When pseudo comes for free

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Outline

Pseudofunctors

Some 2-category theory

Some homotopy theory

Some algebraic homotopy theory

Pseudoalgebras

Cat-operads

Globular operads

(NB: Talk notes available at http://www.dpmms.cam.ac.uk/∼rhgg2)

Pseudofunctors

- **2-Cat** is the category of 2-categories and 2-functors between them.
- $\mathbf{2\text{-}Cat}_\psi$ is the category of 2-categories and pseudofunctors between them.
- We have an identity-on-objects inclusion

$$J\colon \mathbf{2\text{-}Cat} \to \mathbf{2\text{-}Cat}_\psi$$

which has a left adjoint [Blackwell-Kelly-Power 1989]

$$(\ )'\colon \mathbf{2\text{-}Cat}_\psi \to \mathbf{2\text{-}Cat},$$

so that pseudofunctors $\mathscr{K} \to \mathscr{L}$ correspond to 2-functors $\mathscr{K}' \to \mathscr{L}$.

Explicitly, for a 2-category $\mathcal{K}$, we form $\mathcal{K}'$ as follows:

- Ignore the 2-cells and form the free category $FU\mathcal{K}$ on the underlying 1-graph of $\mathcal{K}$;
- Consider $FU\mathcal{K}$ as a locally discrete 2-category, and take the factorisation of the counit map $\varepsilon\colon FU\mathcal{K} \to \mathcal{K}$ as

$$FU\mathcal{K} \xrightarrow{a} \mathcal{K}' \xrightarrow{b} \mathcal{K}$$

  where a is bijective on objects and 1-cells and b is locally fully faithful.

Key observation

- The adjunction $(\ )' \dashv J$ induces a comonad $E = (J\text{–})'$ on **2-Cat**, and we recapture $\mathbf{2\text{-}Cat}_\psi$ as the Kleisli category of this comonad.
- If we could give an intrinsic characterisation of the comonad E in terms of the structure of the "strict" category **2-Cat**, we could obtain the "pseudo" notion for free.
- In fact, by using some ideas from homotopy theory, we can!

Weak factorisation systems

A *weak factorisation system* on a category $\mathscr{C}$ consists of two classes of maps $\mathscr{L}$ and $\mathscr{R}$, closed under retracts and satisfying:

- **Factorisation**: every $\mathscr{C}$-map can be decomposed as an $\mathscr{L}$-map followed by an $\mathscr{R}$-map; and
- **Orthogonality**: $f \,\square\, g$ for every $f \in \mathscr{L}$ and $g \in \mathscr{R}$,

where $f \,\square\, g$ means that for every commutative square

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array}$$

there exists a diagonal fill-in $j\colon B \to C$ making both triangles commute.

Building weak factorisation systems

In a locally presentable category $\mathscr{C}$, any set I of maps generates a weak factorisation system $(\mathscr{L}, \mathscr{R})$ where:

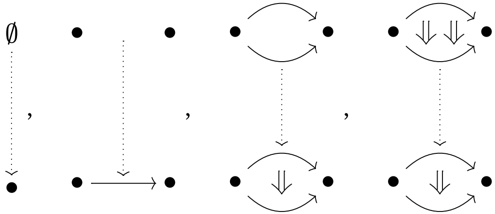
$$\mathscr{R} = \{\, g \colon C \to D \mid f \,\square\, g \text{ for all } f \in I \,\}$$
$$\mathscr{L} = \{\, f \colon A \to B \mid f \,\square\, g \text{ for all } g \in \mathscr{R} \,\}\,.$$

Intuitively:

- Each map $f \colon A \to B$ in I specifies a valid "boundary" shape (A) together with a "cell" (B) which fills it: the map f being the boundary inclusion.
- Each $\mathscr{L}$-map in the resultant weak factorisation system is built by recursively glueing in "I-cells" along their "boundaries" (and then taking retracts).

Example: 2-Cat

[Lack 2002] In **2-Cat** consider the w.f.s. $(\mathscr{L}, \mathscr{R})$ generated by the following set of maps:



- A 2-functor is an $\mathscr{R}$-map iff it is surjective on objects and 1-cells and locally fully faithful.
- A 2-functor is an $\mathscr{L}$-map iff, at the underlying category level, it is obtained by freely adjoining new objects and 1-cells (and then taking a retract).

Cofibrant replacements

Given a w.f.s. $(\mathscr{L}, \mathscr{R})$ on $\mathscr{C}$:

- We call an object $X \in \mathscr{C}$ *cofibrant* if the unique map $0 \to X$ lies in $\mathscr{L}$;
- By a *cofibrant replacement* for an object $X \in \mathscr{C}$, we mean a cofibrant object Y together with an $\mathscr{R}$-map $Y \to X$.

Cofibrant replacements for X correspond to $(\mathscr{L}, \mathscr{R})$ factorisations of the unique map $0 \to X$.

Example: 2-Cat

- For the above w.f.s. $(\mathscr{L}, \mathscr{R})$ on **2-Cat**, a 2-category is cofibrant if and only if its underlying category is free on a graph.
- For any 2-category $\mathscr{K}$, a cofibrant replacement is given by $\mathscr{K}'$ together with the counit map $\mathscr{K}' \to \mathscr{K}$.
- So the comonad $E = (J-)'$ on **2-Cat** is a "cofibrant replacement comonad".
- **Question**: Can we make this notion precise?

Natural weak factorisation systems

A *natural weak factorisation system* [Grandis-Tholen 2006] on a category $\mathscr{C}$ is given by:

- A comonad $\mathbb{L} = (L, \Phi, \Sigma)$ on $\mathscr{C}^{\to}$;
- A monad $\mathbb{R} = (R, \Lambda, \Pi)$ on $\mathscr{C}^{\to}$;
- A distributive law $\Delta\colon LR \Rightarrow RL$.

satisfying some laws:

$$\begin{array}{cccc} \text{dom} \cdot L = \text{dom}, & \text{cod} \cdot L = \text{dom} \cdot R, & \text{cod} \cdot R = \text{cod}; & \\ \text{dom} \cdot \Phi = 1_{\text{dom}}, & \text{cod} \cdot \Phi = \kappa \cdot R, & \text{dom} \cdot \Lambda = \kappa \cdot L, & \text{cod} \cdot \Lambda = 1_{\text{cod}}; \\ \text{and } \text{dom} \cdot \Sigma = 1_{\text{dom}}, & \text{cod} \cdot \Sigma = \text{dom} \cdot \Delta, & \text{dom} \cdot \Pi = \text{cod} \cdot \Delta, & \text{cod} \cdot \Pi = 1_{\text{cod}}. \end{array}$$

Intuitively an "algebraisation" of the notion of weak factorisation system:

- The *property* of being an $\mathscr{L}$-map is replaced with the *structure* of being a coalgebra for the comonad $\mathbb{L}$;
- The *property* of being an $\mathscr{R}$-map is replaced with the *structure* of being an algebra for the monad $\mathbb{R}$;
- The functor part of $\mathbb{L}$ sends a map of $\mathscr{C}$ to the left half of its $(\mathscr{L}, \mathscr{R})$-factorisation;
- The functor part of $\mathbb{R}$ sends a map of $\mathscr{C}$ to the right half of its $(\mathscr{L}, \mathscr{R})$-factorisation;
- Liftings between $\mathbb{L}$-coalgebras and $\mathbb{R}$-algebras are built canonically by interacting their (co)algebraic structure.

Building natural weak factorisation systems

[G. 2007] For a locally presentable category $\mathscr{C}$, any set I of maps in $\mathscr{C}$ generates a natural weak factorisation system $(\mathbb{L}, \mathbb{R}, \Delta)$ where:

- $\mathbb{R}$-algebras are maps $g \colon C \to D$ of $\mathscr{C}$ equipped with a *chosen* lifting against every element of I.
- $\mathbb{L}$-algebras are maps $f \colon A \to B$ of $\mathscr{C}$ equipped with a specification of *how* we recursively glued in "I-cells" along their "boundaries" (think of computads).

[Universal property: this $(\mathbb{L}, \mathbb{R}, \Delta)$ is in a suitable sense *freely* generated by the set I.]

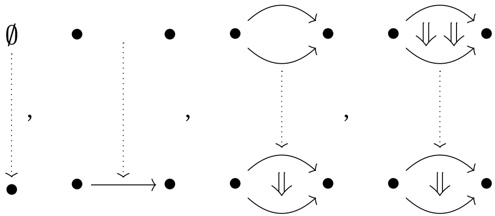
Cofibrant replacements

Given a n.w.f.s. $(\mathbb{L}, \mathbb{R}, \Delta)$ on $\mathscr{C}$:

- A cofibrant structure on X is an $\mathbb{L}$-map structure on the unique map $!\colon 0 \to X$.
- Applying L to a map $!\colon 0 \to X$ sends it to the left half of its $(\mathbb{L}, \mathbb{R})$-factorisation: i.e., to an object EX for which $!\colon 0 \to EX$ is an $\mathbb{L}$-map.
- Thus L restricts to a cofibrant replacement comonad $E\colon \mathscr{C} \to \mathscr{C}$.
- We define the *weak morphism* category $\mathscr{C}_{\mathrm{wk}}$ to be the Kleisli category of E.

Example: 2-Cat

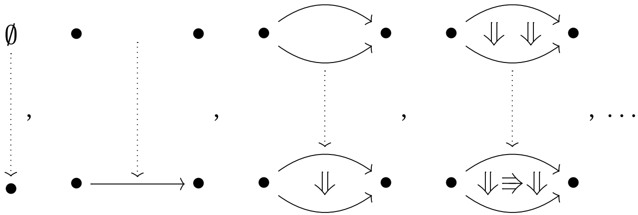
Consider once again the following set of maps in **2**-**Cat**:



- The cofibrant replacement comonad E they generate is the comonad $(J\text{–})'$ from above;
- So the weak morphism category is precisely $\mathbf{2}\text{-}\mathbf{Cat}_{\psi}$.

Example: ω-**Cat**

Let ω-**Cat** be the category of strict ω-categories and strict morphisms. Consider the following set of maps:



and let E be the cofibrant replacement comonad they generate.

- E-**Coalg** is the category of ω-computads;
- Arrows of the weak morphism category ω-$\mathbf{Cat}_{\mathrm{wk}}$ are sensible weak morphisms of ω-categories.

$\mathcal{V}$-operads

- Let $\mathcal{V}$ be a cocomplete monoidal category: $[\mathbb{N}, \mathcal{V}]$ is the monoidal category of $\mathcal{V}$-*collections*, with "substitution" tensor product

$$(F \diamond G)(n) = \sum_{\substack{k, n_1, \ldots, n_k \\ n_1 + \cdots + n_k = n}} F(k) \otimes G(n_1) \otimes \cdots \otimes G(n_k).$$

- $\mathcal{V}$-**Opd** $=$ category of monoids in $[\mathbb{N}, \mathcal{V}]$. We have a monadic adjunction:

$$\mathcal{V}\text{-}\mathbf{Opd} \underset{F}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} [\mathbb{N}, \mathcal{V}].$$

Cat-operads

- **Cat**-**Opd** is the category of **Cat**-operads and operad morphisms;
- **Cat**-**Opd**$_\psi$ is the category of **Cat**-operads and operad pseudomorphisms;
- We have an identity-on-objects inclusion

$$J\colon \mathbf{Cat}\text{-}\mathbf{Opd} \to \mathbf{Cat}\text{-}\mathbf{Opd}_\psi$$

which has a left adjoint [BKP89]

$$(\ )'\colon \mathbf{Cat}\text{-}\mathbf{Opd}_\psi \to \mathbf{Cat}\text{-}\mathbf{Opd},$$

so that operad pseudomorphisms $S \to T$ correspond to operad morphisms $S' \to T$.

Explicitly, for a **Cat**-operad S, we form S' as follows:

- View S as a plain **Set**-operad and form the free **Set**-operad FUS on the underlying collection of S;
- Consider FUS as a discrete **Cat**-operad, and take the factorisation of the counit map $\varepsilon\colon FUS \to S$ as

$$FUS \xrightarrow{a} S' \xrightarrow{b} S$$

  where a is componentwise bijective on objects and b is componentwise fully faithful.

Algebras for Cat-operads

… or, "why we care about operad pseudomorphisms".

- Any category $\mathscr{C}$ induces a **Cat**-operad $\langle \mathscr{C}, \mathscr{C} \rangle$:

$$\langle \mathscr{C}, \mathscr{C} \rangle(n) = \mathbf{Cat}(\mathscr{C}^n, \mathscr{C});$$

- An *algebra* for a **Cat**-operad S is a category $\mathscr{C}$ together with an operad morphism $S \to \langle \mathscr{C}, \mathscr{C} \rangle$.
- A *pseudoalgebra* for a **Cat**-operad S is a category $\mathscr{C}$ together with an operad pseudomorphism $S \to \langle \mathscr{C}, \mathscr{C} \rangle$.

But now

$$\frac{\frac{\text{Pseudo-}S\text{-algebra structures on } \mathscr{C}}{\text{Operad pseudomorphisms } S \to \langle \mathscr{C}, \mathscr{C} \rangle}}{\frac{\text{Operad morphisms } S' \to \langle \mathscr{C}, \mathscr{C} \rangle}{\text{Strict } S'\text{-algebra structures on } \mathscr{C}.}}$$

So *pseudo-S-algebras are just strict S'-algebras*.

Example: monoidal categories

Let S be the terminal **Cat**-operad, $S(n) = 1$ for all $n \in \mathbb{N}$.

- An S-algebra is a strict monoidal category;
- A pseudo-S-algebra is an unbiased monoidal category;
- S' is the operad for unbiased monoidal categories.

We can extend this formalism to deal with "many object" algebras, and then:

- A many-objects S-algebra is a 2-category;
- A many-objects S'-algebra is an unbiased bicategory.

Are "operad pseudomorphisms" an instance of our "weak morphism" formalism?

Yes! Let's give a set of maps generating a suitable n.w.f.s. on **Cat**-**Opd**. First note that:

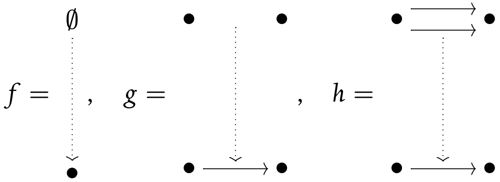
- We have functors $\Sigma_n \colon \mathbf{Cat} \to [\mathbb{N}, \mathbf{Cat}]$ left adjoint to evaluation at n:

$$\Sigma_n(\mathscr{C})(m) = \begin{cases} \mathscr{C} & \text{if } m = n; \\ 0 & \text{otherwise.} \end{cases}$$

- We have a monadic adjunction

$$\mathbf{Cat}\text{-}\mathbf{Opd} \underset{F}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} [\mathbb{N}, \mathbf{Cat}].$$

Now using the maps



in **Cat**, we can give a set I of generating maps in **Cat**-**Opd**:

$$I = \{\, F\Sigma_i(f), F\Sigma_i(g), F\Sigma_i(h) \mid i \in \mathbb{N} \,\}\,.$$

- The cofibrant replacement comonad E generated by these maps is the composite $(J\text{–})'$ from above;
- So the weak morphism category is precisely $\mathbf{Cat}\text{-}\mathbf{Opd}_\psi$.
- (NB: cofibrant replacements for the terminal **Cat**-operad are precisely [Leinster 2004]'s "algebraic notions of bicategory".)

Globular operads

[Batanin 1998; Leinster 2004]

- $\mathbb{G}$ is the *globe category*:

$$0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \underset{t}{\overset{s}{\rightrightarrows}} 2 \dots$$

  satisfying cosource and cotarget equations $ss = ts$ and $st = tt$.
- $\mathbf{GSet} = [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ is the category of *globular sets*.
- We have a monadic adjunction

$$\omega\text{-}\mathbf{Cat} \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \mathbf{GSet},$$

  and write T for the induced monad on $\mathbf{GSet}$.

- The category $\mathcal{N}$ of *globular pasting diagrams* is the category of elements of $T1 \colon \mathbb{G}^{\mathrm{op}} \to \mathbf{Set}$.
- The category **GColl** of *globular collections* is the functor category $[\mathcal{N}, \mathbf{Set}]$; it has a monoidal structure given by substitution of pasting diagrams.
- **GOpd** $=$ category of monoids in **GColl**. We have a monadic adjunction:

$$\mathbf{GOpd} \underset{F}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{GColl}.$$

Algebras for globular operads

- Any globular set $X \in \mathbf{GSet}$ induces a globular operad $\langle X, X \rangle$:

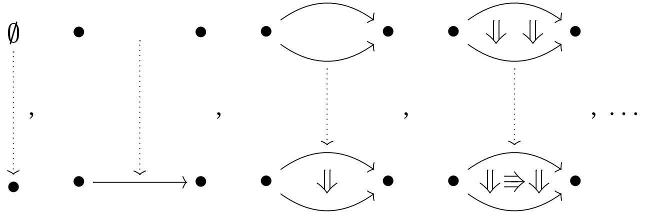
$$\langle X, X \rangle(\pi) = \mathbf{Set}(X^{\hat{\pi}}, X(m)) \quad \text{for } \pi \in T1(m);$$

- An algebra for a globular operad S is a globular set X together with a globular operad morphism $S \to \langle X, X \rangle$.
- If S is the terminal globular operad:

$$S(\pi) = 1 \quad \text{for all } \pi \in \mathcal{N},$$

  then an S-algebra is a strict ω-category.
- But what about weak algebras?

Let's build a set I of generating maps for a n.w.f.s. on **GOpd**. We start with this familiar set of maps in **GSet**:



Taking the preimage of this set under the forgetful functor

$$\Sigma_{T1} : \mathbf{GSet}/T1 \to \mathbf{GSet}$$

gives us a set of maps in $\mathbf{GSet}/T1 \cong \mathbf{GColl}$.

- Explicitly, for every object $\pi \in \mathcal{N}$, we give a map k_π of $\mathbf{GColl} = [\mathcal{N}, \mathbf{Set}]$ by

$$y_{s(\pi)} + y_{t(\pi)} \xrightarrow{\langle y_s, y_t \rangle} y_\pi$$

(where $y_{(-)} \colon \mathcal{N}^{\mathrm{op}} \to [\mathcal{N}, \mathbf{Set}]$ is the Yoneda embedding).
- Now we take I to be the set of maps $\{ Fk_\pi \mid \pi \in \mathcal{N} \}$ in $\mathbf{GOpd}$.

Let E be the cofibrant replacement comonad generated by this set of maps in $\mathbf{GOpd}$, and let $\mathbf{GOpd}_{\mathrm{wk}}$ be the corresponding weak morphism category. So we have an adjunction

$$\mathbf{GOpd} \underset{(\ )'}{\overset{J}{\rightleftarrows}} \mathbf{GOpd}_{\mathrm{wk}}, \quad J \dashv (\ )'.$$

Now we define:

- A weak algebra for a globular operad S is a globular set X together with a weak globular operad morphism $S \to \langle X, X \rangle$.
- Arguing as before, a weak S-algebra is precisely a S'-algebra.
- And we find that, if S is the terminal globular operad:

$$S(\pi) = 1 \quad \text{for all } \pi \in \mathcal{N},$$

  then S' is precisely the "initial globular operad with contraction": that is, (Leinster's) operad for weak ω-categories.
- So a weak S-algebra is a weak ω-category!

Future directions

- We can combine the ideas of the previous two sections to get a category of weak ω-categories and weak morphisms.
- "Unbiased" versions of A_∞-algebras and their morphisms – using a n.w.f.s. on chain complexes which is *enriched* over R-**Mod**.
- Globular computads.

Homotopy limits:

- If $\mathscr{V}$ is a category with a n.w.f.s., we can define a n.w.f.s. on $[\mathscr{A}, \mathscr{V}]$.
- Now define a homotopy limit weighted by $H \in [\mathscr{A}, \mathscr{V}]$ to be a strict limit weighted by the cofibrant replacement H'.
- E.g.: for $\mathscr{V} = \mathbf{Cat}$, we regain the notion of *pseudolimit*.
- E.g.: for $\mathscr{V} = \mathbf{Ch}(R)$, the homotopy colimit of an arrow is its mapping cylinder (again using an enriched n.w.f.s.).

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