Three investigations into linear logic

Richard Garner

January 17, 2005

Abstract

This essay looks into three topics in category theory broadly linked by the theme of *linear logic*, which could perhaps be described as a 'resource-sensitive' version of classical logic: one no longer just worries about proving B from A, but also *how many times* one must invoke A in order to get B.

The first topic describes a method for describing a particular species of category-with-structure which provides an abstract model for the system of linear logic. Previous approaches to this problem have imposed a 'top-down' solution, starting from the desired syntactic goals; here, we provide a 'bottom-up' approach which builds the structure in stages from more elementary components.

The second topic provides a more abstract view of some of the tools used to construct the categories-with-structure of the first section; in particular, it seeks to provide a fresh perspective on several well-known but seemingly ad hoc constructions, a perspective from which they may appear a little more natural.

The third topic describes a new way of describing the structure of 'polycategories', which furnish us with a different, and perhaps more natural, way of describing the system of linear logic.

Contents

Preface

This essay consists in original research, as yet unpublished, carried out in pursuit of my doctoral studies. Where use has been made of others' work, explicit note has been made in the body of the text. A full bibliography is appended. I gratefully acknowledge the support of my supervisor Martin Hyland.

Chapter 1

Overview

This essay investigates three topics in the broad area of *linear logic*. As so often in mathematics, linear logic may be seen as a natural generalisation of several different intuitions. Girard's original paper [Gir87] has it arising as a calculus on certain types of qualitative domains, whilst from a syntactic perspective it can be seen as the result of excising the structural rules of *contraction* and *weakening* from the classical sequent calculus.

However, the most pertinent source of motivation for the purposes of this essay is not *syntax* but *semantics*. Given a logical calculus, we might hope to capture the essence of that calculus by means of some category, whose morphisms $A \to B$ are to be read as 'proofs of B from A' in our calculus. We would consider this enterprise a success if there were a tight correspondence between the morphisms of the category, and the proofs of the calculus; that is, if two proofs of B from A were represented by the same morphism $A \to B$ if and only if they were 'essentially the same proof'. What we mean by 'essentially' in this context is a matter of aesthetics rather than mathematics.

A leading example of this is the correspondence between *intuitionistic classical logic* and *cartesian closed categories with finite coproducts*, in which the logical connectives 'and', 'or' and 'implies' are interpreted categorically by the operations of product, coproduct and function-space. One naturally wonders whether this correspondence can be extended to the full classical logic with negation. The following observation gives us an answer:

Proposition 1 (Joyal). *Let $\mathbb{C}$ be a cartesian closed category, with initial object 0. Consider the functor $\neg = (-) \Rightarrow 0\colon \mathbb{C}^{\text{op}} \to \mathbb{C}$. If $\neg\neg X$ is naturally isomorphic to X for all $X \in \mathbb{C}$, then $\mathbb{C}$ is a preorder category.*

Proof. The functor $\neg$ forms part of an equivalence of categories $\mathbb{C} \simeq \mathbb{C}^{\text{op}}$; hence $\mathbb{C}^{\text{op}}$ is also cartesian closed. So the functor $(-) + 1\colon \mathbb{C} \to \mathbb{C}$ is a right adjoint, and so preserves limits; in particular, $1 + 1$ is terminal, and hence the two injections $\amalg_1, \amalg_2\colon 1 \to 1 + 1$ are the same. Now, given any two morphisms $f, g\colon 1 \to X$ in $\mathbb{C}$, we have

$$1 \xrightarrow{f} X \quad = \quad 1 \xrightarrow{\amalg_1} 1 + 1 \xrightarrow{[f,g]} X \quad = \quad 1 \xrightarrow{\amalg_2} 1 + 1 \xrightarrow{[f,g]} X \quad = 1 \xrightarrow{g} X,$$

and since maps $A \to B$ are in bijection with maps $1 \to (A \Rightarrow B)$, we deduce that $\mathbb{C}$ must be a preorder category. □

So adding classical negation necessarily collapses the categorical semantics back to mere truth-value semantics; we can no longer see *how* we can prove B from A, but merely *if* we can prove B from A. Informally, this collapse happens because maps in a cartesian closed category have a very strong sense of which direction is 'from' and which is 'to'; so strong that the contravariant isomorphism of classical negation cannot be accommodated in a non-degenerate manner.

The story of linear logic can thus be seen as the story of how intuitionistic logic can be altered to allow us to fruitfully reintroduce negation; from a categorical perspective, it is the story of categories in which the distinction between 'source' and 'target' is to some extent blurred. Concretely, these categories are the *$*$-autonomous categories* of [Bar79], but the concept of 'undirectedness' cuts deeper, since many of the intuitions, and hence concepts, of category theory are derived from the 'directed' world of sets, and hence function badly in the context of linear logic.

We can approach $*$-autonomous categories by way of *weakly distributive categories*, which is the topic of the first part of this essay. It shows how we can construct a free such category, starting from the notion of an generic Frobenius algebra (itself a rather 'undirected' structure), and how, by reintroducing negation, we can produce from it a free $*$-autonomous category.

The second part of this essay deals abstractly with the issue of constructing 'undirected' categories, building on the observation that the *glueing* construction for monoidal categories is unsuited to constructing new $*$-autonomous categories from old ones. Solving this problem was one of the main thrusts of [HS03], with its *double glueing construction*; here, we decompose that construction into more elementary pieces, based on the concept of *modules* for monoidal categories.

The final part of this essay develops a new approach to Szabo's *polycategories* [Sza75], which are to weakly distributive categories as multicategories are to monoidal categories. The approach is inspired by the presentation of multicategories as monads in the Kleisli bicategory of a pseudomonad; it develops a distributive law between a pseudomonad and a pseudocomonad such that monads in the Kleisli bicategory of this distributive law are polycategories.

Chapter 2

Frobenius algebras in multiplicative linear logic

2.1 Introduction

We wish to investigate the logical system MLL, and in particular attempt to characterise its proofs. From a categorical viewpoint, we see of MLL as the free $*$-autonomous category on a discrete category X of primitive proposition; and so in order to understand MLL, one may hope to understand the construction of free $*$-autonomous categories. In fact, we shall be more humble, and only attempt to understand the construction of $*$-autonomous categories without units. (The interaction of the units with the binary connectives is a thorny problem; a recent attempt to resolve it may be found in [SL04].)

One may protest that a perfectly adequate description of such categories already exists, given by *proof nets* [Gir87, DR89]. Yet though proof nets may be adequate they are not wholly satisfactory: indeed, that proof nets describe $*$-autonomous categories only becomes apparent after a detour through the sequent calculus MLL. From the perspective of the categorical proof theorist, it would be more appealing to be able to *directly* construct such a free category; and this is what this chapter sets out to do.

2.2 Preliminaries

We assume the reader is reasonably familiar with the sequent calculus MLL, or **multiplicative linear logic** (if not then see [Gir87] or [Gir95] for more details); we give a brief summary here for completeness. Its formulae are built from a set of literals $X = \{A, B, \dots\}$ and their negations $A^\perp, B^\perp, \dots$ using two binary connectives, $\otimes$ and $⅋$, and two nullary connectives I and $\perp$. We extend negation $(-)^\perp$ to arbitrary formulae by the DeMorgan laws:

$$(A \otimes B)^\perp = A^\perp ⅋ B^\perp \qquad (A ⅋ B)^\perp = A^\perp \otimes B^\perp$$
$$I^\perp = \perp \qquad \perp^\perp = I.$$

Proofs are constructed from these formulae according to the following rules of deduction:

$$\frac{}{\vdash A^{\perp}, A}\,\text{id} \qquad \frac{\vdash \Gamma, A \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,\text{cut} \qquad \frac{\vdash \Delta, A, B, \Gamma}{\vdash \Delta, B, A, \Gamma}\,\text{ex}$$

$$\frac{\vdash \Delta, A \quad \vdash B, \Gamma}{\vdash \Delta, A \otimes B, \Gamma}\,\otimes \qquad \frac{\vdash \Delta, A, B}{\vdash \Delta, A \parr B}\,\parr \qquad \frac{}{\vdash I}\,I \qquad \frac{\vdash \Delta}{\vdash \Delta, \perp}\,\perp.$$

As mentioned in the introduction, if we view the set of propositions X as a discrete category, then MLL manifests itself as the free symmetric $*$-autonomous category $SA(X)$ on X, where we recall from [Bar79] that:

Definition 1. A **symmetric $*$-autonomous category** $(\mathbb{C}, \otimes, I, (-)^*)$ is a symmetric monoidal category $(\mathbb{C}, \otimes, I)$ equipped with a functor $(-)^* \colon \mathbb{C} \to \mathbb{C}^{\mathrm{op}}$ such that

- $A^{**} \cong A$ naturally in $A \in \mathbb{C}$;
- $\mathbb{C}(A \otimes B, C^*) \cong \mathbb{C}(B \otimes C, A^*)$ naturally in all variables.

Note that this definition seems to suppress any mention of $\parr$ or $\perp$; but these are recovered by setting $(-) \parr (?) = ((-)^* \otimes (?)^*)^*$ and $\perp = I^*$. If $A \otimes B \cong A \parr B$ and $I \cong \perp$, then we say that $\mathbb{C}$ is **compact closed**. Closely related to $*$-autonomy is the concept of a *weakly distributive category*:

Definition 2. A **symmetric weakly distributive category** $(\mathbb{C}, \otimes, \parr, I, \perp)$ is a category $\mathbb{C}$ such that $(\mathbb{C}, \otimes, I)$ and $(\mathbb{C}, \parr, \perp)$ are symmetric monoidal categories, equipped with a natural family of maps

$$w_{ABC} \colon A \otimes (B \parr C) \to (A \otimes B) \parr C$$

called **weak distributivity** maps, which satisfy axioms making them compatible with the two monoidal structures. (For the full details of these axioms, we refer the reader to [CS97].)

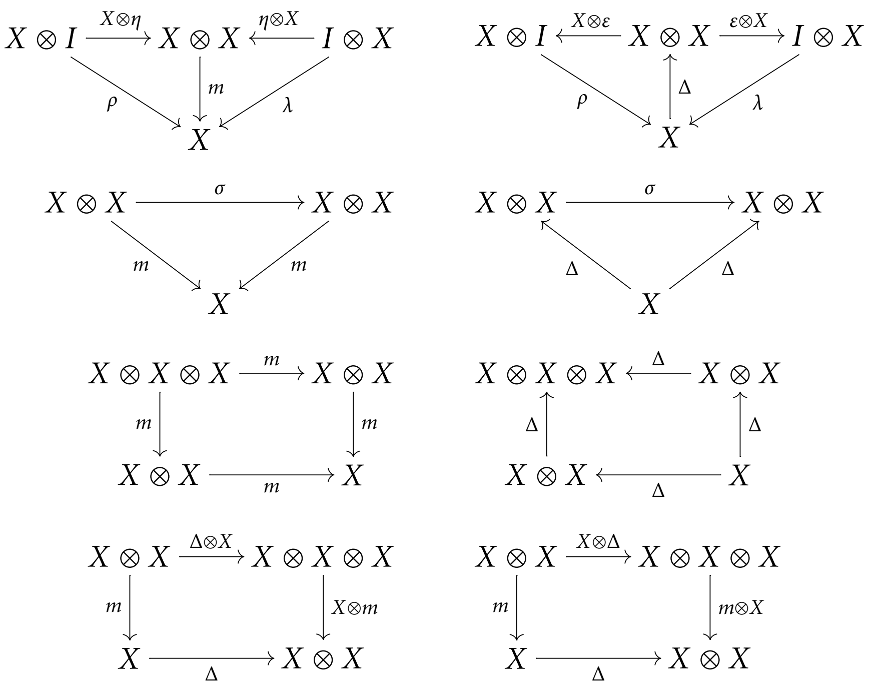
2.3 Frobenius algebras and the Frobenius category

This section reviews a few necessary facts about Frobenius algebras; these can be found in more detail in [Koc04] or [Car95] for example.

Definition 3. A **commutative Frobenius object** in a symmetric monoidal category $(\mathbb{C}, \otimes, I)$ is an object X equipped with morphisms

$$\begin{aligned} m &\colon X \otimes X \to X && \text{(multiplication)},\\ \eta &\colon I \to X && \text{(unit)},\\ \Delta &\colon X \to X \otimes X && \text{(comultiplication), and}\\ \varepsilon &\colon X \to I && \text{(counit)}, \end{aligned}$$

Figure 2.1: Axioms for a symmetric Frobenius object



which make (X, m, η) into a commutative monoid, (X, Δ, ε) into a cocommutative comonoid, and satisfy the **Frobenius laws**:

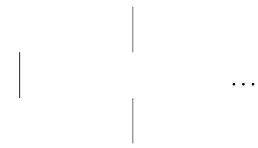
$$(X \otimes m) \circ (\Delta \otimes X) = \Delta \circ m = (m \otimes X) \circ (X \otimes \Delta).$$

The category which interests us is the symmetric monoidal category $\mathbb{F}$ freely generated by a Frobenius object X. Since it costs us nothing and gains us transparency, we may as well assume that $\mathbb{F}$ is strict monoidal; that is, the associativity and identity (though not the symmetry) isomorphisms are taken to be identities. Formally, we have:

Definition 4. Let $\mathbb{G}$ be the strict monoidal category with objects $I, X, \ldots, X^{\otimes n}, \ldots$ and morphisms freely generated (under composition and tensor product) by maps $m\colon X \otimes X \to X$, $\Delta\colon X \to X \otimes X$, $\eta\colon I \to X$, $\varepsilon\colon X \to I$ and $\sigma\colon X \otimes X \to X \otimes X$.

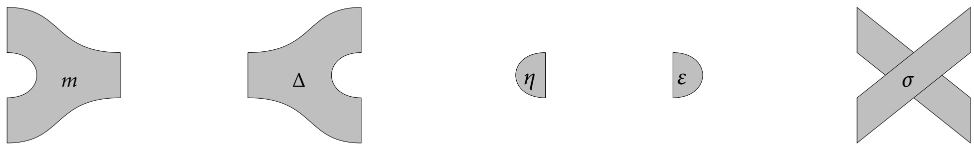
Let $\sim$ be the smallest equivalence relation on $\mathbb{G}$ compatible with composition which makes all the diagrams of Figure 1 commute and makes σ into an involutive natural transformation; then we define $\mathbb{F}$ to be $\mathbb{G}/\!\sim$.

Quotients of categories are usually rather difficult to describe; happily, this is not the case here. Indeed, suppose we interpret $\mathbb{G}$ graphically by drawing objects as:



$I \qquad X \qquad X \otimes X$

and generating morphisms as:



$m\colon X \otimes X \to X, \qquad \Delta\colon X \to X \otimes X, \qquad \eta\colon I \to X, \qquad \varepsilon\colon X \to I, \qquad \sigma\colon X \otimes X \to X \otimes X.$

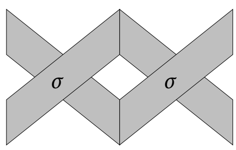
Then horizontal and vertical juxtaposition give us respectively categorical composition and tensor product, and we can describe the equivalence relation $\sim$ explicitly by reference to these diagrams.

Definition 5. Given a morphism $f\colon X^{\otimes n} \to X^{\otimes m}$ of $\mathbb{G}$, view it as a diagram in $\mathbb{R}^3$, with left-hand boundary $X^{\otimes n}$ and right-hand boundary $X^{\otimes m}$. Let cpts_f be its collection of connected components.

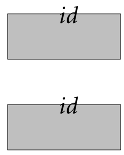
For each $c \in \mathsf{cpts}_f$, we can then ask two questions: firstly, which of the n input copies and m output copies of X lie on its boundary, and secondly, what its genus is. We write in_c, out_c and gen_c for these pieces of data. The **topology of** f, $\mathrm{Top}(f)$ is then the multiset

$$\left[\ (\mathsf{in}_c, \mathsf{out}_c, \mathsf{gen}_c) \mid c \in \mathsf{cpts}_f\ \right].$$

[Note that according to our definition, the two morphisms



and



are topologically equivalent, which might cause a knot theorist some consternation, but is entirely correct in this setting, since we want $\mathbb{F}$ to be symmetric rather than braided.]

Proposition 2. *For $f, g\colon X^{\otimes n} \to X^{\otimes m}$ in $\mathbb{G}$, $f \sim g$ if and only if f and g are topologically equivalent.*

Proof. See, for instance, [Koc04] or [Car95]. □

So we have a crisp solution to our coherence problem: we think of the category $\mathbb{F}$ as being the category $\mathbb{G}$ modulo topological equivalence; thus we work with (diagrams of) morphisms of $\mathbb{G}$ directly and argue topologically to prove commutativity.

Now, $\mathbb{F}$ is in fact already a $*$-autonomous category; indeed, we know it to be symmetric monoidal, and we have a functor $(-)^*\colon \mathbb{F} \to \mathbb{F}^{\mathrm{op}}$ given by the identity on objects and by 'reversal' on morphisms; explicitly, on generating morphisms, it is given by

$$\Delta^* = m, \quad m^* = \Delta, \quad \eta^* = \varepsilon, \quad \varepsilon^* = \eta, \quad \text{and} \quad \sigma^* = \sigma,$$

which has a unique extension to a strict monoidal functor $\mathbb{F} \to \mathbb{F}^{\mathrm{op}}$. Clearly, though, this structure is far from 'free', since we have $\parr = \otimes$; so $\mathbb{F}$ is in fact compact closed.

2.4 An orthogonality category

We now wish to apply the **orthogonality** construction of [HS03] to the category $\mathbb{F}$. We avoid the full generality of that paper by considering only the very well-behaved case we need here.

Definition 6. Let $A \in \mathbb{F}$ and consider maps $u\colon I \to A$ and $x\colon A \to I$. We shall say that u *is orthogonal to* x and write $u \perp x$ just when $x \circ u = \varepsilon \circ \eta$. Further, given $A_p \subset \mathbb{F}(I, A)$, and $A_c \subset \mathbb{F}(A, I)$, we write A_p° and A_c° for the sets

$$A_p^\circ := \left\{\, x\colon A \to I \mid u \perp x \text{ for all } u \in A_p \,\right\}, \text{ and}$$
$$A_c^\circ := \{\, u\colon I \to A \mid u \perp x \text{ for all } x \in A_c \,\}\,.$$

These two operations set up a Galois connection between the powerset of $\mathbb{F}(I, A)$ and the powerset of $\mathbb{F}(A, I)$. We are interested in 'closed' subsets that are paired under this Galois connection; i.e., subsets A_p and A_c such that $A_p^\circ = A_c$ and $A_c^\circ = A_p$.

The category $O(\mathbb{F})$ arises by augmenting objects A of $\mathbb{F}$ with such paired subsets (A_p, A_c). We think of A_p and A_c as respectively being the 'proofs' and 'coproofs' of A. The maps of $O(\mathbb{F})$ are maps between the underlying objects in $\mathbb{F}$, that are required to map proofs to proofs, and coproofs to coproofs. Formally, we have:

Definition 7. The category $O(\mathbb{F})$ has for objects, triples $\mathbf{A} = (A, A_p, A_c)$, where $A \in \mathbb{F}$, $A_p \subset \mathbb{F}(I, A)$ and $A_c \subset \mathbb{F}(A, I)$, subject to $A_p^\circ = A_c$ and $A_c^\circ = A_p$. Its morphisms $f\colon \mathbf{A} \to \mathbf{B}$ are given by maps $f\colon A \to B$ in $\mathbb{F}$, subject to the conditions

$$f \circ u \in B_p \quad \text{for all } u \in A_p, \text{ and}$$
$$x \circ f \in A_c \quad \text{for all } x \in B_c.$$

There is an evident forgetful functor $O(\mathbb{F}) \to \mathbb{F}$ which we shall denote by U.

Proposition 3. *The category $O(\mathbb{F})$ has a $*$-autonomous structure lifting that of $\mathbb{F}$; it is given by:*

$$
\begin{aligned}
(A, A_p, A_c) \otimes (B, B_p, B_c) &= (A \otimes B, (A_p \otimes B_p)^{\circ\circ}, (A_p \otimes B_p)^{\circ}) \\
(A, A_p, A_c) \parr (B, B_p, B_c) &= (A \otimes B, (A_c \otimes B_c)^{\circ}, (A_c \otimes B_c)^{\circ\circ}) \\
(A, A_p, A_c)^* &= (A, A_c, A_p)
\end{aligned}
$$

where we have

$$
\begin{aligned}
A_p \otimes B_p &= \left\{ I \xrightarrow{\cong} I \otimes I \xrightarrow{u \otimes v} A \otimes B \mid u \in A_p, v \in B_p \right\} \\
A_c \otimes B_c &= \left\{ A \otimes B \xrightarrow{u \otimes v} I \otimes I \xrightarrow{\cong} I \mid u \in A_c, v \in B_c \right\}.
\end{aligned}
$$

The tensor unit is given by $\mathbf{I} = (I, \{\mathrm{id}_I\}, \{\eta \circ \varepsilon\})$.

Proof. Immediate from the general theory of [HS03]; we refer the reader to that paper for the details. □

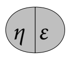
Example 1. Let us set

$$\mathbf{X} := (X, \{\eta\}, \{\varepsilon\})$$

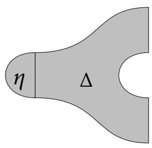
in $O(\mathbb{F})$; now consider the object $((\mathbf{X} \parr \mathbf{X}) \otimes \mathbf{X}) \parr \mathbf{X}$. We wish to calculate its proofs and coproofs: as a first step, we have

$$
\begin{aligned}
\mathbf{X} \parr \mathbf{X} &= (X, \{\eta\}, \{\varepsilon\}) \parr (X, \{\eta\}, \{\varepsilon\}) \\
&= (X \otimes X, \{\varepsilon \otimes \varepsilon\}^{\circ}, \{\varepsilon \otimes \varepsilon\}^{\circ\circ}).
\end{aligned}
$$

To calculate $\{\varepsilon \otimes \varepsilon\}^{\circ}$, we need to find maps $u \colon I \to X \otimes X$ in $\mathbb{F}$ such that $(\varepsilon \otimes \varepsilon) \circ u = \varepsilon \circ \eta$. Graphically, we seek maps $I \to X \otimes X$ in $\mathbb{G}$ which when attached to

gives something topologically equivalent to 

. It's easy to see that any such map must be topologically equivalent to



;

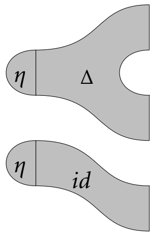
hence $\{\varepsilon \otimes \varepsilon\}^{\circ} = \{\Delta \circ \eta\}$. Likewise, it's immediate that $\{\Delta \circ \eta\}^{\circ} = \{\varepsilon \otimes \varepsilon\}$, so we get

$$\mathbf{X} \parr \mathbf{X} = (X \otimes X, \{\Delta \circ \eta\}, \{\varepsilon \otimes \varepsilon\}).$$

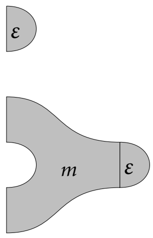
Next, we have:

$$\begin{aligned}(\mathbf{X} \parr \mathbf{X}) \otimes \mathbf{X} &= (X \otimes X, \{\Delta \circ \eta\}, \{\varepsilon \otimes \varepsilon\}) \otimes (X, \{\eta\}, \{\varepsilon\}) \\ &= (X \otimes X \otimes X, \{(\Delta \circ \eta) \otimes \eta\}^{\circ\circ}, \{(\Delta \circ \eta) \otimes \eta\}^{\circ})\end{aligned}$$

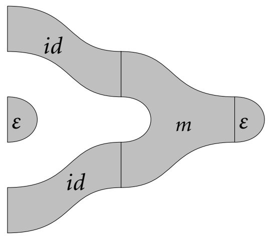
Thus, the coproofs are simply the maps orthogonal to $(\Delta \circ \eta) \otimes \eta$:



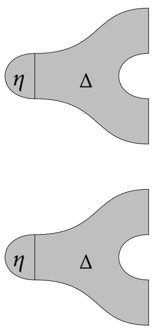
And there are two such, as represented by the following diagrams in $\mathbb{G}$:



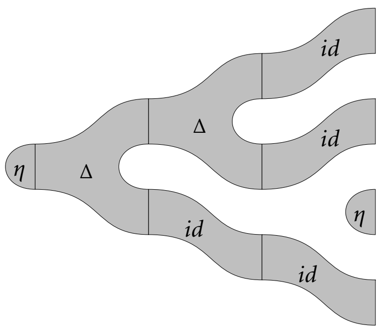
and



So the set of coproofs is $\{\eta \otimes (\eta \circ m), \varepsilon \circ m \circ (X \otimes \varepsilon \otimes X)\}$. And once again, the only map orthogonal to this set is the original map $(\Delta \circ \eta) \otimes \eta$. Continuing after this fashion we find $((\mathbf{X} \parr \mathbf{X}) \otimes \mathbf{X}) \parr \mathbf{X}$ to have two proofs:

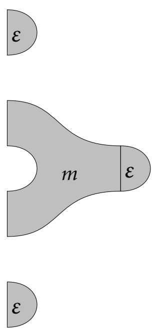


and

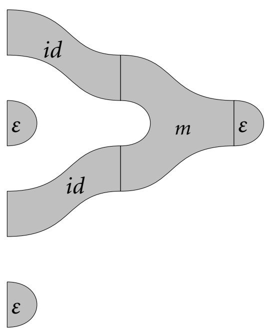


,

and two coproofs:



and



.

Before we continue, we record a few general observations about orthogonality categories that will be used repeatedly in the sequel:

Proposition 4. *Let* $\mathbf{A} = (A, A_p, A_c)$ *and* $\mathbf{B} = (B, B_p, B_c)$ *in* $O(\mathbb{F})$, *and let* $f\colon A \to B$ *in* $\mathbb{F}$. *Then*

$$f \circ u \in B_p \text{ for all } u \in A_p \quad \text{iff} \quad x \circ f \in A_c \text{ for all } x \in B_c.$$

Proof. Both conditions are equivalent to the condition that $f \circ u \circ x = \varepsilon \circ \eta$ for all $u \in A_p$ and $x \in B_c$. □

Proposition 5. *Let* $\mathbf{A} = (A, A_p, A_c)$ *in* $O(\mathbb{F})$. *Then the set* A_p *is in bijection with the set* $O(\mathbb{F})(\mathbf{I}, \mathbf{A})$*; similarly, the set* A_c *is in bijection with the set* $O(\mathbb{F})(\mathbf{A}, \bot)$.

Proof. Let $f\colon I \to A$ in $\mathbb{F}$; then by the previous result, $f \in O(\mathbb{F})(\mathbf{I}, \mathbf{A})$ if and only if $f \circ (-)$ maps $I_p = \{\mathrm{id}_I\}$ into A_p; and this happens if and only if $f \in A_p$. Similarly for the dual result. □

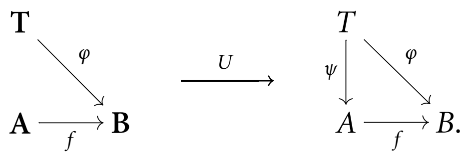
Thus we can freely confuse 'proofs of **A**' (i.e., elements of A_p) with maps $\mathbf{I} \to \mathbf{A} \in O(\mathbb{F})$, and similarly, 'coproofs of **A**' with maps $\mathbf{A} \to \bot$. Now, given a map $f\colon \mathbf{A} \to \mathbf{B}$ in $O(\mathbb{F})$, consider the two maps of sets

$$f \circ (-)\colon A_p \to B_p \quad \text{and} \quad (-) \circ f\colon B_c \to A_c.$$

We shall say that f is **surjective on proofs** (resp., coproofs) if the first (resp., the second) of these maps is surjective.

Proposition 6. *If* $f\colon \mathbf{A} \to \mathbf{B}$ *is surjective on coproofs, then* f *is a cartesian arrow with respect to the forgetful functor* $U\colon O(\mathbb{F}) \to \mathbb{F}$. *Similarly, if* f *is surjective on proofs, then it is cocartesian.*

Proof. Note that U is faithful, so that any lifting of maps from $\mathbb{F}$ to $O(\mathbb{F})$ is necessarily unique. Suppose f is surjective on coproofs, and we have a diagram



Then we need to show that ψ lifts to a map in $O(\mathbb{F})$. For this, by the previous result, it suffices to show that ψ carries coproofs of **A** to coproofs of **T**. Since f is surjective on coproofs, every coproof $x \in A_c$ can be written as $x' \circ f$ for $x' \in B_c$.

So it suffices to show that $x' \circ f \circ \psi \in T_c$ for every $x' \in B_c$. But $x' \circ f \circ \psi = x' \circ \varphi$, and since φ is a map in $O(\mathbb{F})$, it carries coproofs to coproofs; and so we are done. We argue dually if f is surjective on proofs. □

2.5 The unit-free part of $O(\mathbb{F})$

The category $O(\mathbb{F})$ is still slightly degenerate; however, this degeneracy is essentially confined to the units, and so we shall concentrate on the *unit-free* part of it. We thus proceed as follows:

- Let $\mathbb{W}$ denote the free weakly distributive category on one object X, and $\mathbb{W}_m$ the full subcategory thereof determined by the unit-free objects in $\mathbb{W}$.
- Let $U\colon O(\mathbb{F}) \to \mathbb{F}$ be the evident forgetful functor; note that this preserves all the $*$-autonomous structure on the nose.
- Let $V\colon \mathbb{W} \to O(\mathbb{F})$ be the weakly distributive functor uniquely determined by $V(X) = \mathbf{X}$. Note that this induces a functor $UV\colon \mathbb{W} \to \mathbb{F}$, which is the weakly distributive functor uniquely determined by $UV(X) = X$.
- $O(\mathbb{F})_m$ denotes the full subcategory of $O(\mathbb{F})$ determined by those objects in the image of $V|_{\mathbb{W}_m}$.
- $\mathbb{F}_m$ denotes the full subcategory of $\mathbb{F}$ determined by the objects lying in the image of $UV|_{\mathbb{W}_m}$.

In summary, we have:

$$\begin{array}{ccccc} \mathbb{W}_m & \xrightarrow{V|_{\mathbb{W}_m}} & O(\mathbb{F})_m & \hookrightarrow & O(\mathbb{F}) \\ & & \downarrow U & & \downarrow U \\ & & \mathbb{F}_m & \hookrightarrow & \mathbb{F} \end{array} .$$

We claim that restricting our attention to $O(\mathbb{F})_m$ improves matters somewhat. As initial evidence in support of this is the following:

Proposition 7. *Every object $\mathbf{T} \in O(\mathbb{F})_m$ has at least one proof and at least one coproof.*

Proof. We proceed by induction. For the base case, the object $\mathbf{X} = (X, \{\eta\}, \{\varepsilon\})$ clearly satisfies this property. For the inductive step, suppose that $\mathbf{T} = \mathbf{A} \otimes \mathbf{B}$, with $\mathbf{A} = (A, A_p, A_c)$ and $\mathbf{B} = (B, B_p, B_c)$ known to have the desired property. Then, picking any $(f, g) \in A_p \times B_p$, we have

$$I \xrightarrow{\cong} I \otimes I \xrightarrow{f \otimes g} A \otimes B$$

a proof of $\mathbf{T}$ by definition. To construct a coproof of $\mathbf{T}$, let $(j, k) \in A_c \times B_c$; then we can factorise j and k (not necessarily uniquely) as

$$A \xrightarrow{\hat{j}} X \xrightarrow{\varepsilon} I \quad \text{and} \quad B \xrightarrow{\hat{k}} X \xrightarrow{\varepsilon} I$$

respectively. So take the map

$$h := A \otimes B \xrightarrow{\hat{\jmath} \otimes \hat{k}} X \otimes X \xrightarrow{m} X \xrightarrow{\varepsilon} I.$$

We claim that this is a coproof of **T**. Indeed, for any $(f, g) \in A_p \times B_p$, we have:

$$I \xrightarrow{f} A \xrightarrow{\hat{\jmath}} X = I \xrightarrow{\eta} X \quad \text{and} \quad I \xrightarrow{g} B \xrightarrow{\hat{k}} X = I \xrightarrow{\eta} X$$

and hence

$$\begin{aligned} h \circ (f \otimes g) &= I \xrightarrow{f \otimes g} A \otimes B \xrightarrow{\hat{\jmath} \otimes \hat{k}} X \otimes X \xrightarrow{m} X \xrightarrow{\varepsilon} I \\ &= I \xrightarrow{\eta \otimes \eta} X \otimes X \xrightarrow{m} X \xrightarrow{\varepsilon} I \\ &= I \xrightarrow{\eta} X \xrightarrow{\varepsilon} I \end{aligned}$$

as required. The case $\mathbf{T} = \mathbf{A} \parr \mathbf{B}$ follows dually. □

We would like to classify the proofs of a general object **T** of $O(\mathbb{F})_m$. Suppose that $U\mathbf{T} = X^{\otimes n}$, say, and let $f\colon I \to \mathbf{X}^{\otimes n}$ be a proof of **T**. By Proposition 7, **T** has at least one coproof – call it u – satisfying $f \circ u = \varepsilon \circ \eta$, and so we see that:

- f has no connected component of genus > 0, and
- f has no connected component c with $\mathsf{out}_c = \mathsf{in}_c = \emptyset$,

since if either of these were the case then $f \circ u = \varepsilon \circ \eta$ would be impossible. Thus the topology of f is determined by giving a partition $P_1 \amalg \cdots \amalg P_k$ of $\{1, \ldots, n\}$; explicitly,

$$\mathrm{Top}(f) = [\, (\emptyset, P_i, 0) \mid i = 1, \ldots, k \,] .$$

A similar condition hold for the coproofs of **T**. So henceforth we shall implicitly assume that all maps considered satisfy the two bulleted conditions above, unless otherwise stated.

Notation. Let $f\colon I \to X^{\otimes n}$ in $\mathbb{F}$. We say that f **joins** A, for $A \subseteq \{1, \ldots, n\}$ if there is some $c \in \mathsf{cpts}_f$ with $A \subseteq \mathsf{out}_c$. In particular, if f joins $\{i, j\}$, we say that f **joins** i **and** j. Similarly for a map $k\colon X^{\otimes n} \to I$ in $\mathbb{F}$.

Proposition 8. *Let* $\mathbf{T} \in O(\mathbb{F})_m$*, and suppose that there is a proof* f *of* **T** *that joins* A*, and a coproof* u *of* **T** *that joins* B*. Then* $|A \cap B| \leqslant 1$.

Proof. Suppose $|A \cap B| > 1$; then the composite $u \circ f$ has a component of non-zero genus, and hence cannot be equal to $\varepsilon \circ \eta$, which is a contradiction. □

2.6 Co-semisimplicity

We turn our attention now to a special class of objects in $O(\mathbb{F})_m$. Consider the object $\mathbf{T} = \mathbf{X}^{\parr n}$: it is easy to see that $\mathbf{T}$ has a unique proof, namely the map

$$\Delta_n \colon I \xrightarrow{\eta} X \xrightarrow{\Delta} X \parr X \xrightarrow{X \parr \Delta} X \parr X \parr X \to \cdots \to \mathbf{X}^{\parr n}.$$

Generalising this, let us say that an object of $O(\mathbb{F})_m$ is **co-semisimple** if it is of the form

$$\mathbf{X}^{\parr n_1} \otimes \mathbf{X}^{\parr n_2} \otimes \cdots \otimes \mathbf{X}^{\parr n_k}$$

for some natural numbers $k, n_1, \ldots, n_k$. Then we have

Proposition 9. *Any co-semisimple object $\mathbf{T} \in O(\mathbb{F})_m$ has exactly one proof.*

Proof. We have

$$\mathbf{T} = \mathbf{X}^{\parr n_1} \otimes \mathbf{X}^{\parr n_2} \otimes \cdots \otimes \mathbf{X}^{\parr n_k} = (X^{\otimes n}, \{\Delta_{n_1} \otimes \cdots \otimes \Delta_{n_k}\}^{\circ\circ}, \{\Delta_{n_1} \otimes \cdots \otimes \Delta_{n_k}\}^{\circ}).$$

Now, let us define $N_i = \{n_{i-1} + 1, \ldots, n_i\}$ (where $n_0 = 0$). Then $\Delta_{n_1} \otimes \cdots \otimes \Delta_{n_k}$ is the map corresponding to the partition $N_1 \amalg \cdots \amalg N_k$. Now, given any set $A \subset \{1, \ldots, n\}$ of size k such that

$$A \cap N_i = 1 \quad \text{for } i = 1, \ldots, k$$

then the map $u_A \colon X^{\otimes n} \to I$ determined by the partition

$$A \amalg \coprod_{\substack{i \in \{1,\ldots,n\} \\ i \notin A}} \{i\};$$

is easily seen to be orthogonal to $\Delta_{n_1} \otimes \cdots \otimes \Delta_{n_k}$, and hence a coproof of $\mathbf{T}$.

Now, suppose f is a proof of $\mathbf{T}$. Given $a \in N_i$ and $b \in N_j$, where $i \neq j$, we can choose $\{a, b\} \subseteq A$ above, to get a coproof u_A joining a and b; so by Proposition 8, f cannot not join a and b.

So f can only join a and b if $\{a, b\} \subset N_i$ for some i. In fact, in this case f *must* join a and b, or else it's easy to see that $u_A \circ f$ would have at least two disjoint connected components.

Thus we conclude that f joins a and b iff $\{a, b\} \subset N_i$ for some i, and so must be the map $\Delta_{n_1} \otimes \cdots \otimes \Delta_{n_k}$. □

Proposition 10. *Let $\mathbf{T} \in O(\mathbb{F})_m$ with $U\mathbf{T} = X^{\otimes n}$. Then any map $f \colon \mathbf{I} \to \mathbf{T}$ can be factored as:*

$$\begin{array}{ccc} \mathbf{I} & \xrightarrow{\Delta_{n_1} \otimes \cdots \otimes \Delta_{n_k}} & \mathbf{A} \\ & \searrow^{f} & \downarrow g \\ & & \mathbf{T} \end{array}$$

where $\mathbf{A}$ is co-semisimple, $\sum n_i = n$ and Ug is an isomorphism.

Proof. By the remarks above, f is determined by a partition

$$P_1 \amalg \cdots \amalg P_k$$

of $\{1, \ldots, n\}$. But writing $n_i = |P_i|$, and $\Delta_{(n)}$ for the map $\Delta_{n_1} \otimes \cdots \otimes \Delta_{n_k}$, there is an isomorphism $g \in \mathbb{F}$ such that $g \circ \Delta_{(n)}$ is topologically equivalent to f. By the previous result, $\Delta_{(n)}$ is the unique proof of the co-semisimple term

$$\mathbf{A} = \mathbf{X}^{⅋ n_1} \otimes \cdots \otimes \mathbf{X}^{⅋ n_k},$$

and hence a map $\Delta_{(n)} \colon I \to \mathbf{A}$ in $O(\mathbb{F})$. Furthermore, this map is surjective on proofs, and hence cocartesian, and so the factorisation $f = g \circ \Delta_{(n)}$ in $\mathbb{F}$ lifts to a factorisation $f = g \circ \Delta_{(n)}$ in $O(\mathbb{F})$, with Ug an isomorphism. □

2.7 Isomorphisms in $\mathbb{F}$

Our attention is directed by the previous result to maps $g \colon \mathbf{A} \to \mathbf{B}$ in $O(\mathbb{F})$ whose image under U is an isomorphism. The isomorphisms in $\mathbb{F}$ are precisely those maps in the image of

$$S \colon \mathbb{S} \to \mathbb{F}$$

where $\mathbb{S}$ is the free symmetric monoidal category on one object X, and S is the symmetric monoidal functor from $\mathbb{S}$ to $\mathbb{F}$ uniquely determined by $S(X) = X$. Let us write $\mathbb{F}_s$ for the subcategory of $\mathbb{F}_m$ thus determined, and $O(\mathbb{F})_s$ for the pullback of $\mathbb{F}_s \hookrightarrow \mathbb{F}_m$ along U. A typical map in $\mathbb{F}_s$ will be written as $s_\varphi \colon X^{\otimes n} \to X^{\otimes n}$ where φ is a permutation on n letters. Explicitly, s_φ is the map given by

$$\mathrm{Top}(s_\varphi) = \big[\, (\{i\}, \{\varphi(i)\}, 0) \mid i \in \{1, \ldots, n\} \,\big]\,.$$

We say that s_φ **sends** i **to** j if $\varphi(i) = j$.

In particular, we note that all the symmetry and weak distributivity maps in $\mathbb{F}_m$ lie in $\mathbb{F}_s$; and since the functor $U \colon O(\mathbb{F}) \to \mathbb{F}$ preserves all the weakly distributive structure on the nose, we see that all the symmetry and weak distributivity maps in $O(\mathbb{F})_m$ lie in $O(\mathbb{F})_s$. Our main goal is a converse to the above observation, namely:

Proposition. Let $\mathbf{A}, \mathbf{B} \in O(\mathbb{F})_m$ with $\mathbf{A}$ co-semisimple; then every map $s_\varphi \colon \mathbf{A} \to \mathbf{B}$ in $O(\mathbb{F})_s$ can be decomposed as a series of applications of weak distributivities and symmetries.

But we still have some way to go before we can prove this. Now, there is a natural class of maps, built from symmetries and weak distributivities, that we can describe:

Definition 8. Let $\mathbb{C}$ be a weakly distributive category; we write $\mathcal{M}(\mathbb{C})$ for the monoid of endofunctors of $\mathbb{C}$ generated by

$$(-) \otimes C, \quad C \otimes (-), \quad (-) ⅋ C, \quad \text{and} \quad C ⅋ (-)$$

for every $C \in \mathbb{C}$. Given objects $A, B \in \mathbb{C}$, we say that A **is a subterm of** B if there exists $F \in \mathcal{M}(\mathbb{C})$ with $B = F(A)$.

Proposition 11. *Let $\mathbb{C}$ be any weakly distributive category, and let $F \in \mathcal{M}(\mathbb{C})$. Then there are families of maps*

$$\begin{aligned} l_F^{\otimes}(A,B)&\colon A \otimes F(B) \to F(A \otimes B) \\ r_F^{\otimes}(A,B)&\colon F(A) \otimes B \to F(A \otimes B) \\ l_F^{⅋}(A,B)&\colon F(A ⅋ B) \to A ⅋ F(B) \\ r_F^{⅋}(A,B)&\colon F(A ⅋ B) \to F(A) ⅋ B \end{aligned}$$

natural in A and B, built from symmetries and weak distributivities in $\mathbb{C}$.

Proof. We prove only the first; the rest follow by symmetry and duality. We proceed by induction on the form of F. For the base case $F = \mathrm{id}$, we have nothing to do. For the inductive step, we have one of:

$$F(-) = \begin{cases} C \otimes G(-), \\ G(-) \otimes C, \\ C ⅋ G(-) \text{ or} \\ G(-) ⅋ C; \end{cases}$$

accordingly, we have maps

$$A \otimes (C \otimes G(B)) \xrightarrow{\cong} C \otimes (A \otimes G(B)) \xrightarrow{l_G^{\otimes}} C \otimes G(A \otimes B) = F(A \otimes B),$$
$$A \otimes (G(B) \otimes C) \xrightarrow{\cong} (A \otimes G(B)) \otimes C \xrightarrow{r_G^{\otimes}} G(A \otimes B) \otimes C = F(A \otimes B),$$
$$A \otimes (C ⅋ G(B)) \xrightarrow{\text{w.d.}} C ⅋ (A \otimes G(B)) \xrightarrow{l_G^{⅋}} C ⅋ G(A \otimes B) = F(A \otimes B) \text{ and}$$
$$A \otimes (G(B) ⅋ C) \xrightarrow{\text{w.d.}} (A \otimes G(B)) ⅋ C \xrightarrow{r_G^{⅋}} G(A \otimes B) ⅋ C = F(A \otimes B).$$

□

We thus immediately deduce

Proposition 12. *For $F \in \mathcal{M}(O(\mathbb{F})_m)$ and $\mathbf{A}, \mathbf{B} \in O(\mathbb{F})_m$, each of the maps $l_F^{\otimes}(\mathbf{A},\mathbf{B})$, $r_F^{\otimes}(\mathbf{A},\mathbf{B})$, $l_F^{⅋}(\mathbf{A},\mathbf{B})$ and $r_F^{⅋}(\mathbf{A},\mathbf{B})$ constructed in Proposition 11 lies in $O(\mathbb{F})_s$.*

Now, a little notation. Given an object of $\mathbb{F}_m$ (resp., $O(\mathbb{F})_m$), we label its occurrences of X (resp., $\mathbf{X}$) in ascending numerical order from left-to-right, allowing us to distinguish between them easily. For example

$$(((X \otimes X) ⅋ X) \otimes X) \quad \leftrightsquigarrow \quad (((X_1 \otimes X_2) ⅋ X_3) \otimes X_4).$$

We emphasise that this is a purely syntactic convenience, and that we are *not* changing the objects of the category.

Proposition 13. *Let $F, G \in \mathcal{M}(O(\mathbb{F}_m))$, and let $s_\varphi \colon F(X_i \otimes X_{i+1}) \to G(X_j \otimes X_{j+1})$ be a map in $\mathbb{F}_s$ with $\varphi(i) = j$ and $\varphi(i+1) = j+1$. Then we can find a permutation φ' making the following diagrams commute in $\mathbb{F}$:*

$$\begin{array}{ccccc}
F(X_i) & \xrightarrow{F(\Delta)} & F(X_i \otimes X_{i+1}) & \xrightarrow{F(m)} & F(X_i) \\
\downarrow{\scriptstyle s_{\varphi'}} & & \downarrow{\scriptstyle s_\varphi} & & \downarrow{\scriptstyle s_{\varphi'}} \\
G(X_j) & \xrightarrow[G(\Delta)]{} & G(X_j \otimes X_{j+1}) & \xrightarrow[G(m)]{} & G(X_j)
\end{array}$$

Proof. Define φ' by $\varphi'(i) = e(\varphi(\delta(i)))$, where

$$\delta \colon [n-1] \to [n]; \quad i \mapsto \begin{cases} i & \text{if } i \leqslant j \\ i+1 & \text{if } i > j \end{cases} \quad \text{and}$$

$$e \colon [n] \to [n-1]; \quad i \mapsto \begin{cases} i & \text{if } i \leqslant j \\ i-1 & \text{if } i > j \end{cases}$$

Since $s_{\varphi'}$ is built out of symmetry maps, it is natural in each of its $n-1$ variables. In particular, it can be viewed as a natural transformation $F \to G$, in which guise, instantiation at $X \otimes X$ gives us precisely s_φ. So the above diagram is just a naturality square, and hence commutative. □

2.8 The key lemma

A few preliminaries before the key lemma:

Proposition 14. *The maps $m \colon X \otimes X \to X$ and $\Delta \colon X \to X \otimes X$ in $\mathbb{F}$ lift to maps $m \colon \mathbf{X} \otimes \mathbf{X} \to \mathbf{X}$ and $\Delta \colon \mathbf{X} \to \mathbf{X} \parr \mathbf{X}$ in $O(\mathbb{F})$.*

Proof. We simply observe

$$\begin{array}{rcccccccl}
\mathbf{X} \otimes \mathbf{X} & = (& X \otimes X & , & \{\eta \otimes \eta\} & , & \{\varepsilon \circ m\} &) \\
 & & {\scriptstyle m}\downarrow & & {\scriptstyle m\circ(-)}\downarrow & & \uparrow{\scriptstyle (-)\circ m} & \\
\mathbf{X} & = (& X & , & \{\eta\} & , & \{\varepsilon\} &)
\end{array}$$

and

$$\begin{array}{rcccccccl}
\mathbf{X} & = (& X & , & \{\eta\} & , & \{\varepsilon\} &) \\
 & & {\scriptstyle \Delta}\downarrow & & {\scriptstyle \Delta\circ(-)}\downarrow & & \uparrow{\scriptstyle (-)\circ\Delta} & . \\
\mathbf{X} \parr \mathbf{X} & = (& X \otimes X & , & \{\Delta \circ \eta\} & , & \{\varepsilon \otimes \varepsilon\} &)
\end{array}$$

□

Proposition 15. *Let* $\mathbf{T} \in O(\mathbb{F})_m$ *and* $F, G \in \mathcal{M}(O(\mathbb{F})_m)$ *such that* $F(\mathbf{X}_i) \otimes G(\mathbf{X}_j)$ *is a subterm of* $\mathbf{T}$*. Then no proof of* $\mathbf{T}$ *joins* i *and* j*, and some coproof does join* i *and* j*.*

Proof. We have $\mathbf{T} = H(F(\mathbf{X}_i) \otimes G(\mathbf{X}_j))$ for some $H \in \mathcal{M}(O(\mathbb{F})_m)$; so by Proposition 11, we have a map

$$\mathbf{T} = H(F(\mathbf{X}_i) \otimes G(\mathbf{X}_j)) \xrightarrow{l^{\otimes}_G \circ r^{\otimes}_F} HFG(\mathbf{X}_k \otimes \mathbf{X}_{k+1}) \xrightarrow{HFG(m)} HFG(\mathbf{X}_k)$$

in $O(\mathbb{F})_s$. By Proposition 7, the object $HFG(\mathbf{X}_k)$ has at least one coproof. Pick any such $h\colon HFG(X_k) \to I$; then the composite

$$H(F(X_i) \otimes G(X_j)) \xrightarrow{l^{\otimes}_G \circ r^{\otimes}_F} HFG(X_k \otimes X_{k+1}) \xrightarrow{HFG(m)} HFG(X_k) \xrightarrow{h} I$$

is a coproof of $\mathbf{T}$. It is easy to check from the manner of construction of $l^{\otimes}_G$ and $r^{\otimes}_F$ that the first arrow above sends i to k and j to $k+1$; and since the remaining composite joins k and $k+1$, the entire composite joins i and j. But now by Proposition 8, since some coproof of $\mathbf{T}$ joins i and j, no proof of $\mathbf{T}$ can, as required. □

Dually, we have

Proposition 16. *Let* $\mathbf{T} \in O(\mathbb{F})_m$ *and* $F, G \in \mathcal{M}(O(\mathbb{F})_m)$ *with* $F(\mathbf{X}_i) \parr G(\mathbf{X}_j)$ *a subterm of* $\mathbf{T}$*. Then no coproof of* $\mathbf{T}$ *joins* i *and* j*, and some proof does join* i *and* j*.*

Proposition 17 (Key Lemma). *Let* $\mathbf{T} \in O(\mathbb{F})_m$ *with* $\mathbf{T} \neq \mathbf{X}$*. Then either* $\mathbf{T}$ *contains a subterm of the form* $\mathbf{X}_i \otimes \mathbf{X}_{i+1}$*, or* $\mathbf{T}$ *contains a subterm of the form* $\mathbf{X}_i \parr \mathbf{X}_{i+1}$ *such that every proof of* $\mathbf{T}$ *joins* i *and* $i+1$*.*

For the proof of this, we shall need a little notation:

Notation. Let $\mathbf{T}$ be an object of $O(\mathbb{F})_m$. By a **$\otimes$-atom**, we shall mean a subterm $\mathbf{X}_i \parr \mathbf{X}_{i+1} \parr \cdots \parr \mathbf{X}_j$ of $\mathbf{T}$ with $i \leqslant j$; we write $\mathbf{X}_{i,j}$ for such a subterm. A **maximal $\otimes$-atom** is an $\otimes$-atom $\mathbf{X}_{i,j}$ such that neither $\mathbf{X}_{i-1,j}$ nor $\mathbf{X}_{i,j+1}$ are subterms of $\mathbf{T}$, and a **nontrivial $\otimes$-atom** is an $\otimes$-atom $\mathbf{X}_{i,j}$ with $j > i$. For example, the maximal $\otimes$-atoms of

$$(((\mathbf{X}_1 \parr \mathbf{X}_2 \parr \mathbf{X}_3) \otimes \mathbf{X}_4) \parr (\mathbf{X}_5 \otimes (\mathbf{X}_6 \parr \mathbf{X}_7)))$$

are $\mathbf{X}_{1,3}$, $\mathbf{X}_{4,4}$, $\mathbf{X}_{5,5}$ and $\mathbf{X}_{6,7}$, whereas the nontrivial maximal $\otimes$-atoms are $\mathbf{X}_{1,3}$ and $\mathbf{X}_{6,7}$. We shall say that a proof $I \to T$ of $\mathbf{T}$ **completely joins** a given $\otimes$-atom $\mathbf{X}_{i,j}$ just when it joins $\{i, i+1, \ldots, j\}$.

Proof. Suppose that $\mathbf{T}$ contains no subterm of the form $\mathbf{X}_i \otimes \mathbf{X}_{i+1}$. Under this assumption we shall prove the following result: that there exists a nontrivial maximal $\otimes$-atom of $\mathbf{T}$ that every proof of $\mathbf{T}$ completely joins. Clearly the lemma then follows *a fortiori*.

We proceed by induction on the number of $\otimes$ connectives in $\mathbf{T}$. For the base case, we have $\mathbf{T} = \mathbf{X}_{1,n}$, which from above has a unique proof $\Delta_n \colon \mathbf{I} \to \mathbf{T}$. For the inductive step, we may write $\mathbf{T}$ in the form

$$\mathbf{T} = F(\mathbf{X}_{i-n,i} \otimes \mathbf{X}_{i+1,i+m})$$

where $F \in \mathcal{M}(O(\mathbb{F})_m)$ and the left and right hand arguments of the tensor are maximal $\otimes$-atoms. Since $\mathbf{T}$ contains no subterm of the form $\mathbf{X}_i \otimes \mathbf{X}_{i+1}$, we may take it that $\mathbf{X}_{i-n,n}$ is a *nontrivial* $\otimes$-atom, i.e., $n > 1$. Now, we have the following map in $O(\mathbb{F})_m$:

$$\begin{array}{c}
\mathbf{T} = F((\mathbf{X}_{i-n} \parr \cdots \parr \mathbf{X}_i) \otimes (\mathbf{X}_{i+1} \parr \cdots \parr \mathbf{X}_{i+m})) \\
\big\downarrow \text{w.d.} \\
F(\mathbf{X}_{i-n} \parr \cdots \parr (\mathbf{X}_i \otimes \mathbf{X}_{i+1}) \parr \cdots \parr \mathbf{X}_{i+m}) \\
\big\downarrow F(\mathbf{X}_{i-n} \parr \cdots \parr m \parr \cdots \parr \mathbf{X}_{i+m}) \\
\mathbf{T}' = F(\mathbf{X}_{i-n} \parr \cdots \parr \mathbf{X}_i \parr \cdots \parr \mathbf{X}_{i+m-1}),
\end{array}$$

which under U is sent to the map $c = X^{\otimes a} \otimes m \otimes X^{\otimes b} \colon T \to T'$ in $\mathbb{F}$ (for appropriate values of a and b).

By induction there is a nontrivial maximal $\otimes$-atom $\mathbf{X}_{j,k}$ of $\mathbf{T}'$ which every proof of $\mathbf{T}'$ completely joins. If $k < i - n$ then $\mathbf{X}_{j,k}$ is a maximal $\otimes$-atom of $\mathbf{T}$, and c sends p to p for each $j \leqslant p \leqslant k$. So if a proof f of $\mathbf{T}$ did not completely join $\mathbf{X}_{j,k}$, then $c \circ f$ would be a proof of $\mathbf{T}'$ not completely joining $\mathbf{X}_{j,k}$, which is a contradiction. A similar argument holds if $j \geqslant i + m$.

This leaves only the case where $\mathbf{X}_{i-n,i+m-1}$ is a subterm of $\mathbf{X}_{j,k}$. Here, every proof of $\mathbf{T}'$ completely joins $\mathbf{X}_{i-n,i+m-1}$; it follows that every proof of $\mathbf{T}$ completely joins the nontrivial maximal $\otimes$-atom $\mathbf{X}_{i-n,i}$.

Indeed, given $f \in T_p$, if f did not join p to q (for $i - n \leqslant p < q < i$) then $c \circ f$ would be a proof of $\mathbf{T}'$ which did not join p and q, a contradiction. And if f did not join p to i (for $i - n \leqslant p < i$), then, since $c \circ f$ joins p and i, we must have that f joins p and $i + 1$. But this is impossible by Proposition 15. □

2.9 The main result

Proposition 18. *Let* $\mathbf{A}, \mathbf{B} \in O(\mathbb{F})_m$ *with* $\mathbf{A}$ *co-semisimple; then every map* $s_\varphi \colon \mathbf{A} \to \mathbf{B}$ *in* $O(\mathbb{F})_s$ *can be decomposed as a series of applications of weak distributivities and symmetries.*

Proof. We proceed by induction on the number of $\mathbf{X}$'s in $\mathbf{B}$. In the base case $\mathbf{A} = \mathbf{B} = \mathbf{X}$ and there's nothing to do. Otherwise, we proceed following the key lemma.

Case 1: $\mathbf{B}$ contains a subterm $\mathbf{X}_i \otimes \mathbf{X}_{i+1}$, such that $\mathbf{B} = F(\mathbf{X}_i \otimes \mathbf{X}_{i+1})$, say. Then setting $j = \varphi^{-1}(i)$ and $k = \varphi^{-1}(i+1)$, we see by Proposition 16 that $\mathbf{X}_j$ and $\mathbf{X}_k$ cannot appear on opposite sides of a $⅋$ connective in $\mathbf{A}$. Indeed, if they did then some proof f of $\mathbf{A}$ would join j and k, and hence $s_\varphi \circ f$ would be a proof of $\mathbf{B}$ joining i and $i+1$, contradicting Proposition 15.

Thus, $\mathbf{X}_j$ and $\mathbf{X}_k$ must appear on opposite sides of a tensor connective in $\mathbf{A}$; without loss of generality, we assume $j < k$ (if not then applying a symmetry in $O(\mathbb{F})_s$ will sort things out). We compose with symmetries and weak distributivities as follows:

$$\begin{array}{c} \mathbf{A} = \cdots \otimes (\cdots ⅋ \mathbf{X}_j ⅋ \dots) \otimes \cdots \otimes (\cdots ⅋ \mathbf{X}_k ⅋ \cdots) \otimes \cdots \\ \downarrow \cong \\ \cdots \otimes (\cdots ⅋ \mathbf{X}_n) \otimes (\mathbf{X}_{n+1} ⅋ \cdots) \otimes \cdots \\ \downarrow \text{w.d.} \\ \cdots \otimes (\cdots ⅋ (\mathbf{X}_n \otimes \mathbf{X}_{n+1}) ⅋ \cdots) \otimes \cdots \end{array}$$

Since the displayed map is made up of symmetries and weak distributivities it lies in $O(\mathbb{F})_s$, and so is s_ψ for some permutation ψ. Writing G for the functor $\cdots \otimes (\cdots ⅋ (-) ⅋ \cdots) \otimes \cdots$, we thus have the following diagram in $O(\mathbb{F})_m$:

$$\begin{array}{ccccc} \mathbf{A} & \xrightarrow{s_\psi} & G(\mathbf{X}_n \otimes \mathbf{X}_{n+1}) & \xrightarrow{G(m)} & G(\mathbf{X}_n) \\ & \searrow^{s_\varphi} & & & \\ & & F(\mathbf{X}_i \otimes \mathbf{X}_{i+1}) & \xrightarrow[F(m)]{} & F(\mathbf{X}_i) \end{array}$$

We remark that the object $G(\mathbf{X}_n)$ is co-semisimple; hence the top composite is surjective on proofs, and hence cocartesian. Applying U to the above diagram, we see that $s_{\varphi\psi^{-1}}\colon G(X_n \otimes X_{n+1}) \to F(X_i \otimes X_{i+1})$ is a map in $\mathbb{F}_s$ sending n to i and $n+1$ to $i+1$. Hence by Proposition 13, there is a map $s_{\varphi'}$ in $\mathbb{F}_s$, sending n to i and making the following diagram commute:

$$\begin{array}{ccccc} A & \xrightarrow{s_\psi} & G(X_n \otimes X_{n+1}) & \xrightarrow{G(m)} & G(X_n) \\ & \searrow^{s_\varphi} & \downarrow s_{\varphi\psi^{-1}} & & \downarrow s_{\varphi'} \\ & & F(X_i \otimes X_{i+1}) & \xrightarrow[F(m)]{} & F(X_i) \end{array}$$

But since $G(m) \circ s_\psi$ is cocartesian, $s_{\varphi'}$ lifts to a map $G(\mathbf{X}_n) \to F(\mathbf{X}_i)$ in $O(\mathbb{F})_s$.

Since $G(\mathbf{X}_n)$ is co-semisimple and $s_{\varphi'}$ is a map in $O(\mathbb{F})_s$, the inductive hypothesis tells us that $s_{\varphi'}$ can be decomposed as a series of weak distributivities and symmetries; and as such, is a map natural in every variable. In particular, we can view it as a natural transformation $G \to F$, and instantiating it at $\mathbf{X} \otimes \mathbf{X}$ gives us a map $h\colon G(\mathbf{X}_n \otimes \mathbf{X}_{n+1}) \to G(\mathbf{X}_i \otimes \mathbf{X}_{i+1})$ built out of weak distributivities and symmetries in $O(\mathbb{F})_s$, and making the right hand square in the diagram

$$\begin{array}{ccccc} \mathbf{A} & \xrightarrow{s_\psi} & G(\mathbf{X}_n \otimes \mathbf{X}_{n+1}) & \xrightarrow{G(m)} & G(\mathbf{X}_n) \\ & \searrow^{s_\varphi} & \downarrow h & & \downarrow s_{\varphi'} \\ & & F(\mathbf{X}_i \otimes \mathbf{X}_{i+1}) & \xrightarrow[F(m)]{} & F(\mathbf{X}_i) \end{array}$$

commute. But applying U to the diagram, we see that Uh is precisely the map we get from instantiating $s_{\varphi'}$ at $X \otimes X$, i.e., $Uh = s_\varphi \circ s_{\psi^{-1}}$. So $s_\varphi = h \circ s_\psi$ is our desired decomposition in $O(\mathbb{F})_s$, and we are done.

Case 2: $\mathbf{B}$ contains no subterm of the form $\mathbf{X}_i \otimes \mathbf{X}_{i+1}$. Now the key lemma tells us that $\mathbf{B}$ can be written as $F(\mathbf{X}_i \parr \mathbf{X}_{i+1})$ such that every proof of $\mathbf{B}$ joins i and $i+1$. As before, set $j = \varphi^{-1}(i)$ and $k = \varphi^{-1}(i+1)$; then by Proposition 15 we see that $\mathbf{X}_j$ and $\mathbf{X}_k$ cannot appear on opposite sides of a $\otimes$ connective in $\mathbf{A}$; for if they did, then the unique proof u of $\mathbf{A}$ would not join j and k; and then $s_\varphi \circ u$ would be a proof of $\mathbf{B}$ not joining i and $i+1$.

So $\mathbf{X}_j$ and $\mathbf{X}_k$ appear on opposite sides of a par connective; again, without loss of generality we assume that $j < k$. Now we apply a symmetry into $\mathbf{A}$:

$$\begin{array}{c} \cdots \otimes (\cdots \parr \mathbf{X}_j \parr \mathbf{X}_{j+1} \parr \cdots) \otimes \cdots \\ \downarrow \cong \\ \mathbf{A} = \cdots \otimes (\cdots \parr \mathbf{X}_j \parr \cdots \parr \mathbf{X}_k \parr \cdots) \otimes \cdots \end{array}$$

which we denote by s_ψ; so setting $G(-) = \cdots \otimes (\cdots \parr (-) \parr \cdots) \otimes \cdots$ we have the following diagram in $O(F)_m$:

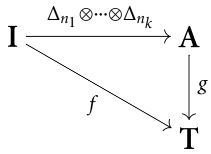
$$\begin{array}{ccccc} G(\mathbf{X}_j) & \xrightarrow{G(\Delta)} & G(\mathbf{X}_j \parr \mathbf{X}_{j+1}) & \xrightarrow{s_\psi} & \mathbf{A} \\ & & & \swarrow_{s_\varphi} & \\ F(\mathbf{X}_i) & \xrightarrow[F(\Delta)]{} & F(\mathbf{X}_i \parr \mathbf{X}_{i+1}) & & \end{array}$$

We proceed as before by applying U to the above diagram, and utilising Proposition 13 to get a map $s_{\varphi'}$ in $\mathbb{F}_s$, sending j to i and making

$$\begin{array}{ccccc} G(X_j) & \xrightarrow{G(\Delta)} & G(X_j \parr X_{j+1}) & \xrightarrow{s_\psi} & A \\ \downarrow s_{\varphi'} & & \downarrow s_{\varphi\psi} & \swarrow_{s_\varphi} & \\ F(X_i) & \xrightarrow[F(\Delta)]{} & F(X_i \parr X_{i+1}) & & \end{array}$$

commute. Now, we observe that $F(\Delta)$ is split mono in $\mathbb{F}$, with left inverse $F(X \otimes \varepsilon)$; from which it follows that the map $F(\Delta)$ in $O(\mathbb{F})$ is surjective on coproofs, and hence cartesian. So $s_{\varphi'}$ lifts to a map in $O(\mathbb{F})_s$. The rest of the proof in this case follows just as before. □

Corollary 19. *Let* $\mathbf{T} \in O(\mathbb{F})_m$ *with* $U\mathbf{T} = X^{\otimes n}$. *Then any map* $f \colon \mathbf{I} \to \mathbf{T}$ *can be factored as:*



where $\mathbf{A}$ *is co-semisimple,* $\sum n_i = n$ *and* g *is built from weak distributivity and symmetry maps.*

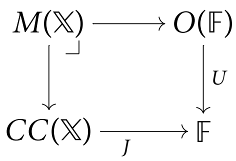
Proof. Follows from the main theorem and Proposition 10. □

2.10 Reintroducing duality

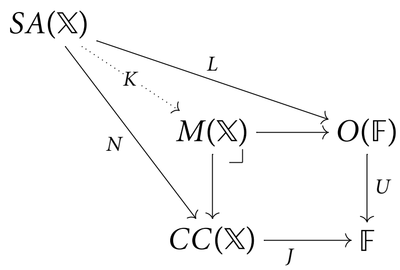
The category $O(\mathbb{F})_m$ is thus fairly close to being a free (unit-free) $*$-autonomous category. We can see in the factorisation diagram of the above corollary, that the vertical arrow g is built from valid maps in a $*$-autonomous category, but that the horizontal arrow $\Delta_{n_1} \otimes \cdots \otimes \Delta_{n_k}$ is not. However, we can resolve this in a straightforward manner.

Notation. Let $\mathbb{X}$ be an arbitrary set, viewed as a discrete category. Then we write $SA(\mathbb{X})$ for the free $*$-autonomous category on $\mathbb{X}$, and $CC(\mathbb{X})$ for the free compact closed category on $\mathbb{X}$.

We observed that $\mathbb{F}$ was a compact closed category, so we determine a functor of compact closed categories $J \colon CC(\mathbb{X}) \to \mathbb{F}$ extending the constant functor $\Delta X \colon \mathbb{X} \to \mathbb{F}$. Now consider the following pullback in **Cat**:



Since both $O(\mathbb{F})$ and $CC(\mathbb{X})$ are $*$-autonomous categories, we can determine a $*$-autonomous functor $L \colon SA(\mathbb{X}) \to O(\mathbb{F})$ extending $\Delta\mathbf{X} \colon \mathbb{X} \to O(\mathbb{F})$ and a $*$-autonomous functor $N \colon SA(\mathbb{X}) \to CC(\mathbb{X})$ extending the natural embedding $\mathbb{X} \to CC(\mathbb{X})$. It's now evident that UL and JN are both $*$-autonomous functors extending the constant functor $\Delta X \colon \mathbb{X} \to \mathbb{F}$, and hence must be the same. Thus we get a unique functor K making the following diagram commute:



We note that to give an object of $M(\mathbb{X})$ is to give an object of $CC(\mathbb{X})$ and an object of $O(\mathbb{F})$ whose image in $\mathbb{F}$ is the same; but this is equivalent to giving an object of $SA(\mathbb{X})$. So the functor K is an isomorphism on objects, and we identify objects of $M(\mathbb{X})$ with their preimage in $SA(\mathbb{C})$.

Proposition 20. *Let $f : I \to A$ be a map in $M(\mathbb{X})$, where A is a unit-free object. Then f is the image under K of a unique map in $SA(\mathbb{X})$.*

Proof. For uniqueness, note that K must be faithful since $N\colon SA(\mathbb{X}) \to CC(\mathbb{X})$ is. It remains only to show existence. Now, to give a map $I \to A$ in $M(\mathbb{X})$ is to give a map $g\colon I \to NA$ of $CC(\mathbb{X})$ and a map $h\colon \mathbf{I} \to LA$ of $O(\mathbb{F})$ with $Jg = Uh$.

In full generality, a map $I \to NA$ in $CC(\mathbb{X})$ is given by a partition of the set of atoms of NA into pairs $\{C, C^\perp\}$, composed with a number of maps of the form

$$\dim a\colon I \to a \otimes a^\perp \xrightarrow{\sigma} a^\perp \otimes a \to I$$

for atoms $a \in \mathbb{X}$. [For the details, we refer the reader to, for example, [Tan97].] However, our map g cannot contain components of the form $\dim a$, as then $Jg = Uf$ would contain more than one connected component, which we know is impossible. So g is given just by a suitable partition of the atoms of NA, and so admits a decomposition

$$I \xrightarrow{\eta_{a_1} \otimes \cdots \otimes \eta_{a_n}} (a_1 \otimes a_1^\perp) \otimes \cdots \otimes (a_n \otimes a_n^\perp) \xrightarrow{s_\varphi} NA$$

where each $a_i \in \mathbb{X}$ with η_{a_i} the unit of the adjunction $a_i \dashv a_i^\perp$, and φ is a suitable permutation in S_{2n}.

By Proposition 10 and the knowledge that $Uf = Jg$, we may decompose f as

$$\mathbf{I} \xrightarrow{\Delta \otimes \cdots \otimes \Delta} \mathbf{X}_{1,2} \otimes \mathbf{X}_{3,4} \otimes \cdots \otimes \mathbf{X}_{2n-1,2n} \xrightarrow{s_\psi} LA$$

and it is clear that we may arrange it that $\psi = \varphi$. Having done this, we see that these two decompositions together give a decomposition of our original map $f : I \to A$ in $M(\mathbb{X})$.

It remains only to give a map in $SA(\mathbb{X})$ corresponding to each part of the decomposition. But since the first part is built from the units of adjunctions in $CC(\mathbb{X})$, we may lift it to the corresponding units of adjunctions in $SA(\mathbb{X})$; and since the second part is built out of weak distributivities and symmetries in $O(\mathbb{F})_m$, we may lift it to the corresponding weak distributivities and symmetries in $SA(\mathbb{X})$. Then it is trivial to verify that this composite gives a map whose image under K is f. □

Chapter 3

Monoidal modules, double glueing and the Chu construction

3.1 Introduction

In the previous chapter, the *orthogonality* construction of [HS03] was used to produce a new $*$-autonomous category from an old one. This construction builds on the *double glueing* construction of the same paper (but see also [Tan97]), which itself generalises the *single glueing* construction. However, whilst single glueing is a conceptually compelling concept, its double glueing counterpart has a whiff of black magic about it; whilst it is routine to see that it works, it is much less clear *why* it should work. One aim of this chapter is to attempt an answer to this question.

Similar accusations have been levelled at another well-known tool for constructing $*$-autonomous categories, the 'Chu construction' of [Bar79] and [Bar96], leading to various attempts to quantify its universality: see [Pav97] or [DS04], for example. By these lights, this chapter could be facetiously subtitled 'Yet another universal property of the Chu construction'.

But foremost, this chapter is about 'monoidal modules'. As monoidal categories are to one-object bicategories, so monoidal modules are to (certain) two-object bicategories. Whilst a lot of this material is well-known (indeed, the basic definition is present in the first few pages of [Bén67]), it is worth being relatively explicit about it; after all, the study of monoidal categories is more than *just* the theory of one-object bicategories; similarly, the theory of monoidal modules has more to it than *just* the theory of two-object bicategories.

A lot of this material is complementary to that of [CKVW98], and indeed the material of both that paper and this chapter can be viewed as specialisations of a general notion of 'profunctors of bicategories'.

3.2 Monoidal actions

The material of this section is essentially folklore; in the literature it appears in [Bén67], and more explicitly in [JK02].

Definition 9. Let $(\mathbb{C}, \otimes, \mathbf{r}, \mathbf{l}, \mathbf{a})$ be a monoidal category and $\mathbb{D}$ an arbitrary category. A **left action** of $\mathbb{C}$ on $\mathbb{D}$ is a strong monoidal functor $F\colon \mathbb{C} \to [\mathbb{D}, \mathbb{D}]$, where the monoidal structure on $[\mathbb{D}, \mathbb{D}]$ is given by composition. To give such a functor is to give a functor $\boxtimes_l\colon \mathbb{C} \times \mathbb{D} \to \mathbb{D}$ together with natural isomorphisms:

$$\lambda_D\colon D \to I \boxtimes_l D$$
$$\alpha_{B,C,D}\colon B \boxtimes_l (C \boxtimes_l D) \to (B \otimes C) \boxtimes_l D$$

subject to the following coherence conditions:

$$\begin{tikzcd} C \boxtimes_l D \arrow[r, "C\boxtimes_l \lambda_D"] \arrow[d, "\mathbf{r}_C \boxtimes_l D"'] & C \boxtimes_l (I \boxtimes_l D) \arrow[dl, "\alpha_{C,I,D}"] \\ (C \otimes I) \boxtimes_l D & \end{tikzcd} \qquad \begin{tikzcd} C \boxtimes_l D \arrow[r, "\lambda_D"] \arrow[d, "\mathbf{l}_C \boxtimes_l D"'] & I \boxtimes_l (C \boxtimes_l D) \arrow[dl, "\alpha_{I,C,D}"] \\ (I \otimes C) \boxtimes D & \end{tikzcd}$$

$$\begin{tikzcd} A \boxtimes_l (B \boxtimes_l (C \boxtimes_l D)) \arrow[r, "\alpha_{A,B,(C\boxtimes_l D)}"] \arrow[d, "A\boxtimes_l \alpha_{B,C,D}"'] & (A \otimes B) \boxtimes_l (C \boxtimes_l D) \arrow[dd, "\alpha_{(A\otimes B),C,D}"] \\ A \boxtimes_l ((B \otimes C) \boxtimes_l D) \arrow[d, "\alpha_{A,(B\otimes C),D}"'] & \\ (A \otimes (B \otimes C)) \boxtimes_l D \arrow[r, "\mathbf{a}_{A,B,C}\boxtimes_l D"'] & ((A \otimes B) \otimes C) \boxtimes_l D. \end{tikzcd}$$

Similarly, a **right action** of $\mathbb{C}$ on $\mathbb{D}$ is a strong monoidal functor $F\colon \mathbb{C} \to [\mathbb{D}, \mathbb{D}]^{\mathrm{rev}}$, where the latter category is monoidal under reversed composition. Explicitly, it is a functor $\boxtimes_r\colon \mathbb{D} \times \mathbb{C} \to \mathbb{D}$ together with natural isomorphisms:

$$\rho_D\colon D \to D \boxtimes_r I$$
$$\beta_{D,C,B}\colon D \boxtimes_r (C \otimes B) \to (D \boxtimes_r C) \boxtimes_r B$$

subject to the following coherence conditions:

$$\begin{tikzcd} D \boxtimes_r C \arrow[r, "D\boxtimes_r \mathbf{l}_C"] \arrow[d, "\rho_D \boxtimes_r C"'] & D \boxtimes_r (I \otimes C) \arrow[dl, "\beta_{D,I,C}"] \\ (D \boxtimes_r I) \boxtimes_r C & \end{tikzcd} \qquad \begin{tikzcd} D \boxtimes_r C \arrow[r, "D\boxtimes_r \mathbf{r}_C"] \arrow[d, "\rho_{D\boxtimes_r C}"'] & D \boxtimes_r (C \otimes I) \arrow[dl, "\beta_{D,C,I}"] \\ (D \boxtimes_r C) \boxtimes_r I & \end{tikzcd}$$

$$\begin{tikzcd} D \boxtimes_r (C \otimes (B \otimes A)) \arrow[r, "\beta_{D,C,(B\otimes A)}"] \arrow[d, "D\boxtimes_l \mathbf{a}_{C,B,A}"'] & (D \boxtimes_r C) \boxtimes_r (B \otimes A) \arrow[dd, "\beta_{(D\boxtimes_r C),B,A}"] \\ D \boxtimes_r ((C \otimes B) \otimes A) \arrow[d, "\beta_{D,(C\otimes B),A}"'] & \\ (D \boxtimes_r (C \otimes B)) \boxtimes_r A \arrow[r, "\beta_{D,C,B}\boxtimes_r A"'] & ((D \boxtimes_r C) \boxtimes_r B) \boxtimes_r A. \end{tikzcd}$$

Given two monoidal categories $\mathbb{C}$ and $\mathbb{E}$, a **two-sided action** of $\mathbb{C}$ and $\mathbb{E}$ on $\mathbb{D}$ consists of a left action $(\boxtimes_l, \lambda, \alpha)$ of $\mathbb{C}$ on $\mathbb{D}$, a right action $(\boxtimes_r, \rho, \beta)$ of $\mathbb{E}$ on $\mathbb{D}$, and natural isomorphisms

$$\tau_{C,D,E} \colon C \boxtimes_l (D \boxtimes_r E) \to (C \boxtimes_l D) \boxtimes_r E$$

such that the following diagrams commute:

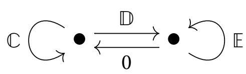
$$\begin{array}{ccc} C \boxtimes_l D & \xrightarrow{C \boxtimes_l \rho_D} & C \boxtimes_l (D \boxtimes_r I) \\ {\scriptstyle \rho_{C \boxtimes_l D}} \downarrow & \swarrow {\scriptstyle \tau_{C,D,I}} & \\ (C \boxtimes_l D) \boxtimes_r I & & \end{array} \qquad \begin{array}{ccc} D \boxtimes_r E & \xrightarrow{\lambda_{D \boxtimes_r E}} & I \boxtimes_l (D \boxtimes_r E) \\ {\scriptstyle \lambda_D \boxtimes_r E} \downarrow & \swarrow {\scriptstyle \tau_{I,D,E}} & \\ (I \boxtimes_l D) \boxtimes_r E & & \end{array}$$

$$\begin{array}{ccc} B \boxtimes_l (C \boxtimes_l (D \boxtimes_r E)) & \xrightarrow{\alpha_{B,C,D \boxtimes_r E}} & (B \otimes C) \boxtimes_l (D \boxtimes_r E) \\ \downarrow {\scriptstyle B \boxtimes_l \tau_{C,D,E}} & & \\ B \boxtimes_l ((C \boxtimes_l D) \boxtimes_r E) & & \downarrow {\scriptstyle \tau_{(B \otimes C),D,E}} \\ \downarrow {\scriptstyle \tau_{B,(C \boxtimes_l D),E}} & & \\ (B \boxtimes_l (C \boxtimes_l D)) \boxtimes_r E & \xrightarrow[\alpha_{B,C,D} \boxtimes_r E]{} & ((B \otimes C) \boxtimes_l D) \boxtimes_r E \end{array}$$

$$\begin{array}{ccc} C \boxtimes_l (D \boxtimes_r (E \otimes F)) & \xrightarrow{\tau_{C,D,E \otimes F}} & (C \boxtimes_l D) \boxtimes_r (E \otimes F) \\ \downarrow {\scriptstyle C \boxtimes_l \beta_{D,E,F}} & & \\ C \boxtimes_l ((D \boxtimes_r E) \boxtimes_r F) & & \downarrow {\scriptstyle \beta_{(C \boxtimes_l D),E,F}} \\ \downarrow {\scriptstyle \tau_{C,(D \boxtimes_r E),F}} & & \\ (C \boxtimes_l (D \boxtimes_r E)) \boxtimes_r F & \xrightarrow[\tau_{C,D,E} \boxtimes_r F]{} & ((C \boxtimes_l D) \boxtimes_r E) \boxtimes_r F. \end{array}$$

We will sometimes adopt 'modular' terminology; if $\mathbb{C}$ has a left action on $\mathbb{D}$ we call $\mathbb{D}$ a **left** $\mathbb{C}$**-module**; similarly we may talk about a **right** $\mathbb{C}$**-module**, a **left** $\mathbb{C}$**-, right** $\mathbb{E}$**-module** or a $\mathbb{C}$-**bimodule**.

Note that we can equivalently present a left $\mathbb{C}$-, right $\mathbb{E}$-module as a two-object bicategory as follows:



Examples 2.

- Any monoidal category $\mathbb{C}$ has a canonical two-sided action on itself, given by $C \boxtimes_l C' = C \boxtimes_r C' = C \otimes C'$.
- If $\mathbb{C}$ acts on $\mathbb{D}$ and $\mathbb{C}'$ is a sub-monoidal category of $\mathbb{C}$ (i.e., a subcategory which is monoidal with the same operations) then $\mathbb{C}'$ acts on $\mathbb{D}$.

- Any strong monoidal functor $F\colon \mathbb{C} \to \mathbb{D}$ induces an action of $\mathbb{C}$ on $\mathbb{D}$ via $C \boxtimes_l D = FC \otimes D$ and $D \boxtimes_r C = D \otimes FC$. The associativity and unit isomorphisms are given in the obvious way using the associativity and unit constraints for $\mathbb{D}$ and the strong monoidal structure of F. We might then think of $\mathbb{D}$ as a '$\mathbb{C}$-algebra'.

- In particular, given a left action of $\mathbb{C}$ on $\mathbb{D}$, we have a strong monoidal functor $\mathbb{C} \to [\mathbb{D}, \mathbb{D}]$ and hence a two-sided action of $\mathbb{C}$ on $[\mathbb{D}, \mathbb{D}]$ given by $(C \boxtimes_l F)(-) = C\boxtimes_l F(-)$ and $F\boxtimes_r C(-) = F(C\boxtimes_l -)$. Similarly, a right action of $\mathbb{C}$ on $\mathbb{D}$ induces a two-sided action of $\mathbb{C}$ on $[\mathbb{D}, \mathbb{D}]$ via $(C \boxtimes_l F)(-) = F(-\boxtimes_r C)$ and $(F \boxtimes_r C)(-) = F(-)\boxtimes_r C$.

- Suppose that $\mathbb{C}$ is a right-closed monoidal category, and that $\mathbb{D}$ is a category enriched in $\mathbb{C}$. If $\mathbb{D}$ admits tensor products with $\mathbb{C}$, so that for all $C \in \mathbb{C}$, $D \in \mathbb{D}$ we have isomorphisms, $\mathbb{C}$-natural in all variables,

$$\mathbb{D}(C \star D, D') \cong [C, \mathbb{D}(D, D')],$$

  then $\star$ induces a left action of the underlying ordinary category $\mathbb{C}_0$ on $\mathbb{D}_0$. In fact, as we shall later see in section 5, this action is *enriched*, since the above isomorphism becomes

$$\mathbb{D}_0(C \star D, D') \cong \mathbb{C}_0(C, \mathbb{D}(D, D')),$$

  so that every functor $(-) \star D$ has a right adjoint $\mathbb{D}(D, -)$.

- If $\mathbb{D}$ is a monoidal category, and T a monoidal comonad on $\mathbb{D}$, then $\mathbb{D}^T$, the category of coalgebras for the comonad, is a monoidal category and the forgetful functor $U\colon \mathbb{D}^T \to \mathbb{D}$ is strict monoidal; hence $\mathbb{D}^T$ acts on $\mathbb{D}$. In particular, if $\mathbb{D}$ is a model of intuitionistic linear logic with ! its exponential, then $\mathbb{D}^!$ acts on $\mathbb{D}$, and since the Kleisli category $\mathbb{D}_!$ is closed under the tensor product on $\mathbb{D}^!$, we have that $\mathbb{D}_!$ also acts on $\mathbb{D}$.

3.3 Maps of modules

Again, the details of this section are folklorish in the sense that they amount to simply doing the obvious thing; however I am unaware of any references in the literature. Comparison with [CKVW98] is instructive, however.

Definition 10. Given $\mathbb{D}$ a left $\mathbb{C}$-module and $\mathbb{D}'$ a left $\mathbb{C}'$-module, we define a **left-modular functor** $(L, K)\colon (\mathbb{C}, \mathbb{D}) \to (\mathbb{C}', \mathbb{D}')$ to be a pair of functors $(L\colon \mathbb{C} \to \mathbb{C}', K\colon \mathbb{D} \to \mathbb{D}')$ together with structure $(m_I, m_{B,C})$ making L a monoidal functor, and structure

$$p_{C,D}\colon LC \boxtimes_l KD \to K(C \boxtimes_l D),$$

natural in C and D, making the following diagrams commute:

$$
\begin{array}{ccc}
KD & \xrightarrow{\lambda_D} & I \boxtimes_l KD \\
{\scriptstyle K\lambda_D}\downarrow & & \downarrow{\scriptstyle m_I \boxtimes_l KD} \\
K(I \boxtimes_l D) & \xleftarrow[p_{I,D}]{} & LI \boxtimes_l KD
\end{array}
\qquad
\begin{array}{ccc}
LB \boxtimes_l (LC \boxtimes_l KD) & \xrightarrow{\alpha_{LB,LC,KD}} & (LB \otimes LC) \boxtimes_l KD \\
{\scriptstyle LB \boxtimes_l p_{C,D}}\downarrow & & \downarrow{\scriptstyle m_{B,C} \boxtimes_l KD} \\
LB \boxtimes_l K(C \boxtimes_l D) & & L(B \otimes C) \boxtimes_l KD \\
{\scriptstyle p_{B,(C \boxtimes_l D)}}\downarrow & & \downarrow{\scriptstyle p_{(B\otimes C),D}} \\
K(B \boxtimes_l (C \boxtimes_l D)) & \xrightarrow[K\alpha_{B,C,D}]{} & K((B \otimes C) \boxtimes_l D).
\end{array}
$$

Given $\mathbb{D}$ a right $\mathbb{E}$-module and $\mathbb{D}'$ a right $\mathbb{E}'$-module, we define similarly a **right-modular functor** $(K, M)\colon (\mathbb{D}, \mathbb{E}) \to (\mathbb{D}', \mathbb{E}')$, with structure $(q_{D,E}, r_{E,F}, r_I)$.

Definition 11. If $\mathbb{D}$ is a left $\mathbb{C}$-, right $\mathbb{E}$-module and $\mathbb{D}'$ a left $\mathbb{C}'$-, right $\mathbb{E}'$-module, then a **two-sided modular functor** $(\mathbb{C}, \mathbb{D}, \mathbb{E}) \to (\mathbb{C}', \mathbb{D}', \mathbb{E}')$ is a triple (L, K, M) equipped with structure $(m_I, m_{B,C}, p_{C,D})$ making (L, K) left-modular, and structure $(q_{D,E}, r_{E,F}, r_I)$ making (K, M) right-modular, subject to diagrams of the following form commuting:

$$
\begin{array}{ccc}
LC \boxtimes_l (KD \boxtimes_r ME) & \xrightarrow{\tau_{LC,KD,ME}} & (LC \boxtimes_l KD) \boxtimes_r ME \\
{\scriptstyle LB \boxtimes_l q_{D,E}}\downarrow & & \downarrow{\scriptstyle p_{C,D} \boxtimes_r ME} \\
LC \boxtimes_l K(D \boxtimes_r E) & & K(C \boxtimes_l D) \boxtimes_r ME \\
{\scriptstyle p_{C,(D \boxtimes_r E)}}\downarrow & & \downarrow{\scriptstyle q_{(C \boxtimes_l D),E}} \\
K(C \boxtimes_l (D \boxtimes_r E)) & \xrightarrow[K\tau_{C,D,E}]{} & K((C \boxtimes_l D) \boxtimes_r E)
\end{array}
.
$$

In the special case where $\mathbb{C} = \mathbb{E}$ and $\mathbb{C}' = \mathbb{E}'$, a **bimodular functor** $(\mathbb{C}, \mathbb{D}) \to (\mathbb{C}', \mathbb{D})$ is a pair (L, K) equipped with structure $(m_I, m_{B,C}, p_{C,D}, q_{D,C})$ such that (L, K, L) is a two-sided modular functor when equipped with structure $(m_I, m_{B,C}, p_{C,D}, q_{D,C}, m_{C,B}, m_I)$.

Viewing our modules as bicategories, these maps correspond to *morphisms of bicategories*. As usual, we shall call a modular functor of any sort **strong** when its structure maps are isomorphisms, and **strict** when they are identities. On the bicategory side, such maps are homomorphisms (respectively, strict homorphisms) of bicategories.

Definition 12. Given left-modular functors $(L, K), (L', K')\colon (\mathbb{C}, \mathbb{D}) \to (\mathbb{C}', \mathbb{D}')$, a **left-modular transformation** $(L, K) \Rightarrow (L', K')$ is a pair of natural transformations $(\alpha\colon L \Rightarrow L', \beta\colon K \Rightarrow K')$ such that α is a monoidal natural transformation, and such that every diagram of the form

$$
\begin{array}{ccc}
LC \boxtimes_l KD & \xrightarrow{p_{LC,KD}} & K(C \boxtimes_l D) \\
{\scriptstyle \alpha_C \boxtimes_l \beta_D}\downarrow & & \downarrow{\scriptstyle \beta_{C \boxtimes_l D}} \\
L'C \boxtimes_l K'D & \xrightarrow[p'_{C,D}]{} & K'(C \boxtimes_l D)
\end{array}
$$

commutes. We define similarly right-modular transformations between right-modular functors.

Definition 13. A **two-sided modular transformation** between two-sided modular functors (L, K, M) and (L', K', M') is a triple (α, β, γ) such that (α, β) is a left-modular transformation and (β, γ) is a right-modular transformation. A **bimodular transformation** between bimodular functors (L, K) and (L', K') is a pair (α, β) such that (α, β, α) is a two-sided modular transformation.

Note that, viewing our modular functors as morphisms of bicategories, the bicategorical notion of transformation is *not* the same as that of modular transformation as given above.

Proposition 21. *We have 2-categories* {**LAct**, **RAct**, **LRAct**, **BAct**} *with:*

- *0-cells {left, right, two-sided, bi}-modules;*
- *1-cells {left, right, two-sided, bi}-modular functors;*
- *2-cells {left, right, two-sided, bi}-modular transformations.*

Proof. We consider the case of **LAct**; the others follow similarly. We have obvious notions of composition inherited from **Cat** × **Cat**, so it suffices to check that we can compose the structural maps for modular functors and that composition of 2-cells respects this. Indeed, given maps

$$(\mathbb{C}_0, \mathbb{D}_0) \xrightarrow{(L_0, K_0)} (\mathbb{C}_1, \mathbb{D}_1) \xrightarrow{(L_1, K_1)} (\mathbb{C}_2, \mathbb{D}_2)$$

with respective structure $(m_I, m_{B,C}, p_{C,D})$ and $(m'_I, m'_{B,C}, p'_{C,D})$, we equip $(L_1 L_0, K_1 K_0)$ with structure

$$m''_I \colon I \xrightarrow{m'_I} L_1 I \xrightarrow{L_1 m_I} L_1 L_0 I$$

$$m''_{B,C} \colon L_1 L_0 B \otimes L_1 L_0 C \xrightarrow{m'_{L_0 B, L_0 C}} L_1(L_0 B \otimes L_0 C) \xrightarrow{L_1 m_{B,C}} L_1 L_0 (B \otimes C)$$

$$p''_{C,D} \colon L_1 L_0 C \boxtimes_l K_1 K_0 D \xrightarrow{p'_{L_0 C, K_0 D}} K_1(L_0 C \boxtimes_l K_0 D) \xrightarrow{K_1 p_{C,D}} K_1 K_0 (C \boxtimes_l D),$$

and it is routine to verify that the required diagrams commute. □

Note that we have a 2-functor **BAct** → **LRAct** which is injective on 0-, 1- and 2-cells, and forgetful 2-functors {**LRAct**, **BAct**} → {**LAct**, **RAct**}.

There is another perspective on these 2-categories. We can see the 2-category **MonCat** of monoidal categories, monoidal functors and monoidal natural transformations as the 2-category of pseudo-algebras, lax algebra maps and algebra transformations for the free strict monoidal category 2-monad T on **Cat**; similarly, there is a 'free strict monoidal module monad' M on **Cat** × **Cat** sending $(\mathbb{C}, \mathbb{D})$ to $(T\mathbb{C}, T\mathbb{C} \times \mathbb{D})$, and we can view **LAct** as the 2-category of pseudo-algebras, lax algebra maps and algebra transformations for this 2-monad; and similarly for the other varieties of module.

Examples 3.

- Monoidal functors $L\colon \mathbb{C} \to \mathbb{D}$ between monoidal categories lift to two-sided modular functors $(L, L, L)\colon (\mathbb{C}, \mathbb{C}, \mathbb{C}) \to (\mathbb{D}, \mathbb{D}, \mathbb{D})$, with structure maps derived in the obvious way from those for L. Likewise, monoidal natural transformations lift to two-sided modular transformations; thus we have a 2-functor $F\colon$ **MonCat** $\to$ **LRAct**.

- Given left $\mathbb{C}$-modules $\mathbb{D}$ and $\mathbb{D}'$, we can contemplate left-modular functors between them of the form $(\mathrm{id}_{\mathbb{C}}, K)$. In our modular terminology, we might refer to these as '$\mathbb{C}$-linear' functors; more traditionally, they are functors $K\colon \mathbb{D} \to \mathbb{D}'$ equipped with a *strength* in the sense of [Koc70]. If the module structures of $\mathbb{D}$ and $\mathbb{D}'$ are derived from an enrichment in $\mathbb{C}$ as in Example 2, then these correspond precisely to (the underlying ordinary functors of) $\mathbb{C}$-enriched functors.

- Recall that a strong monoidal functor $F\colon \mathbb{C} \to \mathbb{D}$ gives an two-sided action of $\mathbb{C}$ on $\mathbb{D}$; then we have a factorisation:

$$(\mathbb{C}, \mathbb{C}, \mathbb{C}) \xrightarrow{(\mathrm{id},F,\mathrm{id})} (\mathbb{C}, \mathbb{D}, \mathbb{C}) \xrightarrow{(F,\mathrm{id},F)} (\mathbb{D}, \mathbb{D}, \mathbb{D})$$

  of two-sided modular functors.

3.4 Coherence for bimodules

We wish to prove a coherence result mirroring the classical coherence result for monoidal categories. Explicitly, we wish to show that every bimodule $(\mathbb{C}, \mathbb{D}) \in$ **BAct** has a bimodular equivalence to a strict bimodule (i.e., one where all structure maps are identities). We shall need the following two basic lemmas which allow us to lift structure from **Cat**$\times$**Cat** to **BAct**.

Proposition 22. *Suppose* $(F, \mathbf{F})\colon (\mathbb{C}, \mathbb{D}) \to (\mathbb{C}', \mathbb{D}')$ *is a strong bimodular functor. Then we can lift an adjunction* $(F, \mathbf{F}) \dashv (G, \mathbf{G})$ *in* **Cat** $\times$ **Cat** *to an adjunction* $(F, \mathbf{F}) \dashv (G, \mathbf{G})$ *in* **BAct**.

Proof. From the remarks from the end of the previous section, **BAct** can be viewed as the 2-category of algebras for a 2-monad on **Cat**$\times$**Cat**; so this is an immediate consequence of [Kel74] ('doctrinal adjunction'). However, it may be informative to give a direct proof. Suppose our adjunction has unit $(\eta, \mathbf{y})$ and counit $(\varepsilon, \mathbf{e})$, and that $(F, \mathbf{F})$ has structure isomorphisms $(m_I, m_{B,C}, p_{C,D}, q_{D,C})$, say. Then we equip $(G, \mathbf{G})$ with structure

$$I \xrightarrow{\eta_I} GFI \xrightarrow{Gm_I^{-1}} GI$$

$$GB \otimes GC \xrightarrow{\eta_{(GB\otimes GC)}} GF(GB \otimes GC) \xrightarrow{Gm_{B,C}^{-1}} G(FGB \otimes FGC) \xrightarrow{G(\varepsilon_B\otimes\varepsilon_C)} G(B \otimes C)$$

$$GC \boxtimes_l \mathbf{G}D \xrightarrow{\mathbf{y}_{(GC\boxtimes_l \mathbf{G}D)}} \mathbf{GF}(GC \boxtimes_l \mathbf{G}D) \xrightarrow{\mathbf{G}p_{C,D}^{-1}} \mathbf{G}(FGC \boxtimes_l \mathbf{FG}D) \xrightarrow{\mathbf{G}(\varepsilon_C\boxtimes_l \mathbf{e}_D)} \mathbf{G}(C \boxtimes_l D)$$

$$\mathbf{G}D \boxtimes_r GC \xrightarrow{\mathbf{y}_{(\mathbf{G}D\boxtimes_r GC)}} \mathbf{GF}(\mathbf{G}D \boxtimes_r GC) \xrightarrow{\mathbf{G}q_{D,C}^{-1}} \mathbf{G}(\mathbf{FG}D \boxtimes_r FGC) \xrightarrow{\mathbf{G}(\mathbf{e}_D\boxtimes_r \varepsilon_C)} \mathbf{G}(D \boxtimes_r C),$$

and routine checking shows that all required diagrams commute. □

Proposition 23. *Suppose that* $(F_1, F_2)\colon (\mathbb{C}, \mathbb{D}) \to (\mathbb{C}', \mathbb{D}')$ *is a strong bimodular functor. Then the factorisation of* (F_1, F_2) *as*

$$(\mathbb{C}, \mathbb{D}) \xrightarrow{e.s.o.} (\mathbb{C}'', \mathbb{D}'') \xrightarrow{\textit{full replete subcategory}} (\mathbb{C}', \mathbb{D}')$$

lifts from **Cat** × **Cat** *to* **BAct**.

Proof. $(\mathbb{C}'', \mathbb{D}'')$ is the full subcategory of $(\mathbb{C}', \mathbb{D}')$ whose objects are those isomorphic to objects in the image of (F_1, F_2); so it suffices to show that $(\mathbb{C}'', \mathbb{D}'')$ is a sub-bimodule of $(\mathbb{C}', \mathbb{D}')$. For this, since $(\mathbb{C}'', \mathbb{D}'')$ is full in $(\mathbb{C}', \mathbb{D}')$, it suffices to show that the objects of $(\mathbb{C}'', \mathbb{D}'')$ are closed under the module operations of $(\mathbb{C}', \mathbb{D}')$.

So let $X, Y \in \mathbb{C}''$ with $X \cong F_1 B$ and $Y \cong F_1 C$, say, and let $Z \in \mathbb{D}''$ with $Z \cong F_2 D$. Now we have $I \cong FI$, so $I \in \mathbb{C}''$, and

$$X \otimes I \cong X \cong F_1 B; \qquad I \otimes Y \cong Y \cong F_1 C;$$
$$Z \boxtimes_r I \cong Z \cong F_2 D; \qquad I \boxtimes_l Z \cong Z \cong F_2 D$$

$$X \otimes Y \cong F_1 B \otimes F_1 C \cong F_1(B \otimes C)$$
$$Y \boxtimes_l Z \cong F_1 C \boxtimes_l F_2 D \cong F_2(C \boxtimes_l D)$$
$$Z \boxtimes_r Y \cong F_2 D \boxtimes_r F_1 C \cong F_2(D \boxtimes_r C)$$

as required. □

Our proof is analogous to that of [JS93]. This proof does not, however, merely reduce to the coherence theorem for bicategories; indeed, if $\mathbb{D}$ is a $\mathbb{C}$-bimodule, then the coherence theorem for bicategories would yield a strict left $\mathbb{C}'$-, right $\mathbb{C}''$-bimodule, for strict monoidal categories $\mathbb{C}'$ and $\mathbb{C}''$; but there would be no reason to suspect that $\mathbb{C}' = \mathbb{C}''$. Hence our proof is only *analogous* and not *the same*. So suppose that $\mathbb{D}$ is a $\mathbb{C}$-bimodule. Let $\mathbb{C}'$ be the following category:

- Objects are pairs of functors $(L\colon \mathbb{C} \to \mathbb{C}, K\colon \mathbb{D} \to \mathbb{D})$ equipped with natural isomorphisms

$$l_{B,C}\colon B \otimes LC \to L(B \otimes C)$$
$$k_{D,C}\colon D \boxtimes_r LC \to K(D \boxtimes_r C);$$

- Maps are pairs of natural transformations $(\alpha\colon L \Rightarrow L', \beta\colon K \Rightarrow K')$ making the following diagrams commute:

$$\begin{array}{ccc} B \otimes LC & \xrightarrow{l_{B,C}} & L(B \otimes C) \\ {\scriptstyle B\otimes\alpha_C}\downarrow & & \downarrow{\scriptstyle \alpha_{B\otimes C}} \\ B \otimes L'C & \xrightarrow[l'_{B,C}]{} & L'(B \otimes C) \end{array} \qquad \begin{array}{ccc} D \boxtimes_r LC & \xrightarrow{k_{D,C}} & K(D \boxtimes_r C) \\ {\scriptstyle D\boxtimes_r\alpha_C}\downarrow & & \downarrow{\scriptstyle \beta_{D\boxtimes_r C}} \\ D \boxtimes_r L'C & \xrightarrow[k'_{D,C}]{} & K'(D \boxtimes_r C). \end{array}$$

Proposition 24. *$\mathbb{C}'$ is a strict monoidal category.*

Proof. We set $I = (\mathrm{id}_{\mathbb{C}}, \mathrm{id}_{\mathbb{D}})$ and set $(L, K) \otimes (H, J) = (HL, JK)$, equipped with the natural isomorphisms

$$B \otimes HLC \xrightarrow{h_{B,LC}} H(B \otimes LC) \xrightarrow{Hl_{B,C}} HL(B \otimes C)$$
$$D \boxtimes_r HLC \xrightarrow{j_{D,LC}} J(D \boxtimes_r LC) \xrightarrow{Jk_{D,C}} JK(D \boxtimes_r C).$$

On maps, given $(\alpha, \beta)\colon (L, K) \Rightarrow (L', K')$ and $(\gamma, \delta)\colon (H, J) \Rightarrow (H', J')$ we take $(\alpha, \beta) \otimes (\gamma, \delta) = (\gamma \circ \alpha, \delta \circ \beta)$ (where $\circ$ is the usual horizontal composite of natural transformations). Routine verification shows that the required coherence diagrams commute. □

Proposition 25. *Each (L, K) in $\mathbb{C}'$, can be equipped with derived natural isomorphisms $\widehat{k}_{C,D}\colon C \boxtimes_l KD \to K(C \boxtimes_l D)$ in such a way that for any map $(\alpha, \beta)\colon (L, K) \to (L', K')$, diagrams of the following form commute:*

$$\begin{array}{ccc} C \boxtimes_l KD & \xrightarrow{\widehat{k}_{C,D}} & K(C \boxtimes_l D) \\ {\scriptstyle C\boxtimes_l \beta_D}\downarrow & & \downarrow{\scriptstyle \beta_{C\boxtimes_l D}} \\ C \boxtimes_l K'D & \xrightarrow[\widehat{k'}_{C,D}]{} & K'(C \boxtimes_l D). \end{array}$$

Proof. We define $\widehat{k}_{C,D}$ by

$$\begin{array}{ccccc} C \boxtimes_l KD & \xrightarrow{C\boxtimes_l K(\rho_D^{-1})} & C \boxtimes_l K(D \boxtimes_r I) & \xrightarrow{C\boxtimes_l k_{D,I}^{-1}} & C \boxtimes_l (D \boxtimes_r LI) \\ & & & & \downarrow{\scriptstyle \tau_{C,D,LI}} \\ K(C \boxtimes_l D) & \xleftarrow[K\rho_{C\boxtimes_l D}]{} & K((C \boxtimes_l D) \boxtimes_r I) & \xleftarrow[k_{C\boxtimes_l D,I}]{} & (C \boxtimes_l D) \boxtimes_r LI. \end{array}$$

whence the required diagrams can be shown to commute. □

So now let $\mathbb{D}'$ be the following category:

- Objects are functors $M\colon \mathbb{C} \to \mathbb{D}$ equipped with natural isomorphisms

$$m_{B,C}\colon B \boxtimes_l MC \to M(B \otimes C);$$

- Maps are natural transformations $\gamma\colon M \Rightarrow M'$ making the following diagrams commute:

$$\begin{array}{ccc} B \boxtimes_l MC & \xrightarrow{m_{B,C}} & L(B \otimes C) \\ {\scriptstyle B\boxtimes_l \gamma_C}\downarrow & & \downarrow{\scriptstyle \gamma_{B\otimes C}} \\ B \boxtimes_l M'C & \xrightarrow[m'_{B,C}]{} & M'(B \otimes C). \end{array}$$

Proposition 26. *$\mathbb{C}'$ has a strict two-sided action on $\mathbb{D}'$.*

Proof. We define on objects

$$(L, K) \boxtimes_l M = ML; \qquad M \boxtimes_r (L, K) = KM.$$

equipped with structure maps

$$B \boxtimes_l MLC \xrightarrow{m_{B,LC}} M(B \otimes LC) \xrightarrow{Ml_{B,C}} ML(B \otimes C)$$
$$B \boxtimes_l KMC \xrightarrow{\widehat{k}_{B,MC}} K(B \boxtimes_l MC) \xrightarrow{Km_{B,C}} KM(B \otimes C).$$

On morphisms, given $(\alpha, \beta): (L, K) \Rightarrow (L', K')$ and $\gamma: M \to M'$, we define $(\alpha, \beta) \boxtimes_l \gamma = \gamma \circ \alpha$ and $\gamma \boxtimes_r (\alpha, \beta) = \beta \circ \gamma$. Again, the required diagrams are easily verified to commute. □

Proposition 27. *There is a full and faithful strong bimodular functor* $(\mathbb{C}, \mathbb{D}) \to (\mathbb{C}', \mathbb{D}')$.

Proof. We have the functor $\otimes: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$; from this we get a functor $L_-: \mathbb{C} \to [\mathbb{C}, \mathbb{C}]$ sending X to $- \otimes X$. Similarly, we have $K_-: \mathbb{C} \to [\mathbb{D}, \mathbb{D}]$ sending X to $- \boxtimes_r X$, and $M_-: D \to [\mathbb{C}, \mathbb{D}]$ sending Y to $- \boxtimes_l Y$. We claim that (L_-, K_-) can be lifted to a functor $F_1: \mathbb{C} \to \mathbb{C}'$. Indeed, if we equip (L_X, K_X) with structure maps

$$l_{B,C}: B \otimes L_X(C) \to L_X(B \otimes C) \quad = \quad \mathbf{a}_{B,C,X}: B \otimes (C \otimes X) \to (B \otimes C) \otimes X$$
$$k_{D,C}: D \boxtimes_r L_X(C) \to K_X(D \boxtimes_r C) \quad = \quad \beta_{D,C,X}: D \boxtimes_r (C \otimes X) \to (D \boxtimes_r C) \boxtimes_r X,$$

then the coherence diagrams for (L_f, K_f) commute as required. Likewise M_- lifts to a functor $F_2: \mathbb{D} \to \mathbb{D}'$, by equipping M_Y with structure maps

$$m_{B,C}: B \boxtimes_l M_Y(C) \to M_Y(B \otimes C) \quad = \quad \alpha_{B,C,X}: B \boxtimes_l (C \boxtimes_l Y) \to (B \otimes C) \boxtimes_l Y.$$

So we have functors $F_1: \mathbb{C} \to \mathbb{C}'$ and $F_2: \mathbb{D} \to \mathbb{D}'$; now we require structure maps

$$m_I: I \Rightarrow (L_I, K_I)$$
$$m_{B,C}: (L_B, K_B) \otimes (L_C, K_C) \Rightarrow (L_{(B \otimes C)}, K_{(B \otimes C)})$$
$$p_{C,D}: (L_C, K_C) \boxtimes_l M_D \Rightarrow M_{C \boxtimes_l D} \text{and}$$
$$q_{D,C}: M_D \boxtimes_r (L_C, K_C) \Rightarrow M_{D \boxtimes_r C},$$

making (F_1, F_2) into a strong bimodular functor. We set

$$m_I = \mathbf{r} \times \rho: (\mathrm{id}_{\mathbb{C}}, \mathrm{id}_{\mathbb{D}}) \Rightarrow (- \otimes I, - \boxtimes_r I)$$
$$m_{B,C} = \mathbf{a}^{-1}_{-,B,C} \times \beta^{-1}_{-,B,C}: ((- \otimes B) \otimes C, (- \boxtimes_r B) \boxtimes_r C) \Rightarrow (- \otimes (B \otimes C), - \boxtimes_r (B \otimes C))$$
$$p_{C,D} = \alpha^{-1}_{-,C,D}: (- \otimes C) \boxtimes_l D \Rightarrow - \boxtimes_l (C \boxtimes_l D)$$
$$q_{D,C} = \tau^{-1}_{-,D,C}: (- \boxtimes_l D) \boxtimes_r C \Rightarrow - \boxtimes_l (D \boxtimes_r C),$$

and the coherence conditions that these structure maps must satisfy are precisely the coherence diagrams for the action of $\mathbb{C}$ on $\mathbb{D}$.

Now we show that (L_-, K_-) is fully faithful. We send $(\alpha, \beta) \in \mathbb{C}\big((L_B, K_B), (L_C, K_C)\big)$ to the map $Y(\alpha, \beta)$ in $\mathbb{C}(B, C)$ given by

$$Y(\alpha,\beta) = B \xrightarrow{\mathsf{l}_B} I \otimes B \xrightarrow{\alpha_I} I \otimes C \xrightarrow{\mathsf{l}_C^{-1}} C.$$

Given $f \in \mathbb{C}(B, C)$, clearly $Y(L_f, K_f) = f$; however, we need to check that $(L_{Y(\alpha,\beta)}, K_{Y(\alpha,\beta)}) = (\alpha, \beta)$. Write $(\widehat{\alpha}, \widehat{\beta})$ for $(L_{Y(\alpha,\beta)}, K_{Y(\alpha,\beta)})$; then we have

$$\begin{array}{ccccccc}
X \otimes B & \xrightarrow{X\otimes \mathsf{l}_B} & X \otimes (I \otimes B) & \xrightarrow{l_{X,I}} & (X \otimes I) \otimes B & \xrightarrow{\mathsf{r}_X^{-1}\otimes B} & X \otimes B \\
\downarrow{\scriptstyle \widehat{\alpha}_X} & & \downarrow{\scriptstyle X\otimes\alpha_I} & & \downarrow{\scriptstyle \alpha_{X\otimes I}} & & \downarrow{\scriptstyle \alpha_X} \\
X \otimes C & \xrightarrow[X\otimes \mathsf{l}_C]{} & X \otimes (I \otimes C) & \xrightarrow[l_{X,I}]{} & (X \otimes I) \otimes C & \xrightarrow[\mathsf{r}_X^{-1}\otimes C]{} & X \otimes C
\end{array}$$

$$\begin{array}{ccccccc}
Y \boxtimes_r B & \xrightarrow{Y\boxtimes_r \mathsf{l}_B} & Y \boxtimes_r (I \boxtimes_r B) & \xrightarrow{k_{Y,I}} & (Y \boxtimes_r I) \boxtimes_r B & \xrightarrow{\mathsf{r}_Y^{-1}\boxtimes_r B} & Y \boxtimes_r B \\
\downarrow{\scriptstyle \widehat{\beta}_Y} & & \downarrow{\scriptstyle Y\boxtimes_r\alpha_I} & & \downarrow{\scriptstyle \beta_{Y\boxtimes_r I}} & & \downarrow{\scriptstyle \beta_Y} \\
Y \boxtimes_r C & \xrightarrow[Y\boxtimes_r \mathsf{l}_C]{} & Y \boxtimes_r (I \boxtimes_r C) & \xrightarrow[k_{Y,I}]{} & (Y \boxtimes_r I) \boxtimes_r C & \xrightarrow[\mathsf{r}_Y^{-1}\boxtimes_r C]{} & Y \boxtimes_r C
\end{array}$$

commuting; but the composites along the top and bottom of each diagram are the identity, and hence we have $(\alpha, \beta) = (\widehat{\alpha}, \widehat{\beta})$ as required; so (L_-, K_-) is full and faithful. Similar considerations show that M_- is also full and faithful. □

Proposition 28. *Every bimodule* $(\mathbb{C}, \mathbb{D})$ *has a bimodular equivalence with a strict bimodule* $(\mathbb{C}'', \mathbb{D}'')$.

Proof. By Proposition 23, we can take the full replete image of $(F_1, F_2)\colon (\mathbb{C}, \mathbb{D}) \to (\mathbb{C}', \mathbb{D}')$ as above; say it is $(\mathbb{C}'', \mathbb{D}'')$. Then the functor $(\mathbb{C}, \mathbb{D}) \to (\mathbb{C}'', \mathbb{D}'')$ is full and faithful and e.s.o., hence an (adjoint) equivalence in **Cat** × **Cat**. Since it is also strong bimodular, Proposition 22 yields an (adjoint) equivalence in **BAct** as desired. □

Thus we have shown that every bimodule $(\mathbb{C}, \mathbb{D})$ in **BAct** has a bimodular equivalence to a strict bimodule; hence also every canonical diagram of associativities we can draw in $(\mathbb{C}, \mathbb{D})$ will commute (since it does in $(\mathbb{C}', \mathbb{D}')$). Henceforth, therefore we will feel free to assume that we are working with *strict* actions where this simplifies matters.

Although we have only proved a coherence result for bimodules, we could have proceeded in a perfectly analogous manner for two-sided modules, by contemplating an analogous subcategory

$$(\mathbb{C}', \mathbb{D}', \mathbb{E}') \subset ([\mathbb{C}, \mathbb{C}], [\mathbb{C}, \mathbb{D}], [\mathbb{D}, \mathbb{D}] \times [\mathbb{E}, \mathbb{E}])$$

and embedding $(\mathbb{C}, \mathbb{D}, \mathbb{E}) \to (\mathbb{C}', \mathbb{D}', \mathbb{E}')$ via

$$(C, D, E) \mapsto (- \otimes C, - \boxtimes_l D, (- \boxtimes_r E, - \otimes E)).$$

3.5 Symmetric actions

Definition 14. If $\mathbb{C}$ is a symmetric monoidal category (with symmetry $\mathbf{s}$, say), then a **symmetric action** of $\mathbb{C}$ on $\mathbb{D}$ is a two-sided action of $\mathbb{C}$ on $\mathbb{D}$ together with natural isomorphisms

$$\sigma_{C,D} \colon C \boxtimes_l D \to D \boxtimes_r C$$

such that the following diagrams commute:

$$\begin{array}{ccc}
B \boxtimes_l (C \boxtimes_l D) & \xrightarrow{B \boxtimes_l \sigma_{C,D}} & B \boxtimes_l (D \boxtimes_r C) \\
{\scriptstyle \alpha_{B,C,D}} \downarrow & & \downarrow {\scriptstyle \tau_{B,D,C}} \\
(B \otimes C) \boxtimes_l D & & (B \boxtimes_l D) \boxtimes_r C \\
{\scriptstyle \sigma_{(B \otimes C),D}} \downarrow & & \downarrow {\scriptstyle \sigma_{B,D} \boxtimes_r C} \\
D \boxtimes_r (B \otimes C) & \xrightarrow[\beta_{D,B,C}]{} & (D \boxtimes_r B) \boxtimes_r C
\end{array} ,
\qquad
\begin{array}{ccc}
B \boxtimes_l (C \boxtimes_l D) & \xrightarrow{B \boxtimes_l \sigma_{C,D}} & B \boxtimes_l (D \boxtimes_r C) \\
{\scriptstyle \alpha_{B,C,D}} \downarrow & & \downarrow {\scriptstyle \tau_{B,D,C}} \\
(B \otimes C) \boxtimes_l D & & (B \boxtimes_l D) \boxtimes_r C \\
{\scriptstyle \mathbf{s}_{B,C} \boxtimes_l D} \downarrow & & \downarrow {\scriptstyle \sigma^{-1}_{C,(B \boxtimes_l D)}} \\
(C \otimes B) \boxtimes_l D & \xrightarrow[\alpha^{-1}_{C,B,D}]{} & C \boxtimes_l (B \boxtimes_l D)
\end{array} ,$$

$$\begin{array}{ccc}
(D \boxtimes_r C) \boxtimes_r B & \xrightarrow{\sigma^{-1}_{B,(D \boxtimes_r C)}} & B \boxtimes_l (D \boxtimes_r C) \\
{\scriptstyle \beta^{-1}_{D,C,B}} \downarrow & & \downarrow {\scriptstyle \tau_{B,D,C}} \\
D \boxtimes_r (C \otimes B) & & (B \boxtimes_l D) \boxtimes_r C \\
{\scriptstyle D \boxtimes_r \mathbf{s}_{C,B}} \downarrow & & \downarrow {\scriptstyle \sigma_{B,D} \boxtimes_r C} \\
D \boxtimes_r (B \otimes C) & \xrightarrow[\beta_{D,B,C}]{} & (D \boxtimes_r B) \boxtimes_r C.
\end{array}$$

Such a $\mathbb{D}$ may also be called a **symmetric** $\mathbb{C}$**-module**.

In terms of coherence, we have:

Proposition 29. *If $\mathbb{D}$ is a symmetric $\mathbb{C}$-module, then the category $(\widehat{\mathbb{C}}, \widehat{\mathbb{D}})$ described in the previous section can be equipped with a symmetry such that the equivalence $(\mathbb{C}, \mathbb{D}) \simeq (\widehat{\mathbb{C}}, \widehat{\mathbb{D}})$ is a symmetric monoidal equivalence.*

Proof. We can transport the structure along the equivalence

$$(\mathbb{C}, \mathbb{D}) \underset{(F,\mathbf{F})}{\overset{(G,\mathbf{G})}{\underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle\top}}}} (\widehat{\mathbb{C}}, \widehat{\mathbb{D}})$$

so the symmetry for $\widehat{\mathbb{C}}$ is given by

$$\mathbf{s}_{C,C'} = C \otimes C' \to GFC \otimes GFC' \to G(FC \otimes FC') \to G(FC' \otimes FC) \to GF(C' \otimes C) \to C' \otimes C$$

and that for $\widehat{\mathbb{D}}$ by

$$\sigma_{C,D} = C \boxtimes_l D \to GFC \boxtimes_l \mathbf{GF}D \to \mathbf{G}(FC \boxtimes_l \mathbf{F}D) \to \mathbf{G}(\mathbf{F}D \boxtimes_r FC) \to \mathbf{GF}(D \boxtimes_r C) \to D \boxtimes_r C.$$

It's straightforward to check that this definition produces a coherent symmetry on $\mathbb{D}$ and makes the above equivalence a symmetric equivalence. ☐

Since not every symmetric monoidal category is equivalent to a strictly symmetric monoidal category, we have no chance of an analogous result here. However, we do have a slightly weaker 'strictification' we can perform.

Proposition 30. *If $\mathbb{C}$ is a symmetric monoidal category, then the left actions of $\mathbb{C}$ on $\mathbb{D}$ are in bijection with the right actions of $\mathbb{C}$ on $\mathbb{D}$. Further, every such bijective pair gives rise canonically to a symmetric action of $\mathbb{C}$ on $\mathbb{D}$.*

Proof. Given a left action $(\boxtimes_l, \lambda, \alpha)$, we define a right action $(\boxtimes_r, \rho, \beta)$ by

$$D \boxtimes_r C = C \boxtimes_l D;$$

$$\begin{array}{ccc} D & \xrightarrow{\rho_D} & D \boxtimes_r I \\ \| & & \| \\ D & \xrightarrow{\lambda_D} & I \boxtimes_l D \end{array} \;;$$

$$\begin{array}{ccccc} (D \boxtimes_r C') \boxtimes_r C & & \xrightarrow{\beta_{D,C',C}} & & D \boxtimes_r (C' \otimes C) \\ \| & & & & \| \\ C \boxtimes_l (C' \boxtimes_l D) & \xrightarrow{\alpha_{C,C',D}} & (C \otimes C') \boxtimes_l D & \xrightarrow{\sigma_{C,C'} \boxtimes_l D} & (C' \otimes C) \boxtimes_l D \end{array} .$$

To make this pair into a symmetric action of $\mathbb{C}$ on $\mathbb{D}$, we define τ by:

$$\begin{array}{ccccccc} C \boxtimes_l (D \boxtimes_r C') & & & \xrightarrow{\tau_{C,D,C'}} & & & (C \boxtimes_l D) \boxtimes_r C' \\ \| & & & & & & \| \\ C \boxtimes_l (C' \boxtimes_l D) & \xrightarrow{\alpha_{C,C',D}} & (C \otimes C') \boxtimes_l D & \xrightarrow{\sigma_{C,C'} \boxtimes_l D} & (C' \otimes C) \boxtimes_l D & \xrightarrow{\alpha^{-1}_{C',C,D}} & C' \boxtimes_l (C \boxtimes_l D) \end{array} ,$$

and σ by $\sigma_{C,D} = \mathrm{id}_{C \boxtimes_l D}$. A diagram chase confirms that these obey the required coherence laws. ☐

Whence we can say that every symmetric action is isomorphic to one in which the left and right actions are the same. To make this statement precise, we first need the right notion of 'is isomorphic to':

Definition 15. If $\mathbb{D}$ is a symmetric $\mathbb{C}$-module, and $\mathbb{D}'$ a symmetric $\mathbb{C}'$-module, then a **symmetric modular functor** $(\mathbb{C}, \mathbb{D}) \to (\mathbb{C}', \mathbb{D}')$ is a bimodular functor (L, K) such that L is a symmetric monoidal functor, and such that

$$\begin{array}{ccc} LC \boxtimes_l KD & \xrightarrow{\sigma_{LC,KD}} & KD \boxtimes_r LC \\ {\scriptstyle p_{C,D}} \downarrow & & \downarrow {\scriptstyle q_{D,C}} \\ K(C \boxtimes_l D) & \xrightarrow[K\sigma_{C,D}]{} & K(D \boxtimes_r C) \end{array}$$

commutes. A symmetric modular transformation is simply a bimodular transformation. We define the 2-category **SAct** with:

- 0-cells symmetric modules;
- 1-cells symmetric modular functors;
- 2-cells symmetric modular transformations.

Proposition 31. *Every symmetric module* $(\mathbb{C}, \mathbb{D}) \in$ **SAct** *is isomorphic to one satisfying* $C \boxtimes_l D = D \boxtimes_r C$.

Proof. We take the module $(\widehat{\mathbb{C}}, \widehat{\mathbb{D}})$ with $\widehat{\mathbb{C}} = \mathbb{C}$ and $\widehat{\mathbb{D}} = \mathbb{D}$, equipped with the actions $\widehat{\boxtimes}_l = \boxtimes_l$ and $\widehat{\boxtimes}_r = \boxtimes_l$ as described in the previous proposition. We equip the identity functor $(\mathrm{id}_{\mathbb{C}}, \mathrm{id}_{\mathbb{D}})\colon (\mathbb{C}, \mathbb{D}) \to (\mathbb{C}, \mathbb{D})$ with the structure of a symmetric modular functor $(\mathbb{C}, \mathbb{D}) \to (\widehat{\mathbb{C}}, \widehat{\mathbb{D}})$ via

$$p_{C,D} = \mathrm{id}_{C\boxtimes_l D}\colon C \boxtimes_l D \to C \boxtimes_l D; \qquad q_{D,C} = \sigma_{C,D}\colon C \boxtimes_l D \to D \boxtimes_r C.$$

It's easy to check that the required diagrams commute, and that we can similarly equip the identity functor with the structure of a symmetric modular functor $(\widehat{\mathbb{C}}, \widehat{\mathbb{D}}) \to (\mathbb{C}, \mathbb{D})$. □

So are justified, in the symmetric case, in essentially ignoring the right action completely, and assuming that it is derived in the canonical manner from the left action. However, we will not simply *define* a symmetric action to be determined by its left action for the following reason: the canonical lifting of a symmetric monoidal category $\mathbb{C}$ into **SAct** is *not* of this form, since it has $C \boxtimes_l C' = C \otimes C' = C \boxtimes_r C'$, rather than $C \boxtimes_l C' = C \otimes C' = C' \boxtimes_r C$.

3.6 Closed modules

We turn now to *closure* for modules, which is comparable to closure for monoidal categories. There is some degree of overlap between the material presented here and [JK02]; compare also the 'starred equipments' of [CKVW98].

Now, since the notation for monoidal categories is not wholly standard in the non-symmetric case, we begin by establishing our conventions.

Definition 16. A monoidal category $\mathbb{C}$ is **left closed** when for each $C \in \mathbb{C}$, the functor $(-)\otimes C$ has a right adjoint which we write as $C \multimap (-)$. It is **right closed** when each functor $C \otimes (-)$ has a right adjoint which we write as $(-) \multimapinv C$.

Note that our notation is that of [Bar95], although we have interchanged the symbols $\multimap$ and $\multimapinv$; his notation is consistent with a preference for right actions, whereas ours is biased in favour of left actions. The units and counits of our adjunctions will thus be:

$$(A \multimap B) \otimes A \to B \qquad A \to B \multimap (A \otimes B)$$
$$A \otimes (B \multimapinv A) \to B \qquad A \to (B \otimes A) \multimapinv B$$

and we obtain maps such as

$$(B \multimap C) \otimes (A \multimap B) \to (A \multimap C) \qquad (A \otimes B) \multimap C \cong A \multimap (B \multimap C)$$
$$(B \multimapinv A) \otimes (C \multimapinv B) \to (C \multimapinv A) \qquad C \multimapinv (B \otimes A) \cong (C \multimapinv B) \multimapinv A$$

Definition 17. Let $\mathbb{D}$ be a left $\mathbb{C}$-module with action $\boxtimes_l$. We say that the module is **enriched** if each functor $(-) \boxtimes_l D \colon \mathbb{C} \to \mathbb{D}$ has a right adjoint, which we write as $D \mathrel{-\!\square}_l (-)$, and **cotensored** if each functor $C \boxtimes_l (-) \colon \mathbb{D} \to \mathbb{D}$ has a right adjoint, which we write as $(-) \mathrel{\square\!-}_l C$.

Similarly, for $\mathbb{D}$ a right $\mathbb{E}$-module, we say it is enriched if each functor $E \boxtimes_r (-) \colon \mathbb{C} \to \mathbb{E}$ has a right adjoint $(-) \mathrel{\square\!-}_r E$, and cotensored if each functor $(-) \boxtimes_r E \colon \mathbb{D} \to \mathbb{D}$ has a right adjoint $E \mathrel{-\!\square}_r (-)$. If $\mathbb{D}$ is a left $\mathbb{C}$-, right $\mathbb{E}$-module, we use the terms **left** and **right** enriched/cotensored to indicate that the component left and right actions are enriched/cotensored; similarly if $\mathbb{D}$ is a $\mathbb{C}$-bimodule.

To summarise, for $\mathbb{D}$ a left $\mathbb{C}$-, right $\mathbb{E}$-module, we have natural isomorphisms:

$\mathbb{C}(A \otimes B, C) \cong \mathbb{C}(A, B \multimap C)$	if $\mathbb{C}$ is left closed;
$\mathbb{D}(C \boxtimes_l D, D') \cong \mathbb{C}(C, D \mathrel{-\!\square}_l D')$	if $\mathbb{D}$ is left enriched;
$\mathbb{D}(D \boxtimes_r E, D') \cong \mathbb{D}(D, E \mathrel{-\!\square}_r D')$	if $\mathbb{D}$ is right cotensored;
$\mathbb{E}(E \otimes F, G) \cong \mathbb{E}(E, F \multimap G)$	if $\mathbb{E}$ is left closed;
$\mathbb{C}(A \otimes B, C) \cong \mathbb{C}(B, C \multimapinv A)$	if $\mathbb{C}$ is right closed;
$\mathbb{D}(C \boxtimes_l D, D') \cong \mathbb{D}(D, D' \mathrel{\square\!-}_l C)$	if $\mathbb{D}$ is left cotensored;
$\mathbb{D}(D \boxtimes_r E, D') \cong \mathbb{E}(E, D' \mathrel{\square\!-}_r D)$	if $\mathbb{D}$ is right enriched; and
$\mathbb{E}(E \otimes F, G) \cong \mathbb{E}(F, G \multimapinv E)$	if $\mathbb{E}$ is right closed.

Viewing our modules as bicategories, asking for these conditions to hold is to ask for the existence of various *right extensions* and *right liftings*.

Definition 18. To say simply that $\mathbb{D}$ is **left closed** is to say that the first four of the above closure properties hold; and to say that it is **right closed** is to say that the latter four hold.

Note in particular that if $\mathbb{D}$ is left enriched, we have maps

$$(D' \mathrel{-\!\square}_l D'') \otimes (D \mathrel{-\!\square}_l D') \to (D \mathrel{-\!\square}_l D'')$$

which say that $\mathbb{D}$ may be viewed as the underlying ordinary category of a category enriched in $\mathbb{C}$. Further, the action $\boxtimes_l$ becomes a tensor for this enrichment, and $\mathrel{\square\!-}_l$ (if it exists) a cotensor; hence the terminology. Similarly, if the right action is enriched, then $\mathbb{D}$ can be viewed as a category enriched in $\mathbb{E}^{\mathrm{rev}}$ (i.e., $\mathbb{E}$ with the tensor product reversed in orientation) with tensors (and possibly cotensors).

Conversely, given a monoidal category $\mathbb{C}$, any category $\mathbb{D}$ enriched in $\mathbb{C}$ and having tensors with $\mathbb{C}$ gives an enriched left action of $\mathbb{C}$ on the underlying ordinary category $\mathbb{D}_0$ of $\mathbb{D}$. However, if $\mathbb{D}$ is enriched with tensors in $\mathbb{C}$ and $\mathbb{E}$, it is not in general true that it becomes a left $\mathbb{C}$-, right $\mathbb{E}^{\mathsf{rev}}$-module; the compatibility of the two actions is by no means guaranteed. In fact, this is a very particular example of a category *enriched from* $\mathbb{C}$ *to* $\mathbb{E}$ [KLSS02].

Proposition 32. *If $\mathbb{D}$ is a symmetric $\mathbb{C}$-module, then it is left enriched (respectively cotensored) if and only if it is right enriched (respectively cotensored).*

Proof. Suppose the left action is enriched. Then we have

$$\mathbb{D}(D \boxtimes_r C, D') \cong \mathbb{D}(C \boxtimes_l D, D') \cong \mathbb{C}(C, D \mathrel{-\!\!\square}_l D')$$

naturally in all variables, so that setting $D' \mathrel{\square\!\!-}_r D = D \mathrel{-\!\!\square}_l D'$ makes the right action enriched. Similarly, if the left action is cotensored, then we have

$$\mathbb{D}(D' \boxtimes_r C, D) \cong \mathbb{D}(C \boxtimes_l D', D) \cong \mathbb{D}(D', D \mathrel{\square\!\!-}_l C)$$

naturally in all variables, so that setting $C \mathrel{-\!\!\square}_r D = D \mathrel{\square\!\!-}_l C$ makes the right action cotensored. □

So we can unambiguously say simply 'enriched' or 'cotensored' when referring to symmetric actions, and are still justified in identifying a symmetric module with its left action. Note also that it thus follows that a symmetric action is left closed if and only if it is right closed, so that we may simply refer to it as 'closed'.

3.7 Cotensored actions

If $\mathbb{C}$ has a cotensored left action on $\mathbb{D}$, then we can derive rather suggestive isomorphisms of the form

$$D \mathrel{\square\!\!-}_l (C' \otimes C) \cong (D \mathrel{\square\!\!-}_l C') \mathrel{\square\!\!-}_l C,$$

from which one might deduce that $\mathrel{\square\!\!-}_l$ is rather like a right action of $\mathbb{C}^{\mathrm{op}}$ on $\mathbb{D}$. In fact, it *is* a right action of $\mathbb{C}^{\mathrm{op}}$ on $\mathbb{D}$; but it turns out to be much more natural to think of it as a right action of $\mathbb{C}$ on $\mathbb{D}^{\mathrm{op}}$, since then any other closed structure the left module $(\mathbb{C}, \mathbb{D})$ might have had is carried over. In detail, we have the following:

Proposition 33. *Given a monoidal category $\mathbb{C}$, cotensored left actions of $\mathbb{C}$ on $\mathbb{D}$ are in bijection with cotensored right actions of $\mathbb{C}$ on $\mathbb{D}^{\mathrm{op}}$; moreover, the property of being an enriched action is preserved under this bijection.*

Proof. Let $\mathbb{D}$ be a cotensored left $\mathbb{C}$-module. We have the following adjunctions $\mathbb{D} \to \mathbb{D}$:

$$A \boxtimes_l (B \boxtimes_l -) \dashv (- \mathrel{\square\!\!-}_l A) \mathrel{\square\!\!-}_l B$$
$$(A \otimes B) \boxtimes_l (-) \dashv (-) \mathrel{\square\!\!-}_l (A \otimes B)$$

Therefore the natural isomorphism $\alpha\colon A \boxtimes_l (B \boxtimes_l -) \to (A \otimes B) \boxtimes_l (-)$ has as an adjoint mate a natural isomorphism $\alpha'\colon (-) \mathbin{\square\!-}_l (A \otimes B) \to (- \mathbin{\square\!-}_l A) \mathbin{\square\!-}_l B$, naturally in A and B. Similarly we have the adjunctions

$$\begin{gathered} \mathrm{id}_{\mathbb{D}} \dashv \mathrm{id}_{\mathbb{D}} \\ I \boxtimes_l (-) \dashv (-) \mathbin{\square\!-}_l I \end{gathered}$$

so the natural isomorphism $\lambda\colon \mathrm{id}_{\mathbb{D}} \to I \boxtimes_l (-)$ has as an adjoint mate a natural isomorphism $\rho\colon (-) \mathbin{\square\!-}_l I \to \mathrm{id}_{\mathbb{D}}$. The coherence diagrams for $\boxtimes_l$ transfer under the adjunction into the following diagrams in $\mathbb{D}$:

$$\begin{array}{ccc} D \mathbin{\square\!-}_l C & \xleftarrow{\rho \mathbin{\square\!-}_l C} & (D \mathbin{\square\!-}_l I) \mathbin{\square\!-}_l C \\ {\scriptstyle D \mathbin{\square\!-}_l \mathbf{l}}\uparrow & \nearrow{\scriptstyle \alpha'} & \\ D \mathbin{\square\!-}_l (I \otimes C) & & \end{array} , \qquad \begin{array}{ccc} D \mathbin{\square\!-}_l C & \xleftarrow{\rho} & (D \mathbin{\square\!-}_l C) \mathbin{\square\!-}_l I \\ {\scriptstyle D \mathbin{\square\!-}_l \mathbf{l}}\uparrow & \nearrow{\scriptstyle \alpha'} & \\ D \mathbin{\square\!-}_l (I \otimes C) & & \end{array} ,$$

$$\begin{array}{ccc} ((D \mathbin{\square\!-}_l C) \mathbin{\square\!-}_l C') \mathbin{\square\!-}_l C'' & \xleftarrow{\alpha' \mathbin{\square\!-}_l C''} & (D \mathbin{\square\!-}_l (C \otimes C')) \mathbin{\square\!-}_l C'' \\ {\scriptstyle \alpha'}\uparrow & & \\ (D \mathbin{\square\!-}_l C) \mathbin{\square\!-}_l (C' \otimes C'') & & \uparrow{\scriptstyle \alpha'} \\ {\scriptstyle \alpha'}\uparrow & & \\ D \mathbin{\square\!-}_l (C \otimes (C' \otimes C'')) & \xleftarrow[D \mathbin{\square\!-}_l \mathbf{a}]{} & D \mathbin{\square\!-}_l ((C \otimes C') \otimes C'') \end{array}$$

which are precisely the required coherence diagrams in $\mathbb{D}^{\mathrm{op}}$ asserting that $(\mathbin{\square\!-}_l^{\mathrm{op}}, \rho^{\mathrm{op}}, \alpha'^{\mathrm{op}})$ is a right action of $\mathbb{C}$ on $\mathbb{D}^{\mathrm{op}}$. So we set $\hat{\boxtimes}_r = \mathbin{\square\!-}_l^{\mathrm{op}}$. Now, to see that this action is cotensored, we set $\widehat{-\!\square}_r = {\boxtimes_l}^{\mathrm{op}}$. For we have

$$\mathbb{D}^{\mathrm{op}}(D \hat{\boxtimes}_r C, D') = \mathbb{D}(D', D \mathbin{\square\!-}_l C) \cong \mathbb{D}(C \boxtimes_l D', D) = \mathbb{D}^{\mathrm{op}}(D, C \mathbin{\widehat{-\!\square}}_r D')$$

as desired. Finally, if the action of $\mathbb{C}$ on $\mathbb{D}$ is enriched, then we define $D \mathbin{\widehat{\square\!-}}_r D' = D \mathbin{-\!\square}_l D'$, and so:

$$\mathbb{D}^{\mathrm{op}}(D \hat{\boxtimes}_r C, D') = \mathbb{D}(D', D \mathbin{\square\!-}_l C) \cong \mathbb{D}(C \boxtimes_l D', D) \cong \mathbb{C}(C, D' \mathbin{-\!\square}_l D) = \mathbb{C}(C, D' \mathbin{\widehat{\square\!-}}_r D)$$

as required. □

Dually, cotensored right actions of $\mathbb{C}$ on $\mathbb{D}$ are in correspondence with cotensored left actions of $\mathbb{C}$ on $\mathbb{D}^{\mathrm{op}}$. Moreover, we have the following two results:

Proposition 34. *Suppose that $\mathbb{D}$ is a left $\mathbb{C}$-, right $\mathbb{E}$-module, cotensored on both sides. Then $\mathbb{D}^{\mathrm{op}}$ is a left $\mathbb{E}$-, right $\mathbb{C}$-module cotensored on both sides.*

Proof. We write $\boxtimes_l$ and $\boxtimes_r$ for the original actions of $\mathbb{C}$ and $\mathbb{E}$ and $\hat{\boxtimes}_l$ and $\hat{\boxtimes}_r$ for those derived from the above proposition; that is, $\hat{\boxtimes}_l = \mathbin{-\!\square}_r^{\mathrm{op}}$, etc. Thus all we require are coherent natural isomorphisms

$$E \mathbin{\hat{\boxtimes}_l} (D \mathbin{\hat{\boxtimes}_r} C) \to (E \mathbin{\hat{\boxtimes}_r} D) \mathbin{\hat{\boxtimes}_l} C$$

in $\mathbb{D}^{\mathrm{op}}$, i.e., natural isomorphisms $(E \mathbin{-\!\square}_r D) \mathbin{\square\!-}_l C \to E \mathbin{-\!\square}_r (D \mathbin{\square\!-}_l C)$ in $\mathbb{D}$. Again, we have adjunctions

$$C \boxtimes_l (- \boxtimes_r E) \dashv E \mathbin{-\!\square}_r (- \mathbin{\square\!-}_l C)$$
$$(C \boxtimes_l -) \boxtimes_r E \dashv (E \mathbin{-\!\square}_r -) \mathbin{\square\!-}_l C$$

so that $\tau_{C,D,E} \colon C \boxtimes_l (D \boxtimes_r E) \to (C \boxtimes_l D) \boxtimes_r E$ has as a mate

$$\hat{\tau}_{E,D,C} \colon (E \mathbin{-\!\square}_r D) \mathbin{\square\!-}_l C \to E \mathbin{-\!\square}_r (D \mathbin{\square\!-}_l C).$$

And proceeding as before, the coherence diagrams for τ in $\mathbb{D}$ transfer under adjunction to coherence diagrams for $\hat{\tau}^{\mathrm{op}}$ in $\mathbb{D}^{\mathrm{op}}$. □

Proposition 35. *Symmetric cotensored actions of* $\mathbb{C}$ *on* $\mathbb{D}$ *are in bijection with symmetric cotensored actions of* $\mathbb{C}$ *on* $\mathbb{D}^{\mathrm{op}}$.

Proof. We know that every symmetric cotensored action of $\mathbb{C}$ on $\mathbb{D}$ gives a two-sided cotensored action of $\mathbb{C}$ on $\mathbb{D}^{\mathrm{op}}$; so it suffices to give coherent natural isomorphisms $C \mathbin{\hat{\boxtimes}_L} D \to D \mathbin{\hat{\boxtimes}_R} C$ in $\mathbb{D}^{\mathrm{op}}$, i.e. natural isomorphisms $D \mathbin{\square\!-}_l C \to C \mathbin{-\!\square}_r D$ in $\mathbb{D}$; and we get these as mates of the isomorphisms $C \boxtimes_l D \to D \boxtimes_r C$ under the adjunctions

$$C \boxtimes_l (-) \dashv (-) \mathbin{\square\!-}_l C$$
$$(-) \boxtimes_r C \dashv C \mathbin{-\!\square}_r (-);$$

and the coherence also transfers across the adjunction. □

The above operations are in fact the object part of certain 2-functors. First, we need to define the sources of these 2-functors:

Definition 19. Given any 2-category $\mathbf{M}$ whose objects are modules, we write $\mathbf{M}_*$ for the sub-2-category with objects the cotensored modules in $\mathbf{M}$, together with all 1- and 2-cells between them.

And second, the targets. The 2-categories we are about to define are a 'different kind of $(-)^{\mathrm{co}}$' peculiar to these particular 2-categories, and encapsulated by the slogan

$$\mathbf{LRAct} \text{ is to } \mathbf{Cat} \times \mathbf{Cat} \times \mathbf{Cat} \quad \text{as} \quad \mathbf{LRAct}^{\circ} \text{ is to } \mathbf{Cat} \times \mathbf{Cat}^{\mathrm{co}} \times \mathbf{Cat}.$$

Definition 20. Given two-sided modular functors (L, K, M) and $(L', K', M')\colon (\mathbb{C}, \mathbb{D}, \mathbb{E}) \to (\mathbb{C}', \mathbb{D}', \mathbb{E}')$, a **two-sided modular cotransformation** $(\alpha, \beta, \gamma)\colon (L, K, M) \Rightarrow (L', K', M')$ is a triple of natural transformations

$$(\alpha\colon L \Rightarrow L', \beta\colon K' \Rightarrow K, \gamma\colon M \Rightarrow M')$$

such that α and γ are monoidal natural transformations, and such that diagrams of the following form commute:

$$\begin{array}{ccc} LC \boxtimes_l KD & \xrightarrow{p_{LC,KD}} & K(C \boxtimes_l D) \\ {\scriptstyle LC\boxtimes_l \beta_D}\uparrow & & \\ LC \boxtimes_l K'D & & \uparrow{\scriptstyle \beta_{C\boxtimes_l D}} \\ {\scriptstyle \alpha_C \boxtimes_l K'D}\downarrow & & \\ L'C \boxtimes_l K'D & \xrightarrow[p'_{C,D}]{} & K'(C \boxtimes_l D) \end{array} \qquad \begin{array}{ccc} KD \boxtimes_r ME & \xrightarrow{q_{KD,MC}} & K(D \boxtimes_r E) \\ {\scriptstyle \beta_D\boxtimes_r ME}\uparrow & & \\ K'D \boxtimes_r ME & & \uparrow{\scriptstyle \beta_{D\boxtimes_r E}} \\ {\scriptstyle K'D \boxtimes_r \gamma_E}\downarrow & & \\ K'D \boxtimes_r M'E & \xrightarrow[q'_{D,E}]{} & K'(D \boxtimes_r E). \end{array}$$

We define analogously left, right, bimodular and symmetric cotransformations, and 2-categories $\{\mathbf{LAct}^\circ, \mathbf{RAct}^\circ, \mathbf{LRAct}^\circ, \mathbf{BAct}^\circ, \mathbf{SAct}^\circ\}$ with:

- 0-cells {left, right, two-sided, bi, symmetric} modules;
- 1-cells {left, right, two-sided, bi, symmetric} modular functors;
- 2-cells {left, right, two-sided, bi, symmetric} modular cotransformations.

Proposition 36. *There are involutary 2-functors*

$$\begin{aligned} \mathrm{op}\colon \{\mathbf{LR}, \mathbf{B}, \mathbf{S}\}\mathbf{Act}_* &\to \{\mathbf{LR}, \mathbf{B}, \mathbf{S}\}\mathbf{Act}_*^\circ \\ \mathrm{op}\colon \mathbf{LAct}_* &\to \mathbf{RAct}_*^\circ \\ \mathrm{op}\colon \mathbf{RAct}_* &\to \mathbf{LAct}_*^\circ \end{aligned}$$

Proof. $\mathrm{op}\colon \mathbf{LRAct}_* \to \mathbf{LRAct}_*^\circ$ is given by:

$$\begin{aligned} (\mathbb{C}, \mathbb{D}, \mathbb{E})^{\mathrm{op}} &= (\mathbb{E}, \mathbb{D}^{\mathrm{op}}, \mathbb{C}) \\ (L, K, M)^{\mathrm{op}} &= (M, K^{\mathrm{op}}, L) \\ (\alpha, \beta, \gamma)^{\mathrm{op}} &= (\gamma, \beta^{\mathrm{op}}, \alpha). \end{aligned}$$

The main point of note is the structure maps for functors. If $(L, K, M)\colon (\mathbb{C}, \mathbb{D}, \mathbb{E}) \to (\mathbb{C}', \mathbb{D}', \mathbb{E}')$ has structure given by $(m_I, m_{B,C}, p_{C,D}, q_{D,E}, r_{E,F}, r_I)$, then the structure for (M, K^{op}, L) is given by $(r_I, r_{F,E}, \hat{q}_{E,D}, \hat{p}_{D,C}, m_{C,B}, m_I)$, where $\hat{q}_{E,D}$ and $\hat{p}_{D,C}$ are the maps

$$\begin{aligned} \hat{q}_{E,D}\colon ME \mathbin{-\!\square}_r KD &\to K(E \mathbin{-\!\square}_r D) \\ \hat{p}_{D,C}\colon KD \mathbin{\square\!-}_l LC &\to K(D \mathbin{\square\!-}_l C) \end{aligned}$$

in $\mathbb{D}'^{\mathrm{op}}$ corresponding to the maps

$$\hat{q}_{E,D} \colon K(E \mathbin{-\!\square}_r D) \to ME \mathbin{-\!\square}_r KD$$
$$\hat{p}_{D,C} \colon K(D \mathbin{\square\!-}_l C) \to KD \mathbin{\square\!-}_l LC$$

in $\mathbb{D}'$ given by

$$K(E \mathbin{-\!\square}_r D) \boxtimes_r ME \xrightarrow{q_{(E \mathbin{-\!\square}_r D),E}} K((E \mathbin{-\!\square}_r D) \boxtimes_r E) \xrightarrow{\mathbf{ev}} KD$$
$$LC \boxtimes_l K(D \mathbin{\square\!-}_l C) \xrightarrow{p_{C,(D \mathbin{\square\!-}_l C)}} K(C \boxtimes_l (D \mathbin{\square\!-}_l C)) \xrightarrow{\mathbf{ev}} KD.$$

under exponential transpose. It is now straightforward to check that the required coherence laws for 1- and 2-cells are satisfied with this definition. We proceed analogously for the other 2-categories of modules. ☐

Example 4. Recall the canonical two-sided action of a monoidal category $\mathbb{C}$ on itself. If $\mathbb{C}$ happens to be monoidal biclosed, then this canonical action is enriched and cotensored, with

$$C \mathbin{-\!\square}_l C' = C \mathbin{-\!\square}_r C' = C \multimap C' \quad \text{and} \quad C' \mathbin{\square\!-}_l C = C' \mathbin{\square\!-}_r C = C' \mathbin{\circ\!-} C.$$

Hence we have an enriched and tensored action of $\mathbb{C}$ on $\mathbb{C}^{\mathrm{op}}$. If we suggestively annotate objects of $\mathbb{C}$ with a + and objects of $\mathbb{C}^{\mathrm{op}}$ with a −, then this action is given by:

$$\begin{aligned} I &= I^+ & B^+ \otimes C^+ &= (B \otimes C)^+ \\ B^+ \multimap C^+ &= (B \multimap C)^+ & C^+ \mathbin{\circ\!-} B^+ &= (C \mathbin{\circ\!-} B)^+ \\ B^+ \boxtimes_l C^- &= (B \multimap C)^- & C^- \boxtimes_r B^+ &= (C \mathbin{\circ\!-} B)^- \\ B^- \mathbin{\square\!-}_r C^- &= (B \multimap C)^+ & C^- \mathbin{-\!\square}_l B^- &= (C \mathbin{\circ\!-} B)^+ \\ B^+ \mathbin{-\!\square}_r C^- &= (B \otimes C)^- & C^- \mathbin{\square\!-}_l B^+ &= (C \otimes B)^- \end{aligned}$$

which one can view as capturing a logical 'algebra of polarities'.

3.8 ∗-autonomy

3.8.1 ∗-autonomous modules

Recall that a (non-symmetric) monoidal category $\mathbb{C}$ is said to be ∗-*autonomous* if it is equipped with an adjoint equivalence $(-)^* \colon \mathbb{C} \to \mathbb{C}^{\mathrm{op}}$ satisfying natural isomorphisms:

$$\mathbb{C}(A \otimes B, C^*) \cong \mathbb{C}(B \otimes C, {}^*A),$$

where we write ${}^*(-)$ for the adjoint inverse of $(-)^*$. For more details and many other equivalent definitions, see [Bar95]. One point which this particular definition obscures is that every ∗-autonomous category is left and right closed, with

$$B \multimap C = (B \otimes {}^*C)^* \quad \text{and} \quad C \mathbin{\circ\!-} B = {}^*(C^* \otimes B).$$

Now we turn to an equivalent notion for modules.

Definition 21. We say that $(\mathbb{C}, \mathbb{D}) \in$ **BAct** is $*$-**autonomous** if it comes equipped with adjoint equivalences

$$^{\circ}(-) \colon \mathbb{C} \to \mathbb{D}^{\mathrm{op}} \dashv\vdash (-)^{\circ} \colon \mathbb{D}^{\mathrm{op}} \to \mathbb{C}$$
$$^{\bullet}(-) \colon \mathbb{D}^{\mathrm{op}} \to \mathbb{C} \dashv\vdash (-)^{\bullet} \colon \mathbb{C} \to \mathbb{D}^{\mathrm{op}}$$

(called the **left star** and **right star** respectively) such that

$$\mathbb{C}(B \otimes C, D^{\circ}) \cong \mathbb{D}(C \boxtimes_l D, {}^{\circ}B),$$
$$\mathbb{D}(C \boxtimes_l D, B^{\bullet}) \cong \mathbb{D}(D \boxtimes_r B, {}^{\circ}C),$$
$$\mathbb{D}(D \boxtimes_r B, C^{\bullet}) \cong \mathbb{C}(B \otimes C, {}^{\bullet}D)$$

naturally in all variables.

It may not be obvious at first glance that this is the correct generalisation of $*$-autonomy; but we shall see that it yields all the properties we would wish from something called '$*$-autonomous'. First:

Proposition 37. *A $*$-autonomous module* $(\mathbb{C}, \mathbb{D})$ *is closed in every respect.*

Proof. We define

$$C_1 \multimap C_2 = (C_1 \boxtimes_l {}^{\circ}C_2)^{\circ} \qquad C_2 \multimapinv C_1 = {}^{\bullet}(C_2^{\bullet} \boxtimes_r C_1)$$
$$D_1 \multimap_l D_2 = (D_1 \boxtimes_r {}^{\bullet}D_2)^{\circ} \qquad D_2 \multimapinv_r D_1 = {}^{\bullet}(D_2^{\circ} \boxtimes_l D_1)$$
$$D \multimapinv_l C = {}^{\circ}(D^{\circ} \otimes C) \qquad C \multimap_r D = (C \otimes {}^{\bullet}D)^{\bullet}.$$

We verify, for example,

$$\begin{aligned} \mathbb{D}(C \boxtimes_l D_1, D_2) &\cong \mathbb{D}(C \boxtimes_l D_1, {}^{\circ}(D_2^{\circ})) \\ &\cong \mathbb{C}(D_2^{\circ} \otimes C, D_1^{\circ}) \\ &\cong \mathbb{D}(D_1, {}^{\circ}(D_2^{\circ} \otimes C)) \end{aligned}$$

as required; similarly for the others. □

Next, we note that we have

$$\mathbb{D}(D, {}^{\circ}I) \cong \mathbb{D}(D \boxtimes_r I, {}^{\circ}I) \cong \mathbb{D}(I \boxtimes_l D, I^{\bullet}) \cong \mathbb{D}(D, I^{\bullet})$$

so that ${}^{\circ}I \cong I^{\bullet}$ by Yoneda. We set $\bot = I^{\bullet}$; and now note that

$$\begin{aligned} &D \multimap_l \bot = \big(D \boxtimes_r {}^{\bullet}(I^{\bullet})\big)^{\circ} \cong D^{\circ} \\ &\bot \multimapinv_l C = {}^{\circ}\big((I^{\bullet})^{\circ} \otimes C\big) \cong {}^{\circ}(({}^{\circ}I)^{\circ} \otimes C) \cong {}^{\circ}C \\ &C \multimap_r \bot = \big(C \otimes {}^{\bullet}(I^{\bullet})\big)^{\bullet} \cong C^{\bullet} \\ &\bot \multimapinv_r D = {}^{\bullet}\big((I^{\bullet})^{\circ} \boxtimes_l D\big) \cong {}^{\bullet}(({}^{\circ}I)^{\circ} \boxtimes_l D) \cong {}^{\bullet}D \end{aligned}$$

so that $\bot$ acts as a 'dualising object'. It follows that the canonical maps

$$\begin{aligned} C &\to (\bot \mathbin{\square\!\!-}_l C) \mathbin{-\!\!\square}_l \bot \\ C &\to \bot \mathbin{\square\!\!-}_r (C \mathbin{-\!\!\square}_r \bot) \end{aligned}$$

are isomorphisms. Conversely, suppose we are given a cotensored and enriched bimodule together with an object $\bot$ such that the two displayed maps are isomorphisms. Then setting $(-)^\circ = (-) \mathbin{-\!\!\square}_l \bot$, etc., we see that we have a pair of equivalences forming putative left and right star operations. We merely need to check the required isomorphisms of homsets; so we calculate

$$\begin{aligned} \mathbb{C}(B \otimes C, D^\circ) &= \mathbb{C}(B \otimes C, D \mathbin{-\!\!\square}_l \bot) \\ &\cong \mathbb{C}\big((B \otimes C) \boxtimes_l D, \bot\big) \\ &\cong \mathbb{C}(B \boxtimes_l (C \boxtimes_l D), \bot) \\ &\cong \mathbb{C}(C \boxtimes_l D, \bot \mathbin{\square\!\!-}_l B) \\ &= \mathbb{C}(C \boxtimes_l D, {}^\circ B) \end{aligned}$$

and similarly for the others. Hence we have shown

Proposition 38. *$*$-autonomous modules are in equivalence with cotensored and enriched modules $(\mathbb{C}, \mathbb{D})$ equipped with an object $\bot \in \mathbb{D}$ such that the natural maps*

$$\begin{aligned} C &\to (\bot \mathbin{\square\!\!-}_l C) \mathbin{-\!\!\square}_l \bot \\ C &\to \bot \mathbin{\square\!\!-}_r (C \mathbin{-\!\!\square}_r \bot) \end{aligned}$$

are isomorphisms.

3.8.2 Cyclic and symmetric $*$-autonomous modules

Recall that a $*$-autonomous monoidal category is said to be **cyclic $*$-autonomous** if it so happens that $(-)^*$ is its own adjoint inverse. In this case we have

$$\mathbb{C}(A \otimes B, C^*) \cong \mathbb{C}(B \otimes C, A^*).$$

We have a notion of cyclic $*$-autonomy for modules too.

Definition 22. A bimodule $(\mathbb{C}, \mathbb{D}) \in$ **BAct** is **cyclic $*$-autonomous** if it is $*$-autonomous and moreover we have $(-)^\bullet$ and $(-)^\circ$ naturally isomorphic to ${}^\circ(-)$ and ${}^\bullet(-)$ respectively. More simply, we may define cyclic $*$-autonomy in terms of a single adjoint equivalence

$$(-)^\bullet \colon \mathbb{C} \to \mathbb{D}^{\mathrm{op}} \dashv\vdash (-)^\circ \colon \mathbb{D}^{\mathrm{op}} \to \mathbb{C}$$

satisfying natural isomorphisms

$$\mathbb{C}(B \otimes C, D^\circ) \cong \mathbb{D}(C \boxtimes_l D, B^\bullet) \cong \mathbb{D}(D \boxtimes_r B, C^\bullet) \cong \mathbb{D}(B \otimes C, D^\circ).$$

Now, it's straightforward to calculate that any symmetric $*$-autonomous category is necessarily cyclic $*$-autonomous. In a similar vein, we have

Proposition 39. *Any symmetric $*$-autonomous module is necessarily cyclic $*$-autonomous.*

Proof. We need to show that there is a natural isomorphism $(-)^\circ \cong {}^\bullet(-)$. We calculate:

$$\begin{aligned}
\mathbb{C}(C, D^\circ) &\cong \mathbb{C}(I \otimes C, D^\circ)\\
&\cong \mathbb{D}(C \boxtimes_l D, {}^\circ I)\\
&\cong \mathbb{D}(D \boxtimes_r C, {}^\circ I)\\
&\cong \mathbb{D}(I \boxtimes_l D, C^\bullet)\\
&\cong \mathbb{D}(D, C^\bullet)\\
&\cong \mathbb{D}(C, {}^\bullet D)
\end{aligned}$$

natural in C and D; hence by Yoneda, we have $(-)^\circ \cong {}^\bullet(-)$; similarly ${}^\circ(-) \cong (-)^\bullet$. □

Hence we can without loss of generality define symmetric $*$-autonomy purely in terms of a left star.

Examples 5.

- If $\mathbb{C}$ is a $*$-autonomous category, then $(\mathbb{C}, \mathbb{C})$ is a $*$-autonomous module, with

  $$(-)^\circ = (-)^\bullet = (-)^* \quad \text{and} \quad {}^\circ(-) = {}^\bullet(-) = {}^*(-).$$

  Clearly if $\mathbb{C}$ is cyclic $*$-autonomous, then so is $(\mathbb{C}, \mathbb{C})$.

- Recall the canonical action of $\mathbb{C}$ on $\mathbb{C}^{\mathrm{op}}$ from Section 5; this is a cyclic $*$-autonomous action. Indeed, we set $(-)^\circ \colon \mathbb{C} \to (\mathbb{C}^{\mathrm{op}})^{\mathrm{op}} = \mathrm{id} = (-)^\bullet \colon (\mathbb{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathbb{C}$, and we confirm:

  $$\begin{aligned}
  \mathbb{C}(B \otimes C, D^\circ) &= \mathbb{C}(B \otimes C, D)\\
  &\cong \mathbb{C}(B, C \multimap D) \cong \mathbb{C}^{\mathrm{op}}(C \multimap D, B) = \mathbb{C}^{\mathrm{op}}(C \boxtimes_l D, B^\bullet)\\
  &\cong \mathbb{C}(C, D \multimapinv B) \cong \mathbb{C}^{\mathrm{op}}(D \multimapinv B, C) = \mathbb{C}^{\mathrm{op}}(D \boxtimes_r B, C^\bullet)
  \end{aligned}$$

  as required.

3.8.3 Maps of $*$-autonomous modules

Before we begin this section, an obvious point needs to be made; namely, that when dealing with involutions it is vital to be clear about which way maps are going. So unless explicitly stated otherwise, all the diagrams and maps in this section will be maps in $\mathbb{C}$ and $\mathbb{D}$ as appropriate, and *not* maps in $\mathbb{C}^{\mathrm{op}}$ or $\mathbb{D}^{\mathrm{op}}$. We also abuse notation by writing K for K^{op}, etc.; but this should not pose a problem.

Definition 23. A $*$-**autonomous modular functor** $(L, K)\colon (\mathbb{C}, \mathbb{D}) \to (\mathbb{C}', \mathbb{D}')$ between $*$-autonomous modules is a bimodular functor (L, K) equipped with natural isomorphisms

$$s_C\colon (LC)^\bullet \to K(C^\bullet) \quad \text{and} \quad t_D\colon (KD)^\circ \to L(D^\circ)$$

(and therefore with derived natural isomorphisms

$$\hat{s}_D\colon {}^\bullet(KD) \to L({}^\bullet D) \quad \text{and} \quad \hat{t}_C\colon {}^\circ(LC) \to K({}^\circ C)\ )$$

such that diagrams of the following form commute:

$$\begin{array}{ccc} \big(K({}^\circ C)\big)^\circ & \xrightarrow{(\hat{t}_C)^\circ} & \big({}^\circ(LC)\big)^\circ \\ {\scriptstyle t_{{}^\circ C}}\downarrow & & \downarrow{\scriptstyle \cong} \\ L\big(({}^\circ C)^\circ\big) & \xrightarrow{\cong} & LC \end{array} \qquad \begin{array}{ccc} \big(L({}^\bullet D)\big)^\bullet & \xrightarrow{(\hat{s}_D)^\bullet} & \big({}^\bullet(KD)\big)^\bullet \\ {\scriptstyle s_{{}^\bullet D}}\downarrow & & \downarrow{\scriptstyle \cong} \\ K\big(({}^\bullet D)^\bullet\big) & \xrightarrow{\cong} & KD. \end{array}$$

[Note that it follows that the dual diagrams

$$\begin{array}{ccc} {}^\circ\big(L(D^\circ)\big) & \xrightarrow{{}^\circ(t_D)} & {}^\circ\big((KD)^\circ\big) \\ {\scriptstyle \hat{t}_{D^\circ}}\downarrow & & \downarrow{\scriptstyle \cong} \\ K\big({}^\circ(D^\circ)\big) & \xrightarrow{\cong} & KD \end{array} \qquad \begin{array}{ccc} {}^\bullet\big(K(C^\bullet)\big) & \xrightarrow{{}^\bullet(s_C)} & {}^\bullet\big((LC)^\bullet\big) \\ {\scriptstyle \hat{s}_{C^\bullet}}\downarrow & & \downarrow{\scriptstyle \cong} \\ L\big({}^\bullet(C^\bullet)\big) & \xrightarrow{\cong} & LC \end{array}$$

also commute.] Similarly, a **cyclic** $*$-**autonomous modular functor** $(L, K)\colon (\mathbb{C}, \mathbb{D}) \to (\mathbb{C}', \mathbb{D}')$ between cyclic $*$-autonomous modules is a bimodular functor (L, K) equipped with natural isomorphisms

$$s_C\colon (LC)^\bullet \to K(C^\bullet)$$

(and therefore with derived natural isomorphisms

$$\hat{s}_D\colon (KD)^\circ \to L(D^\circ)\ )$$

such that diagrams of the following form commute:

$$\begin{array}{ccc} \big(L(D^\circ)\big)^\bullet & \xrightarrow{(\hat{s}_D)^\bullet} & \big((KD)^\circ\big)^\bullet \\ {\scriptstyle s_{D^\circ}}\downarrow & & \downarrow{\scriptstyle \cong} \\ K\big((D^\circ)^\bullet\big) & \xrightarrow{\cong} & KD \end{array}$$

[whence it follows that diagrams of the form

$$\begin{array}{ccc} \big(K(C^\bullet)\big)^\circ & \xrightarrow{(s_C)^\circ} & \big((LC)^\bullet\big)^\circ \\ {\scriptstyle \hat{s}_{C^\bullet}}\uparrow & & \downarrow{\scriptstyle \cong} \\ L\big((C^\bullet)^\circ\big) & \xrightarrow{\cong} & LC \end{array}$$

also commute.]

Now, we need to be somewhat subtle about 2-dimensional maps. They are not naturally modular transformations but rather modular *cotransformations* in the sense of Definition 20. Recall that in the monoidal setting, all natural transformations between $*$-autonomous functors are necessarily invertible; but here, the flexibility to pass to cotransformations allows this degeneracy to be lifted.

Definition 24. A $*$-**autonomous modular cotransformation** $(\alpha, \beta)\colon (L, K) \Rightarrow (L', K')$ between $*$-autonomous modular functors is a modular cotransformation (α, β) such that diagrams of the following form commute:

$$\begin{array}{ccc} (LC)^\bullet & \xrightarrow{s_C} & K(C^\bullet) \\ {\scriptstyle (\alpha_C)^\bullet}\uparrow & & \uparrow{\scriptstyle \beta_{C^\bullet}} \\ (L'C)^\bullet & \xrightarrow[s'_C]{} & K'(C^\bullet) \end{array} \qquad \begin{array}{ccc} (KD)^\circ & \xrightarrow{t_D} & L(D^\circ) \\ {\scriptstyle (\beta_D)^\circ}\downarrow & & \downarrow{\scriptstyle \alpha_{D^\circ}} \\ (K'D)^\circ & \xrightarrow[t'_D]{} & L'(D^\circ). \end{array}$$

And analogously, a cyclic $*$-autonomous modular cotransformation is a modular cotransformation (α, β) such that diagrams of the following form commute:

$$\begin{array}{ccc} (LC)^\bullet & \xrightarrow{s_C} & K(C^\bullet) \\ {\scriptstyle (\alpha_C)^\bullet}\uparrow & & \uparrow{\scriptstyle \beta_{C^\bullet}} \\ (L'C)^\bullet & \xrightarrow[s'_C]{} & K'(C^\bullet). \end{array}$$

Hence we can define a 2-categories $*$-**Act** and **c**$*$-**Act** with

- 0-cells (cyclic) $*$-autonomous modules;
- 1-cells (cyclic) $*$-autonomous modular functors;
- 2-cells (cyclic) $*$-autonomous modular cotransformations.

Note that as a result, $*$-**Act** and **c**$*$-**Act** live naturally inside $\mathbf{BAct}^\circ_*$ rather than $\mathbf{BAct}_*$.

Examples 6.

- If $L\colon \mathbb{C} \to \mathbb{D}$ is a monoidal functor, then the canonical functor $(L, L^{\mathrm{op}})\colon (\mathbb{C}, \mathbb{C}^{\mathrm{op}}) \to (\mathbb{D}, \mathbb{D}^{\mathrm{op}})$ so induced is a cyclic $*$-autonomous functor, with $s_C\colon (LC)^\bullet \to (L^{\mathrm{op}})^{\mathrm{op}}(C^\bullet) = \mathrm{id}_{LC}\colon LC \to LC$. Similarly, a monoidal natural transformation $\alpha\colon L \Rightarrow L'$ lifts to a cyclic $*$-autonomous modular cotransformation $(\alpha, \alpha^{\mathrm{op}})$.

3.9 Comma objects

We recall the *single glueing* construction for monoidal categories:

Proposition 40. *Given a monoidal functor $L\colon \mathbb{C} \to \mathbb{C}'$, the comma object in* **Cat**

$$\begin{array}{ccc} & \mathbb{C}' \downarrow L & \\ {\scriptstyle F}\swarrow & \Leftarrow & \searrow{\scriptstyle G} \\ \mathbb{C} & \xrightarrow[L]{} & \mathbb{C}' \end{array}$$

can be lifted to a diagram in **MonCat** *such that F and G are strict monoidal functors. Further, if $\mathbb{C}$ and $\mathbb{C}'$ are both $\{$left, right$\}$ closed, and $\mathbb{C}'$ has enough pullbacks, then $\mathbb{C}' \downarrow L$ is also $\{$left, right$\}$ closed, and U preserves the structure strictly.*

Proof. The monoidal structure on $\mathbb{C}$ is given by

$$I = I \to LI; \qquad (U \to LB) \otimes (V \to LC) = U \otimes V \to LB \otimes LC \xrightarrow{m_{B,C}} L(B \otimes C).$$

If $\mathbb{C}$ and $\mathbb{C}'$ are left or right closed, then the respective operations for $L \downarrow \mathbb{C}$ are given by the left hand arrows in these pullbacks:

$$(U \to LB) \multimap (V \to LC) : \quad \begin{array}{ccccc} M & & \longrightarrow & & U \multimap V \\ \downarrow & & & & \downarrow \\ L(B \multimap C) & \xrightarrow[m_{B,C}]{} & LB \multimap LC & \longrightarrow & U \multimap LC \end{array}$$

$$(V \to LC) \multimapinv (U \to LB) : \quad \begin{array}{ccccc} M & & \longrightarrow & & V \multimapinv U \\ \downarrow & & & & \downarrow \\ L(C \multimapinv B) & \xrightarrow[m_{C,B}]{} & LC \multimapinv LB & \longrightarrow & LC \multimapinv U. \end{array}$$

□

We extend this result to the modular case:

Proposition 41. *Given a modular functor $(L, K, M)\colon (\mathbb{C}, \mathbb{D}, \mathbb{E}) \to (\mathbb{C}', \mathbb{D}', \mathbb{E}')$, the comma object*

$$\begin{array}{ccc} & (\mathbb{C}' \downarrow L, \mathbb{D}' \downarrow K, \mathbb{E}' \downarrow M) & \\ {\scriptstyle F}\swarrow & \Leftarrow & \searrow{\scriptstyle G} \\ (\mathbb{C}, \mathbb{D}, \mathbb{E}) & \xrightarrow[(L,K,M)]{} & (\mathbb{C}', \mathbb{D}', \mathbb{E}') \end{array}$$

in **Cat**$\times$**Cat**$\times$**Cat** *can be lifted to a diagram in* **LRAct** *such that F and G are strict modular functors. Further, if the modules $\mathbb{D}$ and $\mathbb{D}'$ are both $\{$left, right$\}$ $\{$enriched, cotensored$\}$, and enough pullbacks exist in $\mathbb{C}'$, $\mathbb{D}'$ and $\mathbb{E}'$, then $\mathbb{D}' \downarrow M$ is also $\{$left, right$\}$ $\{$enriched, cotensored$\}$, and U preserves the structure strictly.*

Proof. The monoidal structures on $\mathbb{C}' \downarrow L$ and $\mathbb{E}' \downarrow M$ are given as above, and the left and right actions by:

$$(U \to LC) \boxtimes_l (X \to KD) = U \boxtimes_l X \to LC \boxtimes_l KD \xrightarrow{p_{C,D}} K(C \boxtimes_l D)$$
$$(X \to KD) \boxtimes_r (U \to ME) = X \boxtimes_r U \to KD \boxtimes_r ME \xrightarrow{q_{D,E}} K(D \boxtimes_r E).$$

For the closure properties, we may as well assume that both modules $\mathbb{D}$ and $\mathbb{D}'$ are closed in every way, since there is no dependence between the various forms of closure. We first note that from the structure maps $p_{C,D}$ and $q_{D,E}$ for (L, K, M) we can derive structure maps

$$p_{D,D'} \colon L(D \multimap_l D') \to KD \multimap_l KD' \qquad p_{D,C} \colon K(D \multimapinv_l C) \to KD \multimapinv_l LC$$
$$q_{E,D} \colon K(E \multimap_r D) \to ME \multimap_r KD \qquad q_{D',D} \colon M(D' \multimapinv_r D) \to KD' \multimapinv_r KD.$$

Then we give the various forms of closure by the respective left-hand arrows in the following pullback diagrams:

$$(X \to KD) \multimap_l (Y \to KD') : \qquad \begin{array}{ccccc} Q & \longrightarrow & & \longrightarrow & X \multimap_l Y \\ \downarrow & & & & \downarrow \\ L(D \multimap_l D') & \xrightarrow[p_{D,D'}]{} & KD \multimap_l KD' & \longrightarrow & X \multimap_l KD' \end{array}$$

$$(X \to KD) \multimapinv_l (V \to LC) : \qquad \begin{array}{ccccc} Q & \longrightarrow & & \longrightarrow & X \multimapinv_l V \\ \downarrow & & & & \downarrow \\ K(D \multimapinv_l C) & \xrightarrow[p_{D,C}]{} & KD \multimapinv_l LC & \longrightarrow & KD \multimapinv_l V \end{array}$$

$$(V \to ME) \multimap_r (X \to KD) : \qquad \begin{array}{ccccc} Q & \longrightarrow & & \longrightarrow & V \multimap_r X \\ \downarrow & & & & \downarrow \\ K(E \multimap_r D) & \xrightarrow[q_{E,D}]{} & ME \multimap_r KD & \longrightarrow & V \multimap_r KD \end{array}$$

$$(Y \to KD') \multimapinv_r (X \to KD) : \qquad \begin{array}{ccccc} Q & \longrightarrow & & \longrightarrow & Y \multimap_r X \\ \downarrow & & & & \downarrow \\ M(D' \multimapinv_r D) & \xrightarrow[q_{D',D}]{} & KD' \multimapinv_r KD & \longrightarrow & KD' \multimapinv_r X. \end{array}$$

□

This construction specialises in the obvious way to bimodules $(\mathbb{C}, \mathbb{D})$. In the case of symmetric modules $(\mathbb{C}, \mathbb{D}) \in$ **SAct**, we can equip $(\mathbb{C} \downarrow L)$ with a symmetry as follows:

$$\begin{array}{ccc}
U \otimes V & \xrightarrow{s_{U,V}} & V \otimes U \\
\downarrow & & \downarrow \\
LB \otimes LC & \xrightarrow{s_{LB,LC}} & LC \otimes LB \\
{\scriptstyle m_{B,C}}\downarrow & & \downarrow{\scriptstyle m_{C,B}} \\
L(B \otimes C) & \xrightarrow[Ls_{B,C}]{} & L(C \otimes B)
\end{array},$$

where the bottom square commutes since L is a symmetric monoidal functor. Similarly, we have the required symmetry for the actions:

$$\begin{array}{ccc}
U \boxtimes_l X & \xrightarrow{\sigma_{U,X}} & X \boxtimes_r U \\
\downarrow & & \downarrow \\
LC \boxtimes_l KD & \xrightarrow{\sigma_{LC,KD}} & KD \boxtimes_r LC \\
{\scriptstyle p_{C,D}}\downarrow & & \downarrow{\scriptstyle q_{D,C}} \\
K(C \boxtimes_l D) & \xrightarrow[K\sigma_{C,D}]{} & K(D \boxtimes_r C)
\end{array},$$

where the bottom square commutes since (L, K) is a symmetric modular functor.

Finally, the $*$-autonomous case. In the case of monoidal categories, the glueing construction fails to lift $*$-autonomous structure. However, here we are saved by the differing variances of the 2-cells in $*$-**Act**:

Proposition 42. *Given a $*$-autonomous modular functor $(L, K)\colon (\mathbb{C}, \mathbb{D}) \to (\mathbb{C}', \mathbb{D}')$, if $\mathbb{D}'$ has enough pushouts then the comma object*

$$\begin{array}{ccc}
 & (\mathbb{C}' \downarrow L, K \downarrow \mathbb{D}') & \\
{\scriptstyle F}\swarrow & \Leftarrow & \searrow{\scriptstyle V} \\
(\mathbb{C}, \mathbb{D}) & \xrightarrow[(L,K)]{} & (\mathbb{C}', \mathbb{D}')
\end{array}$$

in **Cat** $\times$ **Cat**^{co} *can be lifted to a diagram in $*$-***Act** *such that F is a strict $*$-autonomous modular functor.*

Proof. We equip $(\mathbb{C}' \downarrow L)$ with monoidal structure as usual. For the actions of $\mathbb{C}' \downarrow L$ on $K \downarrow \mathbb{D}'$, we take the respective right-hand arrows in the following pushout diagrams in

$\mathbb{D}'$:

$$(U \to LC) \boxtimes_l (KD \to Y) : \quad \begin{array}{ccccc} U \boxtimes_l KD & \longrightarrow & LC \boxtimes_l KD & \xrightarrow{p_{C,D}} & K(C \boxtimes_l D) \\ \downarrow & & & & \downarrow \\ U \boxtimes_l Y & & \longrightarrow & & Q \end{array}$$

$$(KD \to X) \boxtimes_r (V \to LC) : \quad \begin{array}{ccccc} KD \boxtimes_r V & \longrightarrow & KD \boxtimes_r LC & \xrightarrow{q_{D,C}} & K(D \boxtimes_r C) \\ \downarrow & & & & \downarrow \\ X \boxtimes_r V & & \longrightarrow & & Q \end{array}$$

and the star operations are given as follows:

$$(KD \to X)^\circ = X^\circ \to (KD)^\circ \to L(D^\circ)$$
$${}^\circ(U \to LC) = K({}^\circ C) \to {}^\circ(LC) \to {}^\circ U$$
$$(U \to LC)^\bullet = K(C^\bullet) \to (LC)^\bullet \to U^\bullet$$
$${}^\bullet(KD \to X) = {}^\bullet X \to {}^\bullet(KD) \to L({}^\bullet D).$$

It is straightforward to check that these operations form two equivalences and that the required isomorphisms of homsets hold. □

Remark. The construction just given for $*$-autonomous modules may look somewhat ad hoc. However, consider instead performing the following actions:

1. apply the 2-functor op: $\mathbf{BAct}^\circ_* \to \mathbf{BAct}_*$;
2. glue along (L, K^{op}) in $\mathbf{BAct}_*$;
3. apply op: $\mathbf{BAct}_* \to \mathbf{BAct}^\circ_*$.

The modules produced by these two methods are easily checked to be isomorphic as closed modules.

Example 7. *Double glueing.* The 'single glueing' construction of Proposition 40 fails to lift $*$-autonomous structure from the base categories to the glued category. To rectify this, Hyland and Schalk, in the paper [HS03], generalise it to a 'double glueing' construction. At its heart are some rather mysterious double pullbacks; but it turns out that these can be demystified using the language of modules. We recall the data required of us for the double glueing construction:

- Symmetric monoidal closed categories $\mathbb{C}$ and $\mathbb{E}$;
- Functors $L: \mathbb{C} \to \mathbb{E}$ and $K: \mathbb{C} \to \mathbb{E}^{\mathrm{op}}$;
- Symmetric monoidal structure $(m_I, m_{R,S})$ for the functor L;

- A 'contraction' $k_{R,S}\colon LR \otimes L(R \otimes S) \to KS$ satisfying axioms.

(We refer the reader to Section 4.2 of that paper for full details.) Now, from the contraction maps $k_{R,S}$ we derive maps

$$p_{C,D} = LC \otimes KD \xrightarrow{LC \otimes K(\mathsf{ev})} LC \otimes K\big(C \otimes (C \multimap D)\big) \xrightarrow{k} K(C \multimap D)$$

$$q_{D,C} = KD \otimes LC \xrightarrow{\mathsf{s}} LC \otimes KD \xrightarrow{p_{C,D}} K(C \multimap D)$$

and, noting that the canonical action of $\mathbb{C}$ on $\mathbb{C}^{\mathrm{op}}$ is given, in the symmetric case, by:

$$C \boxtimes_l D = D \boxtimes_r C = C \multimap D,$$

we see that $(m_I, m_{B,C}, p_{C,D}, q_{D,C})$ provide the required structure for (L, K^{op}) to be a symmetric modular functor $(\mathbb{C}, \mathbb{C}^{\mathrm{op}}) \to (\mathbb{E}, \mathbb{E})$; the axioms required of the 'contraction' in [HS03] are precisely those required of $p_{C,D}$ and $q_{D,C}$ in Section 2 above.

So now we apply Proposition 41 to obtain the symmetric closed module $(\mathbb{E} \downarrow L, \mathbb{E} \downarrow K^{\mathrm{op}})$ and a strict modular functor

$$(F, G)\colon (\mathbb{E} \downarrow L, \mathbb{E} \downarrow K^{\mathrm{op}}) \to (\mathbb{C}, \mathbb{C}^{\mathrm{op}}).$$

Applying Proposition 35, we get the symmetric module $(\mathbb{A}, \mathbb{B}) := (\mathbb{E} \downarrow L, (\mathbb{E} \downarrow K^{\mathrm{op}})^{\mathrm{op}})$ and a strict modular functor

$$(F, G)\colon (\mathbb{A}, \mathbb{B}) \to (\mathbb{C}, \mathbb{C}).$$

Let us spell out in detail what $(\mathbb{A}, \mathbb{B})$ consists of. First, $\mathbb{A}$ has objects $(U \to LR)$; and maps $(U \to LR) \to (V \to LS)$ are pairs $(\varphi\colon U \to V, f\colon R \to S)$ such that

$$\begin{array}{ccc} U & \longrightarrow & LR \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle Lf} \\ V & \longrightarrow & LS \end{array}$$

commutes. $\mathbb{B}$ has objects $(X \to KR)$ and maps $(X \to KR) \to (Y \to KS)$ are pairs $(\psi\colon Y \to X, f\colon R \to S)$ such that

$$\begin{array}{ccc} X & \longrightarrow & KR \\ {\scriptstyle \psi}\uparrow & & \uparrow{\scriptstyle Kf} \\ Y & \longrightarrow & KS \end{array}$$

commutes. The modular structure on $(\mathbb{A}, \mathbb{B})$ is as follows:

$$I = (I \xrightarrow{m_I} LI),$$

$$(U \to LR) \otimes (V \to LS) = U \otimes V \to LR \otimes LS \xrightarrow{m_{R,S}} L(R \otimes S),$$

$$(Z \to KT) \mathbin{\square\!\!-}_l (V \to LS) = Z \otimes V \to KT \otimes LS \xrightarrow{k_{S,T}} K(S \multimap T)$$

$$(V \to LS) \mathbin{-\!\!\square}_r (Z \to KT) = V \otimes Z \to LS \otimes KT \xrightarrow{k_{S,T}} K(S \multimap T)$$

with the remaining operations being given by the left-hand arrows in the following pull-back diagrams:

$$(U \to LR) \boxtimes_l (Y \to KS)\colon \quad \begin{array}{ccccc} Q & \longrightarrow & & \longrightarrow & U \multimap Y \\ \downarrow & & & & \downarrow \\ K(R \otimes S) & \xrightarrow[k_{R,S}]{} & LR \multimap KS & \longrightarrow & U \multimap KS \end{array}$$

$$(X \to KR) \boxtimes_r (V \to LS)\colon \quad \begin{array}{ccccc} Q & \longrightarrow & & \longrightarrow & V \multimap X \\ \downarrow & & & & \downarrow \\ K(R \otimes S) & \xrightarrow[k_{R,S}]{} & LS \multimap KR & \longrightarrow & V \multimap KR \end{array}$$

$$\begin{array}{r} (V \to LS) \multimap (W \to LT) \\ = (W \to LT) \circ\!\!- (V \to LS) \end{array}\colon \quad \begin{array}{ccccc} Q & \longrightarrow & & \longrightarrow & U \multimap V \\ \downarrow & & & & \downarrow \\ L(S \multimap T) & \xrightarrow[m_{S,T}]{} & LS \multimap LT & \longrightarrow & V \multimap LT \end{array}$$

$$\begin{array}{r} (Y \to KS) -\!\!\square_l (Z \to KT) \\ = (Z \to KT) \square\!\!-_r (Y \to KS) \end{array}\colon \quad \begin{array}{ccccc} Q & \longrightarrow & & \longrightarrow & Z \multimap Y \\ \downarrow & & & & \downarrow \\ L(S \multimap T) & \xrightarrow[k_{S,T}]{} & KT \multimap KS & \longrightarrow & Z \multimap KS. \end{array}$$

Comparison with Proposition 30 of [HS03] shows that the pullbacks above are precisely the two sides of the double pullback diagrams there.

The $*$-autonomous case (Section 4.3 of [HS03]) also fits well into this framework. So we now suppose that $\mathbb{C}$ is symmetric $*$-autonomous and take $K(-) = L(-^*)$. Recall that $(\mathbb{E}, \mathbb{E}^{\mathrm{op}})$ is cyclic $*$-autonomous in a canonical way, and in fact symmetric $*$-autonomous if $\mathbb{E}$ is. Now (L, K) becomes in a natural way a $*$-autonomous modular functor from $(\mathbb{C}, \mathbb{C})$ to $(\mathbb{E}, \mathbb{E}^{\mathrm{op}})$; and thus we get the symmetric $*$-autonomous module $(\mathbb{E} \downarrow L, K \downarrow \mathbb{E}^{\mathrm{op}}) \cong (\mathbb{A}, \mathbb{B})$, together with a strict $*$-autonomous functor

$$(F, G)\colon (\mathbb{A}, \mathbb{B}) \to (\mathbb{C}, \mathbb{C});$$

And by the remark after Proposition 42, this has the same closed structure as that spelt out above. We shall see later how to get from $(\mathbb{A}, \mathbb{B})$ to the double glueing category $\mathbb{G}$.

3.10 The cofree monoidal category on a module

3.10.1 The basic case

Our starting point in this section is the well-known construction of 'simple self-dualisation': given a monoidal biclosed $\mathbb{C}$ with finite products, it produces a cyclic $*$-autonomous category $\mathbb{C} \times \mathbb{C}^{\mathrm{op}}$. We produce a similar notion for modules, and characterise it universally. First, we need a mild representability condition on our modules.

Definition 25. We write $\mathbf{LRAct}_+$ for the sub-2-category of $\mathbf{LRAct}$ with objects all modules $(\mathbb{C}, \mathbb{D}, \mathbb{E})$ such that

- $\mathbb{D}$ has finite coproducts, and
- the actions of $\mathbb{C}$ and $\mathbb{E}$ preserve them,

together with 1- and 2-cells between them. Similarly we write $\mathbf{LAct}_+$, $\mathbf{RAct}_+$, $\mathbf{BAct}_+$, $\mathbf{SAct}_+$, etc.

Note in particular that if a module is cotensored, then $C \boxtimes_l (-)$ and $(-) \boxtimes_r E$ are left adjoints and hence automatically preserve finite coproducts. Now, recall from Examples 3 that we have a 2-functor $F\colon \mathbf{MonCat} \to \mathbf{LRAct}$ taking $\mathbb{C}$ to $(\mathbb{C}, \mathbb{C}, \mathbb{C})$. Then our construction is the 'cofree monoidal category' on our module, in that it provides a partial right adjoint to F:

Proposition 43. *There is a 2-functor $G\colon \mathbf{LRAct}_+ \to \mathbf{MonCat}$ such that we have natural isomorphisms of categories:*

$$\mathbf{LRAct}\big(F\mathbb{B}, (\mathbb{C}, \mathbb{D}, \mathbb{E})\big) \cong \mathbf{MonCat}\big(\mathbb{B}, G(\mathbb{C}, \mathbb{D}, \mathbb{E})\big),$$

where we note that $F\mathbb{B}$ need not lie in $\mathbf{LRAct}_+$ for the above isomorphism to hold.

Proof. Suppose $(\mathbb{C}, \mathbb{D}, \mathbb{E}) \in \mathbf{LRAct}_+$. Then we take $G(\mathbb{C}, \mathbb{D}, \mathbb{E})$ to be $\mathbb{C} \times \mathbb{D} \times \mathbb{E}$, equipped with the structure of a monoidal category as follows:

$$I = (I, 0, I)$$
$$(C_1, D_1, E_1) \otimes (C_2, D_2, E_2) = (C_1 \otimes C_2, C_1 \boxtimes_l D_2 + D_1 \boxtimes_r E_2, E_1 \otimes E_2).$$

This is clearly bifunctorial, and associativity and unit isomorphisms are given by

$$\begin{array}{ccc|ccc}
(C, & D, & E) & (C, & D, & E) \\
\downarrow \mathsf{r} & \downarrow \rho & \downarrow \mathsf{r} & \downarrow \mathsf{l} & \downarrow \lambda & \downarrow \mathsf{l} \\
 & D \boxtimes_r I, & & & I \boxtimes_l D, & \\
 & \downarrow \mathsf{inr} & & & \downarrow \mathsf{inl} & \\
(C \otimes I, & C \boxtimes_l 0 + D \boxtimes_r I, & E \otimes I) & (I \otimes C, & I \boxtimes_l D + 0 \boxtimes_r E, & I \otimes E)
\end{array}$$

$$
\begin{array}{ccc}
\big(\, C_1 \otimes (C_2 \otimes C_3), & C_1 \boxtimes_l (C_2 \boxtimes_l D_3 + D_2 \boxtimes_r E_3) + D_1 \boxtimes_r (E_2 \boxtimes_r E_3), & E_1 \otimes (E_2 \otimes E_3) \,\big) \\
 & \downarrow \cong & \\
 & C_1 \boxtimes_l (C_2 \boxtimes_l D_3) + C_1 \boxtimes_l (D_2 \boxtimes_r E_3) + D_1 \boxtimes_r (E_2 \otimes E_3) & \\
\downarrow \mathsf{a} & \downarrow \alpha+\tau+\beta & \downarrow \mathsf{a} \\
 & (C_1 \otimes C_2) \boxtimes_l D_3 + (C_1 \boxtimes_l D_2) \boxtimes_r E_3 + (D_1 \boxtimes_r E_2) \boxtimes_r E_3 & \\
 & \downarrow \cong & \\
\big(\, (C_1 \otimes C_2) \otimes C_3, & (C_1 \otimes C_2) \boxtimes_l D_3 + (C_1 \boxtimes_l D_2 + D_1 \boxtimes_r E_2) \boxtimes_r E_3, & (E_1 \otimes E_2) \otimes E_3 \,\big).
\end{array}
$$

It is routine to check coherence. Next, if $(L, K, M)\colon (\mathbb{C}, \mathbb{D}, \mathbb{E}) \to (\mathbb{C}', \mathbb{D}', \mathbb{E}')$ in $\mathbf{LRAct}_+$, then we make $L \times K \times M$ into a monoidal functor $G(\mathbb{C}, \mathbb{D}, \mathbb{E}) \to G(\mathbb{C}', \mathbb{D}', \mathbb{E}')$ by equipping it with monoidal structure

$$
\begin{array}{ccc}
\big(\, I, & 0, & I \,\big) \\
\downarrow m_I & \downarrow ! & \downarrow r_I \\
\big(\, LI, & KI, & MI \,\big)
\end{array}
$$

$$
\begin{array}{ccc}
\big(\, LC_1 \otimes LC_2, & LC_1 \boxtimes_l KD_2 + KD_1 \boxtimes_r ME_2, & ME_1 \otimes ME_2 \,\big) \\
 & \downarrow p+q & \\
\downarrow m & K(C_1 \boxtimes_l D_2) + K(D_1 \boxtimes_r E_2) & \downarrow r \\
 & \downarrow [K(\mathsf{inl}), K(\mathsf{inr})] & \\
\big(\, L(C_1 \otimes C_2), & K(C_1 \boxtimes_l D_2 + D_1 \boxtimes_r E_2), & M(E_1 \otimes E_2) \,\big).
\end{array}
$$

and on 2-cells, we take $G(\gamma, \delta, \eta) = \gamma \times \delta \times \eta$. Coherence is again straightfoward. Next, we show that $\mathbf{LRAct}\big(F\mathbb{B}, (\mathbb{C}, \mathbb{D}, \mathbb{E})\big) \cong \mathbf{MonCat}\big(\mathbb{B}, G(\mathbb{C}, \mathbb{D}, \mathbb{E})\big)$. Indeed, to give a 1-cell $F\mathbb{B} \to (\mathbb{C}, \mathbb{D}, \mathbb{E})$ is to give functors

$$
L\colon \mathbb{B} \to \mathbb{C}, \quad K\colon \mathbb{B} \to \mathbb{C}, \quad \text{and} \quad M\colon \mathbb{B} \to \mathbb{E}
$$

together with structure maps

$$
\begin{array}{cc}
I \to LI & I \to MI \\
LA \otimes LB \to L(A \otimes B) & MA \otimes MB \to M(A \otimes B) \\
LA \boxtimes_l KB \to K(A \otimes B) & KA \boxtimes_r MB \to K(A \otimes B)
\end{array}
$$

obeying axioms; and this is to give a functor $\langle L, K, M \rangle : \mathbb{B} \to \mathbb{C} \times \mathbb{D} \times \mathbb{E}$ together with structure maps

$$(I, 0, I) \to (LI, KI, MI)$$

$$\begin{array}{ccc}
\big(LA \otimes LB, & LA \boxtimes_l KB + KA \boxtimes_r MB, & MA \otimes MB\big) \\
\downarrow & \downarrow & \downarrow \\
\big(L(A \otimes B), & K(A \otimes B), & M(A \otimes B)\big)
\end{array}$$

obeying axioms; which is to give a monoidal functor $\mathbb{B} \to G(\mathbb{C}, \mathbb{D}, \mathbb{E})$. Finally, giving a 2-cell $(\gamma, \delta, \eta)\colon (L, K, M) \Rightarrow (L', K', M')\colon F\mathbb{B} \to (\mathbb{C}, \mathbb{D}, \mathbb{E})$ is easily seen to be equivalent to giving a monoidal transformation $\left\langle \gamma, \delta, \eta \right\rangle : \langle L, K, M \rangle \Rightarrow \langle L', K', M' \rangle : \mathbb{B} \to G(\mathbb{C}, \mathbb{D}, \mathbb{E})$. Hence we have the desired isomorphisms of hom-categories; and their naturality is straightforward. □

We have a similar result for bimodules:

Proposition 44. *There is a 2-functor* $G'\colon \mathbf{BAct}_+ \to \mathbf{MonCat}$ *such that we have natural isomorphisms of categories:*

$$\mathbf{BAct}\big(F\mathbb{B}, (\mathbb{C}, \mathbb{D})\big) \cong \mathbf{MonCat}\big(\mathbb{B}, G'(\mathbb{C}, \mathbb{D})\big),$$

where F now denotes the natural embedding $\mathbf{MonCat} \to \mathbf{BAct}$. *Further, G' restricts to a 2-functor* $G''\colon \mathbf{SAct}_+ \to \mathbf{SymMonCat}$ *and we have isomorphisms*

$$\mathbf{SAct}\big(F\mathbb{B}, (\mathbb{C}, \mathbb{D})\big) \cong \mathbf{SymMonCat}\big(\mathbb{B}, G''(\mathbb{C}, \mathbb{D})\big),$$

Proof. Entirely analogous; we set $G'(\mathbb{C}, \mathbb{D}) = \mathbb{C} \times \mathbb{D}$, with structure

$$I = (I, 0)$$
$$(C_1, D_1) \otimes (C_2, D_2) = (C_1 \otimes C_2, C_1 \boxtimes_l D_2 + D_1 \boxtimes_r C_2),$$

and the remainder of the proof follows similarly. For the symmetric case, we note that $\mathbb{C} \times \mathbb{D}$ can be made symmetric as follows:

$$\begin{array}{cc}
\big(\, C_1 \otimes C_2, & C_1 \boxtimes_l D_2 + D_1 \boxtimes_r C_2 \,\big) \\
 & \downarrow {\scriptstyle \sigma+\sigma^{-1}} \\
\downarrow {\scriptstyle \mathsf{s}} & D_2 \boxtimes_r C_1 + C_2 \boxtimes_l D_1 \\
 & \downarrow {\scriptstyle [\mathsf{inr},\mathsf{inl}]} \\
\big(\, C_2 \otimes C_1, & C_2 \boxtimes_l D_1 + D_2 \boxtimes_r C_1 \,\big).
\end{array}$$

□

3.10.2 The closed and $*$-autonomous cases

The functors we have defined above are well-behaved with respect to the additional structure we may have on our modules, so long as further mild representability conditions are satisfied.

Proposition 45. *Let* $(\mathbb{C}, \mathbb{D}, \mathbb{E}) \in$ **LRAct**$_+$. *Then*

- *if* $\mathbb{C}$ *has finite products and the module is left closed, then so is* $G(\mathbb{C}, \mathbb{D}, \mathbb{E})$*;*
- *if* $\mathbb{E}$ *has finite products and the module is right closed, then so is* $G(\mathbb{C}, \mathbb{D}, \mathbb{E})$*.*

Proof. For left closure, we define

$$(C_2, D_2, E_2) \multimap (C_3, D_3, E_3) = \big((C_2 \multimap C_3) \times (D_2 \multimap_l D_3), E_2 \multimap_r D_3, E_2 \multimap E_3\big)$$

and for right closure,

$$(C_3, D_3, E_3) \multimapinv (C_1, D_1, E_1) = \big(C_3 \multimapinv C_1, D_3 \multimapinv_l C_1, (D_3 \multimapinv_r D_1) \times (E_3 \multimapinv E_1)\big).$$

We calculate:

$$\begin{aligned}
&\mathbb{C}\times\mathbb{D} \times \mathbb{E}\big((C_1, D_1, E_1) \otimes (C_2, D_2, E_2), (C_3, D_3, E_3)\big) \\
&\quad\cong \mathbb{C}(C_1 \otimes C_2, C_3) \times \mathbb{D}(C_1 \boxtimes_l D_2 + D_1 \boxtimes_r E_2, D_3) \times \mathbb{E}(E_1 \otimes E_2, E_3) \\
&\quad\cong \mathbb{C}(C_1 \otimes C_2, C_3) \times \mathbb{D}(C_1 \boxtimes_l D_2, D_3) \times \mathbb{D}(D_1 \boxtimes_r E_2, D_3) \times \mathbb{E}(E_1 \otimes E_2, E_3) \\
&\quad\cong \mathbb{C}(C_1, C_2 \multimap C_3) \times \mathbb{C}(C_1, D_2 \multimap_l D_3) \times \mathbb{D}(D_1, E_2 \multimap_r D_3) \times \mathbb{E}(E_1, E_2 \multimap E_3) \\
&\quad\cong \mathbb{C}\big(C_1, (C_2 \multimap C_3) \times (D_2 \multimap_l D_3)\big) \times \mathbb{D}(D_1, E_2 \multimap_r D_3) \times \mathbb{E}(E_1, E_2 \multimap E_3) \\
&\quad\cong \mathbb{C} \times \mathbb{D} \times \mathbb{E}\big((C_1, D_1, E_1), (C_2, D_2, E_2) \multimap (C_3, D_3, E_3)\big)
\end{aligned}$$

as required; similarly for right closure. □

Proposition 46. *Let* $(\mathbb{C}, \mathbb{D}) \in$ **BAct**$_+$. *Then if* $\mathbb{C}$ *has finite products and the module is* {*left, right*} *closed, then so is* $G'(\mathbb{C}, \mathbb{D})$*.*

Proof. As before; we define

$$(C_2, D_2) \multimap (C_3, D_3) = \big((C_2 \multimap C_3) \times (D_2 \multimap_l D_3), C_2 \multimap_r D_3\big)$$

and for right closure,

$$(C_3, D_3) \multimapinv (C_1, D_1) = \big((C_3 \multimapinv C_1) \times (D_3 \multimapinv_r D_1), D_3 \multimapinv_l C_1\big).$$

□

Now $*$-autonomy. Note that if $(\mathbb{C}, \mathbb{D})$ is a $*$-autonomous module and $\mathbb{D}$ has coproducts, then the actions necessarily preserve them (since $\mathbb{D}$ is a cotensored module) and $\mathbb{C}$ necessarily has products (since it is equivalent to $\mathbb{D}^{op}$).

Proposition 47. *If* $(\mathbb{C},\mathbb{D}) \in \mathbf{BAct}_+$ *is a (cyclic)* $*$*-autonomous module, then* $G(\mathbb{C},\mathbb{D})$ *is a (cyclic)* $*$*-autonomous category. Further, the closed structure thus induced on* $\mathbb{C}\times\mathbb{D}$ *agrees with that derived by the previous proposition from the canonical closed structure on* $(\mathbb{C},\mathbb{D})$.

Proof. We set $(C,D)^* = (D^\circ, C^\bullet)$ and hence ${}^*(C,D) = ({}^\bullet D, {}^\circ C)$. To check $*$-autonomy, it suffices to check that we have $\mathbb{C}\times\mathbb{D}(X\otimes Y, Z^*) \cong \mathbb{C}\times\mathbb{D}(Y\otimes Z, {}^*X)$. We calculate:

$$\begin{aligned}
&\mathbb{C}\times\mathbb{D}\big((C_1,D_1)\otimes(C_2,D_2),(C_3,D_3)^*\big)\\
&\quad\cong \mathbb{C}(C_1\otimes C_2, D_3^\circ)\times\mathbb{D}(C_1\boxtimes_l D_2 + D_1\boxtimes_r C_2, C_3^\bullet)\\
&\quad\cong \mathbb{C}(C_1\otimes C_2, D_3^\circ)\times\mathbb{D}(C_1\boxtimes_l D_2, C_3^\bullet)\times\mathbb{D}(D_1\boxtimes_r C_2, C_3^\bullet)\\
&\quad\cong \mathbb{D}(C_2\boxtimes_l D_3, {}^\circ C_1)\times\mathbb{D}(D_2\boxtimes_r C_3, {}^\circ C_1)\times\mathbb{C}(C_2\otimes C_3, {}^\bullet D_1)\\
&\quad\cong \mathbb{C}(C_2\otimes C_3, {}^\bullet D_1)\times\mathbb{D}(C_2\boxtimes_l D_3 + D_2\boxtimes_r C_3, {}^\circ C_1)\\
&\quad\cong \mathbb{C}\times\mathbb{D}\big((C_2,D_2)\otimes(C_3,D_3), {}^*(C_1,D_1)\big)
\end{aligned}$$

as required. Clearly, if $(\mathbb{C},\mathbb{D})$ is cyclic $*$-autonomous, then $\mathbb{C}\times\mathbb{D}$ will be too. Now, the closed structure this induces on $\mathbb{C}\times\mathbb{D}$ is given by $Y \multimap Z = (Y\otimes {}^*Z)^*$ and $Z \multimapinv Y = {}^*(Z^*\otimes Y)$. We calculate:

$$\begin{aligned}
((C_1,D_1)\otimes{}^*(C_2,D_2))^* &\cong (C_1\otimes{}^\bullet D_2, C_1\boxtimes_l {}^\circ C_2 + D_1\boxtimes_r {}^\bullet D_2)^*\\
&\cong ((C_1\boxtimes_l {}^\circ C_2)^\circ\times(D_1\boxtimes_r {}^\bullet D_2)^\circ, (C_1\otimes {}^\bullet D_2)^\bullet)\\
&\cong \big((C_1\multimap C_2)\times(D_1 \multimap_{\square l} D_2), C_1 \multimap_{\square r} D_2\big)
\end{aligned}$$

and similarly for left closure. $\square$

Example 8. Given a monoidal closed category $\mathbb{C}$, we have shown that the canonical action of $\mathbb{C}$ on $\mathbb{C}^{\mathrm{op}}$ is cyclic $*$-autonomous. If $\mathbb{C}$ has finite products, then $\mathbb{C}^{\mathrm{op}}$ has finite coproducts, so $(\mathbb{C},\mathbb{C}^{\mathrm{op}}) \in \mathbf{BAct}_+$. Hence we can apply the previous result to see that $\mathbb{C}\times\mathbb{C}^{\mathrm{op}}$ is a cyclic $*$-autonomous category; and the structure on $\mathbb{C}\times\mathbb{C}^{\mathrm{op}}$ is precisely that prescribed by the 'simple self-dualisation' construction. However, even if $\mathbb{C}$ does not have products, $(\mathbb{C},\mathbb{C}^{\mathrm{op}})$ is still a cyclic $*$-autonomous module; it is merely that this structure fails to be representable.

3.11 Modules modulo $\mathbb{C}$

Our principal motivation here is to develop theory that will allow us to complete the explanation of double gluing begun in section 8. Recall that there we produced a symmetric closed module $(\mathbb{A},\mathbb{B}) = (\mathbb{E}\downarrow L, (\mathbb{E}\downarrow K^{\mathrm{op}})^{\mathrm{op}})$ together with a strict modular functor

$(F, G)\colon (\mathbb{A}, \mathbb{B}) \to (\mathbb{C}, \mathbb{C})$. The double glueing category we are after is given by the pullback

$$\begin{array}{ccc} \mathbb{G} & \longrightarrow & \mathbb{B} \\ \big\downarrow & & \big\downarrow{\scriptstyle G} \\ \mathbb{A} & \underset{F}{\longrightarrow} & \mathbb{C}, \end{array}$$

and it is our goal in this section to equip $\mathbb{G}$ with monoidal structure. Essentially, we will do this by extending the cofree construction given above from **Cat** to **Cat**$/\mathbb{C}$.

First we need a notion of 'module in **Cat**$/\mathbb{C}$'. We note that if $\mathbb{C}$ is a monoidal category, then **Cat**$/\mathbb{C}$ can be equipped with the structure of a 2-monoidal 2-category in the sense of [GPS95]; i.e., a monoidal **Cat**-category. Explicitly, this structure is given by $I = 1 \xrightarrow{\Delta I} \mathbb{C}$, the functor picking out the unit object of $\mathbb{C}$, and

$$(\mathbb{A} \to \mathbb{C}) \otimes (\mathbb{B} \to \mathbb{C}) = (\mathbb{A} \times \mathbb{B} \to \mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C}).$$

Now, just as a monoidal category $\mathbb{A}$ is a pseudomonoid in **Cat**, so a pseudomonoid in **Cat**$/\mathbb{C}$ equipped with this 2-monoidal structure is precisely a monoidal category $\mathbb{A}$ equipped with a strict monoidal functor $F\colon \mathbb{A} \to \mathbb{C}$.

Similar considerations lead us to conclude that a *module* $(\mathbb{A}, \mathbb{B})$ in **Cat**$/\mathbb{C}$ should be a module $(\mathbb{A}, \mathbb{B})$ equipped with a strict modular functor $(F, G)\colon (\mathbb{A}, \mathbb{B}) \to (\mathbb{C}, \mathbb{C})$, where the latter is $\mathbb{C}$ viewed as a module over itself. Now, we have a notion of module, but for representability of this structure, we need an analogue of 'category with coproducts'. Since we shall also need an analogue of 'category with products' later, we deal with this at the same time.

Notation. Given a functor $G\colon \mathbb{B} \to \mathbb{C}$, we write $X^{(R)} \in \mathbb{B}$ to indicate that $X \in \mathbb{B}$ and $G(X) = R$. Similarly on maps we write $f^{(\varphi)}$ to indicate $f \in \mathbb{B}$ and $G(f) = \varphi$.

Definition 26. Given a functor $G\colon \mathbb{B} \to \mathbb{C}$, we say that $\mathbb{B}$ has **binary coproducts modulo** $\mathbb{C}$ if the 1-cell Δ

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\Delta} & \mathbb{B} \times_{\mathbb{C}} \mathbb{B} \\ {\scriptstyle G}\big\downarrow & \swarrow{\scriptstyle G} & \\ \mathbb{C} & & \end{array}$$

in **Cat**$/\mathbb{C}$ has a left adjoint. Explicitly, we have a functor $\oplus\colon \mathbb{B} \times_{\mathbb{C}} \mathbb{B} \to \mathbb{B}$ such that $G({B_1}^{(R)} \oplus {B_2}^{(R)}) = R$, and such that

$$\mathbb{B} \times_{\mathbb{C}} \mathbb{B}((B_1, B_2)^{(R)}, (X, X)^{(S)}) \cong \mathbb{B}((B_1 \oplus B_2)^{(R)}, X^{(S)}).$$

Similarly, we say that $\mathbb{B}$ has an **initial object modulo** $\mathbb{C}$ if the 1-cell $! = G$

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{!} & \mathbb{C} \\ {\scriptstyle G}\big\downarrow & \swarrow{\scriptstyle \mathrm{id}} & \\ \mathbb{C} & & \end{array}$$

has a left adjoint. Explicitly, we have a functor $\mathbf{0}\colon \mathbb{C} \to \mathbb{B}$ such that $G(\mathbf{0}(R)) = R$, and such that

$$\mathbb{C}(R, S) \cong \mathbb{B}(\mathbf{0}(R), X^{(S)}).$$

If $\mathbb{B}$ has binary coproducts modulo $\mathbb{C}$ and an initial object modulo $\mathbb{C}$, we say that it has **finite coproducts modulo** $\mathbb{C}$. Similarly, for $\mathbb{B}$ to have **products modulo** $\mathbb{C}$ is for the two functors Δ and $!$ to have *right* adjoints in $\mathbf{Cat}/\mathbb{C}$. If we write $\odot\colon \mathbb{B}\times_{\mathbb{C}}\mathbb{B} \to \mathbb{B}$ and $\mathbf{1}\colon \mathbb{C} \to \mathbb{B}$ for these adjoints, then we have $G({B_1}^{(R)} \odot {B_2}^{(R)}) = R$, $G(\mathbf{1}(R)) = R$,

$$\mathbb{B} \times_{\mathbb{C}} \mathbb{B}((X, X)^{(R)}, (B_1, B_2)^{(S)}) \cong \mathbb{B}(X^{(S)}, (B_1 \odot B_2)^{(S)}), \text{ and}$$
$$\mathbb{C}(R, S) \cong \mathbb{B}(X^{(R)}, \mathbf{1}(S)).$$

Note that for $\mathbb{B}$ to have finite coproducts modulo $\mathbb{C}$, it is a *necessary* condition that each preimage category $G^{-}1(R)$ has finite coproducts; and these provide the object part of the functors $\oplus$ and $\mathbf{0}$. Now, if the map $G\colon \mathbb{B} \to \mathbb{C}$ happens to be a fibration, then it is straightforward to check that it is also a *sufficient* condition.

Similarly, given a cofibration $G\colon \mathbb{B} \to \mathbb{C}$, $\mathbb{B}$ has finite products modulo $\mathbb{C}$ if and only if each fibre $G^{-1}(R)$ has finite products. Almost all naturally occuring examples arise from fibrations and cofibrations in this way.

Examples 9.

- Given a functor $L\colon \mathbb{C} \to \mathbb{E}$, we can form the comma category $\mathbb{E} \downarrow L$. Then the forgetful functor $F\colon \mathbb{E} \downarrow L \to \mathbb{C}$ is a cofibration. Further, if $\mathbb{E}$ has sufficient pullbacks, then each fibre has finite products, given by

$$\mathbf{1}(R) = (LR \to LR); \quad (X \xrightarrow{\varphi} LR) \odot (Y \xrightarrow{\psi} LR) = \begin{array}{ccc} Q & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ Y & \longrightarrow & LR. \end{array}$$

  Further, in this case the forgetful functor also becomes a fibration; and if $\mathbb{E}$ has finite coproducts, then so does each fibre, given by

$$\mathbf{0}(R) = (0 \to LR); \quad (X \xrightarrow{\varphi} LR) \oplus (Y \xrightarrow{\psi} LR) = (X + Y \xrightarrow{[\varphi,\psi]} LR).$$

- The forgetful functor $F\colon \mathbf{Coh} \to \mathbf{Rel}$ is a fibration and a cofibration; indeed, given a coherence space $Y = (S, \frown_Y)$ and $f\colon R \nrightarrow S$, the cartesian lifting f^*Y is $(R, \frown_{f^*Y})$, where

$$r_1 \smile_{f^*Y} r_0 \quad \Leftrightarrow \quad \text{there exist } s_1, s_2 \in S \text{ with } r_1\ f\ s_1,\ r_2\ f\ s_2, \text{ and } s_1 \smile_Y s_2.$$

  Similarly, given $X = (R, \frown_X)$ and $f\colon R \nrightarrow S$, the cocartesian lifting f_*X is $(S, \frown_{f_*X})$, where

$$s_1 \frown_{f_*X} s_0 \quad \Leftrightarrow \quad \text{there exist } r_1, r_2 \in R \text{ with } r_1\ f\ s_1,\ r_2\ f\ s_2, \text{ and } r_1 \frown_X r_2.$$

Further, the fibres have products and coproducts; indeed, $F^{-1}(R)$ is just the lattice of coherence space structures on R.

So now we can state the promised representability result:

Proposition 48. *Suppose we have a modular functor* $(F, G)\colon (\mathbb{A}, \mathbb{B}) \to (\mathbb{C}, \mathbb{C})$ *such that*

- (F, G) *is strict modular (i.e.*

  $$FI = I, \quad F(U \otimes V) = FU \otimes FV, \quad G(U \boxtimes_l Y) = FU \otimes GY, \quad \textit{and} \quad G(X \boxtimes_r V) = GX \otimes FV);$$

- $\mathbb{B}$ *has coproducts modulo* $\mathbb{C}$*, and the actions of* $\mathbb{A}$ *on* $\mathbb{B}$ *preserve them (i.e.,*

  $$U^{(R)} \boxtimes_l (X^{(S)} \oplus Y^{(S)}) \cong (U \boxtimes_l X)^{(R \otimes S)} \oplus (U \boxtimes_l Y)^{(R \otimes S)} \quad \textit{and} \quad U^{(R)} \boxtimes_l \mathbf{0}S \cong \mathbf{0}(R \otimes S),$$

  and similarly for the right action);

then $\mathbb{A} \times_{\mathbb{C}} \mathbb{B}$ *is a monoidal category.*

Proof. Given objects $(U^{(R)}, X^{(R)})$ and $(V^{(S)}, Y^{(S)})$ in $\mathbb{A} \times_{\mathbb{C}} \mathbb{B}$, we note that

$$G(U \boxtimes_l Y) = FU \otimes GY = R \otimes S = GX \otimes FV = G(X \boxtimes_r V),$$

so that $(U \boxtimes_l Y, X \boxtimes_r V) \in \mathbb{B} \times_{\mathbb{C}} \mathbb{B}$. Hence we can form $(U \boxtimes_l Y) \oplus (V \boxtimes_r X)$, and similarly on maps. Now we give the monoidal structure on $\mathbb{A} \times_{\mathbb{C}} \mathbb{B}$ as follows. On objects:

$$(U^{(R)}, X^{(R)}) \otimes (V^{(S)}, Y^{(S)}) = (U \otimes V, (U \boxtimes_l Y) \oplus (X \boxtimes_r V))$$

and on maps

$$\begin{array}{ccccccccc} (U, & X) & & (V, & Y) & & (U \otimes V, & (U \boxtimes_l Y) \oplus (X \boxtimes_r V)) \\ \downarrow {\scriptstyle f_U{}^{(\varphi)}} & \downarrow {\scriptstyle f_X{}^{(\varphi)}} & \otimes & \downarrow {\scriptstyle g_V{}^{(\psi)}} & \downarrow {\scriptstyle g_Y{}^{(\psi)}} & = & {\scriptstyle f_U \otimes g_V} \downarrow & \downarrow {\scriptstyle (f_U \boxtimes_l g_Y) \oplus (f_X \boxtimes_r g_V)} \\ (U', & X') & & (V', & Y') & & (U' \otimes V', & (U' \boxtimes_l Y') \oplus (X' \boxtimes_r V')), \end{array}$$

whilst the unit is

$$I = (I, \mathbf{0}(I)).$$

That the tensor is associative and unital follows from the fact that the actions preserve coproducts modulo $\mathbb{C}$; bifunctoriality follows from that of the actions and of $\oplus$. $\square$

Our result extends in an evident way to the closed and $*$-autonomous cases:

Proposition 49. *Suppose further that*

- $(\mathbb{A}, \mathbb{B})$ *is left (respectively right) closed;*

- *(F, G) is a strict left (respectively right) closed modular functor (i.e.,*

  $$F(B \multimap C) = FB \multimap FC, \quad F(D \multimap_{\square l} E) = GD \multimap GE, \quad G(C \multimap_{\square r} D) = FC \multimap GD,$$

  and similarly for right closure);
- *$\mathbb{A}$ has products modulo $\mathbb{C}$;*

then $\mathbb{A} \times_{\mathbb{C}} \mathbb{B}$ is also left (respectively right) closed.

Proof. Similarly to above, we note that for $(V^{(S)}, Y^{(S)})$ and $(W^{(T)}, Z^{(T)})$, we have

$$F(V \multimap W) = FV \multimap FW = S \multimap T = GY \multimap GZ = F(Y \multimap_{\square l} Z),$$

so that we can form $(V \multimap W) \odot (Y \multimap_{\square l} Z)$, and similarly on maps. Thus we set

$$(V^{(S)}, Y^{(S)}) \multimap (W^{(T)}, Z^{(T)}) = ((V \multimap W) \odot (Y \multimap_{\square l} Z), V \multimap_{\square r} Z),$$

with the by now evident action on maps. To ease notation for the next part, let us write $\mathbb{A}(U, V) \times_{\mathbb{C}} \mathbb{B}(X, Y)$ for $\mathbb{A} \times_{\mathbb{C}} \mathbb{B}\big((U^{(R)}, X^{(R)}), (V^{(S)}, Y^{(S)})\big)$, and similarly for $\mathbb{A}(-,-) \times_{\mathbb{C}} \mathbb{A}(-,-)$, etc. We calculate:

$$\begin{aligned}
\mathbb{A} \times_{\mathbb{C}} \mathbb{B}\big((U, X) \otimes (V, Y), (W, Z)\big) &\cong \mathbb{A}(U \otimes V, W) \times_{\mathbb{C}} \mathbb{B}\big((U \boxtimes_l Y) \oplus (X \boxtimes_r V), Z\big) \\
&\cong \mathbb{A}(U \otimes V, W) \times_{\mathbb{C}} \mathbb{B}(U \boxtimes_l Y, Z) \times_{\mathbb{C}} \mathbb{B}(X \boxtimes_r V, Z) \\
&\cong \mathbb{A}(U, V \multimap W) \times_{\mathbb{C}} \mathbb{A}(U, Y \multimap_{\square l} Z) \times_{\mathbb{C}} \mathbb{B}(X, V \multimap_{\square r} Z) \\
&\cong \mathbb{A}\big(U, (V \multimap W) \odot (Y \multimap_{\square l} Z)\big) \times_{\mathbb{C}} \mathbb{B}(X, V \multimap_{\square r} Z) \\
&\cong \mathbb{A} \times_{\mathbb{C}} \mathbb{B}\big((U, X), (V, Y) \multimap (W, Z)\big),
\end{aligned}$$

and can proceed similarly for right closure. □

Proposition 50. *Suppose that:*

- *$(\mathbb{A}, \mathbb{B})$ is (cyclic) $*$-autonomous;*
- *(F, G) is a strict (cyclic) $*$-autonomous functor (i.e., a strict modular functor such that*

  $$(FA)^* = G(A^\bullet), \quad \textit{and} \quad (GB)^* = F(B^\circ)\ );$$
- *$\mathbb{B}$ has coproducts modulo $\mathbb{C}$;*

then $\mathbb{A} \times_{\mathbb{C}} \mathbb{B}$ is $*$-autonomous.

Proof. We equip $\mathbb{A} \times_{\mathbb{C}} \mathbb{B}$ with $*$-autonomous structure via

$$(U^{(R)}, X^{(R)})^* = (X^\circ, U^\bullet),$$

(and hence ${}^*(U, X) = ({}^\bullet X, {}^\circ U)$), noting that $F(X^\circ) = (GX)^* = (FU)^* = G(U^\bullet)$ as required. Evidently if $(\mathbb{A}, \mathbb{B})$ is cyclic $*$-autonomous, then so will be $\mathbb{A} \times_C \mathbb{B}$. Now the same calculations as previously show that this gives a $*$-autonomous structure on $\mathbb{A} \times_{\mathbb{C}} \mathbb{B}$. □

Examples 10.

- Note that the forgetful functor $F\colon \mathbf{Coh} \to \mathbf{Rel}$ is a strict $*$-autonomous functor; and as evidenced above, **Coh** has products and coproducts modulo $\mathbb{C}$. Hence we may form the following category:
  - **Objects** are triples $(R, \frown_1, \frown_2)$ such that $(R, \frown_1)$ and $(R, \frown_2)$ are coherence spaces;
  - **Maps** $(R, \frown_1, \frown_2) \to (S, \frown_1, \frown_2)$ are relations $f\colon R \leftrightarrow S$ which are maps of coherence spaces $(R, \frown_1) \to (S, \frown_1)$ and $(R, \frown_2) \to (S, \frown_2)$.

  This category is symmetric $*$-autonomous. The tensor product of $(R, \frown_1, \frown_2)$ with $(S, \frown_1, \frown_2)$ is given by $(R \times S, \frown_1, \frown_2)$, where

  $$(r_1, s_1) \frown_1 (r_2, s_2) \quad \text{iff} \quad r_1 \frown_1 r_2 \text{ and } s_1 \frown_1 s_2$$

  $$(r_1, s_1) \frown_2 (r_2, s_2) \quad \text{iff} \quad (\, r_1 \frown_1 r_2 \text{ and } s_1 \frown_2 s_2 \,) \quad \text{or} \quad (\, r_1 \frown_2 r_2 \text{ and } s_1 \frown_1 s_2 \,)$$

  and duality is given by

  $$(R, \frown_1, \frown_2)^* = (R, \smile_2, \smile_1).$$

- We can now conclude our treatment of double glueing. Recall from Example 7 that, given the data for double glueing, we can construct a closed (or $*$-autonomous) symmetric module $(\mathbb{A}, \mathbb{B}) := (\mathbb{E} \downarrow L, (\mathbb{E} \downarrow K^{\mathrm{op}})^{\mathrm{op}})$ and a strict closed (or $*$-autonomous) symmetric modular functor $(F, G)\colon (\mathbb{A}, \mathbb{B}) \to (\mathbb{C}, \mathbb{C})$. From 9, we know that $\mathbb{A}$ has products modulo **Rel** and $\mathbb{B}$ coproducts, and hence we can apply 49 to get the symmetric closed (or $*$-autonomous) category $\mathbb{A} \times_C \mathbb{B}$. It's now immediate to see that what we have are precisely the double glueing categories of [HS03].

Chapter 4

Polycategories

4.1 Introduction

Much has been written over the past few years about the area of *multicategories* and their abstract generalisation. Multicategories were introduced as a logical system by Lambek [Lam69] and put on an algebraic footing by Burroni [Bur71], whose theory has lately been independently rediscovered and popularised by Leinster [Lei04] and Hermida [Her00]. The theory is elegant and puts multicategories on a sound mathematical footing. But multicategories have a poor relation: the *polycategories* of [Sza75]. Here the situation is rather different; no elegant way of describing polycategories has yet been found.

There has been one attempt at such a formulation, given in [Kos03]; however, to these eyes it is not wholly successful, for the following reason. Consider first the case of plain **Set**-based multicategories; Burroni, Leinster and Hermida are united in taking what is essentially a 'fibrational' approach to the algebraic structures concerned. We can see some problems with this: there is a definite issue with moving from non-symmetric multicategories to symmetric multicategories, essentially due to a lack of sufficient structure in the category of sets; and similarly, the concept of '$\mathcal{V}$-enriched multicategory' fails entirely to be captured.

One may rectify both these problems by moving from a 'fibrational' to a 'cofibrational' approach; that is, by working with the bicategory **Mod** of *profunctors*. In this context, we can capture successfully and concisely the notion of symmetric multicategory, and, by replacing **Mod** by $\mathcal{V}$-**Mod**, the notion of '$\mathcal{V}$-enriched (symmetric) multicategory'. This much is essentially already known: see [CT03] or [BD98] for example.

Now, Koslowski's approach to polycategories is an attempt to extend the 'fibrational' approach for multicategories; however, it is in fact much more natural to work in the 'cofibrational' context. There, the composition of maps in polycategories is captured by a certain *pseudo-distributive law* between a pseudomonad (describing the 'source arities') and a pseudocomonad (describing the 'target arities') in the bicategory **Mod**.

In the overview to this essay, we talked of the species of mathematical objects which blur the distinction between source and target; polycategories are very much of this class; there is a strong sense of information flowing 'backwards' as well as 'forwards', so we should not be surprised by the necessity of moving from the 2-category of categories (which like the category of sets, has a very strong sense of directedness) to the bicategory of profunctors.

Now, many of the bald assertions made in the penultimate paragraph hide prodigious amounts of coherence. A subsidiary theme of this chapter will therefore be to consider ways in which we can avoid offending the reader with innumerable coherence pasting diagrams.

Much of the material presented here is still in flux, and many of the 'propositions' will in fact be just that: propositions, pending proofs. However, the statements *without* proofs are usually of the 'abstract nonsense' sort which, if not actually in the literature, at least *ought* be in the literature, whilst those *with* proofs include all the non-trivial combinatorics which lies at the heart of this construction. Therefore I hope not to overly strain the reader's credulity on this front.

4.2 Multicategories

We begin by recapping briefly the theory of multicategories, cast in terms of the bicategory of profunctors. The material here summarises material from [BD98] and [Lei04], amongst others. We begin with some notation:

Notation. We write X^* for the free monoid on a set X, and $\Gamma, \Delta, \Sigma, \Lambda$ for typical elements thereof. We will use comma to denote the concatenation operation on X^*, as in "Γ, Δ"; and we will tend to conflate elements of X with their image in X^*. Given $\Gamma = x_1, \dots, x_n \in X^*$, we define $|\Gamma| = n$, and given $\sigma \in S_n$, write $\sigma\Gamma$ for the element $x_{\sigma(1)}, \dots, x_{\sigma(n)} \in X^*$.

Definition 27. A **symmetric multicategory** $\mathbb{M}$ consists of:

- A set $\mathrm{ob}\,\mathbb{M}$ of **objects**;
- For every $\Gamma \in (\mathrm{ob}\,\mathbb{M})^*$ and $y \in \mathrm{ob}\,\mathbb{M}$, a set $\mathbb{M}(\Gamma; y)$ of **multimaps** from Γ to y (we write a typical element of such as $f : \Gamma \to y$); further, for every $\sigma \in S_{|\Gamma|}$, an **exchange isomorphism**
$$\mathbb{M}(\Gamma; y) \to \mathbb{M}(\sigma\Gamma; y).$$

This data satisfies axioms expressing the fact that exchange isomorphisms compose as expected. Furthermore, we have:

- For every $x \in \mathrm{ob}\,\mathbb{M}$, an **identity map** $\mathrm{id}_x \in \mathbb{M}(x; x)$;
- For every $\Gamma, \Delta_1, \Delta_2 \in (\mathrm{ob}\,\mathbb{M})^*$ and $y, z \in \mathrm{ob}\,\mathbb{M}$, a **composition map**
$$\mathbb{M}(\Gamma; y) \times \mathbb{M}(\Delta_1, y, \Delta_2; z) \to \mathbb{M}(\Delta_1, \Gamma, \Delta_2; z),$$

all subject to axioms expressing that composition is associative and unital, and compatible with the exchange isomorphisms.

Now, this data expresses composition as a *local* operation performed between two multimaps; however, there is an alternative *global* view, where we 'multicompose' a family of multimaps $g_i \colon \Gamma_i \to y_i$ with a multimap $f \colon y_1, \ldots, y_n \to z$. The transit from one view to the other is straightforward: we recover the global composition from the local by performing, in any order, the local compositions of the g_i's with f – and the axioms for local composition ensure that this is uniquely defined. Conversely, we can recover the local composition from the global by setting all but one of the g_i's to be the identity.

We can express this global composition as follows: fix the object set $X = \operatorname{ob}\mathbb{M}$, and consider it as a discrete category. Now, writing T for the symmetric strict monoidal category monad T on **Cat**, we consider the functor category $[(TX)^{\mathrm{op}} \times X, \mathbf{Set}]$. To give an object F of this is to give sets of multimaps as above, together with coherent exchange isomorphisms. Further, this category has a 'substitution' monoidal structure given by

$$(G \otimes F)(\Gamma; z) = \sum_{\substack{k \in \mathbb{N} \\ y_1, \ldots, y_k \in X}} \int^{\Delta_1, \ldots, \Delta_k \in TX} G(y_1, \ldots, y_k; z) \times \prod_{i=1}^{k} F(\Delta_i; y_i) \times TX(\Gamma, \bigotimes_{i=1}^{k} \Delta_i),$$

and

$$I(\Gamma; z) = \begin{cases} \emptyset & \text{if } \Gamma \neq z \\ \{*\} & \text{otherwise;} \end{cases}$$

and to give a multicategory is precisely to give a monoid with respect to this monoidal structure. Indeed, unpacking the above definition, we see that $(G \otimes F)(\Gamma; z)$ can be described as follows. Let $\Delta_1, \ldots, \Delta_k \in (\operatorname{ob}\mathbb{M})^*$ be such that

- $\sum |\Delta_i| = |\Gamma| = n$;
- there exists $\sigma \in S_n$ such that $\sigma\Gamma = \Delta_1, \ldots, \Delta_k$.

Then let $f_i \colon \Delta_i \to y_i$ in F (for $i = 1, \ldots, k$), and let $g \colon y_1, \ldots, y_k \to z$ in G. This gives us a typical element of $(G \otimes F)(\Gamma; z)$, which we visualise as

$$\begin{array}{c} \Gamma \\ \downarrow{\scriptstyle \sigma} \\ \Delta_1, \ldots, \Delta_k \\ \downarrow{\scriptstyle f_1, \ldots, f_k} \\ y_1, \ldots, y_k \\ \downarrow{\scriptstyle g} \\ z. \end{array}$$

At this point we observe that we can express this more abstractly if we are prepared to take the following on trust:

Proposition 51. *T lifts from a monad on* **Cat** *to a pseudomonad on* **Mod**.

Equipped with this, we can form the 'Kleisli bicategory' $Kl(T)$ of the pseudomonad T; its objects are those of **Mod**, and it has $Kl(T)(A, B) = \mathbf{Mod}(A, TB)$, with the obvious composition and identities furnished by the structure of the pseudomonad T. It is a long and tedious calculation with pasting diagrams to show that this genuinely does yield a bicategory, but we can now see that the monoidal structure on $[(TX)^{\mathrm{op}} \times X, \mathbf{Set}]$ described above is just horizontal composition in $Kl(T)(X, X)$. Hence we arrive at an alternative, but equivalent, definition:

Definition 28. A symmetric multicategory is a monad on a discrete object X in the bicategory $Kl(T)$.

4.3 From multicategories to polycategories

We recall now the notion of (symmetric) *polycategory*.

Definition 29. A **symmetric polycategory** $\mathbb{P}$ consists of

- A set $\mathrm{ob}\,\mathbb{P}$ of **objects**,
- For each pair (Γ, Δ) of elements of $(\mathrm{ob}\,\mathbb{P})^*$, a set $\mathbb{P}(\Gamma; \Delta)$ of **polymaps** from Γ to Δ,
- For each $\Gamma, \Delta \in (\mathrm{ob}\,\mathbb{P})^*$, each $\sigma \in S_{|\Gamma|}$ and $\tau \in S_{|\Delta|}$, **exchange isomorphisms**

$$\mathbb{P}(\Gamma; \Delta) \to \mathbb{P}(\sigma\Gamma; \tau\Delta),$$

- For each $x \in \mathrm{ob}\,\mathbb{P}$, an **identity map** $\mathrm{id}_x \in \mathbb{P}(x; x)$, and
- For $\Gamma, \Delta_1, \Delta_2, \Lambda_1, \Lambda_2, \Sigma \in (\mathrm{ob}\,\mathbb{P})^*$, and $x \in \mathrm{ob}\,\mathbb{P}$, **composition maps**

$$\mathbb{P}(\Gamma; \Delta_1, x, \Delta_2) \times \mathbb{P}(\Lambda_1, x, \Lambda_2; \Sigma) \to \mathbb{P}(\Lambda_1, \Gamma, \Lambda_2; \Delta_1, \Sigma, \Delta_2),$$

subject to laws expressing the associativity and unitality of composition, expressing that the exchange isomorphisms compose, and that they are compatible with associativity.

For the full details of this, we refer the reader to [Sza75] or [CS97]. We recover the notion of a **multicategory** if we assert that $\mathbb{P}(\Gamma; \Delta)$ is empty unless Δ is a singleton.

Now, as before, we may shift from a *local* notion of composition to a *global* one; we aim to define a 'polycomposite' of two families of composable polymaps, but first we need to say what we mean by composable.

Definition 30. Let $\mathbf{f} := \{f_m \colon \Lambda_m \to \Sigma_m\}_{1 \leqslant m \leqslant j}$ and $\mathbf{g} := \{g_n \colon \Gamma_n \to \Delta_n\}_{1 \leqslant n \leqslant k}$ be families of polymaps over a set of object X, such that

$$\sum |\Sigma_m| = \sum |\Gamma_n| = l.$$

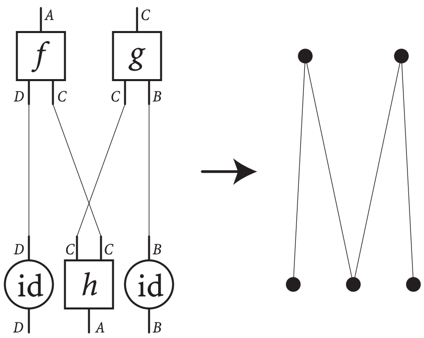
We say that a permutation $\sigma \in S_l$ is a **matching** if $\sigma(\Sigma_1, \ldots, \Sigma_j) = (\Gamma_1, \ldots, \Gamma_k)$.

Informally, this matching shows 'which output has been plugged into which input', and so we can define a composite map $\mathbf{g} \circ_\sigma \mathbf{f}$. However, we would like our global notion of composition to coincide with our local notion; hence, we should be able to generate the global composition maps from repeated application of local compositions. However, not all matchings have this property. Let us define what the 'suitable' matchings are:

Definition 31. Given a matching σ for $\mathbf{f}$ and $\mathbf{g}$, form the bipartite graph G_0 as follows: its two vertex sets are $V_\mathbf{f} = \sum \Sigma_m$ and $V_\mathbf{g} = \sum \Gamma_n$, and we join a vertex of $V_\mathbf{f}$ and a vertex of $V_\mathbf{g}$ just when they are paired under the action of σ.

Now, for each $m \in \{1, \ldots, j\}$, contract the vertices of G_0 corresponding to Σ_m to a point, and similarly for the Γ_n's; this gives us a bipartite multigraph G_1 with vertex set $n + m$; we shall say that the matching σ is **suitable** just when G_1 is acyclic, connected and has no multiple edges.

For example:



and we see that this matching is indeed suitable. It's not hard to prove that

Proposition 52. *A matching σ is suitable if and only if the associated composite map $\mathbf{g} \circ_\sigma \mathbf{f}$ can be formed by repeated local compositions.*

Hence our global notion of composition of polymaps is given by composing a family $\mathbf{f}$ with a family $\mathbf{g}$ along a suitable matching σ.

How can we express this more abstractly? Ideally we would like to express polycategories as monads in a suitable bicategory $\mathscr{B}$. We observe the following:

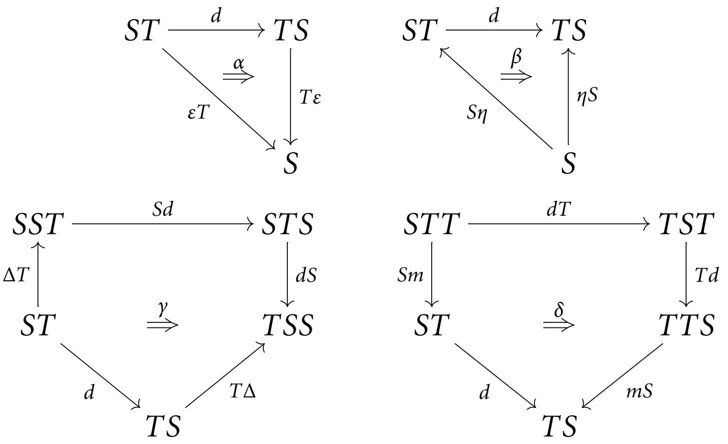
Proposition 53. *T is a pseudocomonad as well as a pseudomonad on* **Mod**.

Proof (sketch). Each component of the natural transformations $\eta\colon 1 \Rightarrow T$ and $\mu\colon TT \Rightarrow T$ for the pseudomonad T on **Mod** has a right adjoint; and these components taken together equip T with counit and comultiplication natural transformations $T \Rightarrow 1$ and $TT \Rightarrow T$. ☐

When regarded in this way, we shall write 'S' as a pseudonym for T to avoid confusion. Now, the structure we have described above should fit into this picture as a *pseudo-distributive law* between the monad T and the comonad S; that is, there should be a pseudo-natural transformation

$$d\colon ST \Rightarrow TS$$

along with invertible modifications



subject to ten coherence laws (see [Tan04]). Now, we already know what d should look like: indeed, in light of the above discussion, we would like to give its component at a discrete category $X \in$ **Mod** by taking for $d_X(\{\Sigma_m\}_{1\leqslant m\leqslant j}; \{\Gamma_n\}_{1\leqslant n\leqslant k})$ the set of admissible matchings of $\{\Sigma_m\}$ with $\{\Gamma_n\}$.

So, supposing we are able to produce such a pseudo-distributive law, how do we produce polycategories out of it? The answer lies in forming the 'two-sided Kleisli bicategory' of the pseudo-distributive law. Since this gadget may not be familiar, we describe it first one dimension down:

Let $\mathbb{C}$ be a category, let T be a monad and S a comonad on $\mathbb{C}$, and let $d\colon ST \to TS$ be distributive law of the monad over the comonad; so the four diagrams above which were filled in with invertible modifications now commute on the nose. Then we define

Definition 32. The two-sided Kleisli category $Kl(d)$ of the distributive law d has:

- **Objects** those of $\mathbb{C}$,
- **Maps** $A \to B$ in $Kl(d)$ given by maps $SA \to TB$ in $\mathbb{C}$,

with

- **Identity maps** $\mathrm{id}_A\colon A \to A$ in $Kl(d)$ given by the map

$$SA \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} TA$$

in $\mathbb{C}$;

- **Composition** for maps $f\colon A \to B$ and $g\colon B \to C$ in $Kl(d)$ given by the map

$$SA \xrightarrow{\Delta_A} SSA \xrightarrow{Sf} STB \xrightarrow{d_B} TSB \xrightarrow{Tg} TTC \xrightarrow{m_C} TC$$

in $\mathbb{C}$.

We claim that such a construction can be emulated in a bicategory, given a pseudo-distributive law of a pseudomonad over a pseudocomonad. At present, no proof of such a fact exists in the literature; but given the strongly analogous results of [Tan04], I ask the reader's indulgence on this point. Hence we are now in a position to set out our goals for the rest of this chapter: to prove the existence of the aforementioned pseudo-distributive law in the least unpleasant way possible.

4.4 The pseudomonad T on Mod

Before we attempt this, we need to describe in a little more detail the pseudomonad T on **Mod**. First, the homomorphism T itself. Its action on objects is the same as for **Cat**; and on morphisms, its action is given as follows: for $f\colon \mathbb{A} \to \mathbb{B}$ in **Mod**,

$$(Tf)(b_1, \ldots, b_n; a_1, \ldots, a_m) = \begin{cases} \sum_{\sigma \in S_n} \prod_{i=1}^{n} f(b_i; a_{\sigma(i)}) & \text{if } m = n; \\ \emptyset & \text{otherwise,} \end{cases}$$

with the evident action thus induced on 2-cells.

Now, for the unit and multiplication maps, recall that given a functor $f\colon \mathbb{A} \to \mathbb{B}$ in **Cat**, we write f_* and f^* for the profunctors:

$$f_*\colon \mathbb{B}^{\mathrm{op}} \times \mathbb{A} \to \mathbf{Set}; \quad f_*(b; a) = \mathbb{B}(b, fa),$$
$$f^*\colon \mathbb{A}^{\mathrm{op}} \times \mathbb{B} \to \mathbf{Set}; \quad f^*(a; b) = \mathbb{B}(fa, b).$$

If we write e and t for the unit and multiplication of the 2-monad T on **Cat**, then the unit η and multiplication m of the pseudomonad T on **Mod** have components $\eta_{\mathbb{A}} = (e_{\mathbb{A}})_*$ and $m_{\mathbb{A}} = (t_{\mathbb{A}})_*$ respectively. Further, when viewed as a pseudocomonad S, the counit ε and comultiplication Δ have components $\varepsilon_{\mathbb{A}} = (e_{\mathbb{A}})^*$ and $\Delta_{\mathbb{A}} = (t_{\mathbb{A}})^*$ respectively. We shall also need the following result:

Proposition 54. *Given $f\colon \mathbb{A} \to \mathbb{B}$ in* **Cat**, *we have $T(f_*) = (Tf)_*$ and $T(f^*) = (Tf)^*$.*

Proof. We have

$$\begin{aligned} T(f_*)(b_1,\ldots,b_n;a_1,\ldots,a_n) &= \sum_{\sigma\in S_n}\prod_{i=1}^{n} f_*(b_i;a_{\sigma(i)}) \\ &= \sum_{\sigma\in S_n}\prod_{i=1}^{n} \mathbb{B}(b_i;f(a_{\sigma(i)})) \\ &= T\mathbb{B}(b_1,\ldots,b_n;f(a_1),\ldots,f(a_n)) \\ &= T\mathbb{B}(b_1,\ldots,b_n;(Tf)(a_1,\ldots,a_n)) \\ &= (Tf)_*(b_1,\ldots,b_n;a_1,\ldots,a_n) \end{aligned}$$

and similarly for f^*. □

The use to which we shall put this is as follows: we will need to consider arrows such as $S\eta_1\colon S1 \to ST1$ in **Mod**, and it is simplest to define this simply as $(Se_1)_*$; and the above proposition allows us to do so, since $S\eta_1 = S(e_1)_* = (Se_1)_*$; similarly for the other structure maps.

4.5 Free monoidal structure at 1

Now, our aim is to build a distributive law $d\colon ST \Rightarrow TS$ on **Mod**; but initally, we will be rather more modest. We wish to extract the combinatorial core of the proof; and to do so, we shall only exhibit the desired data at the terminal category 1. Later we shall utilise the well-behaved nature of T to attempt to lift this structure from 1 to all of **Mod**. First, we shall describe the categories $T1$, T^21 and T^31. We present $T1$ as the category with:

- **Objects** the natural numbers;
- **Maps** $\sigma\colon n \to m$ bijections of n with m.

Note that the unique functor $a_0\colon T1 \to 1$ exhibits $T1$ as a T-algebra. Next, we present T^21 as the following category:

- **Objects** are order-preserving maps $\varphi\colon n_\varphi \to m_\varphi$, where $n_\varphi, m_\varphi \in \mathbb{N}$. We write such an object simply as φ, with the convention that φ has domain and codomain n_φ and m_φ respectively.
- **Maps** $f\colon \varphi \to \psi$ are pairs of bijections $f_n\colon n_\varphi \to n_\psi$ and $f_m\colon m_\varphi \to m_\psi$ such that the following diagram commutes:

$$\begin{array}{ccc} n_\varphi & \xrightarrow{f_n} & n_\psi \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ m_\varphi & \xrightarrow[f_m]{} & m_\psi. \end{array}$$

It may not be immediately obvious that this *is* a presentation of $T^2 1$. The picture is as follows: an object φ of $T^2 1$ is to be thought of as a collection of n_φ points partitioned into m_φ parts in accordance with φ. Given such an object, one can permute internally any of its m_φ parts, or can in fact permute the set of m_φ parts itself; and a typical map describes such a permutation. For example, the objects

$$\begin{aligned} \varphi\colon 5 &\to 4 \\ 1,2,3,4,5 &\mapsto 1,1,3,4,4 \end{aligned} \qquad\qquad \begin{aligned} \psi\colon 5 &\to 4 \\ 1,2,3,4,5 &\mapsto 2,2,3,4,4 \end{aligned}$$

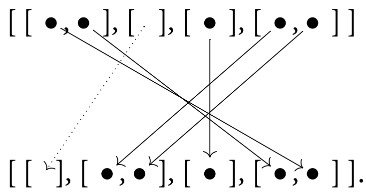
should be visualised as

$$[\,[\,\bullet,\bullet\,]\,,[\;]\,,[\,\bullet\,]\,,[\,\bullet,\bullet\,]\,] \qquad \text{and} \qquad [\,[\;]\,,[\,\bullet,\bullet\,]\,,[\,\bullet\,]\,,[\,\bullet,\bullet\,]\,]$$

respectively, whilst a typical map $\varphi \to \psi$ is given by

$$\begin{aligned} f_n\colon 5 &\to 5 \\ 1,2,3,4,5 &\mapsto 5,4,3,1,2 \end{aligned} \qquad\qquad \begin{aligned} f_m\colon 4 &\to 4 \\ 1,2,3,4 &\mapsto 4,1,3,2 \end{aligned}$$

and should be visualised as



And hence we see that this indeed a valid presentation of $T^2 1$. Note also the two canonical projection functors

$$\begin{aligned} \pi_n\colon T^2 1 &\to T1 \\ \varphi &\mapsto n_\varphi \\ (f_n, f_m) &\mapsto f_n \end{aligned} \qquad\qquad \begin{aligned} \pi_m\colon T^2 1 &\to T1 \\ \varphi &\mapsto m_\varphi \\ (f_n, f_m) &\mapsto f_m. \end{aligned}$$

The first of these, $a_1\colon T^2 1 \to T1$, exhibits $T1$ as a (free) T-algebra, whilst the second is in fact $Ta_0\colon T^2 1 \to T1$. Proceeding similarly, we can present $T^3 1$ as follows:

- **Objects** are diagrams $\varphi = n_\varphi \xrightarrow{\varphi_1} m_\varphi \xrightarrow{\varphi_2} r_\varphi$ in the category of finite ordinals and order preserving maps;
- **Maps** $f\colon \varphi \to \psi$ are triples (f_n, f_m, f_r) of bijections making

$$\begin{array}{ccc} n_\varphi & \xrightarrow{f_n} & n_\psi \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \psi_1} \\ m_\varphi & \xrightarrow{f_m} & m_\psi \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \psi_2} \\ r_\varphi & \xrightarrow[f_r]{} & r_\psi. \end{array}$$

commute.

There are now three canonical projections onto $T^2 1$:

$$\begin{array}{lll} \pi_{nm} \colon T^3 1 \to T^2 1 & \pi_{nr} \colon T^3 1 \to T^2 1 & \pi_{mr} \colon T^3 1 \to T^2 1 \\ \varphi \mapsto (n_\varphi \xrightarrow{\varphi_1} m_\varphi) & \varphi \mapsto (n_\varphi \xrightarrow{\varphi_2\varphi_1} r_\varphi) & \varphi \mapsto (m_\varphi \xrightarrow{\varphi_2} r_\varphi) \\ (f_n, f_m, f_r) \mapsto (f_n, f_m) & (f_n, f_m, f_r) \mapsto (f_n, f_r) & (f_n, f_m, f_r) \mapsto (f_m, f_r), \end{array}$$

the first of which is the map $a_2 \colon T^3 1 \to T^2 1$ exhibiting $T^2 1$ as a T-algebra, the second of which is Ta_1, and the third of which is $T^2 a_0$.

4.6 Spans

Before we can get on to stating and proving the existence of the distributive law, we shall need a few preliminaries about acyclic and connected graphs. We would like to capture the combinatorial essence of these constructions categorically, allowing a smooth presentation of the proof of the distributive law. Happily, this is possible.

The main objects of our attention are *spans in* **FinCard**, i.e., diagrams $n \leftarrow k \to m$ in the category of finite cardinals and all maps. When we write 'span' in future, it should be read as 'span in **FinCard**' unless otherwise stated. We also make use without comment of the evident inclusions **FinOrd** $\to$ **FinCard** and $T1 \to$ **FinCard**.

Now, each span $n \leftarrow k \to m$ determines a (categorist's) graph $k \rightrightarrows n + m$; if we forget the orientation of the edges of this graph, we get a (combinatorialist's) undirected multigraph. We say that a span $n \leftarrow k \to m$ is **acyclic** or **connected** if the associated multigraph is so. Note that the *acyclic* condition includes the assertion that there are no multiple edges.

Proposition 55. *Given a span $n \xleftarrow{\theta_1} k \xrightarrow{\theta_2} m$, the number of connected components of the graph induced by the span is given by the cardinality of r in the pushout diagram*

$$\begin{array}{ccc} k & \xrightarrow{\theta_2} & m \\ {\scriptstyle\theta_1}\downarrow & & \downarrow{\scriptstyle\tau_2} \\ n & \xrightarrow{\tau_1} & r \end{array} \qquad (*)$$

in **FinCard**.

Proof. Given the above pushout diagram, set $n_i = \tau_1^{-1}(i)$ and $m_i = \tau_2^{-1}(i)$ (for $i = 1, \ldots, r$). Now we observe that, for $i \neq j$, we have

$$\theta_1^{-1}(n_i) \cap \theta_2^{-1}(m_j) = \theta_1^{-1}(n_i) \cap \theta_1^{-1}(n_j) = \emptyset,$$

so that induced graph of the span has at least r unconnected parts (with respective vertex sets $n_i + m_i$). On the other hand, if the induced graph G had strictly more than r connected components, we could find vertex sets $v_1, \ldots, v_{r+1}$ which partition $v(G)$, and for which

$$v_i \cap v_j = \emptyset \text{ for } i \neq j. \tag{†}$$

But now define maps $\tau_1 : n \to r+1$ and $\tau_2 : m \to r+1$ by letting $\tau_i(x)$ be the k such that $x \in v_k$. Evidently, then, $\tau_1(\theta_1(x)) = \tau_2(\theta_2(x))$ for all $a \in k$, by equation (†), so we have a commuting diagram

$$\begin{array}{ccc} k & \xrightarrow{\theta_2} & m \\ {\scriptstyle \theta_1}\downarrow & & \downarrow{\scriptstyle \tau_2} \\ n & \xrightarrow[\tau_1]{} & r+1 \end{array}$$

for which the bottom right vertex does not factor through r, contradicting the assumption that r was a pushout. Hence G has precisely r connected components. ☐

Corollary 56. *A span $n \xleftarrow{\theta_1} k \xrightarrow{\theta_2} m$ is connected if and only if the diagram*

$$\begin{array}{ccc} k & \xrightarrow{\theta_2} & m \\ {\scriptstyle \theta_1}\downarrow & & \downarrow \\ n & \longrightarrow & 1 \end{array}$$

is a pushout in **FinCard**.

Proposition 57. *A span $n \xleftarrow{\theta_1} k \xrightarrow{\theta_2} m$ is acyclic if and only, for every monomorphism $\iota\colon k' \hookrightarrow k$,*

$$\begin{array}{ccc} k & \xrightarrow{\theta_2} & m \\ {\scriptstyle \theta_1}\downarrow & & \downarrow \\ n & \longrightarrow & r \end{array} \quad \textit{a pushout implies} \quad \begin{array}{ccc} k' & \xrightarrow{\theta_2\iota} & m \\ {\scriptstyle \theta_1\iota}\downarrow & & \downarrow \\ n & \longrightarrow & r \end{array} \quad \textit{not a pushout.}$$

Proof. Suppose the left hand diagram is a pushout; then the associated graph G of the span has r connected components.

Suppose first that G is acyclic, and $\iota\colon k' \hookrightarrow k$. Then the graph G' associated to the span $n \xleftarrow{\theta_1\iota} k' \xrightarrow{\theta_2\iota} m$ has the same vertices as G but strictly fewer edges; and since G is acyclic, G' must have $> r$ connected components, and hence r cannot be a pushout for the right-hand diagram.

Conversely, if G is not acyclic, then we can remove some edge of G without changing its number of connected components; and thus we obtain some monomorphism $\iota\colon k' \hookrightarrow k$ making the right-hand diagram a pushout. ☐

Proposition 58. *Suppose we have a commuting diagram*

$$\begin{array}{ccc} k & \xrightarrow{\theta_2} & m \\ {\scriptstyle\theta_1}\downarrow & & \downarrow{\scriptstyle\varphi_2} \\ n & \xrightarrow[\varphi_1]{} & r. \end{array} \tag{$*$}$$

Then the spans $m^{(i)} \leftarrow k^{(i)} \rightarrow n^{(i)}$ (for $i = 1, \ldots, r$) induced by pulling back along elements $i\colon 1 \rightarrow r$ are all connected if and only if $(*)$ is a pushout.

Proof. Suppose all the induced spans are connected; then each diagram

$$\begin{array}{ccc} k^{(i)} & \xrightarrow{\theta_2^{(i)}} & m^{(i)} \\ {\scriptstyle\theta_1^{(i)}}\downarrow & & \downarrow \\ n^{(i)} & \longrightarrow & 1 \end{array}$$

is a pushout; hence the diagram

$$\begin{array}{ccc} \sum_i k^{(i)} & \xrightarrow{\sum_i \theta_2^{(i)}} & \sum_i m^{(i)} \\ {\scriptstyle\sum_i \theta_1^{(i)}}\downarrow & & \downarrow \\ \sum_i n^{(i)} & \longrightarrow & r \end{array}$$

is also a pushout, whence it follows that $(*)$ is itself a pushout.

Conversely, if $(*)$ is a pushout, then pulling this back along the map $i\colon 1 \rightarrow r$ yields another pushout in **FinCard**, so that each induced span is connected. ☐

Proposition 59. *Let G be a graph with finite edge and vertex sets. Any two of the following conditions implies the third:*

- *G is acyclic;*
- *G is connected;*
- $|v(G)| = |e(G)| + 1$.

Proof.

- If G is acyclic and connected, then it is a tree, and so $|v(G)| = |e(G)| + 1$;
- if G is connected with $|v(G)| = |e(G)| + 1$, then it is minimally connected, hence a tree, and so acyclic;
- if G is acyclic with $|v(G)| = |e(G)| + 1$, then it is maximally acyclic, hence a tree, and so connected.

□

Corollary 60. *A span $n \xleftarrow{\theta_1} k \xrightarrow{\theta_2} m$ is acyclic and connected if and only if the diagram*

$$\begin{array}{ccc} k & \xrightarrow{\theta_2} & m \\ {\scriptstyle\theta_1}\downarrow & & \downarrow \\ n & \longrightarrow & 1 \end{array}$$

is a pushout in **FinCard**, *and $n + m = k + 1$.*

Corollary 61. *Suppose we have a commuting diagram*

$$\begin{array}{ccc} k & \xrightarrow{\theta_2} & m \\ {\scriptstyle\theta_1}\downarrow & & \downarrow{\scriptstyle\varphi_2} \\ n & \xrightarrow[\varphi_1]{} & r. \end{array} \qquad (*)$$

then the induced spans $m^{(i)} \leftarrow k^{(i)} \rightarrow n^{(i)}$ (for $i = 1, \ldots, r$) are acyclic and connected if and only if $(*)$ is a pushout and $m + n = k + r$.

4.7 The distributive law at 1

Recall that we are using S as a pseudonym for T when viewed as a pseudocomonad rather than a pseudomonad on **Mod**. In particular, the action of S and T on objects is the same, so we may write $TS1$, $ST1$, $STS1$, etc., to refer to $T^2 1$ or $T^3 1$. We are now ready to give the component of our distributive law at 1; we define $d_1 \colon TS1^{\mathrm{op}} \times ST1 \to \mathbf{Set}$ as follows:

- **On objects**: elements $f \in d_1(\varphi; \psi)$ are bijections f_n fitting into the diagram

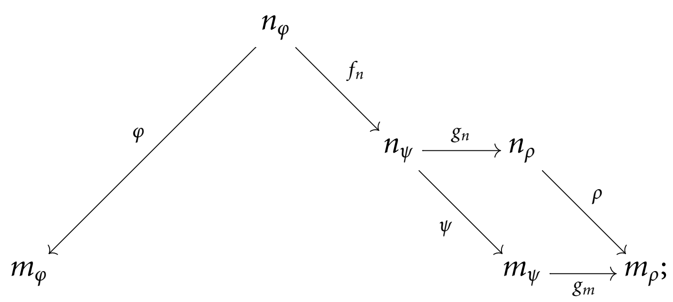
$$\begin{array}{ccc} n_\varphi & \xrightarrow{f_n} & n_\psi \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ m_\varphi & & m_\psi \end{array}$$

  such that the span $m_\varphi \xleftarrow{\varphi} n_\varphi \xrightarrow{\psi \circ f_n} m_\psi$ is acyclic and connected.

- **On maps**: Let $g \colon \psi \to \rho$ in $ST1$ and let $f \in d_1(\varphi; \psi)$. Then we give an element $g \circ f \in d_1(\varphi; \rho)$ by

$$\begin{array}{ccc} n_\varphi & \xrightarrow{g_n \circ f_n} & n_\rho \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\rho} \\ m_\varphi & & m_\rho \end{array}$$

This action is evidently functorial, but we still need to check that it really does yield an element of $d_1(\varphi; \rho)$; that is, we need the associated span to be acyclic and connected. But this span is the top path of the diagram



and therefore also the bottom path, since the right-hand square commutes. But since g_m is an isomorphism, the graph induced by the span $m_\varphi \xleftarrow{\varphi} n_\varphi \xrightarrow{\psi f_n} m_\psi$ is isomorphic to the graph induced by the span $m_\varphi \xleftarrow{\varphi} n_\varphi \xrightarrow{g_m \psi f_n} m_\rho$, and hence the latter is acyclic and connected since the former is. So we have a well-defined left action of $ST1$ on d_1; proceeding similarly we give a well-defined right action of $TS1$ on d_1.

Now, we shall also need the distributive law at $S1$; for the moment, we shall simply state what it should be, and later on we will check its consistency. So, we define the functor $d_{S1} \colon TSS1^{\mathrm{op}} \times STS1 \to \mathbf{Set}$ as follows:

- **On objects**: elements $f \in d_{S1}(\varphi; \psi)$ are pairs of bijections f_n and f_m fitting in the diagram

$$\begin{array}{ccc} n_\varphi & \xrightarrow{f_n} & n_\psi \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \psi_1} \\ m_\varphi & \xrightarrow[f_m]{} & m_\psi \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \psi_2} \\ r_\varphi & & r_\psi \end{array}$$

  such that the span $r_\varphi \xleftarrow{\varphi_2} m_\varphi \xrightarrow{\psi_2 \circ f_m} r_\psi$ is acyclic and connected.

- **On maps**: Let $g \colon \psi \to \rho$ in $STS1$ and let $f \in d_{S1}(\varphi; \psi)$. Then we give an element

$g \circ f \in d_{S1}(\varphi; \rho)$ by

$$\begin{array}{ccc} n_\varphi & \xrightarrow{g_n \circ f_n} & n_\rho \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \rho_1} \\ m_\varphi & \xrightarrow[g_m \circ f_m]{} & m_\rho \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \rho_2} \\ r_\varphi & & r_\rho. \end{array}$$

For this to be an element of $d_{S1}(\varphi; \rho)$, we need firstly that

$$\begin{array}{ccccc} n_\varphi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\rho \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \rho_1} \\ m_\varphi & \xrightarrow[f_m]{} & m_\psi & \xrightarrow[g_m]{} & m_\rho, \end{array}$$

commutes; and indeed it does, the left hand square since $f \in d_{S1}(\varphi; \psi)$ and the right hand square since g is a map in $STS1$. Furthermore, we need that the span it induces is acyclic and connected. But this span is the top path of the diagram

$$\begin{array}{ccccccc} & & m_\varphi & & & & \\ & {\scriptstyle \varphi_2}\swarrow & & \searrow{\scriptstyle f_m} & & & \\ & & & & m_\psi & \xrightarrow{g_m} & m_\rho \\ & & & & & {\scriptstyle \psi_2}\searrow & & \searrow{\scriptstyle \rho_2} \\ r_\varphi & & & & & & r_\psi & \xrightarrow[g_r]{} & r_\rho; \end{array}$$

and therefore also the bottom path, since the right-hand square commutes. So as before, the graph induced by this span is acyclic and connected, since g_r is an isomorphism. So we have a well-defined left action, and can give a right action similarly.

Similarly, we state the definition of the functor $Sd_1 \colon STS1^{\mathrm{op}} \times SST1 \to \mathbf{Set}$.

- **On objects**: elements $f \in Sd_1(\varphi; \psi)$ are pairs of bijections $f_n \colon n_\varphi \to n_\psi$ and $f_r \colon r_\varphi \to r_\psi$ fitting in the diagram

$$\begin{array}{ccc} n_\varphi & \xrightarrow{g_n} & n_\psi \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \psi_1} \\ m_\varphi & & m_\psi \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \psi_2} \\ r_\varphi & \xrightarrow[g_r]{} & r_\psi \end{array}$$

such that for each $i = 1, \ldots, r_\psi$, the induced spans

$$\begin{array}{ccccc} & & n_\varphi^{(i)} & & \\ & \swarrow^{\varphi_1^{(i)}} & & \searrow^{g_n^{(i)}} & \\ m_\varphi^{(i)} & & & & n_\psi^{(i)} \\ & & & & \downarrow{\scriptstyle \psi_1^{(i)}} \\ & & & & m_\psi^{(i)} \end{array}$$

are acyclic and connected.

Let us clarify what the induced spans referred to above actually are. We have the commuting diagram

$$\begin{array}{ccccc} n_\varphi & \xrightarrow{g_n} & n_\psi & \xrightarrow{\psi_1} & m_\psi \\ {\scriptstyle \varphi_1}\downarrow & & & & \downarrow{\scriptstyle \psi_2} \\ m_\varphi & \xrightarrow[\varphi_2]{} & r_\varphi & \xrightarrow[g_r]{} & r_\psi \end{array} \qquad (*)$$

and the induced spans are the result of pulling this diagram back along elements $i\colon 1 \to r_\psi$. By the results of the previous section, these spans are all acyclic and connected if and only if $(*)$ is a pushout and $r_\psi + n_\varphi = m_\varphi + m_\psi$.

- **On maps**: Let $g\colon \psi \to \rho$ in $SST1$ and let $f \in Sd_1(\varphi; \psi)$. Then we give an element $g \circ f \in Sd_1(\varphi; \rho)$ by

$$\begin{array}{ccc} n_\varphi & \xrightarrow{g_n \circ f_n} & n_\rho \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \rho_1} \\ m_\varphi & & m_\rho \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \rho_2} \\ r_\varphi & \xrightarrow[g_r \circ f_r]{} & r_\rho . \end{array}$$

  As before, this makes

$$\begin{array}{ccccc} n_\varphi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\rho \\ {\scriptstyle \varphi_2\varphi_1}\downarrow & & \downarrow{\scriptstyle \psi_2\psi_1} & & \downarrow{\scriptstyle \rho_2\rho_1} \\ r_\varphi & \xrightarrow[f_r]{} & r_\psi & \xrightarrow[g_r]{} & r_\rho \end{array}$$

  commute as required; it remains to show that the spans it induces are acyclic and connected. By the above remarks, it will do to show firstly that $r_\rho + n_\varphi = m_\varphi + m_\rho$, which is true since $r_\psi + n_\varphi = m_\varphi + m_\psi$, $r_\psi = r_\rho$ and $m_\psi = m_\rho$, and secondly that

$$\begin{array}{ccccccc} n_\varphi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\rho & \xrightarrow{\rho_1} & m_\rho \\ {\scriptstyle \varphi_1}\downarrow & & & & & & \downarrow{\scriptstyle \rho_2} \\ m_\varphi & \xrightarrow[\varphi_2]{} & r_\varphi & \xrightarrow[f_r]{} & r_\psi & \xrightarrow[g_r]{} & r_\rho \end{array}$$

is a pushout; for which we observe that both squares in

$$\begin{array}{ccccccc} n_\varphi & \xrightarrow{f_n} & n_\psi & \xrightarrow{\psi_1} & m_\psi & \xrightarrow{g_m} & m_\rho \\ {\scriptstyle \varphi_1}\downarrow & & & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \rho_2} \\ m_\varphi & \xrightarrow[\varphi_2]{} & r_\varphi & \xrightarrow[f_r]{} & r_\psi & \xrightarrow[g_r]{} & r_\rho \end{array}$$

are pushouts, and hence the outer square is as well; and since $g_m\psi_1 = \rho_1 g_n$ we are done. The right action we give similarly.

4.8 The mediating 2-cells at 1

Now we have defined the distributive law, we need to produce the component of the invertible modifications α, β, γ and δ at 1. We note that the functors $Se_1 \colon S1 \to ST1$ and $e_{S1} \colon S1 \to TS1$ are given by

$$\begin{aligned} Se_1 \colon n &\mapsto (n \xrightarrow{\text{id}} n) \\ f &\mapsto (f, f) \end{aligned} \qquad\qquad \begin{aligned} e_{S1} \colon n &\mapsto (n \xrightarrow{!} 1) \\ f &\mapsto (f, !) \end{aligned}$$

and hence $S\eta_1 \colon ST1^{\text{op}} \times S1 \to \mathbf{Set}$ and $\eta_{S1} \colon TS1^{\text{op}} \times S1 \to \mathbf{Set}$ are given by:

$$\begin{aligned} S\eta_1(\varphi; n) &= ST1(\varphi, (n \xrightarrow{\text{id}} n)) \\ \eta_{S1}(\varphi; n) &= TS1(\varphi, (n \xrightarrow{!} 1)) \end{aligned}$$

Proposition 62. *Consider the diagram*

$$\begin{array}{ccc} S1 & \xrightarrow{S\eta_1} & ST1 \\ & {\scriptstyle \eta_{S1}}\searrow & \downarrow{\scriptstyle d_1} \\ & & TS1 \end{array}$$

in **Mod**. *There is a natural isomorphism α mediating the centre of this diagram.*

Proof. The lower side of this diagram has

$$\eta_{S1}(\varphi; n) = TS1(\varphi, (n \xrightarrow{!} 1)) \cong \begin{cases} S1(n_\varphi, n) & \text{if } m_\varphi = 1; \\ \varnothing & \text{otherwise,} \end{cases}$$

naturally in φ and n, whilst the upper side has

$$(d_1 \otimes S\eta_1)(\varphi; n) = \int^{\psi \in ST1} ST1(\psi, (n \xrightarrow{\text{id}} n)) \times d_1(\varphi; \psi),$$

which is isomorphic to $d_1(\varphi; (n \xrightarrow{\text{id}} n))$, naturally in φ and n. Now, any element f of $d_1(\varphi; (n \xrightarrow{\text{id}} n))$, given by

$$\begin{array}{ccc} n_\varphi & \xrightarrow{f_n} & n \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \text{id}} \\ m_\varphi & & n \end{array}$$

say, must satisfy $m_\varphi + n = n_\varphi + 1$; but since $n = n_\varphi$, this can only happen if $m_\varphi = 1$; and in this case, the diagram

$$\begin{array}{ccc} n_\varphi & \xrightarrow{f_n} & n \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle !} \\ m_\varphi & \xrightarrow[!]{} & 1 \end{array}$$

is necessarily a pushout. Hence

$$d_1(\varphi; (n \xrightarrow{\text{id}} n)) \cong \begin{cases} S1(n_\varphi, n) & \text{if } m_\varphi = 1; \\ \emptyset & \text{otherwise,} \end{cases}$$

naturally in φ and n; so we are done. □

Likewise, $T\varepsilon_1 \colon T1^{\text{op}} \times TS1 \to \mathbf{Set}$ and $\varepsilon_{T1} \colon T1^{\text{op}} \times ST1 \to \mathbf{Set}$ are given by:

$$T\varepsilon_1(n; \varphi) = TS1((n \xrightarrow{\text{id}} n), \varphi)$$
$$\varepsilon_{T1}(n; \varphi) = ST1((n \xrightarrow{!} 1), \varphi),$$

and there is a similar natural isomorphism β mediating the diagram

$$\begin{array}{ccc} ST1 & \xrightarrow{\varepsilon_{T1}} & T1 \\ {\scriptstyle d_1}\downarrow & \nearrow{\scriptstyle T\varepsilon_1} & \\ TS1 & & \end{array}$$

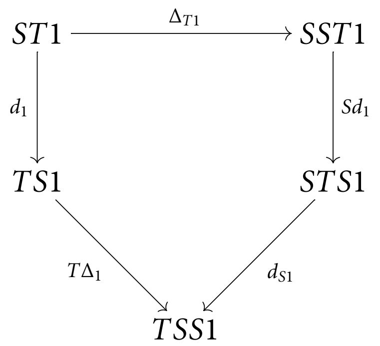
Next, we must produce the more complicated mediating 2-cells for the multiplication. The functors $t_{T1} \colon SST1 \to ST1$ and $Tt_1 \colon TSS1 \to TS1$ in **Cat** are given by

$$m_{T1} \colon (n_\varphi \xrightarrow{\varphi_1} m_\varphi \xrightarrow{\varphi_2} r_\varphi) \mapsto (n_\varphi \xrightarrow{\varphi_1} m_\varphi) \qquad Tm_1 \colon (n_\varphi \xrightarrow{\varphi_1} m_\varphi \xrightarrow{\varphi_2} r_\varphi) \mapsto (n_\varphi \xrightarrow{\varphi_2 \varphi_1} r_\varphi)$$
$$(f_n, f_m, f_r) \mapsto (f_n, f_m) \qquad\qquad (f_n, f_m, f_r) \mapsto (f_n, f_r)$$

and hence $\Delta_{T1}\colon SST1^{\text{op}} \times ST1 \to \textbf{Set}$ and $T\Delta_1\colon TSS1^{\text{op}} \times TS1 \to \textbf{Set}$ are given by:

$$\Delta_{T1}(\varphi;\psi) = ST1((n_\varphi \xrightarrow{\varphi_1} m_\varphi),\psi)$$
$$T\Delta_1(\varphi;\psi) = TS1((n_\varphi \xrightarrow{\varphi_2\varphi_1} r_\varphi),\psi)$$

Proposition 63. *Consider the diagram*



in **Mod***. There is a natural isomorphism* γ *mediating the centre of this diagram.*

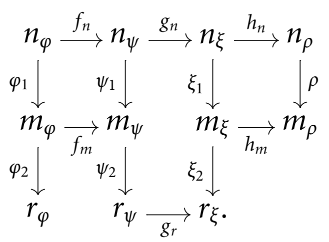
Proof. The lower side of this diagram has

$$(T\Delta_1 \otimes d_1)(\varphi;\rho) = \int^{\psi\in TS1} d_1(\psi;\rho) \times TS1\big((n_\varphi \xrightarrow{\varphi_2\varphi_1} r_\varphi),\psi\big),$$

which is isomorphic to $d_1\big((n_\varphi \xrightarrow{\varphi_2\varphi_1} r_\varphi);\rho\big)$, naturally in φ and ρ, whilst the upper side of this diagram has

$$K(\varphi;\rho) = (d_{S1} \otimes Sd_1 \otimes \Delta_{T1})(\varphi;\rho) = \int^{\substack{\psi\in STS1,\\ \xi\in SST1}} ST1((n_\xi \xrightarrow{\xi_1} m_\xi),\rho) \times Sd_1(\psi;\xi) \times d_{S1}(\varphi;\psi).$$

So, we need to set up an isomorphism between $K(\varphi;\rho)$ and $d_1((n_\varphi \xrightarrow{\varphi_2\varphi_1} r_\varphi);\rho)$ which is natural in φ and ρ. Suppose first of all that we are given an element x of $K(\varphi;\rho)$; we may represent x by elements $f \in d_{S1}(\varphi;\psi)$, $g \in Sd_1(\psi;\xi)$, and $h \in ST1((n_\xi \xrightarrow{\xi_1} m_\xi),\rho)$:



We send x to the element $\hat{x}$ of $d_1\big((n_\varphi \xrightarrow{\varphi_2\varphi_1} r_\varphi); \rho\big)$ given by

$$\begin{array}{ccc} n_\varphi & \xrightarrow{h_n g_n f_n} & n_\rho \\ {\scriptstyle \varphi_2\varphi_1}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ r_\varphi & & m_\rho. \end{array}$$

Note that this element is independent of the representation of x that we chose, and this assignation is natural in φ and ρ; but for it to be well-defined, we need still to check that the span $r_\varphi \xleftarrow{\varphi_2\varphi_1} n_\varphi \xrightarrow{\rho h_n g_n f_n} m_\rho$ is acyclic and connected. For this, we observe first that in the following diagram

$$\begin{array}{ccccccccc} n_\varphi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\xi & \xrightarrow{\xi_1} & m_\xi & \xrightarrow{h_n} & m_\rho \\ {\scriptstyle \varphi_1}\big\downarrow & & {\scriptstyle \psi_1}\big\downarrow & & & & \big\downarrow{\scriptstyle \xi_2} & & \big\downarrow \\ m_\varphi & \xrightarrow[f_m]{} & m_\psi & \xrightarrow[\psi_2]{} & r_\psi & \xrightarrow[g_r]{} & r_\xi & & \\ {\scriptstyle \varphi_2}\big\downarrow & & & & \big\downarrow & & \big\downarrow & & \\ r_\varphi & & \longrightarrow & & 1 & \longrightarrow & 1 & \longrightarrow & 1 \end{array}$$

each of the smaller squares is a pushout; and hence the outer square is also a pushout. But the top edge is $h_n \xi_1 g_n f_n = \rho h_n g_n f_n$, so that the square

$$\begin{array}{ccc} n_\varphi & \xrightarrow{\rho h_n g_n f_n} & n_\rho \\ {\scriptstyle \varphi_2\varphi 1}\big\downarrow & & \big\downarrow \\ r_\psi & \longrightarrow & 1 \end{array}$$

is a pushout as required. Furthermore, the following equalities hold:

$$\begin{aligned} r_\varphi + r_\psi &= m_\varphi + 1 \\ m_\psi + m_\xi &= n_\psi + r_\xi \\ m_\psi &= m_\varphi \\ m_\rho &= m_\xi \\ r_\psi &= r_\xi \\ n_\psi &= n_\varphi, \end{aligned}$$

whence we have $m_\rho + r_\varphi = n_\varphi + 1$. So the span $r_\varphi \xleftarrow{\varphi_2\varphi_1} n_\varphi \xrightarrow{\rho h_n g_n f_n} m_\rho$ is acyclic and connected as required.

Conversely, suppose we are given an element k of $d_1((n_\varphi \xrightarrow{\varphi_2\varphi_1} r_\varphi);\rho)$:

$$\begin{array}{ccc} n_\varphi & \xrightarrow{k_n} & n_\rho \\ {\scriptstyle \varphi_2\varphi_1}\downarrow & & \downarrow{\scriptstyle \rho} \\ r_\varphi & & m_\rho; \end{array}$$

then we take the following pushout:

$$\begin{array}{ccc} n_\varphi & \xrightarrow{\rho k_n} & m_\rho \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle i_2} \\ m_\varphi & \xrightarrow[i_1]{} & r. \end{array}$$

Now, the map i_1 in this pushout square need not be order-preserving; but it has a (non-unique) factorisation as $m_\varphi \xrightarrow{\alpha_1} r_1 \xrightarrow{\sigma_1} r$, where α_1 is order-preserving and σ_1 a bijection. Similarly, we can factorise i_2 as $m_\rho \xrightarrow{\alpha_2} r_2 \xrightarrow{\sigma_2} r$ with α_2 is order-preserving and σ_2 a bijection. [Note that it follows that each of the diagrams

$$\begin{array}{ccc} n_\varphi & \xrightarrow{\rho k_n} & m_\rho \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \sigma_1^{-1} i_2} \\ m_\varphi & \xrightarrow[\alpha_1]{} & r_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} n_\varphi & \xrightarrow{\rho k_n} & m_\rho \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ m_\varphi & \xrightarrow[\sigma_2^{-1} i_1]{} & r_2 \end{array}$$

is also a pushout.] Now we send k to the element $\hat{k}$ of $K(\varphi;\rho)$ represented by the following:

$$\begin{array}{ccccccc} n_\varphi & \xrightarrow{\mathrm{id}} & n_\varphi & \xrightarrow{k_n} & n_\rho & \xrightarrow{\mathrm{id}} & n_\rho \\ {\scriptstyle \varphi_1}\downarrow & & {\scriptstyle \varphi_1}\downarrow & & {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ m_\varphi & \xrightarrow[\mathrm{id}]{} & m_\varphi & & m_\rho & \xrightarrow[\mathrm{id}]{} & m_\rho \\ {\scriptstyle \varphi_2}\downarrow & & {\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \alpha_2}\downarrow & & \\ r_\varphi & & r_1 & \xrightarrow[\sigma_2^{-1}\sigma_1]{} & r_2. & & \end{array}$$

We need to check that this is a valid element of $K(\varphi;\rho)$. Clearly all squares commute in the diagram above, but we also need to check the acyclic and connected conditions.

We check first connectedness; for the middle map, the diagram

$$\begin{array}{ccccc} n_\varphi & \xrightarrow{k_n} & n_\rho & \xrightarrow{\rho} & m_\rho \\ {\scriptstyle \varphi_1}\downarrow & & & & \downarrow{\scriptstyle \alpha_2} \\ m_\varphi & \xrightarrow[\alpha_1]{} & r_1 & \xrightarrow[\sigma_2^{-1}\sigma_1]{} & r_2 \end{array} \quad = \quad \begin{array}{ccc} n_\varphi & \xrightarrow{\rho k_n} & m_\rho \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ m_\varphi & \xrightarrow[\sigma_2^{-1} i_1]{} & r_2 \end{array}$$

is indeed a pushout, so the induced spans for the middle map are connected. For the left-hand map, consider the diagram

$$\begin{array}{ccc}
n_\varphi & \xrightarrow{\rho k_n} & m_\rho \\
{\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \sigma_1^{-1} i_2} \\
m_\varphi & \xrightarrow[\alpha_1]{} & r_1 \\
{\scriptstyle \varphi_2}\downarrow & & \downarrow \\
r_\varphi & \longrightarrow & 1;
\end{array}$$

the outer square and the upper square are both pushouts, and hence so is the lower square; so the left-hand span is connected.

And now acyclicity. For the middle map, we need that, given any monomorphism $\iota \colon n'_\varphi \hookrightarrow n_\varphi$, the diagram

$$\begin{array}{ccc}
n'_\varphi & \xrightarrow{\rho k_n \iota} & m_\rho \\
{\scriptstyle \varphi_1 \iota}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\
m_\varphi & \xrightarrow[\sigma_2^{-1} i_1]{} & r_2
\end{array}$$

is no longer a pushout. But suppose it were; then in the diagram

$$\begin{array}{ccc}
n'_\varphi & \xrightarrow{\rho k_n \iota} & m_\rho \\
{\scriptstyle \varphi_1 \iota}\downarrow & & \downarrow{\scriptstyle \sigma_1^{-1} i_2} \\
m_\varphi & \xrightarrow[\alpha_1]{} & r_1 \\
{\scriptstyle \varphi_2}\downarrow & & \downarrow \\
r_\varphi & \longrightarrow & 1
\end{array}$$

the upper and lower squares would be pushouts, hence making the outer edge a pushout; but this contradicts the acyclicity of the span $r_\varphi \leftarrow n_\varphi \rightarrow m_\rho$. So the induced spans for the middle map are acyclic. Thus we now know that the following equations hold:

$$\begin{aligned}
m_\varphi + m_\rho &= n_\varphi + r_2 \\
r_\varphi + m_\rho &= n_\varphi + 1 \\
r_1 &= r_2,
\end{aligned}$$

and so can deduce that $r_1 + r_\varphi = m_\varphi + 1$, as required for the left-hand span to be acyclic.

It remains to check that these two assignations are mutually inverse. It is evident, given

$k \in d_1((n_\varphi \xrightarrow{\varphi_2\varphi_1} r_\varphi); \rho)$, that $\hat{\hat{k}} = k$. For the other direction, we send

$$x = \begin{array}{ccccccc} n_\varphi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\xi & \xrightarrow{h_n} & n_\rho \\ {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\psi_1}\downarrow & & {\scriptstyle\xi_1}\downarrow & & \downarrow{\scriptstyle\rho} \\ m_\varphi & \xrightarrow[f_m]{} & m_\psi & & m_\xi & \xrightarrow[h_m]{} & m_\rho \\ {\scriptstyle\varphi_2}\downarrow & & {\scriptstyle\psi_2}\downarrow & & {\scriptstyle\xi_2}\downarrow & & \\ r_\varphi & & r_\psi & \xrightarrow[g_r]{} & r_\xi. & & \end{array} \quad \text{to} \quad \hat{\hat{x}} = \begin{array}{ccccccc} n_\varphi & \xrightarrow{\text{id}} & n_\varphi & \xrightarrow{k_n} & n_\rho & \xrightarrow{\text{id}} & n_\rho \\ {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ m_\varphi & \xrightarrow[\text{id}]{} & m_\varphi & & m_\rho & \xrightarrow[\text{id}]{} & m_\rho \\ {\scriptstyle\varphi_2}\downarrow & & {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & \\ r_\varphi & & r_1 & \xrightarrow[\sigma_2^{-1}\sigma_1]{} & r_2. & & \end{array}$$

We claim that these two diagrams represent the same element of $K(\varphi; \rho)$. Indeed, note that in the diagram

$$\begin{array}{ccccccccc} n_\varphi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\xi & \xrightarrow{\xi_1} & m_\xi & \xrightarrow{h_m} & m_\rho \\ {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\psi_1}\downarrow & & & & {\scriptstyle\xi_2}\downarrow & & \downarrow{\scriptstyle g_r^{-1}\xi_2 h_m^{-1}} \\ m_\varphi & \xrightarrow[f_m]{} & m_\psi & \xrightarrow[\psi_2]{} & r_\psi & \xrightarrow[g_r]{} & r_\xi & \xrightarrow[g_r^{-1}]{} & r_\psi \end{array}$$

each of the smaller squares is a pushout, and hence the outer edge is. But the upper edge is $h_m\xi_1 g_n f_n = \rho h_n g_n f_n = \rho k_n$, so that the diagram

$$\begin{array}{ccc} n_\varphi & \xrightarrow{\rho k_n} & m_\rho \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle g_r^{-1}\xi_2 h_m^{-1}} \\ m_\varphi & \xrightarrow[\psi_2 f_m]{} & r_\psi \end{array}$$

is a pushout. Since r_1 is also a pushout for this diagram, it follows that there is an isomorphism $\beta_1 \colon r_1 \to r_\psi$ such that $\beta_1\alpha_1 = \psi_2 f_m$; hence the following diagram commutes:

$$\begin{array}{ccc} n_\varphi & \xrightarrow{f_n} & n_\psi \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\psi_1} \\ m_\varphi & \xrightarrow{f_m} & m_\psi \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\psi_2} \\ r_1 & \xrightarrow[\beta_1]{} & r_\psi \end{array}$$

Similarly, we see that

$$\begin{array}{ccc} n_\varphi & \xrightarrow{\rho k_n} & m_\rho \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\xi_2 h_m^{-1}} \\ m_\varphi & \xrightarrow[g_r\psi_2 f_m]{} & r_\xi \end{array}$$

is a pushout, and so there is an isomorphism $\beta_2 \colon r_\xi \to r_2$ such that $\beta_2\xi_2 h_m^{-1} = \alpha_2$, i.e.,

$\beta_2\xi_2 = \alpha_2 h_m$. Hence the following diagram commutes:

$$\begin{array}{ccc}
n_\xi & \xrightarrow{h_n} & n_\rho \\
{\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \rho} \\
m_\xi & \xrightarrow{h_m} & m_\rho \\
{\scriptstyle \xi_2}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\
r_\xi & \xrightarrow[\beta_2]{} & r_2.
\end{array}$$

Furthermore, we have $r_1 \xrightarrow{\beta_1} r_\psi \xrightarrow{g_r} r_\xi \xrightarrow{\beta_2} r_2 = r_1 \xrightarrow{\sigma_1} r \xrightarrow{\sigma_2^{-1}} r_2$, since each of these objects is a pushout of the same span, and the isomorphisms between them are isomorphisms of pushouts. Thus, using an evident notation for the internal actions, we have

$$x = \begin{array}{ccccccc}
n_\varphi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\xi & \xrightarrow{h_n} & n_\rho \\
{\scriptstyle \varphi_1}\downarrow & & {\scriptstyle \psi_1}\downarrow & & {\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \rho} \\
m_\varphi & \xrightarrow[f_m]{} & m_\psi & & m_\xi & \xrightarrow[h_m]{} & m_\rho \\
{\scriptstyle \varphi_2}\downarrow & & {\scriptstyle \psi_2}\downarrow & & {\scriptstyle \xi_2}\downarrow & & \\
r_\varphi & & r_\psi & \xrightarrow[g_r]{} & r_\xi. & &
\end{array}
\equiv
\begin{array}{ccccccccccc}
n_\varphi & \xrightarrow{\mathrm{id}} & n_\varphi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\xi & \xrightarrow{h_n} & n_\rho & \xrightarrow{\mathrm{id}} & n_\rho \\
{\scriptstyle \varphi_1}\downarrow & & {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \psi_1} & & {\scriptstyle \xi_1}\downarrow & & {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\
m_\varphi & \xrightarrow[\mathrm{id}]{} & m_\varphi & \xrightarrow{f_m} & m_\psi & & m_\xi & \xrightarrow{h_m} & m_\rho & \xrightarrow[\mathrm{id}]{} & m_\rho \\
{\scriptstyle \varphi_2}\downarrow & & {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \psi_2} & & {\scriptstyle \xi_2}\downarrow & & {\scriptstyle \alpha_2}\downarrow & & \\
r_\varphi & & r_1 & \xrightarrow[\beta_1]{} & r_\psi & \xrightarrow[g_r]{} & r_\xi & \xrightarrow[\beta_2]{} & r_2. & &
\end{array}$$

$$\equiv \begin{array}{ccccccc}
n_\varphi & \xrightarrow{\mathrm{id}} & n_\varphi & \xrightarrow{k_n} & n_\rho & \xrightarrow{\mathrm{id}} & n_\rho \\
{\scriptstyle \varphi_1}\downarrow & & {\scriptstyle \varphi_1}\downarrow & & {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\
m_\varphi & \xrightarrow[\mathrm{id}]{} & m_\varphi & & m_\rho & \xrightarrow[\mathrm{id}]{} & m_\rho \\
{\scriptstyle \varphi_2}\downarrow & & {\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \alpha_2}\downarrow & & \\
r_\varphi & & r_1 & \xrightarrow[\sigma_2^{-1}\sigma_1]{} & r_2. & &
\end{array} = \hat{\hat{x}}.$$

So the assignations $x \mapsto \hat{x}$ and $k \mapsto \hat{k}$ are mutually inverse as required. It now follows that the assignation $d_1((n_\varphi \xrightarrow{\varphi_2\varphi_1} r_\varphi); \rho) \to K(\varphi; \rho)$ is natural in φ and ρ, since its inverse is. This completes the proof. □

Entirely analgously, $m_{S1}\colon TS1^{\mathrm{op}} \times TTS1 \to \mathbf{Set}$ and $Sm_1\colon ST1^{\mathrm{op}} \times STT1 \to \mathbf{Set}$ are given by:

$$m_{S1}(\varphi; \psi) = ST1((n_\varphi \xrightarrow{\varphi_1} m_\varphi), \psi)$$
$$Sm_1(\varphi; \psi) = TS1((n_\varphi \xrightarrow{\varphi_2\varphi_1} r_\varphi), \psi)$$

and we obtain

Proposition 64. *Consider the diagram*

$$\begin{array}{ccccc} ST1 & & \xleftarrow{Sm_1} & & STT1 \\ \downarrow{\scriptstyle d_1} & & & & \downarrow{\scriptstyle d_{T1}} \\ TS1 & & & & TST1 \\ & \nwarrow{\scriptstyle m_{S1}} & & \swarrow{\scriptstyle Td_1} & \\ & & TTS1 & & \end{array}$$

in **Mod**. *There is a natural isomorphism δ mediating the centre of this diagram.*

4.9 Clubs and pro-clubs

We now wish to show how we can lift this pseudo-distributive law from 1 to the whole of **Mod**. Due to the ongoing nature of this research, the results of this section are of a highly speculative nature, and proofs will be sketchy.

First, we recall some background about Kelly's *clubs*. These date back to the 1970s, but a more modern perspective is offered by [Kel92] or [Web05]. The results that we summarise below are drawn from the latter.

Definition 33. Let $\mathbb{C}$ and $\mathbb{D}$ be a finitely complete categories. We say that a natural transformation $\alpha\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$ is **cartesian** if all its naturality squares are pullbacks.

Now, given functors $F, M\colon \mathbb{C} \to \mathbb{D}$ and a cartesian natural transformation $\alpha\colon F \Rightarrow M$, it is easy to see that F and α are determined up to isomorphism by the value of $F1$ and the map $\alpha_1\colon F1 \to M1$; indeed, FX and α_X are determined (up to isomorphism) by the pullback

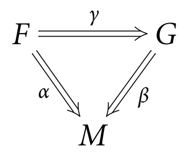
$$\begin{array}{ccc} FX & \xrightarrow{F!} & F1 \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ MX & \xrightarrow[M!]{} & M1 \end{array}$$

and $Ff\colon FX \to FY$ by the pullback

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ MX & \xrightarrow[Mf]{} & MY. \end{array}$$

More formally, we define the category $[\mathbb{C}, \mathbb{D}] \,/_c\, M$ with

- **Objects** (α, F) being cartesian natural transformations $\alpha\colon F \Rightarrow M$;
- **Maps** $\gamma\colon (\alpha, F) \to (\beta, G)$ being natural transformations $\gamma\colon F \Rightarrow G$ such that



  commutes. Note that it follows that γ is itself a cartesian natural transformation.

and the above observations show that

Proposition 65. *There is an equivalence of categories:*

$$\mathbb{D}/G1 \simeq [\mathbb{C}, \mathbb{D}] \,/_c\, G.$$

Now, suppose that $M\colon \mathbb{C} \to \mathbb{C}$ comes equipped with a monad structure (M, η, μ); then M becomes a monoid in the strict monoidal category $[\mathbb{C}, \mathbb{C}]$, and hence the glued category $[\mathbb{C}, \mathbb{C}]/M$ acquires strict monoidal structure:

$$\left(F \overset{\alpha}{\Longrightarrow} M\right) \otimes \left(G \overset{\beta}{\Longrightarrow} M\right) = \left(FG \overset{\alpha\beta}{\Longrightarrow} MM \overset{\mu}{\Longrightarrow} M\right)$$

$$I = \left(\mathrm{id}_{\mathbb{C}} \overset{\eta}{\Longrightarrow} M\right)$$

with the evident action on maps. Given a monoidal category $\mathbb{C}$, we say that $\mathbb{D} \hookrightarrow \mathbb{C}$ is a **sub-monoidal category** of $\mathbb{C}$ if the inclusion $\mathbb{D} \hookrightarrow \mathbb{C}$ is a strict monoidal functor.

Definition 34. A monad (M, η, μ) is a **club** on $\mathbb{C}$ if $[\mathbb{C}, \mathbb{C}] \,/_c M$ is a sub-monoidal category of $[\mathbb{C}, \mathbb{C}]/M$.

Proposition 66 ([Kel92]). *A monad (M, η, μ) is a club if and only if η and μ are cartesian natural transformations and M preserves cartesian natural transformations into M (i.e., $M\alpha\colon MF \Rightarrow MM$ is cartesian whenever $\alpha\colon F \Rightarrow M$ is).*

In particular, we may consider the case $\mathbb{C} = \mathbf{Cat}$; in this case the equivalence of categories exhibited above enriches to an equivalence of 2-categories which is locally an isomorphism of categories. Now, consider again the equivalence

$$\mathbf{Cat}/M1 \simeq [\mathbf{Cat}, \mathbf{Cat}] \,/_c\, M.$$

We should like to replace both sides by something rather more general.

Definition 35. Let $\mathbb{S}$ be a small category; then the bicategory $\mathbf{Mod} \mathbin{/\!\!/} \mathbb{S}$ is given by:

- **Objects** $(F, \mathbb{C})$ are functors $\mathbb{C} \xrightarrow{F} \mathbb{S}$;

- **1-cells** $(K, \alpha)\colon (F, \mathbb{C}) \to (G, \mathbb{D})$ are pairs (K, α) where $K\colon \mathbb{C} \to \mathbb{D}$ is an arrow of **Mod**, and α is a 2-cell $\alpha\colon G_* \otimes K \to F_*$ in $\mathbf{Mod}(\mathbb{C}, \mathbb{S})$:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{K} & \mathbb{D} \\ & {\scriptstyle F_*}\searrow \;\; \overset{\alpha}{\Leftarrow} \;\; \swarrow{\scriptstyle G_*} & \\ & \mathbb{S}. & \end{array}$$

  Since $G_* \dashv G^*$, we may equivalently give a 2-cell $\alpha'\colon K \to G^* \otimes F_*$, which amounts to giving a family of maps $\alpha_{dc}\colon K(d; c) \to \mathbb{S}(Gd, Fc)$, natural in d and c.

- **2-cells** $\gamma\colon (K, \alpha) \to (L, \beta)$ are 2-cells $\gamma\colon K \to L$ in **Mod** such that

$$\begin{array}{ccc} K & \xrightarrow{\gamma} & L \\ & {\scriptstyle \alpha'}\searrow \quad \swarrow{\scriptstyle \beta'} & \\ & G^* \otimes F_* & \end{array}$$

  commutes in $\mathbf{Mod}(\mathbb{C}, \mathbb{D})$. Explicitly, to give such is to give maps $\gamma_{dc}\colon K(d; c) \to L(d; c)$ in $\mathcal{V}$, natural in d and c, such that

$$K(d; c) \xrightarrow{\gamma_{dc}} L(d; c) \xrightarrow{\beta_{dc}} \mathbb{S}(Gd, Fc) = K(d; c) \xrightarrow{\alpha_{dc}} \mathbb{S}(Gd, Fc).$$

Composition of 1- and 2-cells is inherited in the evident way from **Mod**.

So on the left hand side, we would like to replace $\mathbf{Cat}/M1$ by $\mathbf{Mod} \mathbin{/\!/} M1$; however, this still leaves us with the question of what should replace the right-hand side. We sketch one possible answer:

Definition 36. We say that an 2-functor $F\colon \mathbf{Cat} \to \mathbf{Cat}$ is **equippable** if we can find a homomorphism $\hat{F}\colon \mathbf{Mod} \to \mathbf{Mod}$ and invertible modifications l and r as follows:

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{(-)_*} & \mathbf{Mod} \\ {\scriptstyle F}\downarrow & \overset{l}{\Rightarrow} & \downarrow{\scriptstyle F} \\ \mathbf{Cat} & \xrightarrow[(-)_*]{} & \mathbf{Mod} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{Cat} & \xrightarrow{(-)^*} & \mathbf{Mod} \\ {\scriptstyle F}\downarrow & \overset{r}{\Rightarrow} & \downarrow{\scriptstyle F} \\ \mathbf{Cat} & \xrightarrow[(-)^*]{} & \mathbf{Mod}. \end{array}$$

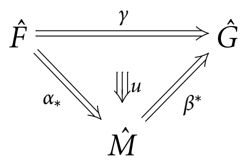
If we fix a particular choice of $\hat{F}$, l and r, we say that F is **equipped**.

Conjecture 1. *If F and G are equipped 2-functors on* **Cat** *and $\alpha\colon F \to G$ is a cartesian natural transformation between them, then α lifts to a strong transformation $\alpha_*\colon \hat{F} \Rightarrow \hat{G}\colon$* **Mod** $\to$ **Mod** *with components $(\alpha_*)_{\mathbb{C}} = (\alpha_{\mathbb{C}})_*$; it also lifts to a strong transformation $\alpha^*\colon \hat{G} \Rightarrow \hat{F}\colon$* **Mod** $\to$ **Mod** *with components $(\alpha^*)_{\mathbb{C}} = (\alpha_{\mathbb{C}})^*$.*

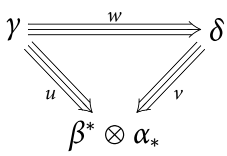
In support of this conjecture, we note that a rather similar proposition is proven in Section 3 of the paper [Her01]. If this conjecture holds, we can give:

Definition 37. Suppose that M is an equipped 2-functor $M\colon \mathbf{Cat} \to \mathbf{Cat}$. We write $[\mathbf{Mod}, \mathbf{Mod}] /\!\!/_c M$ for the following bicategory:

- **Objects** (F, α) are equipped 2-functors $F\colon \mathbf{Cat} \to \mathbf{Cat}$ together with cartesian natural transformations $\alpha\colon F \Rightarrow M$;
- **1-cells** $(\gamma, u)\colon (F, \alpha) \to (G, \beta)$ consist of a strong transformation $\gamma\colon \hat{F} \Rightarrow \hat{G}$ and a modification $u\colon \gamma \Rrightarrow \beta^* \otimes \alpha_*$:



  in $[\mathbf{Mod}, \mathbf{Mod}]$;
- **2-cells** $w\colon (\gamma, u) \to (\delta, v)$ are modifications $w\colon \gamma \Rrightarrow \delta$ such that



  commutes in $[\mathbf{Mod}, \mathbf{Mod}](\hat{F}, \hat{G})$.

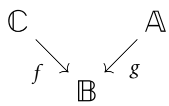
Conjecture 2. *Let M be an equipped 2-functor; then there is a biequivalence*

$$\mathbf{Mod} /\!\!/ M1 \simeq [\mathbf{Mod}, \mathbf{Mod}] /\!\!/_c M.$$

We sketch briefly why this should be plausible. First, to each object of $\mathbf{Mod} /\!\!/ M1$, we can assign, via the equivalence $\mathbf{Cat}/M1 \simeq [\mathbf{Cat}, \mathbf{Cat}]/_c M$, a cartesian natural transformation $\alpha\colon F \Rightarrow M$. We claim:

Conjecture 3. *If M is equipped, and $\alpha\colon F \Rightarrow M$ is a cartesian natural transformation, then we can equip F.*

Outline of putative proof. We consider a special class of equippable 2-functors; those that preserve *codiscrete cofibrations*. Recall that a cospan



in **Cat** is a **cofibration** from $\mathbb{A}$ to $\mathbb{C}$ if and only if:

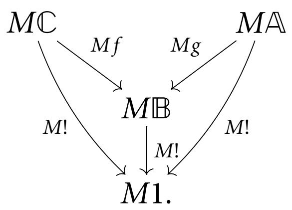
- f and g are full and faithful;

- given $b \in \operatorname{Im} f$ and $a \in \operatorname{Im} g$, $\mathbb{B}(a, b) = \emptyset$;

furthermore, it is **codiscrete** if the map $[f, g] \colon \mathbb{C} + \mathbb{A} \to \mathbb{B}$ is bijective on objects. Now, a codiscrete cofibration from $\mathbb{A}$ to $\mathbb{C}$ gives rise to a profunctor $f^* \otimes g_* \colon \mathbb{A} \to \mathbb{C}$; conversely a profunctor $\mathbb{A} \to \mathbb{C}$ gives rise to a codiscrete cofibration by taking its lax colimit; this gives rise to an equivalence of categories $\mathbf{CodCofib}(\mathbb{A}, \mathbb{C}) \simeq \mathbf{Prof}(\mathbb{A}, \mathbb{C})$.

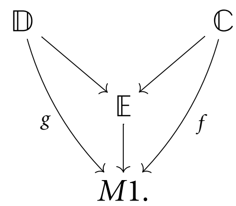
Now, if M preserves codiscrete cofibrations, then we see that the action of $\hat{M}$ on arrows of **Mod** can be read off from its action on a representing codiscrete cofibration in **Cat**. It is not clear whether all equippable 2-functors arise in this way, but it is certainly the case for the 2-functor T we want to consider. We aim to proceed similarly for the rest of the structure.

So now, suppose we are given an arrow $h \colon \mathbb{A} \to \mathbb{C}$ in **Mod**. Let $\mathbb{A} \xrightarrow{f} \mathbb{B} \xleftarrow{g} \mathbb{C}$ be its associated cofibration; then consider the diagram



By pulling this diagram back along the arrow $\alpha_1 \colon F1 \to M1$, we get a cospan $F\mathbb{A} \to F\mathbb{B} \leftarrow F\mathbb{C}$. Now, in the displayed diagram, the top part is a cofibration (since M preserves such), and it's easy to see that pulling a cofibration back along $\alpha_1 \colon F1 \to M1$ produces another cofibration. This defines $\hat{F}$ on arrows of **Mod**. The action on 2-cells follows in the evident way. Note that it isn't at all clear that $\hat{F}$ will preserve composition of profunctors. We choose to elide this point at present. □

Hence to each object of $\mathbf{Mod} \mathbin{/\!\!/} M1$ we can assign an object of $[\mathbf{Mod}, \mathbf{Mod}] \mathbin{/\!\!/_c} M$. On morphisms, we proceed similarly. Indeed, suppose we are given a 1-cell $(f, \mathbb{C}) \to (g, \mathbb{D})$ in $\mathbf{Mod} \mathbin{/\!\!/} M1$. Note that to give a 1-cell $(f, \mathbb{C}) \to (g, \mathbb{D})$ in $\mathbf{Mod} \mathbin{/\!\!/} M1$ is equivalently to give a commuting diagram



in **Cat** such that the top cospan is a codiscrete cofibration.

Let $\alpha \colon F \Rightarrow M$ and $\beta \colon G \Rightarrow M$ be the associated objects of $[\mathbf{Mod}, \mathbf{Mod}] \mathbin{/\!\!/_c} M$; we seek a strong transformation $\gamma \colon \hat{F} \Rightarrow \hat{G}$. To give its component $\gamma_{\mathbb{A}}$ at $\mathbb{A}$, we observe that $\hat{F}\mathbb{A}$ is given by pulling back $f \colon \mathbb{C} \to M1$ along $M! \colon M\mathbb{A} \to M1$, and similarly for G;

hence we can produce a codiscrete cofibration from $F\mathbb{A}$ to $G\mathbb{A}$, and hence a profunctor $\gamma_{\mathbb{A}}\colon F\mathbb{A} \to G\mathbb{A}$, by pulling back the displayed diagram along $M!\colon M\mathbb{A} \xrightarrow{M!} M1$.

So now, suppose that we are given an equipped 2-functor M such that $\hat{M}$ has the structure of a pseudomonad. It follows that $\hat{M}$ is a pseudomonoid in the monoidal bicategory $[\mathbf{Mod}, \mathbf{Mod}]$, and hence that the lax slice bicategory $[\mathbf{Mod}, \mathbf{Mod}]/\hat{M}$ becomes a monoidal bicategory. There is an evident inclusion

$$[\mathbf{Mod}, \mathbf{Mod}] /\!\!/_c M \to [\mathbf{Mod}, \mathbf{Mod}]/\hat{M}.$$

Definition 38. We say that M is a **pro-club** if $[\mathbf{Mod}, \mathbf{Mod}] /\!\!/_c M$ is a sub-monoidal bicategory of $[\mathbf{Mod}, \mathbf{Mod}]/\hat{M}$.

Note that it follows that if M is a pro-club, then $\mathbf{Mod} /\!\!/ M1$ acquires the structure of a monoidal bicategory. Indeed, we have a monoidal bicategory structure on $[\mathbf{Mod}, \mathbf{Mod}] /\!\!/_c M$ which transports along the biequivalence of this bicategory with $\mathbf{Mod} /\!\!/ M1$.

Conjecture 4. *The equipped 2-functor T is a pro-club.*

4.10 Lifting the distributive law from 1

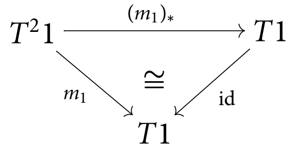
If we are prepared to accept that something like the results of the previous section are true, we may now conclude our treatment of the pseudo-distributive law d. In fact, let us state precisely what we wish to assume is true:

Conjecture 5. *The bicategory $\mathbb{M} = \mathbf{Mod} /\!\!/ T1$ has the structure of a monoidal bicategory extending the monoidal structure on $\mathbf{Cat}/T1$. Furthermore, there is a homomorphism of monoidal bicategories*

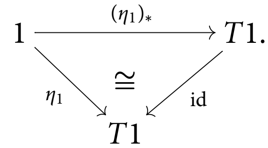
$$\Phi\colon \mathbb{M} \to [\mathbf{Mod}, \mathbf{Mod}].$$

Moreover, the object $T = \mathrm{id}\colon T1 \to T1$ is a pseudomonoid and a pseudocomonoid in $\mathbb{M}$ (again, in this guise we will write it as S), and this structure is carried by Φ to the pseudomonad and pseudocomonad structure on the homomorphism $T\colon \mathbf{Mod} \to \mathbf{Mod}$.

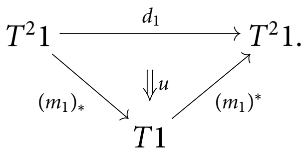
Now, we claim we can recast all our pseudo-distributivity diagrams at 1 as diagrams in $\mathbb{M}$. Observe first that since the monoidal structure on $\mathbb{M}$ extends that of $\mathbf{Cat}/T1$, we already know its action on objects: in particular, the object $T \otimes T$ is given by $T^2 1 \xrightarrow{m_1} T1$, etc. Furthermore, we can see what the maps $T \otimes T \to T$ and $1 \to T$ in $\mathbb{M}$ making T into a pseudomonoid must be:



and



and similarly for the maps making S into a pseudocomonoid. We do, however, need to check that we can lift the map $d_1 \colon TS1 \to ST1$ from **Mod** to $\mathbb{M}$. To do so, we need to provide a 2-cell u fitting into the diagram:



In other words, we need to provide maps $d_1(\varphi; \psi) \to T1(n_\varphi; n_\psi)$ which are natural in φ and ψ; and there is an evident choice of such, since $d_1(\varphi; \psi)$ is simply a collection of bijections $n_\varphi \to n_\psi$ satisfying certain conditions; in fact, we see that d_1 is a subobject of $(m_1)^* \otimes (m_1)_*$ in $\mathbf{Mod}(T^2 1, T^2 1)$.

Furthermore, it is evident from an examination of the proofs that the mediating 2-cells α, β, γ and δ we constructed above commute with this assignation, and hence lift to 2-cells in $\mathbf{Mod} /\!\!/ T1$. Thus we can lift all our pseudo-commutative diagrams from above to pseudo-commutative diagrams in $\mathbf{Mod} /\!\!/ T1$.

The final point is to observe that we can recast the maps in these diagrams as maps involving the monoidal bicategory structure of $\mathbf{Mod} /\!\!/ T1$; so for instance, the maps Td_1 and d_{T1} are identified with the maps $T \otimes d_1$ and $d_1 \otimes T$. Therefore, under the homomorphism of monoidal bicategories Φ, this structure transports to mediating modifications for a pseudo-distributive law between T and S as homomorphisms $\mathbf{Mod} \to \mathbf{Mod}$.

Thus far, we have not checked any of the ten coherence laws for a pseudo-distributivity: there are indications that by careful use of the two-cell u, we may be able to avoid checking these explicitly, but this aspect is still under consideration. However, we hope to have conveyed to the reader at least a feel for how this pseudo-distributive law might be set up, and hence how we might put the theory of polycategories on a sound footing.

Chapter 5

Future directions

All three of the topics presented in this essay have scope for extension. In the first section, we saw how to construct a free $*$-autonomous category *without units*. One would naturally like to remove the qualifier from this statement; and it seems plausible that the proof-net based approach of [SL04] could be adapted to the Frobenius algebra framework described above.

The second section as it stands may not appear obviously to lead anywhere; but particularly towards its end, it describes tools for constructing new $*$-autonomous categories from old which have not been explored in any great detail. Indeed, mathematical objects such as the *bistructures* of [CPW00] seem as if they ought to arise from constructions such as these; similarly, the *Dialectica* interpretations for linear logic developed in [dP89] should be explicable by means of a 2-dimensional version of these sorts of glueing constructions.

Finally, the third section has in a sense already mapped out its future directions; but beyond these lie further possibilities. One such, though little more than a pipe-dream at present, would be a polycategory-like presentation of the full linear logic system – that is, of a $*$-automous category with finite products and coproducts. Explicitly, one would expect there to be a distributive law between the 'free symmetric strict monoidal category with products' pseudomonad and the 'free symmetric strict monoidal category with coproducts' pseudocomonad on **Mod**. At present, the technology is not in place to describe the 'free monoidal category with products', so this remains a rather distant prospect; but an enticing prospect nonetheless.

Bibliography

[Bar79] Michael Barr. *$*$-autonomous categories*, volume 752 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979. With an appendix by Po Hsiang Chu.

[Bar95] Michael Barr. Nonsymmetric *-autonomous categories. *Theoret. Comput. Sci.*, 139(1-2):115–130, 1995.

[Bar96] Michael Barr. The Chu construction. *Theory Appl. Categ.*, 2:No. 2, 17–35 (electronic), 1996.

[BD98] John C. Baez and James Dolan. Higher-dimensional algebra. III. n-categories and the algebra of opetopes. *Adv. Math.*, 135(2):145–206, 1998.

[Bén67] Jean Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, Berlin, 1967.

[Bur71] Albert Burroni. T-catégories (catégories dans un triple). *Cahiers Topologie Géom. Différentielle*, 12:215–321, 1971.

[Car95] Sean Carmody. *Cobordism Categories*. PhD thesis, Cambridge, 1995.

[CKVW98] A. Carboni, G. M. Kelly, D. Verity, and R. J. Wood. A 2-categorical approach to change of base and geometric morphisms. II. *Theory Appl. Categ.*, 4:No. 5, 82–136 (electronic), 1998.

[CPW00] Pierre-Louis Curien, Gordon Plotkin, and Glynn Winskel. Bistructures, bidomains, and linear logic. In *Proof, language, and interaction*, Found. Comput. Ser., pages 21–54. MIT Press, Cambridge, MA, 2000.

[CS97] J. R. B. Cockett and R. A. G. Seely. Weakly distributive categories. *J. Pure Appl. Algebra*, 114(2):133–173, 1997.

[CT03] Maria Manuel Clementino and Walter Tholen. Metric, topology and multicategory—a common approach. *J. Pure Appl. Algebra*, 179(1-2):13–47, 2003.

[dP89] Valeria C. V. de Paiva. A Dialectica-like model of linear logic. In *Category theory and computer science (Manchester, 1989)*, volume 389 of *Lecture Notes in Comput. Sci.*, pages 341–356. Springer, Berlin, 1989.

[DR89] Vincent Danos and Laurent Regnier. The structure of multiplicatives. *Arch. Math. Logic*, 28(3):181–203, 1989.

[DS04] Brian Day and Ross Street. Quantum categories, star autonomy, and quantum groupoids. In *Galois theory, Hopf algebras, and semiabelian categories*, volume 43 of *Fields Inst. Commun.*, pages 187–225. Amer. Math. Soc., Providence, RI, 2004.

[Gir87] Jean-Yves Girard. Linear logic. *Theoret. Comput. Sci.*, 50(1):101, 1987.

[Gir95] Jean-Yves Girard. Linear logic: its syntax and semantics. In *Advances in linear logic (Ithaca, NY, 1993)*, volume 222 of *London Math. Soc. Lecture Note Ser.*, pages 1–42. Cambridge Univ. Press, Cambridge, 1995.

[GPS95] R. Gordon, A. J. Power, and Ross Street. Coherence for tricategories. *Mem. Amer. Math. Soc.*, 117(558):vi+81, 1995.

[Her00] Claudio Hermida. Representable multicategories. *Adv. Math.*, 151(2):164–225, 2000.

[Her01] Claudio Hermida. From coherent structures to universal properties. *J. Pure Appl. Algebra*, 165(1):7–61, 2001.

[HS03] Martin Hyland and Andrea Schalk. Glueing and orthogonality for models of linear logic. *Theoret. Comput. Sci.*, 294(1-2):183–231, 2003. Category theory and computer science.

[JK02] G. Janelidze and G. M. Kelly. A note on actions of a monoidal category. *Theory Appl. Categ.*, 9:61–91 (electronic), 2001/02. CT2000 Conference (Como).

[JS93] André Joyal and Ross Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993.

[Kel74] G. M. Kelly. Doctrinal adjunction. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 257–280. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[Kel92] G. M. Kelly. On clubs and data-type constructors. In *Applications of categories in computer science (Durham, 1991)*, volume 177 of *London Math. Soc. Lecture Note Ser.*, pages 163–190. Cambridge Univ. Press, Cambridge, 1992.

[KLSS02] Max Kelly, Anna Labella, Vincent Schmitt, and Ross Street. Categories enriched on two sides. *J. Pure Appl. Algebra*, 168(1):53–98, 2002.

[Koc70] Anders Kock. Monads on symmetric monoidal closed categories. *Arch. Math. (Basel)*, 21:1–10, 1970.

[Koc04] Joachim Kock. *Frobenius algebras and 2D topological quantum field theories*, volume 59 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004.

[Kos03] J. Koslowski. A monadic approach to polycategories. *Electronic Notes in Theoretical Computer Science*, 69:1–26, 2003.

[Lam69] Joachim Lambek. Deductive systems and categories. II. Standard constructions and closed categories. In *Category Theory, Homology Theory and their Applications, I (Battelle Institute Conference, Seattle, Wash., 1968, Vol. One)*, pages 76–122. Springer, Berlin, 1969.

[Lei04] Tom Leinster. Operads in higher-dimensional category theory. *Theory Appl. Categ.*, 12:No. 3, 73–194 (electronic), 2004.

[Pav97] Duško Pavlović. Chu. I. Cofree equivalences, dualities and $*$-autonomous categories. *Math. Structures Comput. Sci.*, 7(1):49–73, 1997.

[SL04] Lutz Straßburger and François Lamarche. On proof nets for multiplicative linear logic with units. In Jerzy Marcinkowski and Andrzej Tarlecki, editors, *Computer Science Logic, CSL 2004*, volume 3210 of *Lecture Notes in Computer Science*, pages 145–159. Springer-Verlag, 2004.

[Sza75] M. E. Szabo. Polycategories. *Comm. Algebra*, 3(8):663–689, 1975.

[Tan97] Audrey Tan. *Full Completeness Results for Models of Linear Logic*. PhD thesis, Cambridge, 1997.

[Tan04] Miki Tanaka. *Pseudo-distributive laws*. PhD thesis, Edinburgh, 2004.

[Web05] Mark Weber. Generic morphisms, parametric representations and weakly cartesian monads. *Theory Appl. Categ.*, 13:191–234 (electronic), 2005.