Sequences, Series and Power Series

A sequence is a set of numbers occurring in order.

For example,

$$x_n = \frac{1}{n}$$

represents the sequences

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

Formally, a sequence is a function of the form

$$f : \mathbb{N} \longrightarrow \mathbb{R},$$

where $x_n = f(n)$.

Limits

Consider the sequence

$$x_n = \frac{1}{n}.$$

As n gets large, then x_n gets very close to 0.

We say that when n goes to infinity then x_n converges to zero.

$$x_n \xrightarrow[n \to \infty]{} 0$$

$$\lim_{n \to \infty} x_n = 0.$$

Definition

We say that a sequence x_n converges to a limit $x \in \mathbb{R}$

$$\lim_{n \to \infty} x_n = x$$

if, given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x - x_n| < \varepsilon$$

whenever $n > N$.

Example. $x_n = \frac{1}{n}$

$$\lim_{n \to \infty} x_n = 0.$$

Indeed

$$|x_n - 0| = \frac{1}{n} < \varepsilon$$

whenever $n > N = \frac{1}{\varepsilon}$.

Example

$$\lim_{n \to \infty} \frac{n+2}{n} = 1 .$$

Indeed

$$\left| \frac{n+2}{n} - 1 \right| = \frac{2}{n} < \varepsilon$$

whenever $n > N = 2\frac{1}{\cancel{2}\varepsilon}$.

Question

$$\lim_{n \to \infty} \frac{2n+3}{3n+4} = ?$$

Question

$$\lim_{n \to \infty} (-1)^n = ?$$

Theorem 1.1

Suppose that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

Then

a) $\lim_{n\to\infty} (x_n + y_n) = x + y$;

b) $\lim_{n\to\infty} (x_n y_n) = x \cdot y$;

c) if $y \neq 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$

d) if $a \in \mathbb{R}$, then $\lim_{n\to\infty} a x_n = a \cdot x$

Examples

a) $$\lim_{n\to\infty} n^\alpha = \begin{cases} 0 & \text{if } \alpha < 0 \\ \infty & \end{cases}$$

b) $$\lim_{n\to\infty} x^n = 0$$

if $|x| < 1$.

c) $$\lim_{n\to\infty} \frac{n^a}{b^n} = 0$$

for every $a \in \mathbb{R}$ and $b > 1$.

Example

$$\lim_{n\to\infty} \frac{10n^2+2n}{3n^2+2n+3} =$$

$$= \lim_{n\to\infty} \frac{10+\frac{2}{n}}{3+\frac{2}{n}+\frac{3}{n^2}} =$$

$$= \frac{10 + \lim_{n\to\infty} \frac{2}{n}}{3 + \lim_{n\to\infty}\left(\frac{2}{n}+\frac{3}{n^2}\right)} =$$

$$= \frac{10+0}{3+0+0} = \frac{10}{3}$$

Examples (More)

$$\lim_{n\to\infty} \sqrt{n+1}-\sqrt{n} = \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$$

$$\lim_{n\to\infty} \sqrt{n^2+n+1}-n =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}+n+1-\cancel{n^2}}{\sqrt{n^2+n+1}+n} =$$

$$= \lim_{n\to\infty} \frac{n+1}{\sqrt{n^2+n+1}+n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1}$$

$$= \frac{1}{\sqrt{1+0+0}+1} = \frac{1}{2}$$

Theorem 1.2

(Squeezing principle)

Suppose that

$$y_n \leq a_n \leq x_n$$

and that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = X.$$

Then $\lim_{n\to\infty} a_n = X$.

Example. Consider

$$a_n = \frac{1}{n}\sin n^{n^3}.$$

Then

$$-\frac{1}{n} \leq a_n \leq \frac{1}{n}$$

and so $\lim_{n\to\infty} a_n = 0$.

Definition

A sequence x_n is said to be bounded if there exists a number $M \in \mathbb{R}$ such that

$$|x_n| \leq M$$

for every $n \in \mathbb{N}$.

Theorem 1.3

A convergent sequence is bounded.

Example.

The sequence $x_n = (-1)^n$ is bounded but not convergent.

Theorem 1.4

Suppose that x_n is an increasing sequence.

If x_n is bounded then x_n converges as $n \to \infty$.

If x_n is not bounded then

$$\lim_{n\to\infty} x_n = +\infty.$$

Theorem 1.5

Suppose that x_n is a decreasing ...

Definition.

Suppose that

$$x_1, x_2, x_3, x_4, \dots$$

is a sequence. Next suppose that $n_1 < n_2 < n_3 < \dots < n_p < \dots$ is an infinite sequence of natural numbers. Then the sequence

$$x_{n_1}, x_{n_2}, x_{n_3}, \dots, x_{n_p}, \dots$$

is called subsequence of the original sequence.

Examples

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \dots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots$$

$$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$$

Theorem 1.5
Suppose that

$$\lim_{n\to\infty} x_n = x.$$

Then for any subsequence

$$\lim_{p\to\infty} x_{n_p} = x.$$

Therem 1.6
Every bounded sequence has a convergent subsequence.

Series

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \dots$$

Examples

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$$

Partial sum

$$S_N = \sum_{n=1}^{N} x_n = x_1 + x_2 + \dots + x_N .$$

Definition

$$\sum_{n=1}^{\infty} x_n = \lim_{N \to \infty} S_N .$$