Big)^2.$$

By the same argument as in (2.7) if we put $z=|z|e^{i\theta}$, then $\|p_z(\,\cdot\,,y)\|^2_{L^2}=\|p_{|z|\cos\theta}(\,\cdot\,,y)\|^2_{L^2}$. Hence by Lemma 2.1

$$\begin{aligned}|F_y(z)| &\le e^{-R^m|z|\cos\theta}\mu(B(y,1/R))\|p_{|z|\cos\theta}(\,\cdot\,,y)\|^2_{L^2}\\ &\le Ce^{-R^m|z|\cos\theta}\frac{\mu(B(y,1/R))}{\mu(B(y,\sqrt[m]{|z|\cos\theta}))}\le Ce^{-R^m|z|\cos\theta}\Big(1+\frac{R^{-m}}{|z|\cos\theta}\Big)^{d/m}\\ &\le CR^{-d}(|z|\cos\theta)^{-d/m}.\end{aligned}$$

Similarly for $\theta=0$ by Lemma 2.1

$$|F_y(|z|)|\le CR^{-d}|z|^{-d/m}\exp\Big(-\frac{br^{m/(m-1)}}{|z|^{1/(m-1)}}\Big).$$

Now let us recall the following version of Phragmen-Lindelöf Theorem

Lemma 4.2 ([15, Lemma 9])**.** *Suppose that function F is analytic in $\{z\in\mathbf{C}\colon\Re e\,z>0\}$ and that*

$$|F(|z|e^{i\theta})|\le a_1(|z|\cos\theta)^{-\beta_1}$$
$$|F(|z|)|\le a_1|z|^{-\beta_1}\exp(-a_2|z|^{-\beta_2})$$

for some $a_1,a_2>0$, $\beta_1\ge 0$, $\beta_2\in(0,1]$, all $|z|>0$ and all $\theta\in(-\pi/2,\pi/2)$. Then

$$|F(|z|e^{i\theta})|\le a_1 2^{\beta_1}(|z|\cos\theta)^{-\beta_1}\exp\Big(-\frac{a_2\beta_2}{2}|z|^{-\beta_2}\cos\theta\Big)$$

for all $|z| > 0$ *and all* $\theta \in (-\pi/2, \pi/2)$.

Now if $|z|e^{i\theta} = (1+i\tau)R^{-m}$, then $|z| = R^{-m}(1+|\tau|^2)^{1/2}$, $\cos\theta = (1+|\tau|^2)^{-1/2}$ and $|z|\cos\theta = R^{-m}$. Putting $a_1 = CR^{-d}$, $a_2 = br^{m/(m-1)}$, $\beta_1 = d/m$ and $\beta_2 = 1/(m-1)$ in Lemma 4.2 we conclude that

$$|F_y((1+i\tau)R^{-m})| \le C' \exp\Big(- b'(rR/(1+|\tau|))^{m/(m-1)}\Big).$$

Hence

$$\mu(B(y,1/R)) \int_{X-B(y,r)} |p_{(1+i\tau)R^{-m}}(x,y)|^2 \,\mathrm{d}\mu(x) \le C \exp\Big(- b'(rR/(1+|\tau|))^{m/(m-1)}\Big).$$

Finally, we have

$$\begin{aligned} &\int_X |p_{(1+i\tau)R^{-m}}(x,y)|^2 \rho(x,y)^s \,\mathrm{d}\mu(x) \\ &\quad= \sum_{k\ge 0} \int_{k(1+|\tau|)R^{-1}\le \rho(x,y) \le (k+1)(1+|\tau|)R^{-1}} |p_{(1+i\tau)R^{-m}}(x,y)|^2 \rho(x,y)^s \,\mathrm{d}\mu(x) \\ &\quad\le (1+|\tau|)^s R^{-s} \sum_{k\ge 0} (k+1)^s \int_{X-B(y,k(1+|\tau|)R^{-1})} |p_{(1+i\tau)R^{-m}}(x,y)|^2 \,\mathrm{d}\mu(x) \\ &\quad\le C\mu(B(y,1/R))^{-1} R^{-s} (1+|\tau|)^s. \end{aligned}$$

□

Lemma 4.3. *(a) Suppose that* A *satisfies (3.1) for some* $p \in [2,\infty]$ *and that* $R > 0$, $s > 0$. *Then for any* $\varepsilon > 0$ *there exists a constant* $C = C(s,\varepsilon)$ *such that*

$$\int_X |K_{F(\sqrt[m]{A})}(x,y)|^2 (1+R\rho(x,y))^s d\mu(x) \le C\mu(B(y,R^{-1}))^{-1} \|\delta_R F\|^2_{W^p_{s/2+\varepsilon}} \tag{4.2}$$

for all Borel functions F *such that* $\operatorname{supp} F \subseteq [R/4, R]$.

(b) Suppose that A *satisfies (3.4) for some* $p \in [2,\infty]$ *and* $N > 8$ *is a natural number. For* $\xi \in C_c^\infty([-1,1])$ *we define the function* ξ_N *by the formula* $\xi_N(\lambda) = N\xi(N\lambda)$. *Then for any* $s > 0$, $\varepsilon > 0$ *and function* $\xi \in C_c^\infty([-1,1])$ *there exists a constant* $C = C(s,\varepsilon,\xi)$ *such that*

$$\int_X |K_{F*\xi_{N^{\kappa-1}}(\sqrt[m]{A})}(x,y)|^2 (1+N\rho(x,y))^s \,\mathrm{d}\mu(x) \le C\mu(B(y,1/N))^{-1} \|\delta_N F\|^2_{W^p_{s/2+\varepsilon}} \tag{4.3}$$

for all Borel functions F *such that* $\operatorname{supp} F \subseteq [N/4, N]$.

Proof. In virtue of the Fourier inversion formula

$$G(A/R^m)e^{-A/R^m} = \frac{1}{2\pi} \int_{\mathbf{R}} \exp((i\tau - 1)R^{-m}A) \widehat{G}(\tau) \,\mathrm{d}\tau$$

and so

$$K_{F(\sqrt[m]{A})}(x,y) = \frac{1}{2\pi} \int_{\mathbf{R}} \widehat{G}(\tau) p_{(1-i\tau)R^{-m}}(x,y) \,\mathrm{d}\tau,$$

where $G(\lambda) = [\delta_R F](\sqrt[m]{\lambda})e^{\lambda}$. Hence by Lemma 4.1 and Lemma 2.1

$$
\begin{aligned}
(4.4)\qquad & \Big(\int_X |K_{F(\sqrt[m]{A})}(x,y)|^2(1+R\rho(x,y))^s \,\mathrm{d}\mu(x)\Big)^{1/2} \\
\le\ & \int_{\mathbf{R}} |\widehat{G}(\tau)| \Big(\int_X |p_{(1-i\tau)R^{-m}}(x,y)|^2(1+R\rho(x,y))^s \,\mathrm{d}\mu(x)\Big)^{1/2} \mathrm{d}\tau \\
\le\ & C\mu(B(y,1/R))^{-1/2} \int_{\mathbf{R}} |\widehat{G}(\tau)|(1+|\tau|)^{s/2}\,\mathrm{d}\tau \\
\le\ & C\mu(B(y,1/R))^{-1/2} \Big(\int_{\mathbf{R}} |\widehat{G}(\tau)|^2(1+\tau^2)^{\frac{s+\varepsilon+1}{2}}\Big)^{1/2} \Big(\int_{\mathbf{R}} (1+\tau^2)^{\frac{-1-\varepsilon}{2}}\Big)^{1/2} \\
\le\ & C\mu(B(y,1/R))^{-1/2} \|G\|_{W^2_{(s+1+\varepsilon)/2}}.
\end{aligned}
$$

However, $\operatorname{supp} F \subseteq [R/4, R]$ and $\operatorname{supp} \delta_R F \subseteq [1/4, 1]$ so

$$
(4.5)\qquad \|G\|_{W^2_{(s+1+\varepsilon)/2}} \le C\|\delta_R F\|_{W^2_{(s+1+\varepsilon)/2}} \le C\|\delta_R F\|_{W^p_{(s+1+\varepsilon)/2}}
$$

for all $p \ge 2$. From (4.4) and (4.5) we obtain a multiplier result in which the required order of differentiability of the function $\delta_R F$ is $1/2$ greater than that of Lemma 4.3. To get rid of this additional $1/2$ we use an interpolation argument as in [37]. First we note that (4.2) is equivalent to the following estimates

$$
(4.6)\qquad \int_X |K_{\delta_{1/R}H(\sqrt[m]{A})}(x,y)|^2(1+R\rho(x,y))^s d\mu(x) \le C\mu(B(y,R^{-1}))^{-1}\|H\|^2_{W^p_{s/2+\varepsilon}}
$$

for all bounded Borel functions H such that $\operatorname{supp} H \subset [1/4, 1]$. Now we define the linear operator $K_{y,R}\colon L^p([1/4,1]) \to L^2(X,\mu)$ by the formula

$$
K_{y,R}(H) = K_{\delta_{1/R}H(\sqrt[m]{A})}(\,\cdot\,, y).
$$

By (3.1)

$$
\Big\|K_{y,R}\Big\|^2_{L^p([1/4,1]) \to L^2(X,\mu)} \le C\mu(B(y,R^{-1}))^{-1}.
$$

Next we put $L^2_{y,s,R} = L^2(X, \mu_{y,s,R})$, where $\mathrm{d}\mu_{y,s,R}(x) = (1+R\rho(x,y))^s\,\mathrm{d}\mu(x)$ and by $W^p_a([1/4,1])$ we denote the space of all Borel functions F such that $\operatorname{supp} F \subseteq [1/4,1]$ and $\|F\|_{W^p_a} = \|(\Delta+1)^a F\|_{L^p(\mathbf{R})} < \infty$. By (4.4) and (4.5)

$$
\Big\|K_{y,R}\Big\|^2_{W^p_{(s+1+\varepsilon)/2}([1/4,1]) \to L^2_{y,s,R}} \le C\mu(B(y,R^{-1}))^{-1}.
$$

By interpolation, for every $\theta \in (0,1)$ there exists a constant C such that

$$
\|\delta_{1/R}H(\sqrt[m]{A})(\,\cdot\,,y)\|_{L^2_{y,s\theta,R}} \le C\mu(B(y,1/R))^{-1/2}\|H\|_{[L^P,W^p_{(s+1+\varepsilon)/2}]_{[\theta]}}.
$$

In particular, for all $s > 0$, $\varepsilon' > 0$ and $\theta \in (0,1)$

$$
\|\delta_{1/R}H(\sqrt[m]{A})(\,\cdot\,,y)\|_{L^2_{y,s\theta,R}} \le C\mu(B(y,1/R))^{-1/2}\|H\|_{W^{s\theta/2+\theta/2+\varepsilon'}}.
$$

Hence by putting $s' = s/\theta$ in this inequality and taking θ small enough we obtain

$$
\|\delta_{1/R}H(\sqrt[m]{A})(\,\cdot\,,y)\|_{L^2_{y,s',R}} \le C\mu(B(y,1/R))^{-1/2}\|H\|_{W^p_{s'/2+\varepsilon''}}
$$

for all $s' > 0$ and $\varepsilon'' > 0$. This proves (4.6) and (4.2).

The main idea of the proof of (4.3) is similar to that of the proof of (4.2). First we can state (4.3) in the following way

$$\int_X |K_{\xi_{N^{\kappa-1}}*\delta_{1/N}H(\sqrt[m]{A})}(x,y)|^2(1+N\rho(x,y))^s \,\mathrm{d}\mu(x) \le C\mu(B(y,1/N))^{-1}\|H\|^2_{W^p_{s/2+\varepsilon}} \tag{4.7}$$

for all bounded Borel functions H such that $\operatorname{supp} H \subseteq [1/4,1]$. Now if $N>8$ and $\operatorname{supp} H \subseteq [1/4,1]$ then $\operatorname{supp}(\xi_N * H) \subseteq [1/8,2]$. Moreover,

$$|\xi_N * H(\lambda)|^p \le \|\xi_N\|^p_{L^{p'}} \int_{\lambda-1/N}^{\lambda+1/N} |H(\lambda')|^p \,\mathrm{d}\lambda', \tag{4.8}$$

so

$$\begin{aligned} \|\xi_N * H\|_{N,p} &= \left(\frac{1}{N}\sum_{i=1-N}^{2N} \sup_{\lambda\in[\frac{i-1}{N},\frac{i}{N})} |\xi_N * H(\lambda)|^p\right)^{1/p} \\ &\le \frac{\|\xi_N\|_{L^{p'}}}{N^{1/p}}\left(\sum_{i=1}^{N}\int_{(i-2)/N}^{(i+1)/N} |H(\lambda')|^p \,\mathrm{d}\lambda'\right)^{1/p} \le \frac{3\|\xi_N\|_{L^{p'}}}{N^{1/p}}\|H\|_{L^p} \le C\|H\|_{L^p}. \end{aligned} \tag{4.9}$$

Therefore by (3.4)

$$\begin{aligned} \int_X |K_{\xi_{N^{\kappa-1}}*\delta_{1/N}H(\sqrt[m]{A})}(x,y)|^2 \,\mathrm{d}\mu(x) &= \int_X |K_{\delta_{1/N}[\xi_{N^\kappa}*H](\sqrt[m]{A})}(x,y)|^2 \,\mathrm{d}\mu(x) \\ &\le C\mu(B(y,N^{-1}))^{-1}\|\xi_{N^\kappa} * H\|^2_{N^\kappa,p} \le C\mu(B(x,N^{-1}))^{-1}\|H\|^2_{L^p} \end{aligned} \tag{4.10}$$

for all Borel functions H such that $\operatorname{supp} H \subseteq [1/4,1]$. Next, putting $F=\xi_{N^{\kappa-1}} * \delta_{1/N}H$ in (4.4) we get

$$\begin{aligned} &\left(\int_X |K_{\xi_{N^{\kappa-1}}*\delta_{1/N}H(\sqrt[m]{A})}(x,y)|^2(1+N\rho(x,y))^s \,\mathrm{d}\mu(x)\right)^{1/2} \\ &\qquad\qquad \le C\mu(B(x,1/N))^{-1/2}\|G\|_{W^2_{(s+1+\varepsilon)/2}}, \end{aligned} \tag{4.11}$$

where $G(\lambda)=[\xi_{N^\kappa} * H](\sqrt[m]{\lambda})e^{\lambda}$. However, $\operatorname{supp}(\xi_N * H)\subseteq[1/8,2]$ and

$$\|G\|_{W^2_s} \le \|G\|_{W^p_s} \le \|\xi_{N^\kappa} * H\|_{W^p_s} \le C\|H\|_{W^p_s} \tag{4.12}$$

for all $p\ge 2$. Now we define operator $\widetilde{K}_{y,N}\colon L^\infty([1/4,1]) \to L^2(X,\mu)$ by the formula

$$\widetilde{K}_{y,N}(H) = K_{y,N}(\xi_{N^\kappa} * H) = K_{\xi_{N^{\kappa-1}}*\delta_{1/N}H(\sqrt[m]{A})}(\,\cdot\,,y).$$

In virtue of (4.10), (4.11) and (4.12)

$$\left\|\widetilde{K}_{y,N}\right\|^2_{L^p([1/4,1])\to L^2(X,\mu)} \le C\mu(B(y,N^{-1}))^{-1}$$

and

$$\left\|\widetilde{K}_{y,N}\right\|^2_{W^p_{(s+1+\varepsilon)/2}([1/4,1])\to L^2_{y,s,N}} \le C\mu(B(y,N^{-1}))^{-1}.$$

Thus by interpolation

$$\|\xi_{N^{\kappa-1}} * \delta_{1/N}H(\sqrt[m]{A})(\,\cdot\,,y)\|_{L^2_{y,s,N}} \le C\mu(B(y,1/N))^{-1/2}\|H\|_{W^p_{s/2+\varepsilon'}}$$

for all $s>0$ and $\varepsilon'>0$. This proves (4.7) and (4.3). □

The following lemma is a consequence of Assumption 2.2

Lemma 4.4. *Suppose that (2.2) holds and $s > d$. Then*

$$\int_{X-B(y,r)} (1+R\rho(x,y))^{-s}\,\mathrm{d}\mu(x) \le C\mu(B(y,1/R))(1+rR)^{d-s}. \tag{4.13}$$

Proof. Assume that $rR \ge 1$. Then

$$\begin{aligned}\int_{X-B(y,r)} (1+R\rho(x,y))^{-s}\,\mathrm{d}\mu(x) &\le \sum_{k\ge 0}\int_{2^k r\le\rho(x,y)\le 2^{k+1}r} (R\rho(x,y))^{-s}\,\mathrm{d}\mu(x)\\ &\le \sum_{k\ge 0}(2^k rR)^{-s}\mu(B(y,2^{k+1}r)) \le C\sum_{k\ge 0}(2^k rR)^{d-s}\mu(B(y,1/R)) \qquad (4.14)\\ &\le (rR)^{d-s}\mu(B(y,1/R)).\end{aligned}$$

If $rR < 1$ we estimate the integral over X by the sum of the integrals over $B(y,1/R)$ and $X-(B(y,1/R)$. Putting $r = 1/R$ in (4.14) we obtain

$$\begin{aligned}&\int_X (1+R\rho(x,y))^{-2s}\,\mathrm{d}\mu(x) \qquad (4.15)\\ &\le \int_{\rho(x,y)\ge 1/R} (R\rho(x,y))^{-s}\,\mathrm{d}\mu(x) + \mu(B(y,1/R)) \le C\mu(B(y,1/R)).\end{aligned}$$

□

To prove that operator is of weak type (1,1) we usually use estimates for the gradient of the kernel. The following theorem replaces the gradient estimates in our proof of Theorem 3.1.

Theorem 4.5. *Suppose that $\|F\|_{L^\infty} \le C_1$, and that*

$$\sup_{r\in\mathbf{R}^+}\sup_{y\in X}\int_{X-B(y,r)} |K_{F(1-\Phi_r)(\sqrt[m]{A})}(x,y)|\,\mathrm{d}\mu(x) \le C_1, \tag{4.16}$$

where $\Phi_r(\lambda) = \exp(-(\lambda r)^m)$. Then

$$\|F(\sqrt[m]{A})\|_{L^1(X,\mu)\to L^{1,\infty}(X,\mu)} \le CC_1.$$

For a very simple proof of Theorem 4.5 see [20, Theorem 2]. See [11, 18] for other variants of the proof. See also [22, 25] and [13].

Proof of Theorem 3.1. First note that $\sup_{t>0}\|\eta\,\delta_t F\|_{W^p_s} \sim \sup_{t>0}\|\eta\,\delta_t G\|_{W^p_s}$, where $G(\lambda) = F(\sqrt[m]{\lambda})$. Therefore we can replace $F(A)$ by $F(\sqrt[m]{A})$ in the proof. Then we choose a function ω in $C_c^\infty(\mathbf{R}_+)$ supported in $[1/4,1]$ such that

$$\sum_{n\in\mathbf{Z}}\omega(2^n\lambda) = 1 \qquad \forall\lambda\in\mathbf{R}^+, \tag{4.17}$$

and let ω_n denote the function $\omega(2^{-n}\cdot)$. Then

$$F(1-\Phi_r)(\sqrt[m]{A}) = \sum_{n\in\mathbf{Z}}\omega_n F(1-\Phi_r)(\sqrt[m]{A}).$$

By Lemma 4.3 and Lemma 4.4 for any $d/2 < s' < s$

$$\begin{aligned}(4.18)\qquad &\int_{X-B(y,r)} |K_{\omega_n F(1-\Phi_r)(\sqrt[m]{A})}(x,y)|\,\mathrm{d}\mu(x)\\ &\leq \Big(\int_X |K_{\omega_n F(1-\Phi_r)(\sqrt[m]{A})}(x,y)|^2(1+2^n\rho(x,y))^{2s'}\,\mathrm{d}\mu(x)\Big)^{1/2}\\ &\times \Big(\int_{X-B(y,r)} |(1+2^n\rho(x,y))^{-2s'}d\mu(x)\Big)^{1/2}\\ &\leq C(1+2^n r)^{d/2-s'}\|\delta_{2^n}[\omega_n F(1-\Phi_r)]\|_{W^p_s}.\end{aligned}$$

Now for any Sobolev space $W^p_s(\mathbf{R})$, if k is an integer greater than s, then

$$\|\delta_{2^n}[\omega_n F(1-\Phi_r)]\|_{W^p_s} \leq C\|\delta_{2^n}[\omega_n F]\|_{W^p_s}\|\delta_{2^n}[1-\Phi_r]\|_{C_k([1/4,1])} \leq \frac{C2^n r}{1+2^n r}\|\delta_{2^n}[\omega_n F]\|_{W^p_s}.$$

Finally

$$\begin{aligned}(4.19)\qquad &\sup_{y\in X}\int_{X-B(y,r)} |K_{F(1-\Phi_r)(\sqrt[m]{A})}(x,y)|\,\mathrm{d}\mu(x)\\ &\leq C\sum_n \frac{2^n r}{1+2^n r}\,(1+2^n r)^{d/2-s'}\|\delta_{2^n}[\omega_n F]\|_{W^p_s} \leq C\sup_{n\in\mathbf{Z}}\|\delta_{2^n}[\omega_n F]\|_{W^p_s}\end{aligned}$$

as required to prove Theorem 3.1. ☐

Proof of Theorem 3.2. Note that by (3.5) for any F such that $\operatorname{supp} F \subset [0,2]$

$$\|F(\sqrt[m]{A})\|_{L^1(X,\mu)\to L^1(X,\mu)} \leq C\|F\|_{L^\infty}.$$

Hence we can assume that $\operatorname{supp} F \subset [1,\infty]$ and consider only $n > 0$ in (4.18). Let

$$\widetilde{F} = \sum_{n>0}(\omega_n F) * \xi_{2^{n(\kappa-1)}}.$$

By repeating the proof of Theorem 3.1 and using (4.3) in place of (4.2) we can prove that

$$(4.20)\qquad \sup_{y\in X}\int_{X-B(y,r)} |K_{\widetilde{F}(1-\Phi_r))(\sqrt[m]{A})}(x,y)|\,\mathrm{d}\mu(x) \leq C\sup_{n>0}\|\delta_{2^n}[\omega_n F]\|_{W^p_s}.$$

Therefore to prove Theorem 3.2 it is enough to show that

$$\|F-\widetilde{F}(\sqrt[m]{A})\|_{L^1(X,\mu)\to L^1(X,\mu)} \leq C\sup_{t>1}\|\eta\,\delta_t F\|_{W^p_s}.$$

We write H_n for $\omega_n F - (\omega_n F) * \xi_{2^{n(\kappa-1)}}$. Since $\operatorname{supp} H_n \subseteq [-1, 2^n+1]$, it follows from (3.5) that

$$\|H_n(\sqrt[m]{A})\|^2_{L^1(X,\mu)\to L^1(X,\mu)} \leq C2^{n(d\kappa+\varepsilon)}\|\delta_{2^n}H_n\|^2_{2^{n\kappa},p}$$

Everything then boils down to estimating $\|\cdot\|_{2^{n\kappa},p}$ norm of $\delta_{2^n}H_n$. We make the following claim

Proposition 4.6. *Suppose that* $\xi \in C_c^\infty$ *is a function such that* $\operatorname{supp}\,\xi \subset [-1,1]$, $\xi \geq 0$, $\widehat{\xi}(0) = 1$ *and* $\widehat{\xi}^{(k)}(0) = 0$ *for all* $1 \leq k \leq [s]+2$. *Next assume that* $\operatorname{supp}\,G \subset [0,1]$. *Then*

$$\|G - G * \xi_N\|_{N,p} \leq CN^{-s}\|G\|_{W^p_s}.$$

for all $s > 1/p$.

In virtue of Proposition 4.6 and (3.5) it then follows that

$$
\begin{aligned}
\|H_n(\sqrt[m]{A})\|^2_{L^1(X,\mu)\to L^1(X,\mu)} &\le C2^{n(d\kappa+\varepsilon)}\|\delta_{2^n}[\omega_n F] - \xi_{2^{n\kappa}} * \delta_{2^n}[\omega_n F]\|^2_{2^{n\kappa},p} \\
&\le C2^{n(d\kappa+\varepsilon)}2^{-2ns\kappa}\|\delta_{2^n}[\omega_n F]\|^2_{W^p_s} \qquad (4.21)
\end{aligned}
$$

Finally

$$
\begin{aligned}
\|F(\sqrt[m]{A}) - \widetilde{F}(\sqrt[m]{A})\|_{L^1\to L^1} \le \sum_{n>0}\|H_n(\sqrt[m]{A})\|_{L^1\to L^1} &\le C\sum_{n>0}2^{n((d/2-s)\kappa+\varepsilon)}\|\delta_{2^n}[\omega_n F]\|_{W^p_s} \\
&\le C\sup_{n>0}\|\delta_{2^n}[\omega_n F]\|_{W^p_s}
\end{aligned}
$$

as required. □

Proof of the Proposition 4.6. Proposition 4.6 is proved in [13]. For readers convenience we repeat the proof here. We write ζ_s for the function on $\mathbf{R}$ defined by the condition that

$$\widehat{\zeta_s} = \big(1-\widehat{\xi}\big)|\cdot|^{-s}.$$

Observe first that

$$\sum_{i\in\mathbf{Z}}\sup_{t\in[i-1,i]}|\zeta_s * H|^p \le C\|H\|^p_{L^p} \qquad \forall H\in L^p(\mathbf{R}). \tag{4.22}$$

Indeed, Fourier analysis shows that $|\zeta_s(t)| \le C_1|t|^{s-1}$ when $|t|\le 1$ and $|\zeta_s(t)| \le C_2|t|^{s-k-1}$ when $|t|\ge 1$. Therefore we may write ζ_s as $\sum_{j\in\mathbf{Z}}\xi_{s,j}(\cdot - j)$, where $\operatorname{supp}\xi_{s,j}\subseteq[-1,1]$ and $\sum_{j\in\mathbf{Z}}\|\xi_{j,s}\|_{L^{p'}} < \infty$ (this is where we require that $s > 1/p$). The argument of (4.8) and (4.9) then shows that (4.22) holds. The proof of our claim is now straightforward. Indeed let H be a function such that $\delta_N H = G$. Then

$$
\begin{aligned}
\|\delta_N H - \xi_N * \delta_N H]\|^p_{N,p} &= N^{-1}\textstyle\sum_{i=1-N}^{2N}\sup_{t\in[\frac{i-1}{N},\frac{i}{N}]}|[H-\xi * H](Nt)|^p \\
&\le N^{-1}\textstyle\sum_{i=-\infty}^{\infty}\sup_{t\in[i-1,i]}|\zeta_s * I_s H(t)|^p,
\end{aligned}
$$

where ζ_s is as above and $\widehat{(I_sF)} = |\cdot|^s\widehat{F}$. Therefore, by (4.22),

$$\|\delta_N[H-\xi * H]\|_{N,p} \le C\,N^{-1/p}\|I_sH\|_{L^p} \le C\,N^{-s}\|\delta_N H\|_{W^p_s}.$$

□

Remark. It is easy to see that $E_A(0) = \chi_{\{0\}}(A)$ is bounded on L^q for all $q\in[1,\infty]$. But we do not have to show it to prove Theorem 3.1. Indeed, $(1-\Phi_r)\chi_{\{0\}}(\lambda) = 0$ so $\chi_{\{0\}}(A)$ is of weak type $(1,1)$ by Theorem 4.5. Note that if (3.1) holds for $p<\infty$, then pointwise spectrum is empty and hence $E_A(0) = 0$ (see (3.3)). $E_A(0)=0$ also if $\mu(\widetilde{X}) = \infty$. Indeed

$$\|K_{E_A(0)}(\,\cdot\,,y)\|^2_{L^2(X,\mu)} \le C\inf_{R>0}\mu(B(y,R^{-1}))^{-1}\|\chi_{\{0\}}\|^2_{L^\infty} = 0.$$

If $E_A(0) = 0$, then one can skip $|F(0)|$ in (3.2). If for $c>0$, $E_A([0,c]) = 0$, then we can assume that $\operatorname{supp}\eta\subset(0,c)$ and skip $\|F\|_{L^\infty}$ in (3.6). Note however that (3.2) without $|F(0)|$ is false for the Laplace-Beltrami operators on compact manifolds.

5. Plancherel measure

Our next aim is to discuss examples of operators which satisfy (3.1) or (3.4). First we would like to introduce the concept of the Plancherel measure corresponding to the considered operator A.

Lemma 5.1. *If we define the measure $\nu_{A,y}$ by the formula*

$$\int_0^\infty F(\lambda)\,\mathrm{d}\nu_{A,y}(\lambda) = \int_0^\infty F(\lambda)e^{2\lambda^m} m\lambda^{m-1}\,\mathrm{d}(E_A(\lambda^m)p_1(\,\cdot\,,y),p_1(\,\cdot\,,y)), \tag{5.1}$$

then

$$\|K_{F(\sqrt[m]{A})}(\,\cdot\,,y)\|^2_{L^2(X,\mu)} = \int_0^\infty |F(\lambda)|^2\,\mathrm{d}\nu_{A,y}(\lambda).$$

Proof. (See also [10, Proposition 3]).

$$\|K_{F(\sqrt[m]{A})}(\,\cdot\,,y)\|_{L^2} = \int_0^\infty \mathrm{d}(E_A(\lambda)K_{F(\sqrt[m]{A})}(\,\cdot\,,y),K_{F(\sqrt[m]{A})}\,\cdot\,,y))$$

$$= \int_0^\infty e^{2\lambda}\,\mathrm{d}(E_A(\lambda)\exp(-A)(K_{F(\sqrt[m]{A})}(\,\cdot\,,y)),\exp(-A)(K_{F(\sqrt[m]{A})}(\,\cdot\,,y)))$$

$$= \int_0^\infty e^{2\lambda}\,\mathrm{d}(E_A(\lambda)F(\sqrt[m]{A})p_1(\,\cdot\,,y),F(\sqrt[m]{A})p_1(\,\cdot\,,y))$$

$$= \int_0^\infty |F(\sqrt[m]{\lambda})|^2 e^{2\lambda}\,\mathrm{d}(E_A(\lambda)p_1(\,\cdot\,,y),p_1(\,\cdot\,,y))$$

$$= \int_0^\infty |F(\lambda)|^2 e^{2\lambda^m} m\lambda^{m-1}\,\mathrm{d}(E_A(\lambda^m)p_1(\,\cdot\,,y),p_1(\,\cdot\,,y)).$$

□

Following Christ [10] we call the measure $\nu_{A,y}$ the Plancherel measure of the operator A. Now we put $\mathrm{d}\nu_{A,y,R}(\lambda) = \chi_{[0,1]}(\lambda)\,\mathrm{d}\widetilde{\nu}_{A,y,R}(\lambda)$, where

$$\int_0^\infty \delta_R F(\lambda)\,\mathrm{d}\widetilde{\nu}_{A,y,R}(\lambda) = \int_0^\infty F(\lambda)\,\mathrm{d}\nu_{A,y}(\lambda).$$

By Lemma 2.2

$$\nu_{A,y,R}([0,1]) \le \mu(B(y,R^{-1}))^{-1}. \tag{5.2}$$

Now if ν is a positive Borel measure on the interval $[0,1]$, then for $1/p' + 1/p'' = 1$ and $p' \in (1,\infty]$ we put

$$\|\nu\|_{L^{p'}([0,1])} = \|\Lambda_\nu\|_{L^{p''}([0,1])\to\mathbf{C}},$$

where $\Lambda_\nu(F) = \int_0^1 F\,\mathrm{d}\nu$. In other words if $\|\nu\|_{L^{p'}}$ is finite, then $\mathrm{d}\nu(\lambda) = \alpha(\lambda)\,\mathrm{d}\lambda$ and $\|\nu\|_{L^{p'}} = \|\alpha\|_{L^{p'}}$. Now we can state (3.1) in the following way

Lemma 5.2. *Suppose that $1/p' + 2/p = 1$ and $p \in [2,\infty)$. Then (3.1) holds for p if and only if*

$$\|\nu_{A,y,R}\|_{L^{p'}} \le C\mu(B(y,R^{-1}))^{-1}$$

for all $y \in X$.

The proof of Lemma 5.2 is straightforward so we skip it.

5.1. **Operator A_1+A_2 acting on $L^2(X_1\times X_2,\mu_1\times\mu_2)$.** Suppose that $(\widetilde{X}_1,\mu_1,\rho_1,A_1)$ and $(\widetilde{X}_2,\mu_2,\rho_2,A_2)$ satisfy Assumption 2.1 and 2.2 for some positive constants d_1 and d_2 and that $m_1=m_2$. Now we consider the space $\widetilde{X}=\widetilde{X}_1\times\widetilde{X}_2$ with the measure $\mu=\mu_1\times\mu_2$ and the metric $\rho((x_1,x_2),(y_1,y_2))=\max(\rho_1(x_1,y_1)\rho_2(x_2,y_2))$. Denote by A_1+A_2 the operator $A_1\otimes 1+1\otimes A_2$. It generates a semigroup whose kernel p_t is given by the formula

$$p_t((x_1,x_2),(y_1,y_2))=p_t^{[1]}(x_1,y_1)p_t^{[2]}(x_2,y_2),$$

where $p^{[1]}$ and $p^{[2]}$ are the heat kernels corresponding to A_1 and A_2 respectively. Note that $\mu(B((x_1,x_2),r))=\mu_1(B(x_1,r))\mu_2(B(x_2,r))$ and that $(\widetilde{X},\mu,\rho)$ and A_1+A_2 satisfy Assumptions 2.1 and 2.2 with $d=d_1+d_2$ and $m=m_1=m_2$. Now if we define measure $\nu'_{A,y}$ by the formula $\nu'_{A,y}([0,\lambda^m])=\nu_{A,y}([0,\lambda])$, then (see (5.1))

$$\|K_{F(A)}(\,\cdot\,,y)\|^2_{L^2(X,\mu)}=\int_0^\infty |F(\lambda)|^2\,\mathrm{d}\nu'_{A,y}(\lambda).$$

In the following setting it is more convenient to consider measure $\nu'_{A,y}$ instead of $\nu_{A,y}$.

Lemma 5.3. *We have*

$$\nu'_{A_1+A_2,(y_1,y_2)}=\nu'_{A_1,y_1}*\nu'_{A_2,y_2}. \tag{5.3}$$

Proof. To prove Lemma 5.3 it is enough to show that for all functions $F\in C_c(\mathbf{R})$

$$\int_0^\infty F\,\mathrm{d}\nu'_{A_1+A_2,(y_1,y_2)}=\int_0^\infty\int_0^\infty F(\lambda_1+\lambda_2)\,\mathrm{d}\nu'_{A_1,y_1}(\lambda_1)\,\mathrm{d}\nu'_{A_2,y_2}(\lambda_2). \tag{5.4}$$

However, to show (5.4) for all $F\in C_c([0,\infty))$ it is enough to prove that (5.4) holds for all functions $(F_t)_{t>0}$, where $F_t(\lambda)=e^{-t\lambda}$. Now

$$\begin{aligned}
\int_0^\infty F_{2t}\,\mathrm{d}\nu'_{A_1+A_2,(y_1,y_2)}&=\int_0^\infty\int_0^\infty |p_t((x_1,x_2),(y_1,y_2))|^2\,\mathrm{d}\mu_1(x_1)\,\mathrm{d}\mu_2(x_2)\\
&=\int_0^\infty |p_t^{[1]}(x_1,y_1)|^2\,\mathrm{d}\mu_1(x_1)\int_0^\infty |p_t^{[2]}(x_2,y_2)|^2\,\mathrm{d}\mu_2(x_2)\\
&=\int_0^\infty F_{2t}(\lambda_1)\,\mathrm{d}\nu'_{A_1,y_1}(\lambda_1)\int_0^\infty F_{2t}(\lambda_2)\,\mathrm{d}\nu'_{A_2,y_2}(\lambda_2)\\
&=\int_0^\infty\int_0^\infty F_{2t}(\lambda_1+\lambda_2)\,\mathrm{d}\nu'_{A_1,y_1}(\lambda_1)\,\mathrm{d}\nu_{A_2,y_2}(\lambda_2)
\end{aligned}$$

as required. □

It is sometimes convenient to consider the following variation of condition (3.1)

$$\|K_{F(A)}(\,\cdot\,,y)\|^2_{L^2(X,\mu)}\le C\mu(B(y,R^{-1}))^{-1}\|\delta_{R^m}F\|_{L^p} \tag{5.5}$$

for some $p\in[2,\infty]$ and for any $R>0$ and all Borel functions F such that $\operatorname{supp}F\subseteq[0,R^m]$. Note that (3.1) follows from (5.5). However, if we put $X=\mathbf{R}$ and $A=-\mathrm{d}^2/\mathrm{d}x^2$, then $\mathrm{d}\nu_{A,y}(\lambda)=1/\pi\,\mathrm{d}\lambda$ and $\mathrm{d}\nu'_{A,y}(\lambda)=1/(2\pi)\lambda^{-1/2}\,\mathrm{d}\lambda$. Hence in this case, condition (3.1) holds for all $p\in[2,\infty]$ whereas (5.5) is true only for $p>4$. Let us also consider the following variation of condition (3.4)

$$\|K_{F(A)}(\,\cdot\,,y)\|^2_{L^2(X,\mu)}\le C\mu(B(y,N^{-1}))^{-1}\|\delta_{N^m}F\|_{N^\kappa,p} \tag{5.6}$$

for some $p\in[2,\infty]$ and for all $N\in Z_+$ and all functions $F\in C_c((-N^m,2N^m))$. Note that (3.4) follows from (5.6).

Note that in the following theorem we cannot replace (5.5) by (3.1) or (5.6) by (3.4).

Theorem 5.4. *Suppose that (5.5) (or (5.6)) holds for A_i and $p_1, p_2 \in [2, \infty]$ (and for $\kappa_1 = \kappa_2$). Then the operator $A_1 + A_2$ acting on $L^2(X_1 \times X_2)$ satisfies (5.5) and so (3.1) (or (5.6) and (3.4)) with $\kappa = \kappa_1 = \kappa_2$ respectively) for $p = \max(2, (1/p_1 + 1/p_2)^{-1})$.*

Proof. Note that (5.5) holds if and only if for $1/p' + 2/p = 1$ we have

$$\|\nu'_{A,y,R}\|_{L^{p'}} \le C\mu(B(y, R^{-1}))^{-1},$$

where $\nu'_{A,y,R} = \chi_{[0,1]}\widetilde{\nu'}_{A,y,R}$ and

$$\int \delta_{R^m} F(\lambda)\, \mathrm{d}\widetilde{\nu'}_{A,y,R}(\lambda) = \int F(\lambda)\, \mathrm{d}\nu'_{A,y}(\lambda)$$

(see Lemma 5.2). Next by Lemma 5.3

$$\mathrm{d}\nu'_{A_1+A_2,(y_1,y_2),R}(\lambda) = \chi_{[0,1]}(\lambda)\, \mathrm{d}(\nu'_{A_1,y_1,R} * \nu'_{A_1,y_1,R})(\lambda)$$

and by Young inequality

$$\begin{aligned}\|\nu'_{A_1+A_2,(y_1,y_2),R}\|_{L^{p'}} &\le \|\nu'_{A_1,y_1,R}\|_{L^{p'_1}} \|\nu'_{A_2,y_2,R}\|_{L^{p'_2}} \\ &\le C\mu(B(y_1, R^{-1}))^{-1}\mu(B(y_2, R^{-1}))^{-1} = C\mu(B((y_1, y_2), R^{-1}))^{-1},\end{aligned}$$

where $1 + 1/p' = 1/p'_1 + 1/p'_2$. Now if $(1/p_1 + 1/p_2) \le 1/2$, $1/p'_1 + 2/p_1 = 1$, $1/p'_2 + 2/p_2 = 1$, $1/p' + 2/p = 1$ and $1 + 1/p' = 1/p'_1 + 1/p'_2$, then $1/p = 1/p_1 + 1/p_2$. Finally to prove Theorem 5.4 in the case $(1/p'_1 + 1/p'_2) > 1/2$ it is enough to note that if $p < \tilde{p}$ and condition (5.5) holds for p, then (5.5) also holds for $\tilde{p}$.

Note that (5.6) holds if and only if for $1/p' + 2/p = 1$ we have (compare Lemma 5.2)

$$\|N^\kappa \nu'_{A,y,N} * \chi_{[0,N^{-\kappa}]}\|_{L^{p'}} \le C\mu(B(y, N^{-1}))^{-1}.$$

Now

$$\begin{aligned}\|N^\kappa \nu'_{A_1+A_2,(y_1,y_2),N}(\lambda) * \chi_{[0,N^{-\kappa}]}\|_{L^{p'}} &\le C\|N^{2\kappa}\nu'_{A_1+A_2,(y_1,y_2),N}(\lambda) * \chi_{[0,N^{-\kappa}]} * \chi_{[0,N^{-\kappa}]}\|_{L^{p'}} \\ &\le C\|N^\kappa \nu'_{A_1,y_1,N} * \chi_{[0,N^{-\kappa}]}\|_{L^{p'_1}} \|N^\kappa \nu'_{A_2,y_2,N} * \chi_{[0,N^{-\kappa}]}\|_{L^{p'_2}},\end{aligned}$$

where $1 + 1/p' = 1/p'_1 + 1/p'_2$. The rest of the proof is the same as for condition (5.5). □

Corollary 5.5. *Suppose that operator A of order 2 (i.e. $m = 2$) acting on $L^2(X, \mu)$ satisfies Assumptions 2.1 and 2.2. Then operator $A' = A - \partial^2$ (or $A'' = A - \partial_1^2 - \partial_2^2$) acting on $L^2(X \times \mathbf{R})$ ($L^2(X \times \mathbf{R}^2)$ respectively) satisfies (3.1) for $p = 4 + \varepsilon$ for all $\varepsilon > 0$ ($p = 2$ respectively).*

6. RIESZ MEANS

As we explained in the introduction, one of the main goal of investigating Theorem 3.1 was to study the Bochner-Riesz summability for $d/2 \ge \alpha > (d-1)/2$. We noted earlier that Theorem 3.1 with $p = 2$ implies the Riesz summability for all $\alpha > (d-1)/2$ on $L^q(X)$, $q \in (1, \infty)$. However, one can obtain only weak type $(1,1)$ estimates in virtue of Theorem 3.1 and formally Theorem 3.1 does not imply continuity and convergence of the Riesz means on $L^1(X, \mu)$. For the sake of completeness let us describe how to modify, or actually simplify, the proof of Theorem 3.1 and 3.2 to prove that (3.1), or (3.4) and (3.5) with $p \in [2, \infty]$ imply the uniform continuity of the Riesz means of order greater than $(d/2 - 1/p)$ on all spaces $L^q(X, \mu)$ for $q \in [1, \infty]$.

Proposition 6.1. *Suppose that A satisfies condition (3.1), or (3.4) and (3.5) for some $p \in [2,\infty]$. Next suppose that $H \in C_c((a,b))$, where $1/4 < a, b < 4$. Then for any $s > d/2$ there exists a constant C independent of $R > 0$ such that*

$$\|\delta_{1/R}H(A)\|_{L^1(X,\mu)\to L^1(X,\mu)} \le C\|H\|_{W^p_s}.$$

Proof. First we consider the case when A satisfies condition (3.4) and (3.5). Then without loosing generality one can assume that $R = N \in Z_+$. Next (see (4.18) and (4.15))

$$\begin{aligned}\Big(\int |K_{\xi_{N^{\kappa-1}}*\delta_{1/N}H(\sqrt[m]{A})}(x,y)|\,\mathrm{d}\mu(x)\Big)^2 \\ \le \int |K_{\xi_{N^{\kappa-1}}*\delta_{1/N}H(\sqrt[m]{A})}(x,y)|^2(1+N\rho(x,y))^{2s'}\,\mathrm{d}\mu(x) \\ \times \int |(1+N\rho(x,y))^{-2s'}d\mu(x) \le C\|H\|^2_{W^p_s}\end{aligned}$$

for all $2d < s' < s$. By (3.5) and Proposition 4.6 (see (4.21))

$$\begin{aligned}\|\delta_{1/N}[H - H*\xi_{N^\kappa}](\sqrt[m]{A})\|^2_{L^1(X,\mu)\to L^1(X,\mu)} &\le C'N^{d\kappa+\varepsilon}\|H-\xi_{N^\kappa}*H\|^2_{N^\kappa,p} \\ &\le C'N^{(d\kappa+\varepsilon)}N^{-2s\kappa}\|H\|^2_{W^p_s} \le C\|H\|^2_{W^p_s}.\end{aligned}$$

Now if the operator A satisfies condition (3.1), then (see (4.18) and (4.15))

$$\|\delta_{1/R}H(A)\|_{L^1(X,\mu)\to L^1(X,\mu)} \le \sup_{y\in X}\int |K_{\delta_{1/R}H(\sqrt[m]{A})}(x,y)|\,\mathrm{d}\mu(x) \le C\|H\|_{W^p_s}$$

as required. ☐

Theorem 6.2. *Suppose that A satisfies condition (5.5) for some $p \in [2,\infty]$ and that $s > d/2$. Then for any function $H \in C_c((-1,1))$*

$$\|\delta_{1/R}H(A)\|_{L^q(X,\mu)\to L^q(X,\mu)} \le C\|H\|_{W^p_s} \tag{6.1}$$

for all $R > 0$ and $q \in [1,\infty]$. Hence if $H(0) = 1$ and $H \in W^p_s \cap C_c((-1,1))$, then

$$\lim_{R\to\infty}\|\delta_{1/R}H(A)f - f\|_{L^q(X,\mu)} = 0 \tag{6.2}$$

for all $f \in L^q$ and $q \in [1,\infty)$.

Proof. First note that if we put $G(\lambda) = H(\lambda)e^\lambda$, then (see (4.4))

$$\Big(\int_X |K_{\delta_{1/R^m}H(A)}(x,y)|^2(1+R\rho(x,y))^s\,\mathrm{d}\mu(x)\Big)^{1/2} \le C\mu(B(y,1/R))^{-1/2}\|G\|_{W^2_{(s+1+\varepsilon)/2}}.$$

Now if $\operatorname{supp} H \subseteq [-1,1]$ and $p \ge 2$, then

$$\|G\|_{W^2_{(s+1+\varepsilon)/2}} \le \|H\|_{W^2_{(s+1+\varepsilon)/2}} \le \|H\|_{W^p_{(s+1+\varepsilon)/2}}.$$

By repeating the proof of (4.2) one can show that

$$\int_X |K_{\delta_{1/R^m}H(A)}(x,y)|^2(1+R\rho(x,y))^s d\mu(x) \le C\mu(B(y,R^{-1}))^{-1}\|H\|^2_{W^p_{s/2+\varepsilon}} \tag{6.3}$$

for all functions $H \in C_c((-1,1))$. Finally (6.1) follows from (6.3) (see the proof of Proposition 6.1). The proof that (6.2) follows from (6.1) is standard so we skip it. ☐

Corollary 6.3. *Suppose that operator A satisfies condition (3.1), or (3.4) and (3.5) for some $p \in [2,\infty]$. Then for any $\alpha > d/2 - 1/p$ and $q \in [1,\infty]$*

$$\sup_{R>0} \|\sigma_R^\alpha(A)\|_{L^q(X,\mu)\to L^q(X,\mu)} \le C < \infty.$$

Hence for any $q \in [1,\infty)$ and $f \in L^q(X,\mu)$

$$\lim_{R\to\infty} \|\sigma_R^\alpha(A)f - f\|_{L^q(X,\mu)\to L^q(X,\mu)} = 0,$$

where σ_R^α is defined by (1.2).

Proof. Suppose that $\alpha > s > d/2 - 1/p$. Then there exist functions $\operatorname{supp}\sigma' \subseteq [-2/3, 2/3]$ and $\operatorname{supp}\sigma'' \subseteq [1/3, 1]$ such that

$$\sigma_1^\alpha = \sigma' + \sigma''$$

and $\|\sigma'\|_{W_s^\infty} < \infty$, $\|\sigma''\|_{W_s^p} < \infty$. By Lemma 2.2 condition (5.5) is always true for $p = \infty$ so Corollary 6.3 follows from Proposition 6.1 and Theorem 6.2. □

Remark We noted in the introduction that Theorem 1.1 implies the Riesz summability for $\alpha > (d-1)/2$ (see also [10, pp. 74]). Actually to prove the Riesz summability for all L^q, $q \in (1,\infty)$ and $\alpha > (d-1)/2$ it is enough to show that

$$\|F(A)\|_{L^1(X,\mu)\to L^{1,\infty}(X,\mu)} \le C_s \sup_{t>0} \|\eta\, \delta_t F\|_{W_s^1} \tag{6.4}$$

for all $s > (d+1)/2$ and for any bounded Borel function F (see also [25, Theorem (2.4)]). Using the estimate (6.4) one can obtain examples of singular integral operators. It is usually very difficult to prove continuity of singular integral operators for general measure spaces (see e.g. [52]). Hence in the case of a general measure space it is substantially more difficult to obtain (6.4) than the Riesz summability for $\alpha > (d-1)/2$. However if we consider only spaces with the doubling condition (or their open subspaces), then (6.4) and the sharp Riesz summability are essentially equivalent. To avoid easy but tedious detailed discussion of the relation between (6.4) and the Riesz summability let us only mention that for $k \in Z_+ \cup \{0\}$ and $F \in C_c^k((0,R))$

$$F(A) = (-1)^k/(k-1)! \int_0^R F^{(k)}(\lambda)\lambda^{k-1}\sigma_\lambda^{(k-1)}(A)\, \mathrm{d}\lambda$$

so

$$\|F(A)\|_{L^1(X,\mu)\to L^1(X,\mu)} \le C\|\delta_R F\|_{W_k^1} \sup_R \|\sigma_R^{(k-1)}(A)\|_{L^1(X,\mu)\to L^1(X,\mu)} \tag{6.5}$$

(see [24] and (7.21)). Then one can use Theorem 4.5 to show that (6.4) essentially follows from (6.5). However for any $s' > s + 1/2$ we have $W_{s'}^1 + W_s^\infty \subsetneq W_s^2$. Therefore it seems that Theorem 1.1 is still a substantially stronger result than both Theorem 1.6 and the Bochner-Riesz summability for $\alpha > (d-1)/2$ even if we consider only spaces with the doubling condition. (See also the remark in Section 7.5).

7. Examples

To motivate introduction of the Plancherel type estimates we discuss several examples of operators which satisfy conditions (3.1) or (3.4) with some $p \in [2,\infty)$. First we describe how to use Theorems 3.1 and 3.2 to prove Theorem 1.1 and to obtain spectral multipliers theorems for elliptic operators on compact manifolds. The new and the most interesting results which we describe here concern Schrödinger operators with positive potential (compare [26]).

7.1. **Homogeneous groups.** First let us show that Theorem 1.1 is a straightforward consequence of Theorem 3.1.

Proof of Theorem 1.1. It is well known that the operator L defined by (1.3) and underlying group G satisfy Assumptions 2.1 and 2.2. Therefore it is enough to show (3.1) for $p = 2$. Now if L is a left-invariant operator acting on a unimodular Lie group G then $K_{F(\sqrt{L})}(x,y) = F(\sqrt[m]{L})(zx,zy)$ for all $x,y,z \in G$. Hence the measure $\nu_{L,y}$ does not depend on y. If in addition the operator L is homogeneous, then $K_{\delta_{1/t}F(\sqrt{L})}(x,y) = t^d K_{F(\sqrt{L})}(\tilde{\delta}_t x, \tilde{\delta}_t y)$ and so

$$\|K_{\delta_{1/t}F(\sqrt{L})}(\,\cdot\,,y)\|^2_{L^2(\mathbf{G})} = t^d \|K_{F(\sqrt{L})}(\,\cdot\,,\tilde{\delta}_t y)\|^2_{L^2(\mathbf{G})}.$$

Hence $\int_{\mathbf{R}_+} |\delta_{1/t}F(\lambda)|^2 \,\mathrm{d}\mu(\lambda) = t^d \int_{\mathbf{R}_+} |F(\lambda)|^2 \,\mathrm{d}\mu(\lambda)$ for any F and so

$$\mathrm{d}\nu_{L,y}(\lambda) = C\lambda^{d-1}\,\mathrm{d}\lambda. \tag{7.1}$$

(see also [10, Proposition 3]). Therefore $\|\nu_{L,y,R}\|_{L^\infty} = CR^d = C\mu(B(y,R^{-1}))^{-1}$ and Theorem 1.1 follows from Lemma 5.2 and Theorem 3.1. □

Corollary 7.1. *Let L be a positive definite self-adjoint left invariant operator on a homogeneous group $\mathbf{G}$. Suppose that the operator L is homogeneous of order m, i.e. $\tilde{\delta}_t L = t^m L$ and that*

$$|K_{\exp(-L)}(x,y)| = |K_{\exp(-L)}(e,x^{-1}y)| \le C\exp(-c|x^{-1}y|^{m/(m-1)}) \tag{7.2}$$

where C,c are positive constants and $|\cdot|$ is homogeneous norm on G (see [23]). Then for $s > d/2$ and for any Borel function $F\colon [0,\infty) \to \mathbf{C}$

$$\|F(L)\|_{L^1(\mathbf{G})\to L^{1,\infty}(\mathbf{G})} \le C\sup_{t>0} \|\eta\,\delta_t F\|_{W^2_s},$$

where d is the homogeneous dimension of $\mathbf{G}$.

Proof. Assumptions 2.1 and 2.2 follow from homogeneity of the operator L and group $\mathbf{G}$ and from (7.2). Hence to finish the proof it is enough to note that by homogeneity of L (7.1) still holds. □

Now let us describe another generalisation of Theorem 1.1. Let $(\tilde{\delta}_t)_{t>0}$ be a family of dilation on $\mathbf{G}$. As we said earlier the operator L defined by (1.3) is homogeneous if $\tilde{\delta}_t X_i = tX_i$. Now we say that L is 'quasi-homogeneous' if $\tilde{\delta}_t X_i = t^{d_i'} X_i$ for some $d_i' \ge 1$. For example on any two-step nilpotent Lie group any operator L defined by (1.3) is 'quasi-homogeneous' for some family of dilations. L is also 'quasi-homogeneous' if $L = \sum Y_i^2$ where Y_i is a homogeneous basis of Lie algebra of $\mathbf{G}$ (see (1.4)).

Theorem 7.2. *Suppose that L is a quasi-homogeneous operator acting on a homogeneous group and that $s > d/2 = \max(d_0, d_\infty)/2$. Then for any Borel function F*

$$\|F(L)\|_{L^1(\mathbf{G})\to L^{1,\infty}(\mathbf{G})} \le C\sup_{t>0} \|\eta\,\delta_t F\|_{W^2_s}.$$

Theorem 7.2 is proved in [48]. Here we note that Theorem 7.2 is a straightforward consequence of Theorem 3.1, Lemma 5.2 and the following result

Theorem 7.3. *Suppose that the operator L is quasi-homogeneous and let $\nu_{L,y}$ be the measure defined by (5.1). Then $d\nu_{L,y} = \alpha(\lambda)d\lambda$, where*

$$\alpha(\lambda) \le C_n \begin{cases} \lambda^{d_\infty - 1} & \text{if } \lambda \le 1 \\ \lambda^{d_0 - 1} & \text{if } \lambda > 1. \end{cases} \tag{7.3}$$

Theorem 7.3 is proved in [48, Theorem 1].

7.2. **Compact manifolds.** For a general positive definite elliptic operator on a compact manifold, Assumption 2.2 holds by general elliptic regularity theory. Further, one has the Avakumovič–Agmon–Hörmander theorem.

Theorem 7.4. *Let A be a positive definite elliptic pseudo-differential operator of order m on a compact manifold X of dimension d. Then*

$$\|\chi_{[R,R+1)}(A^{1/m})\|^2_{L^1(X)\to L^2(X)} \le C\, R^{d-1} \qquad \forall R \in \mathbf{R}^+. \tag{7.4}$$

Theorem 7.4 was proved by Hörmander in [30]; see also [1, 4, 31] and [54, §5.1]. This theorem has the following useful consequence.

Lemma 7.5. *Condition (3.4) with $p = 2$ and $\kappa = 1$ holds for positive definite elliptic pseudo-differential operators on compact manifolds.*

Proof. By the spectral theorem,

$$\sup_{y\in X} \|K_{F(\sqrt[m]{A})}(\,\cdot\,, y)\|_{L^2(X)} \le \Big(\sum_{l=1}^{N} \|\chi_{[l-1,l)}\, F(\sqrt[m]{A})\|^2_{L^1\to L^2}\Big)^{1/2} \le CN^{d/2}\|\delta_N F\|_{N,2},$$

as required. ☐

The importance of the estimate (7.4) for multiplier theorems was noted by C.D. Sogge [53], who used it to establish the convergence of the Riesz means up to the critical exponent $(d-1)/2$ (see also [8]). The following theorem is due to Seeger and Sogge [46] (see also Hebisch [28]).

Theorem 7.6. *Suppose that $s > d/2$ and that A is a self-adjoint, positive definite elliptic differential operator of order $m \ge 2$ acting on a compact Riemannian manifold X of dimension d. Then*

$$\|F(A)\|_{L^1(X)\to L^{1,\infty}(X)} \le C\Big(\sup_{t\ge 1} \|\eta\, \delta_t F\|_{W^2_s} + \|F\|_{L^\infty}\Big)$$

for any Borel function $F\colon [0,\infty) \to \mathbf{C}$.

Proof. This result is a consequence of Theorem 3.2, Lemma 3.3 and Lemma 7.5. ☐

Theorem 7.6 applied to an elliptic operator on a compact Lie group gives a stronger result than Theorem 1.2. One can say that for elliptic operators on a compact Lie group Theorem 1.1 holds. However, we do not know if the Avakumovič–Agmon–Hörmander condition holds for sub-elliptic operators on a compact Lie group (see also [13]). Hence Theorem 1.2 gives the strongest known result for sub-elliptic operators on a compact Lie group.

7.3. **Laplace operators on irregular domains with Dirichlet boundary conditions.** Let X be a connected open subset of $\mathbf{R}^d$. Note that if X is irregular then X is not necessarily a homogeneous space. Thus the following result gives examples of singular integral multipliers on spaces without the doubling conditions (see also [21]).

Theorem 7.7. *Suppose that Δ_X is the Laplace operator with Dirichlet boundary condition on $X \subset \mathbf{R}^d$. Then for any $s > d/2$*

$$\|F(\Delta_X)\|_{L^1(\mathbf{R}^d)\to L^{1,\infty}(\mathbf{R}^d)} \le C \sup_{t>0} \|\eta\, \delta_t F\|_{W^\infty_s}. \tag{7.5}$$

Proof. Note that

$$0 \le K_{\exp(-t\Delta_X)}(x,y) \le (4\pi t)^{-d/2} \exp(-|x-y|^2/4t)$$

(see e.g. [14, Example 2.1.8]). Hence $\widetilde{X} = \mathbf{R}^n$ and Δ_X satisfy Assumptions 2.1 2.2 and Theorem 7.7 follows from Lemma 2.2 and Theorem 3.1. □

Remark. A natural question arises: *does (3.1) or (3.4) hold for any $p < \infty$?.* This question is open. However, if X is compact and ∂X is smooth, then (compare (7.4))

$$\begin{aligned} \int_X \|K_{\chi_{[R,R+1)}(\sqrt{A})}(\,\cdot\,,y)\|^2_{L^2(X,\mu)}\, \mathrm{d}\mu(y) &= \int_X K_{\chi_{[R,R+1)}(\sqrt{A})}(x,x)\, \mathrm{d}\mu(x) \\ &= \Lambda(R+1) - \Lambda(R) \le CR^{d-1} \end{aligned} \tag{7.6}$$

where $\Lambda(R)$ denotes the number of eigenvalues of $\sqrt{A}$ which are $\le R$ (see [32, § 17.5, § 29.3]; see also [54, §5 Notes]). Condition (7.6) is called the Weyl asymptotic or the sharp Weyl formula. Note that if $\widetilde{X} = \mathbf{R}^n$, then $\mu(B(x,r)) = c_n r^n$ and our Plancherel condition (3.4) with $p = 2$ is equivalent to the Avakumovič–Agmon–Hörmander condition i.e. $\sup_x \|K_{\chi_{[R,R+1)}(\sqrt{A})}(\,\cdot\,,x)\|^2_{L^2(X)} = \|K_{\chi_{[R,R+1)}(\sqrt{A})}(x,x)\|_{L^\infty(X)} \le CR^{d-1}$ (see Lemma 7.5). The sharp Weyl formula i.e. (7.6) holds if and only if $\|K_{\chi_{[R,R+1)}(\sqrt{A})}(x,x)\|_{L^1(X)} \le CR^{d-1}$. Thus if $\mu(X) < \infty$, then the Plancherel estimates (3.4) with $p = 2$ are stronger than the sharp Weyl formula (7.6). Although it seems that (7.6) does not imply the sharp Hörmander-type spectral multiplier the sharp Weyl formula itself is regarded as an important topic (see [32, § 17.5, § 29.3], [54, §4.2], see also [1, 4, 17, 34, 38]). Note that in the case of group invariant operators on compact Lie groups the Plancherel estimates and the sharp Weyl formula are equivalent.

The following corollary gives examples of operators which satisfy the Plancherel estimates and the sharp Weyl formula.

Corollary 7.8. *Condition (3.4) with $p = 2$, $\kappa = 1$ and so the sharp Weyl formula hold for Δ_X, where $X = X' \times (0,1)^2 \subset \mathbf{R}^d$ and X' is an arbitrary connected bounded open subset of $\mathbf{R}^{d-2}$. Hence*

$$\|F(\Delta_X)\|_{L^1(X)\to L^{1,\infty}(X)} \le C\big(\sup_{t\ge 0} \|\eta\, \delta_t F\|_{W^2_s} + \|F\|_{L^\infty}\big) \tag{7.7}$$

for any Borel function $F\colon [0,\infty) \to \mathbf{C}$. (7.7) holds also for $X = X' \times \mathbf{R}^2 \subset \mathbf{R}^d$, where X' is an arbitrary connected open subset of $\mathbf{R}^{d-2}$.

Proof. Corollary 7.8 follows from Theorems 5.4, 3.2 and 3.1. The fact that Δ_X coincides with $\Delta_{X'} + \Delta_{(0,1)^2}$ or $\Delta_{X'} + \Delta_{\mathbf{R}^2}$ can be shown by using the associated quadratic forms and the well known fact that functions in $C^\infty_c(X)$ can be approximated by functions of the type $f \otimes g$ where $f \in C^\infty_c(X')$ and $g \in C^\infty_c(0,1)$ or $C^\infty_c(\mathbf{R}^2)$. □

7.4. **Schrödinger operators.** Let X be a connected and complete Riemannian manifold. We consider the Schrödinger operator $A = -\Delta + V$ where $V\colon X \to \mathbf{R}$, $V \in L^1_{\text{loc}}(X)$ and $V \geq 0$. The operator A is defined by the quadratic form technique. If $p_t(x,y)$ denotes the heat kernel corresponding to A then as a consequence of the Trotter product formula

$$|p_t(x,y)| \leq p'_t(x,y) \tag{7.8}$$

where $p'_t(x,y)$ denotes the heat kernel corresponding to Δ.
More generally, (7.8) holds for the heat kernel $p_t(x,y)$ of the magnetic Schrödinger operator $A_{Y,V}$ associated with the quadratic form

$$(A_{Y,V}f,f) = \int\limits_{\mathbf{R}^n} \left(|grad\, f(x) + if(x)Y|^2 + V(x)|f(x)|^2\right) \mathrm{d}x \ , \tag{7.9}$$

where Y is a real vector field such that $|Y|^2 \in L^1_{\text{loc}}(\mathbf{R}^n)$ and $0 \leq V \in L^1_{\text{loc}}(\mathbf{R}^n)$ (see [51, Theorem 2.3]). This result can be extended with a similar proof to the situation of magnetic Schrödinger operators acting on complete Riemannian manifolds with a vector filed $Y \in C^1$.

We start our discussion of Schrödinger operators with positive potentials with the following lemma (compare Lemma 3.3)

Lemma 7.9. *Let $A = -\Delta_d + V$, where $V \in L^1_{\text{loc}}(\mathbf{R}^d)$ and $V \geq 0$. Suppose that for some $\kappa > 0$ and any $\varepsilon > 0$*

$$\int_{\mathbf{R}^d} (1+V(x))^{d(1-\kappa)/2-\varepsilon}\,\mathrm{d}x < \infty \tag{7.10}$$

Then condition (3.4) implies (3.5).

Remark. It is not difficult to see that one does not have to assume that $\kappa \geq 1$ is a natural number in Theorem 3.2. More precisely, we just replace N^κ by its integer part $[N^\kappa]$ in the statement of conditions (3.4) and (3.5). We assume that κ is a natural number in Theorem 3.2 only to simplify notation since in all cases for which we know how to prove (3.4) for some $p < \infty$, $\kappa = 1$ or $\kappa = 2$. Note that if one studies the operator $A = -\Delta + x^4$, then one has to put $\kappa = 3/2$.

Proof. To prove Lemma 7.9 it is enough to show that if $A = -\Delta + V$, where $V \in L^1_{\text{loc}}(\mathbf{R}^d)$ and $V \geq 0$, then for any $c > 0$

$$\|(A+1)^{-c/2}\|_{L^2(\mathbf{R}^d)\to L^1(\mathbf{R}^d)} < C\int_{\mathbf{R}^d} (1+V(x))^{-c}\,\mathrm{d}x. \tag{7.11}$$

Indeed, suppose that (3.4) holds. Then we put $c = (d(\kappa-1)+\varepsilon)/2$ in (7.11) and by (7.10)

$$\begin{aligned}
&\|F(\sqrt{A})\|^2_{L^1(\mathbf{R}^d)\to L^1(\mathbf{R}^d)} \\
&\quad\leq \|F(\sqrt{A})(A+1)^{(d(\kappa-1)+\varepsilon)/4}\|^2_{L^1(\mathbf{R}^d)\to L^2(\mathbf{R}^d)} \|(A+1)^{-(d(\kappa-1)+\varepsilon)/4}\|^2_{L^2(\mathbf{R}^d)\to L^1(\mathbf{R}^d)} \\
&\quad\leq CN^d \|\delta_N[F(\lambda)(1+\lambda)^{(d(\kappa-1)+\varepsilon)/2}]\|^2_{N^\kappa,p} \leq CN^{d\kappa+\varepsilon}\|\delta_N F\|^2_{N^\kappa,p}
\end{aligned}$$

for any Borel function F such that $\operatorname{supp} F \subseteq [0,N]$.

To prove (7.11) we put $M_g(f) = fg$ and $M = M^{1/2}_{1+V}$. Then we note that

$$\|(A+1)^{1/2}f\|^2_{L^2(\mathbf{R}^d)} = \langle (A+1)f,f\rangle \geq \langle M^2 f,f\rangle = \|Mf\|^2_{L^2(\mathbf{R}^d)}.$$

For any quadratic forms B_1 and B_2, if $B_1 \geq B_2 \geq 0$, then $B_1^\alpha > B_2^\alpha$ for all $\alpha \in [0,1]$. Hence $\langle (A+1)^\alpha f, f\rangle \geq \langle M^{2\alpha} f, f\rangle$ and

$$\|M^\alpha (A+1)^{-\alpha/2}\|^2_{L^2(\mathbf{R}^d)\to L^2(\mathbf{R}^d)} < \infty \tag{7.12}$$

for all $\alpha \in [0,1]$. Further we note that for the Riesz transform $M(A+1)^{-1/2}$ we have

$$\|M(A+1)^{-1/2}\|_{L^q(\mathbf{R}^d)\to L^q(\mathbf{R}^d)} < \infty \tag{7.13}$$

for all $q \in (1,2]$. (7.13) is proved in [49] and the proof of (7.13) is a minor modification of the proof of [11, Theorem 1.1]. Finally by Hölder's inequality for any $s \geq (1/q_2 - 1/q_1)^{-1}$ and any function $V > 0$

$$\|M^{-1}_{1+V}\|^s_{L^{q_1}(\mathbf{R}^d)\to L^{q_2}(\mathbf{R}^d)} < C\int_{\mathbf{R}^d} (1+V(x))^{-s}\,\mathrm{d}x \tag{7.14}$$

and to finish the proof of Lemma 7.9 it is enough to note that

$$(A+1)^{-c/2} = (M^{-1}M(A+1)^{-1/2})^{-[c]}M^{[c]-c}M^{c-[c]}(A+1)^{([c]-c)/2}.$$

□

7.5. **Harmonic oscillator acting on $L^2(\mathbf{R})$.** The one dimensional harmonic oscillator is the operator acting on $L^2(\mathbf{R})$ given by formula

$$A = -\,\mathrm{d}^2/\,\mathrm{d}x^2 + x^2 = \Delta + x^2.$$

As an application of Theorem 3.2 we obtain the following result

Theorem 7.10. *If $A = \Delta + x^2$, then for any $s > 1/2$ and any Borel function F*

$$\|F(A)\|_{L^1(\mathbf{R})\to L^{1,\infty}(\mathbf{R})} \leq C_s \sup_{t>1} \|\eta\, \delta_t F\|_{W^4_s}. \tag{7.15}$$

Proof. Let us note that in virtue of Theorem 3.2 and Lemma 7.9 it is enough to prove (3.4) for $\kappa = 2$ and for $p = 4+\varepsilon$ for all $\varepsilon > 0$. (3.4) follows from (5.6) and one can state (5.6) for $\kappa = 2$ and $p = 4+\varepsilon$ in the following way

$$\|K_{F(A)}(\,\cdot\,,y)\|^2_{L^2(\mathbf{R})} \leq CN\|\delta_{N^2}F\|^2_{N^2,4+\varepsilon} \tag{7.16}$$

for any function $F \in C_c([0,N^2])$. Or replacing N^2 by N in (7.16) we have to show that

$$\|K_{F(A)}(\,\cdot\,,y)\|^2_{L^2(\mathbf{R})} \leq CN^{1/2}\|\delta_N F\|^2_{N,4+\varepsilon} \tag{7.17}$$

for any $F \in C_c([0,N])$. To prove (7.17) we recall well known estimates for the Hermite functions. By h_k we denote the k-th Hermite function. The Hermite functions form an orthonormal basis of $L^2(\mathbf{R})$ and $Ah_k = (2k+1)h_k$. Moreover (see [40, (2.3), pp. 435])

$$|h_k(x)| \leq C \begin{cases} (\sqrt[3]{2k+1} + |2k+1-x^2|)^{-1/4} & \text{when } x^2 \leq 4k \\ \exp(-cx^2) & \text{when } x^2 > 4k. \end{cases} \tag{7.18}$$

We are going to prove (7.17) only for $y^2 = N$ as the proof for other $y \in \mathbf{R}$ is similar or simpler. First we note that

$$\|K_{F(A)}(\,\cdot\,,y)\|^2_{L^2(\mathbf{R})} = \sum_{k=1}^{[N/2]} \|K_{\chi_{[2k-1,2k+1)}F(A)}(\,\cdot\,,y)\|^2_{L^2(\mathbf{R})}$$

$$= N\Big(\sum_{k=1}^{[N/2]} \frac{|F(2k+1)|^2|h_k(y)|^2}{N}\Big) \leq CN\|\delta_N F\|^2_{N,2p}\Big(\sum_{k=1}^{[N/2]} \frac{|h_k(y)|^{2p'}}{N}\Big)^{1/p'}, \tag{7.19}$$

where $1/p + 1/p' = 1$. Now, if $y^2 = N$, then by (7.18)

$$\sum_{k=1}^{[N/2]} \frac{|h_k(y)|^{2p'}}{N} \le C \sum_{k=1}^{[N/2]} \frac{(|N-2k|+1)^{-p'/2}}{N} \le CN^{-p'/2} \tag{7.20}$$

for all $p' < 2$. Hence (7.17) follows from (7.19) and (7.20). $\square$

Remark. Theorem 7.10 is stronger than [61, Theorem 4.2.1]. In [60, Theorem 5.1] Thangavelu proved the Riesz means convergence for the harmonic oscillator for $\alpha > 1/6$. Using Thangavelu's result and Theorem 4.5 one can show that for $p = 1$ and $s > 7/6$

$$\|F(A)\|_{L^1(\mathbf{R}) \to L^{1,\infty}(\mathbf{R})} \le C_s \sup_{t>1} \|\eta\, \delta_t F\|_{W^p_s}. \tag{7.21}$$

Note that $W^4_{1/2} \not\subseteq W^1_{7/6}$ and $W^4_{1/2} \not\supseteq W^1_{7/6}$. This means that Thangavelu's result i.e. [60, Theorem 5.1] and Theorem 7.10 are independent i.e., neither of them follows from the other. Note also that in virtue of Theorem 1.3, (7.21) does not hold for $s < 1/p + 1/6$. Using the interpolation technique, (7.21) with $p = 1$ and (7.15) one can show that (7.21) holds for $s > 8/(9p) + 5/18$. We do not know if (7.21) is true when $p < 4$ and $1/p + 1/6 < s \le 8/(9p) + 5/18$.

Sketch of the proof of (7.21). Suppose that $\operatorname{supp} F \subset [0, R]$, $s \ge 0$ and let $W^s F$ be a Weyl fractional derivative of F of order s (see e.g. [24]). (For $s \in Z_+$ $W^s F = (-1)^s F^{(s)}$). Then

$$F(A) = \frac{1}{\Gamma(s)} \int_0^\infty W^s F(\lambda) \lambda^{s-1} \sigma_\lambda^{(s-1)}(A)\, \mathrm{d}\lambda$$

so for any $\varepsilon > 0$

$$\|F(A)\|_{L^1(X,\mu) \to L^1(X,\mu)} \le CC^\sigma_s \|F\|_{AC^s} = CC^\sigma_s \|\delta_R F\|_{AC^s} \le CC^\sigma_s \|\delta_R F\|_{W^1_{s+\varepsilon}},$$

where $\|F\|_{AC^s} = \int_0^\infty |W^s F(\lambda)| \lambda^{s-1}\, \mathrm{d}\lambda$ and $C^\sigma_s = \sup_r \|\sigma_r^{(s-1)}(A)\|_{L^1(X,\mu) \to L^1(X,\mu)}$ (see e.g. [24]). Thus if $\operatorname{supp} F \subseteq [2^{n-2}, 2^n]$, then for $s > 7/6$

$$\sup_{y \in X} \int_{X - B(y,r)} |K_{F(\sqrt[m]{A})}(x,y)|\, \mathrm{d}\mu(x) \le C \|\delta_{2^n} \omega_n F\|_{W^1_s}. \tag{7.22}$$

Using (7.22), (4.18) and the Mauceri Meda interpolation trick (see the proof of Lemma 4.3) we can show that for any $s > 7/6$ there exists $\varepsilon > 0$ such that

$$\sup_{y \in X} \int_{X - B(y,r)} |K_{\omega_n F(\sqrt[m]{A})}(x,y)|\, \mathrm{d}\mu(x) \le C(1 + 2^n r)^{-\varepsilon} \|\delta_{2^n} \omega_n F\|_{W^1_s}.$$

Finally we obtain (7.21) using Theorem 4.5 in the same way as in the proof of Theorem 3.1. $\square$

7.6. **Harmonic oscillator acting on $L^2(\mathbf{R}^d)$, $d \ge 2$.** In this section we study spectral multipliers for the operator A_d on R^d, $d \ge 2$, where $A_d = \Delta_d + |x|^2 = -\sum_{j=1}^d \partial_j^2 + \sum_{j=1}^d x_j^2$. We noted that there is an essential difference between spectral multiplier theorems for $\Delta_1 + x^2$ (Theorem 7.10) and Δ_1 (Theorem 1.1). There is not such a difference between spectral multiplier theorems for $\Delta_d + |x|^2$ and Δ_d if $d \ge 2$. Therefore it is quite surprising that Theorem 7.11 is an obvious consequence of (7.17) and Theorem 7.10.

Theorem 7.11. *Suppose that* $A_d = \Delta_d + |x|^2$. *Then for any* $s > d/2$ *and any Borel function* F

$$\|F(A)\|_{L^1(\mathbf{R}^d)\to L^{1,\infty}(\mathbf{R}^d)} \le C_s \sup_{t>1} \|\eta\, \delta_t F\|_{W^2_s}. \tag{7.23}$$

Proof. By Theorem 5.4 and (7.17) $A_d = \Delta_d + |x|^2$ satisfies condition (5.6) and so (3.4) for any $p > 2$ if $d = 2$ and for any $p \ge 2$ if $d > 2$. Hence Theorem 7.11 follows from Theorem 3.2 and Lemma 7.9. □

Remark. Theorem 7.11 is substantially stronger than [61, Theorems 3.3.2 and 4.2.1].

7.7. **Laguerre Expansion.** For $a \ge 1/2$, we denote by A_a the operator

$$\langle A_a f, f\rangle = \int_0^\infty |f'(x)|^2 + (x^2 + (a^2 - 1/4)x^{-2})|f(x)|^2 \,\mathrm{d}x. \tag{7.24}$$

for $f \in C_c^\infty(\mathbf{R}_+)$. With some abuse of notation we will also denote by A_a the Friedrich's extension of this operator. We put

$$f_k^a(x) = \Big(\frac{\Gamma(k+1)}{\Gamma(k+a+1)}\Big)^{1/2} L_k^a(x^2)e^{-x^2/2}x^{a+1/2},$$

where L_k^a are the Laguerre polynomials. It is well known that (f_k^a) is an orthonormal basis of $L^2(\mathbf{R}_+, \mathrm{d}x)$ and that $A_a f_k^a = (4k + 2a + 2)f_k^a$ (see [36]). Finally For $a_j \ge 1/2$, $j = 1, \ldots, d$ we define the operator $A_{(a_1,\ldots,a_d)} = A_{a_1} + \ldots + A_{a_d}$ by the formula

$$\langle A_{(a_1,\ldots,a_d)} f, f\rangle = \sum_{j=1}^d \int_{\mathbf{R}^d_+} |\partial_{x_j} f(x)|^2 + (x_j^2 + (a_j^2 - 1/4)x_j^{-2})|f(x)|^2 \,\mathrm{d}x. \tag{7.25}$$

for $f \in C_c^\infty(\mathbf{R}^d_+)$. Again we will denote also by $A_{(a_1,\ldots,a_d)}$ the Friedrich's extension of this operator. The following theorem is generalisation of Theorem 7.10 and Theorem 7.11

Theorem 7.12. *Suppose that* $a \ge 1/2$ *and* A_a *is defined by (7.24). Then for any* $s > 1/2$ *and any Borel function* F

$$\|F(A_a)\|_{L^1(\mathbf{R})\to L^{1,\infty}(\mathbf{R})} \le C_s \sup_{t>1} \|\eta\, \delta_t F\|_{W^4_s}. \tag{7.26}$$

Next suppose that $d > 1$, $a_j \ge 1/2$ *for* $j = 1, \ldots, d$ *and that* $A_{(a_1,\ldots,a_d)}$ *is defined by (7.25). Then for any* $s > d/2$ *and any Borel function* F

$$\|F(A_{(a_1,\ldots,a_d)})\|_{L^1(\mathbf{R}^d)\to L^{1,\infty}(\mathbf{R}^d)} \le C_s \sup_{t>1} \|\eta\, \delta_t F\|_{W^2_s}. \tag{7.27}$$

Proof. The proofs of Assumptions 2.1 and 2.2 for the operators A_a and $A_{(a_1,\ldots,a_d)}$ are standard. It is well known that $f_k^a(x) = \mathcal{L}_k^a(x^2)(2x)^{1/2}$, where $\mathcal{L}_k^a$ are the Laguerre functions. Hence by [40, (2.5), pp. 435]

$$|f_k^a(x)| \le C \begin{cases} (\sqrt[3]{2k+1} + |2k+1-x^2|)^{-1/4} & \text{when } x^2/2 \le 2k+1 \\ \exp(-cx^2) & \text{when } x^2/2 > 2k+1 \end{cases} \tag{7.28}$$

for all $a \ge 1/2$.

Inspecting the proofs of Theorem 7.10 and Theorem 7.11 we see that to show (3.4) we use only (7.18) Thus to obtain Theorem 7.12 we repeat the proof of Theorem 7.10 and Theorem 7.11 using (7.28) instead of (7.18).

The proof of condition (3.5) is an easy modification of the proof of Lemma 7.9 so we skip it. □

Remark. Theorem 7.12 is substantially stronger than [61, Theorems 6.4.2 and 6.4.3]. See also [61, Theorem 6.4.1].

7.8. **Perturbation of harmonic oscillator.** For $d = 1, 2, 3$ Theorems 7.10 and 7.11 hold also for small perturbation of the harmonic oscillator.

Theorem 7.13. *Suppose that $s > d/2$ and $A_{V,d} = \Delta_d + x^2 + V(x)$, where $d = 1, 2, 3$ and $|V(x)| < c < 1$. Then for any Borel function F.*

$$\|F(A)\|_{L^1(\mathbf{R}^d)\to L^{1,\infty}(\mathbf{R}^d)} \le C_s \sup_{t>1} \|\eta\, \delta_t F\|_{W^p_s}, \tag{7.29}$$

where $p = 4$ for $d = 1$ and $p = 2$ for $d = 2, 3$.

Proof. First we note that

$$\|K_{\chi_{[d+2k-1,d+2k+1)}(A_{V,d})}(\,\cdot\,,y)\|_{L^2(\mathbf{R}^d)} \le 2\|K_{(A_{V,d}-d-2k+1)^{-1}}(\,\cdot\,,y)\|_{L^2(\mathbf{R}^d)} \tag{7.30}$$

However

$$\begin{aligned} \|K_{(A_{V,d}-d-2k+1)^{-1}}(\,\cdot\,,y)\|_{L^2(\mathbf{R}^d)} &\le \|K_{(A_{0,d}-d-2k+1)^{-1}}(\,\cdot\,,y)\|_{L^2(\mathbf{R}^d)} \\ &\quad\times \sum_{l=0}^{\infty} \|M_V(A_d-d-2k+1)^{-1}\|^l_{L^2(\mathbf{R}^d)\to L^2(\mathbf{R}^d)} \\ &\le 1/(1-c)\|K_{(A_d-d-2k+1)^{-1}}(\,\cdot\,,y)\|_{L^2(\mathbf{R}^d)}, \end{aligned} \tag{7.31}$$

where $A_d = \Delta_d + x^2$ and $M_V f = Vf$. For $k \in Z_+$ we put

$$a_{d,k}(y) = \|K_{\chi_{[d+2k-1,d+2k+1)}(A_{V,d})}(\,\cdot\,,y)\|^2_{L^2(\mathbf{R}^d)} \quad \text{and}$$
$$b_{d,k}(y) = \|K_{\chi_{[d+2k-1,d+2k+1)}(A_{0,d})}(\,\cdot\,,y)\|^2_{L^2(\mathbf{R}^d)}.$$

To finish the proof of Theorem 7.13 it is enough to show that (see Sections 7.5 and 7.6)

$$\sum_{k=0}^{N} a_{1,k}(y)^{p'} \le C_{p'} N^{1-p'/2} \tag{7.32}$$

for all $1 \le p' < 2$,

$$\sum_{k=0}^{N} a_{2,k}(y)^{p'} \le C_{p'} N \tag{7.33}$$

for all $1 \le p' < \infty$ and

$$a_{3,k}(y) \le Ck^{1/2}. \tag{7.34}$$

However, by (7.30) and (7.31)

$$a_{d,k_1}(y) \le C \sum_{k_2=0}^{\infty} \frac{b_{d,k_2}(y)}{1+(k_1-k_2)^2}. \tag{7.35}$$

By (7.35) to prove (7.32), (7.33) and (7.34) for $a_{d,k}$ it is enough to note that (7.32), (7.33) and (7.34) hold for $b_{d,k}$. (see (7.20) and Section 7.6). For example for $d=3$, we have $b_{d,k}(y) \le Ck^{1/2}$ and

$$a_{d,k_1}(y) \le Ck_1^{1/2} \sum_{k_2=0}^{2k_1} 1/(1+(k_1-k_2)^2) + C \sum_{k_2=2k_1+1}^{\infty} k_2^{-3/2} \le Ck_1^{1/2}$$

□

Remark. We do not know if [60, Theorem 5.1] and (7.21) with p=1 and $s > 7/6$ hold in the setting of Theorem 7.13.

7.9. **Twisted Laplace operator.** Consider the twisted Laplacian on R^d, $d = 2l$, $l \in \mathbf{Z}_+$.

$$L_d = \Delta_x + \Delta_y + 1/4(|x|^2 + |y|^2) - i/2 \sum_{j=1}^{l} \Big(x_j \partial_{y_j} - y_j \partial_{x_j} \Big), \tag{7.36}$$

where $(x,y) \in \mathbf{R}^l \times \mathbf{R}^l$ and $\Delta_x = -\sum_{j=1}^l \partial_{x_j}^2$, $\Delta_y = -\sum_{j=1}^l \partial_{y_j}^2$. In virtue of results obtained in [57] the critical index for convergence of the Riesz means of the twisted Laplace operators on the space $L^1(\mathbf{R}^d)$ is equal $(d-1)/2$. Now we prove the following singular integral version of this result

Theorem 7.14. *Suppose that L_d is the twisted Laplace operator defined by (7.36). Then for any $s > d/2 = l$ and any Borel function F*

$$\|F(L_d)\|_{L^1(\mathbf{R}^d)\to L^{1,\infty}(\mathbf{R}^d)} \le C_s \sup_{t>1} \|\eta\, \delta_t F\|_{W^2_s}. \tag{7.37}$$

Proof. We prove that (3.4) and (3.5) hold for $\kappa = 2$ and $p = 2$. To prove (3.4) it is enough to show that

$$\|K_{F(L_d)}(\,\cdot\,,y)\|^2_{L^2(\mathbf{R}^d)} \le CN^{d/2} \|\delta_N F\|^2_{N,2} \tag{7.38}$$

for any bounded Borel function F such that $\operatorname{supp} F \subset [0,N]$. It is proved in [57] that (compare (7.4))

$$\|\chi_{[r-1,r)}(L_d)\|^2_{L^1\to L^2} \le C\, r^{d/2-1}. \tag{7.39}$$

Hence (compare Lemma 7.5)

$$\|F(L_d)\|_{L^1\to L^2} \le \big(\sum_{i=1}^{N} \|\chi_{[i-1,i)}\, F(L_d)\|^2_{L^1\to L^2} \big)^{1/2} \le CN^{d/2} \|\delta_N F\|_{N,2}.$$

This proves (7.38) and (3.4). To prove (3.5) we note that

$$\|F(\sqrt{L}_d)\|_{L^1(\mathbf{R}^d)\to L^1(\mathbf{R}^d)} \le \sup_{y\in\mathbf{R}^d} \|K_{F(\sqrt{L}_d)}(\,\cdot\,,y)\|_{L^1(\mathbf{R}^d)}.$$

However, because of the convolution structure of the operator L_d

$$\sup_{y\in\mathbf{R}^d} \|K_{F(\sqrt{L}_d)}(\,\cdot\,,y)\|_{L^1(\mathbf{R}^d)} = \|K_{F(\sqrt{L}_d)}(\,\cdot\,,0)\|_{L^1(\mathbf{R}^d)}$$

(see [61, § 1.2]). Next we note that

$$L_d^{d+\varepsilon} K_{F(\sqrt{L}_d)}(\,\cdot\,,0) = A_d^{d+\varepsilon} K_{F(\sqrt{L}_d)}(\,\cdot\,,0),$$

where $A_d = \Delta_d + 1/4(|x|^2 + |y|^2)$. Thus

$$\begin{aligned}\|F(\sqrt{L}_d)\|_{L^1(\mathbf{R}^d)\to L^1(\mathbf{R}^d)} &= \|K_{F(\sqrt{L}_d)}(\,\cdot\,,0)\|_{L^1(\mathbf{R}^d)} \le C\|A_d^{(d+\varepsilon)/4}K_{F(\sqrt{L}_d)}(\,\cdot\,,0)\|_{L^2(\mathbf{R}^d)} \\ &= C\|L_d^{(d+\varepsilon)/4}K_{F(\sqrt{L}_d)}(\,\cdot\,,0)\|_{L^2(\mathbf{R}^d)} \le CN^d\|\delta_N[F(\lambda)\lambda^{(d+\varepsilon)/2}]\|^2_{N^2,2} \\ &\le N^{2d+\varepsilon}\|\delta_N F\|^2_{N^2,2}\end{aligned}$$

for any Borel function F such that $\operatorname{supp} F \subseteq [0,N]$. □

7.10. **Scattering operators.** The next example seems to be one of the most interesting example of operators satisfying (3.1) which we study here. We are going to investigate the operators $\Delta_3 + V(x) = -(\partial_1^2 + \partial_2^2 + \partial_3^2) + V(x)$, where $V(x) \ge 0$ is a compactly supported function and

$$\frac{1}{4\pi}\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{V(y)}{|x-y|}\,\mathrm{d}y < 1. \tag{7.40}$$

In addition we assume that V is in the Rollnik class, which means that

$$\int_{\mathbf{R}^6}\frac{|V(x)||V(y)|}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Theorem 7.15. *Suppose that $V(x) \ge 0$ is in the Rollnik class and that V satisfies (7.40). Then $\Delta_3 + V(x)$ satisfies (3.1) for $p = 2$. Hence*

$$\|F(A)\|_{L^1(\mathbf{R}^3)\to L^{1,\infty}(\mathbf{R}^3)} \le C_s \sup_{t>0}\|\eta\,\delta_t F\|_{W^2_s}$$

for any $s > 3/2$ and all Borel functions F.

Proof. For $x, k \in R^3$ we denote by $u(x,k) = e^{i\langle x,k\rangle} + v(x,k)$ the solution of the Lippmann-Schwinger equation (see [43, §XI.6, pp. 98])

$$u(x,k) = e^{i\langle x,k\rangle} - \frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}V(y)u(k,y)\,\mathrm{d}y.$$

Now if we define the operator $B_{|k|,V}$ by the formula

$$B_{|k|,V}(f)(x) = -\frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}V(y)f(y)\,\mathrm{d}y,$$

then by (7.40)

$$\|B_{|k|,V}\|_{L^\infty(\mathbf{R}^3)\to L^\infty(\mathbf{R}^3)} = c < 1. \tag{7.41}$$

Let

$$v(x,k) = \sum_{l=1}^{\infty} B^l_{|k|,V}(e^{i\langle\,\cdot\,,k\rangle})(x).$$

Then by (7.41) the function $u(x,k) = e^{i\langle x,k\rangle} + v(x,k)$ is the solution of Lippmann-Schwinger equation and

$$|u(x,k)| \le \frac{1}{1-c} \le C < \infty. \tag{7.42}$$

Next, for $f \in L^2(\mathbf{R}^3) \cap L^1(\mathbf{R}^3)$ we define the distorted Fourier transform of function f by the formula

$$\boldsymbol{\Phi}_V(f)(k) = (2\pi)^{-3/2} \int_{\mathbf{R}^3} \overline{u(x,k)} f(x)\, \mathrm{d}x. \tag{7.43}$$

By [43, Theorem XI.41, pp. 99]

$$\int_{\mathbf{R}^3} |\boldsymbol{\Phi}_V(f)(k)|^2 \, \mathrm{d}k = \int_{\mathbf{R}^3} |f(x)|^2 \, \mathrm{d}x \tag{7.44}$$

and

$$\boldsymbol{\Phi}_V((\Delta_3 + V)f)(k) = |k|^2 \boldsymbol{\Phi}_V(f)(k). \tag{7.45}$$

By (7.43) and (7.44)

$$K_{F(\Delta_3+V)}(x,y) = \int_{\mathbf{R}^3} F(k) u(x,k) \overline{u(y,k)} \, \mathrm{d}k$$

and so

$$\|K_{F(\Delta_3+V)}(.,y)\|_{L^2(\mathbf{R}^3)} \le C \int_{k \in \mathbf{R}^3} |F(|k|)|^2 \, \mathrm{d}k = C' \int_0^\infty |F(\lambda)|^2 \lambda^2 \, \mathrm{d}\lambda. \tag{7.46}$$

Now (3.1) follows from (7.46). □

7.11. **Fractals.** Many interesting examples of spaces and operators satisfying Assumptions 2.1 and 2.2 are described in the theory of Brownian Motion on fractals (see for example [35]). Here we would like mention only the Laplace operator on the Sierpinski Gasket. For the Laplace operator on the Sierpinski Gasket Assumptions 2.1 and 2.2 holds with $d = \log 3/(\log 5 - \log 3) = 2.1506601\ldots$ and $m = d + 1 = \log 5/(\log 5 - \log 3)$ (see [5, 58]). Applying Lemma 2.2 and Theorem 3.1 we obtain

Corollary 7.16. *Suppose that A is the Laplacian on the Sierpinski Gasket. Then for any $\varepsilon > 0$*

$$\|A^{i\alpha}\|_{L^1 \to L^{1,\infty}} \le C_\varepsilon (1 + |\alpha|)^{d/2+\varepsilon} \tag{7.47}$$

where $d = \log 3/(\log 5 - \log 3)$.

We do not know if the lower bounds corresponding to (7.47) hold (compare (1.7)).

We leave formulation of the results concerning the Bochner-Riesz summability and spectral multipliers for the Laplacian on the Sierpinski Gasket to interested readers. For other related results see [58].

8. MISCELLANEOUS

8.1. **Estimates on the holomorphic functional calculus.** For $\theta > 0$ we put $\Sigma(\theta) = \{z \in \mathbf{C} - \{0\} \colon |\arg z| < \theta\}$. Let F be a bounded holomorphic function on $\Sigma(\theta)$. By $\|F\|_{\theta,\infty}$ we denote the supremum of F on $\Sigma(\theta)$. We are interested in finding sharp bounds, in terms of θ, of the norm of $F(A)$ as the operator acting on $L^p(X,\mu)$. It is known (see [12, Theorem 4.10]) that these bounds on the holomorphic functional calculus when θ tends to 0 are related to spectral multiplier theorems for A.

It is easy to check, using the Cauchy formula that there exists a constant C independent of F and θ such that

$$\sup_{\lambda>0} |\lambda^k F^{(k)}(\lambda)| \le \frac{C}{\theta^k}\|F\|_{\theta,\infty}, \ \forall k \in \mathbf{Z}_+. \tag{8.1}$$

For any $\varepsilon > 0$ $\sup_{t>0} \|\eta\, \delta_t F\|_{W^\infty_{k-\varepsilon}} \le C \sup_{\lambda>0} |\lambda^k F^{(k)}(\lambda)|$ and so by (8.1) and interpolation

$$\sup_{t>0} \|\eta\, \delta_t F\|_{W^\infty_s} \le \frac{C_\varepsilon}{\theta^{s+\varepsilon}}\|F\|_{\theta,\infty}. \tag{8.2}$$

Applying now (8.2), Theorem 3.1 and interpolation we obtain the following proposition (see also [12, Theorem 4.10]

Proposition 8.1. *For all* $p \in (1, \infty)$, *we have*

$$\|F(A)\|_{L^p \to L^p} \le \frac{C_\varepsilon}{\theta^{d|\frac{1}{p}-\frac{1}{2}|+\varepsilon}}\|F\|_{\theta,\infty} \tag{8.3}$$

for every $\theta > 0$.

Similar estimates were shown in [18] (see Corollary 6.4 and Theorem 6.6) in the case where the volume is polynomial and in [19] in the case of Lie groups of polynomial growth. The estimates given in these papers are similar to (8.3) but with $d+2$ in place of d. Hence we improved these results.

8.2. **The case** $m = 2$**.** In this paper we use the Gaussian bounds for the heat kernel to obtain spectral multiplier results. Actually, most of spectral multiplier theorems rely on the Gaussian bound for the corresponding heat kernel (see [2, 3, 6, 10, 18, 19, 21, 25, 28, 26, 33, 37, 24]). For $m = 2$ the Gaussian bounds for the heat kernel (2.4) are essentially equivalent to the finite speed propagation of the corresponding wave equation (see [47, Theorem 3]. The finite speed propagation property is used in e.g. [7, 13, 48, 50] to study spectral multiplier theorems. The wave equation technique seems to be more complicated than the heat kernel approach. It is also impossible to use the wave equation technique to investigate m-th order differential operator. However, results which use the finite speed property are more precise. For example it is proved in [50] that if $m = 2$, then

$$\|L^{i\alpha}\|_{L^1 \to L^{1,\infty}} \le C(1 + |\alpha|)^{d/2}. \tag{8.4}$$

Using Theorem 3.1 we can only show that for any $\varepsilon > 0$

$$\|L^{i\alpha}\|_{L^1(X,\mu) \to L^{1,\infty}(X,\mu)} \le C_\varepsilon(1 + |\alpha|)^{d/2+\varepsilon}$$

We do not know if (8.4) holds for any $m \ne 2$.

Therefore it seems that the wave equation approach and the heat kernel method are essentially different and they are of independent interest in the theory of spectral multipliers.

Finally let us mention that the most precise spectral multiplier results can be obtained when we use the Fourier transform technique (see e.g. [9, 8, 53, 54, 59]). This technique can be used to obtain spectral multipliers for Fourier's multipliers on $\mathbf{R}^d$ or for elliptic (pseudo)-differential operators on compact manifolds. But it seems to be difficult to use the Fourier transform technique in a more general setting. In some situations, like for example operators with irregular measurable coefficients it seems to be impossible to apply the Fourier transform technique at all.

8.3. **The case $d_0 \neq d_\infty$.** Our last remark concerns the situation where the volume has polynomial growth. For d_0 and d_∞ in $[0,\infty)$, we define $V_{d_0,d_\infty} : \mathbf{R}^+ \to \mathbf{R}^+$ by the formula

$$V_{d_0,d_\infty} = \begin{cases} t^{d_0} & \text{when } t \le 1 \\ t^{d_\infty} & \text{when } t \ge 1. \end{cases}$$

We assume that

$$CV_{d_0,d_\infty}(r) \le \mu(B(x,r)) \le C'V_{d_0,d_\infty}(r),\ \forall x \in X, r > 0 \tag{8.5}$$

Note that (2.2) holds in this situation for $d = \max(d_0, d_\infty)$. If $d_0 > d_\infty$, then it is possible to obtain obtain a little bit more precise version of Theorem 3.1.

Theorem 8.2. *Suppose that $d_0 > d_\infty$ and that for some $2 \le p \le \infty$ A satisfies (3.1). Assume also that*

$$\sup_{t \le 1} \|\eta\, \delta_t F\|_{W^p_s} < \infty, \quad \textit{for some} \ \ s > d_\infty/2$$

and

$$\sup_{t > 1} \|\eta\, \delta_t F\|_{W^p_s} < \infty, \quad \textit{for some } s > d_0/2.$$

Then $F(A)$ is of weak type $(1,1)$ and $F(A)$ extends to a bounded operator on $L^q(X,\mu)$ for all $q \in (1,\infty)$.

Proof. If ω_n are the same functions as in the proof of Theorem 3.1, then we put

$$F(\lambda) = F_{d_\infty}(\lambda) + F_{d_0}(\lambda) = \sum_{-\infty}^{0} \omega_n(\lambda)F(\lambda) + \sum_{1}^{\infty} \omega_n(\lambda)F(\lambda).$$

By Theorem 3.1 $F_{d_0}(A)$ is of weak type $(1,1)$. Hence one only has to show that $F_{d_\infty}(A)$ is of weak type $(1,1)$. However we note that for $s > d_\infty$

$$\int_{\rho(x,y)\ge r} (1 + 2^n\rho(x,y))^{-s} d\mu(x) \le \mu(B(y,2^{-n}))(1 + r2^n)^{d_\infty - s}.$$

for all $r > 0$ and $n \le 0$. The rest of the proof is just a repetition of the proof of Theorem 3.1 so we skip it. □

Also in the case $d_0 < d_\infty$ it is possible to obtain a result slightly stronger than just Theorem 3.1 (see [2] and [48, Theorem 2]). But the difference between this stronger version and Theorem 3.1 is not significant so we do not discuss details here.

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