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DMTH237 Discrete Mathematics II

Week 5&6: Fridays — 29 March, 5 April 2019
NON-DETERMINISTIC FINITE STATE MACHINES

NDFSA acceptance

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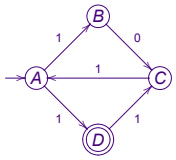
Well, let's ask: what strings can be accepted by this NDFSA?
First we have to modify our definition of 'accepted'.

Definition (non-deterministic FSA)

A string is accepted by a non-deterministic FSA if it is possible to terminate in an accepting state.

What can be accepted? $(101 + 111)^*(101 + 111)^*1(101 + 111)^*1$



1011 could terminate in *B*, or in the black-hole, or in *D*. Going multiply around the top loop accepts strings matching regular expression: $(101)^*1$.

1111 if it just goes around the bottom loop.
Any sequence of 1s matching the regular expression $(111)^*1$ will be accepted.
Going around either loop, many times, in any order: e.g., '1011111011011'.

Reference: Rosen 7th ed. p.873; 8th ed. p.912

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Non-Deterministic Finite State Machines (NDFSMs)

NDFSA: easier to design

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The point of considering non-deterministic machines is that, in many cases, they are much easier to design than deterministic ones.

Try to construct a deterministic FSA to accept the language $(101 + 111)^*1$.

It is not so easy.

- So, the advantage of non-deterministic FSAs over deterministic ones is that they are easier to design.
- But how do we implement them?
  - use a random number generator whenever we have a choice;
  - work in parallel — follow multiple paths simultaneously;
  - need a method for converting non-deterministic FSAs into deterministic ones.

Multiple Transitions

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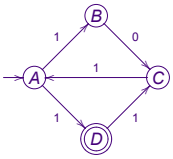
Do you notice anything peculiar about this FSA?

There seems to be a mistake because there are two transitions labelled '1' coming out of state *A*.



If we are in state *A* and we happen to read a '1', do we go to state *B* or to state *D*?
We appear to have a choice.

Does this mean we have to toss a coin so that sometimes we go to state *B* but on other occasions, with the same input string, we go to *D*?

There is no mistake.
This is an example of what is known as a **non-deterministic FSA**.

It appears to incorporate the element of chance.
But what possible use could there be for a machine whose behaviour is unpredictable?

Reference: Rosen 7th ed. p.873; 8th ed. p.911; §13.3.5

State changes w/o input

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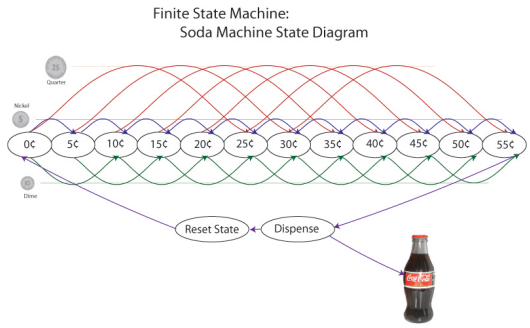

Null transitions

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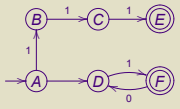
Having multiple transitions, for the same input, is not the only way an FSA can be non-deterministic. It can have **null transitions**.

Consider the follow state diagram for an acceptor.

Example



Note the two transitions coming out of state *A*.
We have not forgotten to write the input character that produces the transition from *A* to *D*.

It can be made spontaneously, *without* the need to read any input. **Accepts**: $111 + (10)^*1$.

However there is danger in using an arrow without a label for a null transition.
If we simply forget to label an arrow it could be interpreted wrongly.
To avoid this, we use the null-string symbol 'λ' to represent a null transition.

Run the machine on the string '10101'. When we start with a null transition ...
— it is accepted because it is possible to end in an accepting state.

scanning for nulls

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Working with a transition table is much more reliable than using a state diagram as it is not so easy to miss a null in a table.
We use the same system to complete the nulls as to find accessible states.

Example (scanning for nulls)

	0	1	λ
A	B	A	BD
B	F	C	F
C		F	
D	AC		A
E		ABC	
F	C		BE

- *A*: which has null transitions; so ...
- *A*, *B*, *D* : ... write down its nulls.
- *A*, *B*, *D* : Look at next state in the list; it has null to *F*
- *A*, *B*, *D*, *F* : ... so add *F* to the list.
- *A*, *B*, *D*, *F* : Next state, has null to *A*, already listed;
- *A*, *B*, *D*, *F*, *E* : *F* adds *E* to the list.
- *A*, *B*, *D*, *F*, *E* : no more to add.

Hence there can be null transitions from state *A* to all of *BDEF*. Similarly from *B* to *EF*, and from *D* to *ABEF*, and from *F* to *BE*, as already stated. This completes the λ-column.

NDFSA: Multiple Initial States

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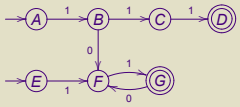
This next non-deterministic machine offers a choice of where to start.

Example



It accepts the strings '111' and those such as '101', '10101', '1010101', '101010101', ... by starting at *A*.

It also accepts strings such as '11', '1101', '110101', ... by starting at *E*.

The language accepted by this machine is: $111 + 10(10)^*1 + 1(10)^*1$.
This could be written using more parentheses as: $1(1 + (\lambda + 0)(10)^*)1$,
which while shorter is not necessarily any clearer.
This is even shorter and remains quite clear: $111 + (10 + 1)(10)^*1$.

Removing the Nulls

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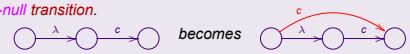
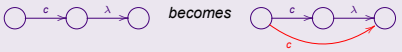
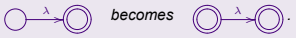
Suppose we have a complete NDFSA.
We now remove any null transitions as follows.

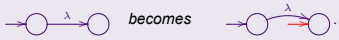
Procedure (removing the nulls)

If there is a null transition from A to B and a non-null transition from B to C then add a non-null transition from A to C associated with the same input character as the original non-null transition.

In state diagrams: 

and similarly 

while for accepting states 

With an initial state 

Having done this, each null transition is now completely redundant, so can be simply removed.

Completing the Nulls

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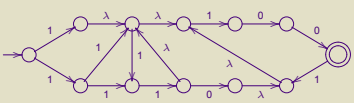
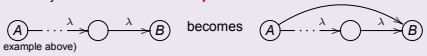
In the following NDFSA it is possible to move spontaneously along several null transitions one after the other.

Example



Definition (completing the nulls)

If there is a sequence of null transitions from state *A* to state *B* add a direct null transition from *A* to *B* (unless there is already a direct null from *A* to *B*).
We then say that the NDFSA is **complete**.



(see example above)

removing nulls, example

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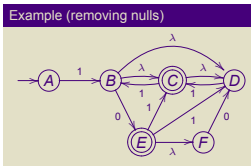
Consider the following complete NDFSA.

Example (removing nulls)



	0	1	λ	
→ A		BCD		
B	EF		CD	
C		BCD	D	*
D		CD		
E		CD	F	*
F	D			

	0	1	
→ A		B	
B	E	BC	*
C		BC	*
D		C	
E	D	CD	*
F	D		

In the transition table, augment any state names that can λ-transition by including their target states. Now when in state *B* reading a '1' we could use the '1' directly from *B* (though in this case there is no appropriate 1-transition) or first move by a λ to either *C* or *D*, and then use the '1'. Take the union of sets in the '1' column for the *B*, *C* and *D* rows.
This is: $\emptyset \cup \{B, C, D\} \cup \{C, D\} = \{B, C, D\}$, which we write as *BCD*.
Similarly state *C* can λ-transition to *D*, so take the union of rows *C* and *D*.
Similarly state *E* can λ-transition to *F*, so take the union of rows *E* and *F*.
Row *A* is unchanged, as are rows *D* and *F*.

The resulting NDFSA has no nulls, which have all been replaced by multiple transitions.

incomplete NDFSA

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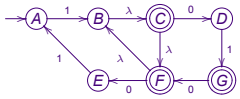
Consider this incomplete NDFSA.



	0	1	λ		0	1	λ		0	1	
→A		B				B				BCF	
B			C				CF	*	DE		*
C	D		F	⟹ complete nulls	D		BF	* ⟹ remove nulls	DE		*
D		G				G				G	
E		A				A				A	
F	E		B	*	E		BC	*	DE		*
G	F			*	F			*	BCF		*

Note how states *B*, *C* and *F* can all null-transition to each other, and that *B*, *C* become accepting because they can null-transition to an accepting state. Upon removing nulls these three states all transition equivalently.
We now have multiple transitions. Next we'll see how to combine these.

Summary of conversion NDFSA → FSA

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Given the state diagram for a non-deterministic FSA there are six stages in converting it into a reduced deterministic FSA presented in standard form.

1. construct the state table
2. complete the nulls
3. remove the nulls
4. remove multiple transitions
5. combine equivalent states
6. put into standard form

Combining Multiple Transitions

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Suppose now we have a non-deterministic FSA, *M*, with all its null transitions removed, but having multiple transitions from a state for the same input character.
We transform this non-deterministic machine *M* into a deterministic one (we shall call this M^Δ, with the Greek delta representing 'deterministic') by considering sets of states as the states of the deterministic machine.

Definition (deterministic machine M^Δ)

If an NDFSA *M* has a set of states *S*, the new machine M^Δ is defined to have

- $\wp(S)$, the power set of *S*, as its set of states;
- if *X* is a subset of *S* and *c* is an input character, then $M^\Delta(X, c)$ is defined to be the subset of all states in *S* that can be reached from some state in *X* by reading the character *c*.
  In other words: $M^\Delta(X, c) = \bigcup_{x \in X} M(x, c) \in \wp(S)$.
- A subset $X \in \wp(S)$ is accepting in M^Δ if there is $x \in X$ such that *x* is accepting in *M*.

The resulting FSA, M^Δ, will clearly be deterministic because from each set of states of the original machine *M* there is exactly one subset of states of *M* that we can reach for a given input character. So we will now have a deterministic FSA that is equivalent to *M*, in the sense that it accepts precisely the same language as *M*.

Reference: Rosen 7th ed. p.874; 8th ed. p.913, Theorem 1

A worked example

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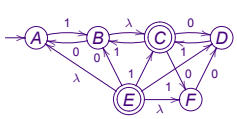

	0	1	λ		0	1	
→A		B				B	
B	A		C		ADF	B	*
C	DF	B		*	DF	B	*
D		C				C	
E	B	CD	AF	*	BD	BCD	*
F	D				D		

	0	1	
→0	1	2	A
1	1	1	∅
2	3	2	* B
3	4	5	ADF
4	1	6	D
5	3	2	* BC
6	7	2	* C
7	4	6	DF

	0	1	≡0	0	1	≡1	0	1	≡2	0	1	≡3
0	1	2	0	0	1	1	1	2	0	1	2	0
1	1	1	0	0	0	1	1	1	1	1	1	1
2	3	2	1	0	1	2	0	2	2	3	2	2
3	4	5	0	0	1	0	0	2	3	0	2	3
4	1	6	0	0	1	0	1	2	0	1	2	0
5	3	2	1	0	1	2	0	2	2	3	2	2
6	7	2	1	0	1	2	0	2	2	3	2	2
7	4	6	0	0	1	0	0	2	3	0	2	3

Reduced

	0	1	
→0	1	2	
1	1	1	
2	3	2	*
3	0	2	

1. construct state table; 2. complete the nulls — nothing to do; 3. remove the nulls;
4. remove multiple transitions; 5. combine multiple transitions;
6. put into standard form — nothing to do.

Convert NDFSA to FSA

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Convert the following non-deterministic FSA to a deterministic one:

Example (NDFSA → FSA)

	0	1	
→A	CD	D	*
B	AD	B	
C	AC	C	
D	A		

	0	1	
→0	1	2	A
1	3	4	CD
2	0	5	D
3	6	1	AC
4	3	4	C
5	5	5	∅
6	6	1	ACD

	0	1	
→0	1	2	* A
1	3	4	CD
2	0	5	D
3	6	1	* AC
4	3	4	C
5	5	5	∅
6	6	1	* ACD

We don't need all $2^4 = 16$ subsets in the power set. Just identify the subsets that we actually need, starting from the initial state. Fill in the table, numbering consecutively the subsets of states that can reached. The empty set corresponds to a 'black-hole'. When the table is complete identify the accepting states as those subsets including an accepting state in the NDFSA.

Index of 'jargon' terms

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