Week 5 SGTA Exercises : Linear systems, matrices, vectors
Minimisation of FSMs & FSAs

MACQUARIE University

Solutions — for instructors initially, perhaps later released to students

SGTA questions for you

This section contains the problems you should attempt at home in preparation for your SGTA class.

Algebra: Linear systems, Matrices, Vectors

1. Solve each of these systems. (a) $\begin{cases} a+2b+3c=1 \\ 4a+3b+2c=0 \\ 3a+5b+7c=0 \end{cases}$ **Solution**: Create the augmented matrix of coefficients, then reduce it; *viz.* $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 3 & 2 & 0 \\ 3 & 5 & 7 & 0 \end{array}\right) \xrightarrow[\text{III}-2\text{I}]{\text{II}-4\text{I}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -5 & -10 & -4 \\ 0 & -1 & -2 & -3 \end{array}\right) \xrightarrow{\text{II}-5\text{III}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 11 \\ 0 & -1 & -2 & -3 \end{array}\right)$.

Now the second row stands for the equality of numbers $0 = 11$, which is clearly false; therefore this system of equations, and hence also the original system, can have no solution.

(b) $\begin{cases} p+2q+3r+s+2t=1 \\ 2p+5q+6r+3s+7t=0 \\ 3p+9q+8r+7s+6t=0 \end{cases}$ **Solution:** $\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 2 & 1 \\ 2 & 5 & 6 & 3 & 7 & 0 \\ 3 & 9 & 8 & 7 & 6 & 0 \end{array}\right) \xrightarrow[\text{III}-3\text{I}]{\text{II}-2\text{I}} \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 3 & -2 \\ 0 & 3 & -1 & 4 & 0 & -3 \end{array}\right) \xrightarrow[\text{III}-3\text{II}]{\text{I}-2\text{II}}$

$\xrightarrow[\text{III}-3\text{II}]{\text{I}-2\text{II}} \left(\begin{array}{ccccc|c} 1 & 0 & 3 & -1 & -4 & 5 \\ 0 & 1 & 0 & 1 & 3 & -2 \\ 0 & 0 & -1 & 1 & -9 & 3 \end{array}\right) \xrightarrow[\text{III}\times(-1)]{} \left(\begin{array}{ccccc|c} 1 & 0 & 3 & -1 & -4 & 5 \\ 0 & 1 & 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -1 & 9 & -3 \end{array}\right) \xrightarrow{\text{I}-3\text{III}} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & -31 & 14 \\ 0 & 1 & 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -1 & 9 & -3 \end{array}\right)$ has achieved reduced row-echelon form, so we can read off the solutions. Using m and n for the free parameters, we have: $p = 14-2m+31n$, $q = -2-m-3n$, $r = -3+m-9n$, $s = m$, $t = n$, with m and n arbitrary real numbers.

You should substitute these formulæ into the original system of equations, to check that they work.

2. Carry out each of these computations without using a calculator, treating all numbers as exact.

(a) $3\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} - 2\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$ **Solution**: $3\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} - 2\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 9 & 12 \\ 15 & 18 \end{pmatrix} - \begin{pmatrix} 8 & 10 \\ 12 & 14 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(b) $(3,1,6,2)\cdot(9,-8,-7,6)$ **Solution**: $(3,1,6,2)\cdot(9,-8,-7,6) = 27-8-42+12 = 39-50 = -11$.

(c) $\begin{pmatrix} 2 & 3 \\ 5 & 4 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 7 & 1 \end{pmatrix}$ **Solution:** $\begin{pmatrix} 2 & 3 \\ 5 & 4 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} 6+21 & 4+3 \\ 15+28 & 10+4 \\ 3+42 & 2+6 \end{pmatrix} = \begin{pmatrix} 27 & 7 \\ 43 & 14 \\ 45 & 8 \end{pmatrix}$

(d) $\begin{pmatrix} 48596735462 & -94857362514 \\ -39286754153 & 59408362556 \end{pmatrix} + \begin{pmatrix} 1.2345 & 2.3456 \\ 3.4567 & 4.5678 \end{pmatrix} + \begin{pmatrix} -48596735462 & 94857362514 \\ 39286754153 & -59408362556 \end{pmatrix}$

Solution: $\ldots = \begin{pmatrix} 1.2345 & 2.3456 \\ 3.4567 & 4.5678 \end{pmatrix}$ since the 1st and 3rd matrices are negatives of each other, so cancel exactly.

What would you expect to happen if you tried doing (d) using a calculator, or working to some fixed precision?

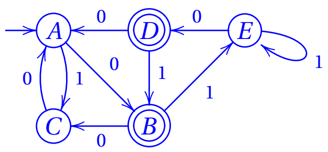
Solution: If working to some lesser degree of precision, when the 2nd matrix is added to the first, some of those decimal places may well get lost or rounded away. Depending upon just what software is being used, you may get an answer of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or one of many other non-zero matrix answers with floating-point numbers as entries.

Finite State Machines & Acceptors

3. Draw the state diagram for the FSA shown at right. Identify whether there are any inaccessible states. Determine whether there are any equivalent states.
Do a reduction, if applicable, and put the machine into *standard form* if this concept has been discussed already in lectures.

	0	**1**	
→ *A*	*B*	*C*	
B	*C*	*E*	*
C	*A*	*C*	
D	*A*	*B*	*
E	*D*	*E*	

Solution:



	0	1	$\equiv_0$	0	1	$\equiv_1$	0	1	$\equiv_2$	0	1	
$\to A$	B	C	0	1	0	0	1	2	0	1	2	
B	C	E	1	0	0	1	2	0	1	2	4	$*$
C	A	C	0	0	0	2	0	2	2	0	2	
D	A	B	1	0	1	3	0	1	3	0	1	$*$
E	D	E	0	1	0	0	3	0	4	3	4	

For standard form states 3 & 4 swap.

Questions for the SGTA class

Algebra: Linear systems, Matrices, Vectors

4. Find a, b and c such that the graph of $y = ax^2 + bx + c$ goes through the points $(-1, 10)$, $(1, 3)$, and $(2, 8)$.

Solution: The three points give rise to three linear equations for the unknowns: $\begin{cases} a - b + c = 10 \\ a + b + c = 3 \\ 4a + 2b + c = 8\,. \end{cases}$

Setting up the augmented matrix $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 1 & 1 & 1 & 3 \\ 4 & 2 & 1 & 8 \end{array}\right)$ and solving by the usual methods, yields a final answer of: $a = 17/6$, $b = -7/2$ and $c = 11/3$.

CHECKING: with $y(x) = \frac{17}{6}x^2 - \frac{7}{2}x + \frac{11}{3} = \frac{1}{6}(17x^2 - 21x + 22)$ then we have that $y(-1) = \frac{1}{6}(17 + 21 + 22) = \frac{1}{6} \times 60 = 10$, $y(1) = \frac{1}{6}(17 - 21 + 22) = \frac{1}{6} \times 18 = 3$ and $y(2) = \frac{1}{6}(68 - 42 + 22) = \frac{1}{6} \times 48 = 8$, all being as required.

5. (a) Show that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions of $A\boldsymbol{x} = \boldsymbol{0}$ then so is $3\boldsymbol{u} + 4\boldsymbol{v}$.

Solution: $A(3\boldsymbol{u} + 4\boldsymbol{v}) = 3A\boldsymbol{u} + 4A\boldsymbol{u} = (3 + 4)\boldsymbol{0} = \boldsymbol{0}$, since every component in $\boldsymbol{0}$ is 0. Thus $3\boldsymbol{u} + 4\boldsymbol{v}$ is also a solution of the matrix equation $A\boldsymbol{x} = \boldsymbol{0}$.

(b) Show that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions of $A\boldsymbol{x} = \boldsymbol{b}$ then so is $4\boldsymbol{u} - 3\boldsymbol{v}$; explain why $4\boldsymbol{u} + 3\boldsymbol{v}$ would *not* be a solution when $\boldsymbol{b} \neq \boldsymbol{0}$.

Solution: This time we get that $A(4\boldsymbol{u} - 3\boldsymbol{v}) = 4A\boldsymbol{u} - 3A\boldsymbol{v} = (4 - 3)\boldsymbol{b} = \boldsymbol{b}$, hence is also a solution.
However, $A(4\boldsymbol{u} + 3\boldsymbol{v}) = (4 + 3)\boldsymbol{b} = 7\boldsymbol{b}$, so is not a solution when $\boldsymbol{b} \neq \boldsymbol{0}$.
To get a solution from a linear combination of two others, we need the sum of the coefficients to be exactly 1.

6. What are the inverses of the following matrices? Verify your results.

$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{A}^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{B}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{C}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Languages & Machines

7. Identify and remove any inaccessible states from the following FSAs, then apply the minimisation procedure to locate any equivalent states, thereby reducing the machine.

	0	1	
$\to A$	E	C	
B	A	D	$*$
C	A	E	$*$
D	E	B	
E	C	C	

Solution: reduces to

	0	1	
$\to A$	E	C	
E	C	C	
C	A	E	$*$

	0	1	
$\to A$	A	E	
B	D	B	
C	E	E	$*$
D	C	E	$*$
E	D	B	$*$

Solution: does not reduce; viz.

	0	1	$\equiv_0$	0	1	$\equiv_1$	0	1	$\equiv_2$	0	1	
$\to A$	A	E	0	0	1	0	0	3	0	0	4	
B	D	B	0	1	0	1	2	1	1	3	1	
C	E	E	1	1	1	2	3	3	2	4	4	$*$
D	C	E	1	1	1	2	2	3	3	2	4	$*$
E	D	B	1	1	0	3	2	1	4	3	1	$*$

Note that states B and E transition the same way, but are *inequivalent* since one is accepting, the other not.

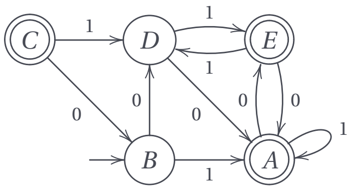
	0	**1**	
A	A	E	
$\to B$	D	B	
C	E	E	$*$
D	C	E	$*$
E	D	B	$*$

Solution: can drop state A, then

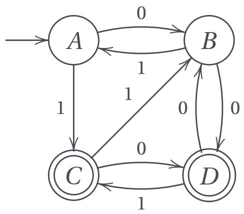
	0	**1**	$\equiv_0$	**0**	**1**	$\equiv_1$	**0**	**1**	$\equiv_2$	**0**	**1**	
$\to B$	D	B	0	1	0	0	1	0	0	1	0	
C	E	E	1	1	1	1	2	2	2	3	3	$*$
D	C	E	1	1	1	1	1	2	1	2	3	$*$
E	D	B	1	1	0	2	1	0	3	1	0	$*$

with all states separate, noting that although states B and E transition the same way, they are *inequivalent* as one is accepting, the other not.

8. Reduce the following FSAs, expressing your answers as a state table.



(a)



(b)

Solution: The first gives state table and minimisation:

	0	**1**	$\equiv_0$	**0**	**1**	$\equiv_1$	**0**	**1**	
$\to B$	D	A	0	0	1	0	1	2	
D	A	E	0	1	1	1	2	3	
A	E	A	1	1	1	2	3	2	$*$
E	A	D	1	1	0	3	2	1	

where clearly state C is inaccessible, and the order of states was chosen to give *standard form*. (If not covered yet, you'll soon find out what *standard form* means.)

The second gives a state table already minimal:

	0	**1**	$\equiv_0$	**0**	**1**	$\equiv_1$	**0**	**1**	
$\to A$	B	C	0	0	1	0	1	2	
B	D	A	0	1	0	1	3	0	
C	D	B	1	1	0	2	3	1	$*$
D	B	C	1	0	1	3	1	2	$*$

which is already in *standard form*.

Further exercises to do at home

Finite State Machines & Acceptors

9. (a) Let $\boldsymbol{A} = \begin{pmatrix} 6 & -4 \\ 9 & -6 \end{pmatrix}$. Show that $\boldsymbol{A}^2$ is the zero matrix.

Solution: We have $\boldsymbol{A}^2 = \begin{pmatrix} 6 & -4 \\ 9 & -6 \end{pmatrix}\begin{pmatrix} 6 & -4 \\ 9 & -6 \end{pmatrix} = \begin{pmatrix} 36-36 & -24+24 \\ 54-54 & -36+36 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \boldsymbol{0}$.

(b) Find all 2×2 matrices $\boldsymbol{B} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that $\boldsymbol{B}^2$ is the zero matrix.

Solution: We have $\boldsymbol{B}^2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha^2+\beta\gamma & \beta(\alpha+\delta) \\ \gamma(\alpha+\delta) & \beta\gamma+\delta^2 \end{pmatrix}$; so with $\delta = -\alpha$ and $\gamma = -\alpha^2\beta^{-1}$ we'll have $\boldsymbol{B}^2 = \boldsymbol{0}$, for any real numbers α and $\beta \neq 0$. Also there are solutions with $\alpha = \delta = 0$ and $\beta\gamma = 0$, which allows $\beta = 0$ with $\gamma \neq 0$, and of course the case where $\boldsymbol{B} = \boldsymbol{0}$ itself.

10. Find the inverse of the following matrix, and verify your result.

$$\boldsymbol{D} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \textbf{Solution:} \quad \boldsymbol{D}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It's easy to check that $\boldsymbol{D}\boldsymbol{D}^{-1} = \boldsymbol{I} = \boldsymbol{D}^{-1}\boldsymbol{D}$.

11. Reduce each of the following FSAs, and put them into *standard form*.

	0	1	
A	B	A	*
B	J	K	
C	C	H	
→D	F	K	
E	G	F	
F	K	I	*
G	E	I	
H	H	C	
I	K	F	*
J	B	J	*
K	E	I	

	0	1	
A	D	B	*
B	E	B	
→C	D	A	*
D	C	D	
E	B	A	*

	0	1	
A	G	G	*
B	C	H	*
→C	B	I	*
D	I	G	
E	G	G	*
F	B	C	*
G	F	D	
H	B	B	*
I	B	C	

	0	1	
→A	B	E	
B	E	C	
C	E	B	*
D	D	F	
E	C	E	*
F	A	D	*

Solution:

	0	1	
A	B	A	*
B	J	K	
C	C	H	
→D	F	K	
E	G	F	
F	K	I	*
G	E	I	
H	H	C	
I	K	F	*
J	B	J	*
K	E	I	

⟼

	0	1	$\equiv_0$	0	1	$\equiv_1$	0	1	$\equiv_2$
→D	F	K	0	1	0	0	1	2	0
F	K	I	1	0	1	1	2	1	1
K	E	I	0	0	1	2	2	1	2
E	G	F	0	0	1	2	2	1	2
I	K	F	1	0	1	1	2	1	1
G	E	I	0	0	1	2	2	1	2

⟼

	0	1	
→0	1	2	
1	2	1	*
2	2	1	

after removing inaccessible states, and re-ordering for standard form, before the final minimisation. Note how it can be convenient to reorder states early.

	0	1	
A	D	B	*
B	E	B	
→C	D	A	*
D	C	D	
E	B	A	*

becomes

	0	1	$\equiv_0$	0	1	$\equiv_1$	0	1	$\equiv_2$
→C	D	A	1	0	1	0	1	2	0
D	C	D	0	1	0	1	0	1	1
A	D	B	1	0	0	2	1	1	2
B	E	B	0	1	0	1	0	1	1
E	B	A	1	0	1	0	1	2	0

reducing to

	0	1	
→0	1	2	*
1	0	1	
2	1	1	*

	0	1	
A	G	G	*
B	C	H	*
→C	B	I	*
D	I	G	
E	G	G	*
F	B	C	*
G	F	D	
H	B	B	*
I	B	C	

has inaccessible states:

	0	1	$\equiv_0$	0	1	$\equiv_1$	0	1	$\equiv_2$	0	1
→C	B	I	1	1	0	0	1	2	0	1	2
B	C	H	1	1	1	1	0	1	1	0	3
I	B	C	0	1	1	2	1	0	2	1	0
H	B	B	1	1	1	1	1	1	3	1	1

The re-ordering after dropping inaccessible states has put this machine into standard form order.

	0	1	
→A	B	E	
B	E	C	
C	E	B	*
D	D	F	
E	C	E	*
F	A	D	*

has inaccessible states:

	0	1	$\equiv_0$	0	1	$\equiv_1$	0	1
→A	B	E	0	0	1	0	0	1
B	E	C	0	1	1	1	2	3
E	C	E	1	1	1	2	3	2
C	E	B	1	1	0	3	2	1

is in standard form.