# Two-dimensional locally cartesian closed categories

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# Outline

# Motivation

We have many examples of locally cartesian closed categories, e.g. from topos theory:

- $\mathbf{Set}$;
- $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ for a small $\mathbf{C}$;
- $\mathbf{Sh}(\mathscr{J})$ for a small site $\mathscr{J}$.

Each has a natural 2-categorical (or groupoid-enriched) analogue:

- $\mathbf{Gpd}$;
- $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathbf{Gpd})$ for a small 2-category $\mathbf{C}$;
- $\mathbf{Stack}(\mathscr{J})$ for a small site $\mathscr{J}$.

In what sense are these 2-categories locally cartesian closed?

# Example: Gpd

- **Gpd** is cartesian closed as a 2-category: for each $X \in \mathbf{Gpd}$, we have a 2-adjunction

$$\mathbf{Gpd} \underset{[X,-]}{\overset{(-)\times X}{\underset{\perp}{\rightleftarrows}}} \mathbf{Gpd}.$$

- It is *not* locally cartesian closed as a mere category: e.g., consider the groupoid

$$\mathbf{Iso} = a \underset{u^{-1}}{\overset{u}{\rightleftarrows}} b$$

  and functor

$$a \colon 1 \to \mathbf{Iso}.$$

  The pullback functor $a^* \colon \mathbf{Gpd}/\mathbf{Iso} \to \mathbf{Gpd}$ has no right adjoint.

However it is *locally cartesian closed as a bicategory*.

For $\mathbf{A} \in \mathbf{Gpd}$, define $\mathbf{Gpd}//\mathbf{A}$ to have:

- **Objects** functors $g\colon \mathbf{G} \to \mathbf{A}$;
- **1-cells** pseudocommutative triangles

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow{\quad k \quad} & \mathbf{H} \\ & {}_{g}\searrow \quad \cong \quad \swarrow_{h} & \\ & \mathbf{A} & \end{array}$$

- **2-cells** compatible natural isos $k \cong k'$.

**Proposition (Street, 1980)**

*For each* $f\colon \mathbf{A} \to \mathbf{B}$ *in* **Gpd**, *the pullback 2-functor*

$$f^*\colon \mathbf{Gpd}/\!/\mathbf{B} \to \mathbf{Gpd}/\!/\mathbf{A}$$

*has a right biadjoint.*

Very weak result!

Perhaps we can do better, using the fact that:

- If $f\colon \mathbf{A} \to \mathbf{B}$ is a *groupoid fibration* ( = Grothendieck fibration = Conduché fibration = prefibration = isofibration) then $f^*\colon \mathbf{Gpd}/\mathbf{B} \to \mathbf{Gpd}/\mathbf{A}$ has a right 2-adjoint.

But how to formalise this?

# Example: $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathbf{Gpd})$

Let $\mathbf{C}$ be a small category. $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathbf{Gpd})$ is 2-category with:

- **Objects** being functors $X \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Gpd}$;
- **1-cells** being pseudo-natural transformations $f \colon X \Rightarrow Y$;
- **2-cells** being modifications $\alpha \colon f \Rrightarrow g$.

$\mathbf{St}(\mathbf{C}^{\mathrm{op}}, \mathbf{Gpd})$ is sub-2-category whose 1-cells are the *2-natural* transformations $X \Rightarrow Y$.

$\mathbf{St}(\mathbf{C}^{\mathrm{op}}, \mathbf{Gpd})$ is cartesian closed as a 2-category by usual Yoneda argument. But what about $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathbf{Gpd})$?

Important fact: the inclusion functor

$$\mathbf{St}(\mathbf{C}^{\mathrm{op}}, \mathbf{Gpd}) \hookrightarrow \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathbf{Gpd})$$

has a right adjoint

$$(-)^* \colon \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathbf{Gpd}) \to \mathbf{St}(\mathbf{C}^{\mathrm{op}}, \mathbf{Gpd}).$$

(Consequence of Blackwell-Kelly-Power, or directly by Yoneda lemma).

So for $Y, Z\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Gpd}$, have that:

$$\begin{aligned}\mathbf{Ps}(- \times Y, Z) &\cong \mathbf{St}(- \times Y, Z^*)\\ &\cong \mathbf{St}(-, [Y, Z^*]).\end{aligned}$$

Moreover, $[Y, Z^*]$ is *coflexible*: i.e., have a pseudonatural equivalence

$$\mathbf{St}(-, [Y, Z^*]) \underset{p}{\overset{i}{\underset{\simeq}{\rightleftarrows}}} \mathbf{Ps}(-, [Y, Z^*])$$

satisfying $pi = \mathrm{id}$. This is because:

- Each $Z^*$ is coflexible (Blackwell-Kelly-Power);
- Coflexible objects form an exponential ideal in **St** (check directly).

Hence for each $X, Y, Z \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Gpd}$, have an equivalence of categories

$$\mathbf{Ps}(X \times Y, Z) \underset{\mathsf{app}}{\overset{\mathsf{abs}}{\underset{\simeq}{\rightleftarrows}}} \mathbf{Ps}(X, [Y, Z^*])$$

satisfying $\mathsf{app}.\mathsf{abs} = \mathrm{id}$.

How can we make sense of this all?

# Categorical logic

Robert Seely showed in 1984 that we have:

$$\begin{array}{c}\text{Extensional Martin-Löf}\\\text{type theories}\end{array} \underset{\textit{syntax}}{\overset{\textit{semantics}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \begin{array}{c}\text{Locally cartesian}\\\text{closed categories.}\end{array}$$

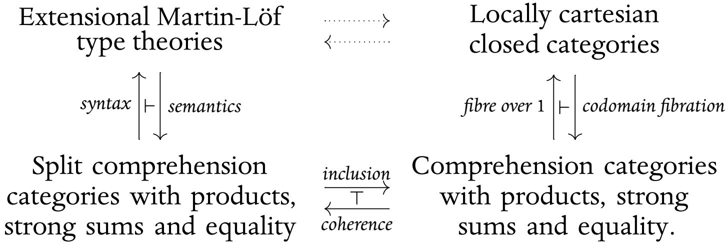
Our plan is to extend this to:

$$\begin{array}{c}\text{2-dimensional Martin-Löf}\\\text{type theories}\end{array} \underset{\textit{syntax}}{\overset{\textit{semantics}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \begin{array}{c}\text{Locally cartesian}\\\text{closed 2-categories}\end{array}$$

by defining the notion on the left and deducing the one on the right.

Martin Hofmann pointed out in 1994 an inaccuracy in Seely's work arising from the failure of a certain fibration to be split.

More accurate picture is:



We will generalise the left-hand side of this to:

2-dimensional Martin-Löf
type theories

*syntax* ↑ ↓ *semantics*

Split comprehension
2-categories with products,
strong sums and equality;

But first we recall how the one-dimensional case works.

# Martin-Löf type theory

Sequent calculus with four forms of judgement:

- $A$ type;
- $a : A$;
- $A = B$ type;
- $a = b : A$.

Can also have judgements under hypotheses; so if $A$ type, we can have

- $x : A \vdash B(x)$ type;
- $x : A \vdash f(x) : B(x)$;
- $x : A \vdash B(x) = C(x)$ type;
- $x : A \vdash f(x) = g(x) : B(x)$.

And so on; in general can have things like

$$x : A,\ y : B(x),\ z : C(x, y) \vdash f(x, y, z) : D(x, y, z).$$

These come with inference rules for:

- Weakening, contraction and exchange;
- *Substitution*; for example:

$$\frac{x : A \vdash f(x) : B \qquad y : B \vdash C(y)\ \mathsf{type}}{x : A \vdash C(f(x))\ \mathsf{type}}$$

  or

$$\frac{x : A \vdash f(x) : B \qquad y : B \vdash g(y) : C(y)\ \mathsf{type};}{x : A \vdash g(f(x)) : C(f(x))}$$

- *Logical operations* as follows.

# Dependent sums

$$\frac{A \text{ type} \qquad x : A \vdash B(x) \text{ type}}{\Sigma x : A.\, B(x) \text{ type}}\ \Sigma\text{-FORM}; \qquad \frac{a : A \qquad b : B(a)}{\langle a, b\rangle : \Sigma x : A.\, B(x)}\ \Sigma\text{-INTRO};$$

$$\frac{\begin{array}{c} z : \Sigma x : A.\, B(x) \vdash C(z) \text{ type} \\ x : A,\ y : B(x) \vdash d(x,y) : C(\langle x, y\rangle) \qquad s : \Sigma x : A.\, B(x) \end{array}}{E(C, d, s) : C(s)}\ \Sigma\text{-ELIM};$$

$$\frac{\begin{array}{c} z : \Sigma x : A.\, B(x) \vdash C(z) \text{ type} \\ x : A,\ y : B(x) \vdash d(x,y) : C(\langle x, y\rangle) \qquad a : A \qquad b : B(a) \end{array}}{E(C, d, \langle a, b\rangle) = d(a,b) : C(\langle a, b\rangle)}\ \Sigma\text{-COMP}.$$

# Dependent products

$$\frac{A \text{ type} \qquad x : A \vdash B(x) \text{ type}}{\Pi x : A.\, B(x) \text{ type}}\ \Pi\text{-}\textsc{form};$$

$$\frac{x : A \vdash f(x) : B(x)}{\lambda x : A.\, f(x) : \Pi x : A.\, B(x)}\ \Pi\text{-}\textsc{abs}; \qquad \frac{M : \Pi x : A.\, B(x) \qquad a : A}{M \cdot a : B(a)}\ \Pi\text{-}\textsc{app};$$

$$\frac{x : A \vdash f(x) : B(x) \qquad a : A}{\big(\lambda x : A.\, f(x)\big) \cdot a = f(a) : B(a)}\ \Pi\text{-}\beta.$$

# Identity types

$$\frac{A \text{ type} \qquad a, b : A}{\mathrm{Id}_A(a, b) \text{ type}}\ \text{Id-FORM}; \qquad \frac{a : A}{r(a) : \mathrm{Id}_A(a, a)}\ \text{Id-INTRO};$$

$$\frac{\begin{array}{c} x, y : A,\ z : \mathrm{Id}_A(x, y) \ \vdash\ C(x, y, z) \text{ type} \\ x : A \ \vdash\ d(x) : C(x, x, r(x)) \qquad a, b : A \qquad p : \mathrm{Id}_A(a, b) \end{array}}{J_C(d, a, b, p) : C(a, b, p)}\ \text{Id-ELIM};$$

$$\frac{\begin{array}{c} x, y : A,\ z : \mathrm{Id}_A(x, y) \ \vdash\ C(x, y, z) \text{ type} \\ x : A \ \vdash\ d(x) : C(x, x, r(x)) \qquad a : A \end{array}}{J_C(d, a, a, r(a)) = d(a) : C(a, a, r(a))}\ \text{Id-COMP}.$$

Intuition: to each type $A$ we can associate an weak $\omega$-groupoid $\mathscr{A}$ with:

- Objects being elements $x : A$;
- 1-cells $f\colon x \to y$ being elements $f : \mathrm{Id}_A(x, y)$;
- 2-cells $\alpha\colon f \Rightarrow g\colon x \to y$ being elements $\alpha : \mathrm{Id}_{\mathrm{Id}_A(x,y)}(f, g)$;

and so on.

# Extensional Martin-Löf type theory

The above is the *intensional* version of Martin-Löf type theory.

The *extensional* version adds two inference rules:

$$\frac{A \text{ type} \qquad a, b : A \qquad p : \mathrm{Id}_A(a, b)}{a = b : A} \text{ Id-REFL-1};$$

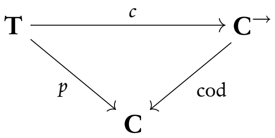
$$\frac{A \text{ type} \qquad a, b : A \qquad p : \mathrm{Id}_A(a, b)}{p = r(a) : \mathrm{Id}_A(a, b)} \text{ Id-REFL-2};$$

which force the weak $\omega$-groupoid $\mathscr{A}$ associated to $A$ to be discrete.

# Categorical semantics

A (split) *comprehension category* (Jacobs 1993) consists in:

- A category $\mathbf{C}$;
- A (split) fibration $p\colon \mathbf{T} \to \mathbf{C}$;
- A full and faithful functor




sending cartesian morphisms in $\mathbf{T}$ to pullback squares in $\mathbf{C}$.

Given $\Gamma \in \mathbf{C}$ and $A \in \mathbf{T}(\Gamma)$, we write the image of $A$ under $c$ as

$$\pi_\Gamma\colon \Gamma.A \to \Gamma$$

in $\mathbf{C}$, and call it a *dependent projection*.

# Example: extensional syntactic model

We construct a category **Ctxt** from the syntax of extensional Martin-Löf type theory:

- **Objects** are *contexts* of types

$$\Gamma = \big(x_1 : C_1,\ x_2 : C_2(x_1),\ \ldots,\ x_n : C_n(x_1, \ldots, x_{n-1})\big);$$

- **Morphisms** $f : \Gamma \to \Delta$ are *context morphisms* given by collections of judgements

$$x_1 : C_1,\ \ldots,\ x_n : C_n(x_1, \ldots, x_{n-1}) \ \vdash\ f_1(x_1, \ldots, x_n) : D_1$$
$$x_1 : C_1,\ \ldots,\ x_n : C_n(x_1, \ldots, x_{n-1}) \ \vdash\ f_2(x_1, \ldots, x_n) : D_2\big(f_1(x_1, \ldots, x_n)\big)$$
$$\ldots$$

which we abbreviate as

$$\boldsymbol{x} : \Gamma \ \vdash\ f(\boldsymbol{x}) : \Delta.$$

Now obtain (split) indexed category $\mathbf{Type}\colon \mathbf{Ctxt}^{\mathrm{op}} \to \mathbf{Cat}$.

The category $\mathbf{Type}(\Gamma)$ has:

- **Objects** being judgements

$$x : \Gamma \ \vdash \ A(x) \ \mathsf{type};$$

- **Morphisms** $A \to B$ being judgements

$$x : \Gamma, \ \ y : A(x) \ \vdash \ f(x, y) : B(x).$$

Given $f\colon \Delta \to \Gamma$, the functor $f^*\colon \mathbf{Type}(\Gamma) \to \mathbf{Type}(\Delta)$ sends:

$$y : \Gamma \ \vdash \ A(y) \ \mathsf{type} \qquad \text{to} \qquad x : \Delta \ \vdash \ A(f(x)) \ \mathsf{type}.$$

Thus obtain a split fibration $p\colon \mathbf{Type} \to \mathbf{Ctxt}$.

And now have a functor $c\colon \mathbf{Type} \to \mathbf{Ctxt}^{\to}$ sending

$$x : \Gamma \;\vdash\; A(x) \text{ type}$$

to the context morphism

$$(x : \Gamma,\;\; y : A(x)) \to (x : \Gamma)$$

which projects away the last variable.

## Sum types

A split comprehension category has *sums* if:

- For each $\Gamma \in \mathbf{C}$ and $A \in \mathbf{T}(\Gamma)$, the reindexing functor

  $$(-) \times_\Gamma A := \mathbf{T}(\pi_\Gamma) \colon \mathbf{T}(\Gamma) \to \mathbf{T}(\Gamma.A)$$

  has a left adjoint $\Sigma_A$.

- These left adjoints satisfy the *Beck-Chevalley condition*.

Let $B \in \mathbf{T}(\Gamma.A)$ and consider the unit map

$$\eta_{A,B} \colon B \to \Sigma_A(B) \times_\Gamma A$$

of the adjunction $\Sigma_A \dashv (-) \times_\Gamma A$. It is equivalently a map

$$\begin{array}{ccc}
\Gamma.A.B & \xrightarrow{\;i_{A,B}\;} & \Gamma.\Sigma_A B \\
{\scriptstyle \pi_{\Gamma.A}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_\Gamma} \\
\Gamma.A & \xrightarrow[\;\pi_\Gamma\;]{} & \Gamma
\end{array}$$

in $\mathbf{C}$.

Say that we have *strong sums* just when each $i_{A,B}$ is an isomorphism.

# Example: extensional syntactic model

Let $\Gamma \in \mathbf{Ctxt}$ and $A \in \mathbf{Type}(\Gamma)$; then the functor

$$\Sigma_A \colon \mathbf{Type}(x : \Gamma,\ y : A(x)) \to \mathbf{Type}(x : \Gamma)$$

sending

$$x : \Gamma,\ y : A(x) \ \vdash\ B(x, y)\ \mathsf{type}$$

to

$$x : \Gamma \ \vdash\ \Sigma y : A(x).\, B(x, y)$$

equips the syntactic comprehension category with strong sums.

# Product types

A split comprehension category has *products* if:

- For each $\Gamma \in \mathbf{C}$ and $A \in \mathbf{T}(\Gamma)$, the reindexing functor

  $$(-) \times_\Gamma A := \mathbf{T}(\pi_\Gamma) \colon \mathbf{T}(\Gamma) \to \mathbf{T}(\Gamma.A)$$

  has a right adjoint $\Pi_A$.

- These right adjoints satisfy the *Beck-Chevalley condition*.

# Example: extensional syntactic model

Let $\Gamma \in \mathbf{Ctxt}$ and $A \in \mathbf{Type}(\Gamma)$; then the functor

$$\Pi_A : \mathbf{Type}(x : \Gamma,\ y : A(x)) \to \mathbf{Type}(x : \Gamma)$$

sending

$$x : \Gamma,\ y : A(x) \ \vdash\ B(x, y) \ \mathsf{type}$$

to

$$x : \Gamma \ \vdash\ \Pi y : A(x).\, B(x, y)$$

equips the syntactic comprehension category with products.

# 2-dimensional Martin-Löf type theory

Let us say that a type $A$ is *discrete* just when the Id-reflection rules

$$\frac{A \ \mathsf{type} \qquad a, b : A \qquad p : \mathrm{Id}_A(a, b)}{a = b : A} \ \text{Id-}\textsc{refl}\text{-1};$$

$$\frac{A \ \mathsf{type} \qquad a, b : A \qquad p : \mathrm{Id}_A(a, b)}{p = r(a) : \mathrm{Id}_A(a, b)} \ \text{Id-}\textsc{refl}\text{-2};$$

obtain at the type $A$.

- So the intensional theory says that *no types need be discrete*;
- The extensional theory says that *all types are discrete*;
- Our 2-dimensional theory will say that *all identity types are discrete*.

Explicitly, we augment the intensional theory with the rules:

$$\frac{A\ \mathsf{type} \qquad a, b : A \qquad p, q : \mathrm{Id}_A(a,b) \qquad s : \mathrm{Id}_{\mathrm{Id}_A(a,b)}(p,q)}{p = q : \mathrm{Id}_A(a,b)}\ \text{Id-}\textsc{disc}\text{-1};$$

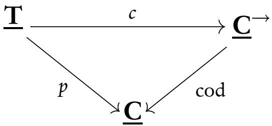
$$\frac{A\ \mathsf{type} \qquad a, b : A \qquad p, q : \mathrm{Id}_A(a,b) \qquad s : \mathrm{Id}_{\mathrm{Id}_A(a,b)}(p,q)}{s = r(p) : \mathrm{Id}_{\mathrm{Id}_A(a,b)}(p,q)}\ \text{Id-}\textsc{disc}\text{-2}.$$

These force the weak $\omega$-groupoid $\mathscr{A}$ associated to $A$ to be a common-or-garden groupoid.

## 2-categorical semantics

A (split) *comprehension 2-category* consists in:

- A 2-category $\underline{\mathbf{C}}$;
- A (split) 2-fibration $p\colon \underline{\mathbf{T}} \to \underline{\mathbf{C}}$;
- A 2-fully faithful 2-functor




sending 2-cartesian morphisms in $\underline{\mathbf{T}}$ to 2-pullback squares in $\underline{\mathbf{C}}$.

# Revision of 2-fibrations

A 2-functor $p\colon \underline{\mathbf{T}} \to \underline{\mathbf{C}}$ is a (split) **2-fibration** (Hermida, 1999) just when:

- The underlying ordinary functor $p\colon \mathbf{T} \to \mathbf{C}$ is a (split) fibration;
- Each cartesian morphism of $\underline{\mathbf{T}}$ has an obvious 2-dimensional universal property (say it is *2-cartesian*);
- Each $p_{y,z}\colon \underline{\mathbf{T}}(y,z) \to \underline{\mathbf{C}}(py,pz)$ is a fibration;
- For all $f\colon x \to y$ in $\underline{\mathbf{T}}$ the whiskering functor

$$(-) \circ f\colon \underline{\mathbf{T}}(y,z) \to \underline{\mathbf{T}}(x,z)$$

  sends cartesian 2-cells to cartesian 2-cells.

# Example: 2-dimensional syntactic model

First we must extend the category of contexts **Ctxt** to a 2-category $\underline{\mathbf{Ctxt}}$. What are the 2-cells?

- Easy for 2-cells into a context of length 1. Given $f, g \colon \Gamma \to (y : B)$ in $\underline{\mathbf{Ctxt}}$, a 2-cell $\alpha \colon f \Rightarrow g$ is given by

$$x : \Gamma \vdash p(x) : \mathrm{Id}_B(f(x), g(x)).$$

For longer contexts, need more theory. Given $A$ type and $x : A \vdash C(x)$ type, we can define "substitution" operations

$$\frac{a, b : A \qquad c : C(a) \qquad p : \mathrm{Id}_A(a, b)}{p_*(c) : C(b)}$$

by Id-elimination.

So now:

- Given $f, g\colon \Gamma \to (y : B,\ z : C(y))$ in **<u>Ctxt</u>** looking like

  $$x : \Gamma \ \vdash\ f_1(x) : B, \qquad x : \Gamma \ \vdash\ f_2(x) : C(f_1(x))$$

  and

  $$x : \Gamma \ \vdash\ g_1(x) : B, \qquad x : \Gamma \ \vdash\ g_2(x) : C(g_1(x))$$

  a 2-cell $\alpha\colon\ f \Rightarrow g$ will be given by

  $$x : \Gamma \ \vdash\ p_1(x) : \mathrm{Id}_B(f_1(x), g_1(x))$$
  $$x : \Gamma \ \vdash\ p_2(x) : \mathrm{Id}_{C(f_1(x))}\big(p_1(x)_*(f_2(x)),\ g_2(x)\big).$$

And so on.

Now extend the split indexed category $\mathbf{Type}\colon \mathbf{Ctxt}^{\mathrm{op}} \to \mathbf{Cat}$ to a *split indexed 2-category*; which amounts to a trihomomorphism

$$\underline{\mathbf{Type}}\colon \underline{\mathbf{Ctxt}}^{\mathrm{op}} \to \mathbf{Gray}$$

which is strictly functorial on 1-cells. Thus we must:

- Extend each fibre category $\mathbf{Type}(\Gamma)$ to a 2-category $\underline{\mathbf{Type}}(\Gamma)$ – straightforward;
- Extend each substitution functor $f^*\colon \mathbf{Type}(\Gamma) \to \mathbf{Type}(\Delta)$ to a 2-functor – straightforward;
- Give a pseudo-natural transformation

$$\overline{\alpha}\colon f^* \Rightarrow g^*\colon \underline{\mathbf{Type}}(\Gamma) \to \underline{\mathbf{Type}}(\Delta)$$

for each $\alpha\colon f \Rightarrow g\colon \Delta \to \Gamma$ in $\underline{\mathbf{Ctxt}}$.

Eg: let $\alpha\colon f \Rightarrow g\colon (x : A) \to (y : B)$ in $\underline{\mathbf{Ctxt}}$ be given by

$$x : A \vdash f(x) : B, \qquad x : A \vdash g(x) : B,$$
$$x : A \vdash \alpha(x) : \mathrm{Id}_B(f(x), g(x)).$$

To give

$$\overline{\alpha}\colon f^* \Rightarrow g^*\colon \underline{\mathbf{Type}}(B) \to \underline{\mathbf{Type}}(A)$$

we must give, for each $C \in \mathbf{Type}(B)$, a component 1-cell

$$\overline{\alpha}_C\colon C(f(-)) \to C(g(-))$$

in $\mathbf{Type}(A)$; which we take to be

$$x : A,\ y : C(f(x)) \vdash \alpha(x)_*(y) : C(g(x)).$$

Thus obtain a split 2-fibration $p\colon \underline{\mathbf{Type}} \to \underline{\mathbf{Ctxt}}$.

And may now extend the functor $c\colon \mathbf{Type} \to \mathbf{Ctxt}^{\to}$ to a (2-fully faithful) 2-functor $\underline{\mathbf{Type}} \to \underline{\mathbf{Ctxt}}^{\to}$.

## Digression on two-dimensional adjoints

To talk about sums and products, we need a suitably weak notion of adjoint.

- Let $H\colon \mathbf{A} \to \mathbf{B}$ be a functor. A **right strict pseudoinverse** for $H$ is:
  - $K\colon \mathbf{B} \hookrightarrow \mathbf{A}$ with $HK = \mathrm{id}_{\mathbf{B}}$; and
  - $\phi\colon \mathrm{id}_{\mathbf{A}} \cong KH$ with $\phi K = \mathrm{id}_K$ and $H\phi = \mathrm{id}_H$.
- $H$ has a right strict pseudoinverse only if fully faithful and surjective on objects;
- Then right strict pseudoinverses for $H \longleftrightarrow$ sections for $\mathrm{ob}\, H\colon \mathrm{ob}\, \mathbf{A} \to \mathrm{ob}\, \mathbf{B}$.

A **right strict left biadjoint** for $G\colon \underline{\mathbf{C}} \to \underline{\mathbf{D}}$ is:

- For each $d \in \underline{\mathbf{D}}$, some $Fd \in \underline{\mathbf{C}}$ and $\eta_d\colon d \to GFd$ in $\underline{\mathbf{D}}$;
- A right strict pseudoinverse for each functor

$$\underline{\mathbf{C}}(Fd, c) \xrightarrow{G} \underline{\mathbf{D}}(GFd, Gc) \xrightarrow{\mathbf{D}(\eta_d, Gc)} \underline{\mathbf{D}}(d, Gc).$$

A **right strict right biadjoint** for $G\colon \underline{\mathbf{C}} \to \underline{\mathbf{D}}$ is:

- For each $d \in \underline{\mathbf{D}}$, some $Hd \in \underline{\mathbf{C}}$ and $\varepsilon_d\colon GHd \to d$ in $\underline{\mathbf{D}}$;
- A right strict pseudoinverse for each functor

$$\underline{\mathbf{C}}(c, Hd) \xrightarrow{G} \underline{\mathbf{D}}(Gc, GHd) \xrightarrow{\mathbf{D}(Gc, \varepsilon_d)} \underline{\mathbf{D}}(Gc, d).$$

# Sum types

A split comprehension 2-category has *sums* if:

- For each $\Gamma \in \underline{\mathbf{C}}$ and $A \in \underline{\mathbf{T}}(\Gamma)$, the reindexing 2-functor

  $$(-) \times_{\Gamma} A := \underline{\mathbf{T}}(\pi_{\Gamma})\colon \underline{\mathbf{T}}(\Gamma) \to \underline{\mathbf{T}}(\Gamma.A)$$

  has a right strict left biadjoint $\Sigma_A$.

- These left biadjoints satisfy some kind of *Beck-Chevalley condition* (not sure what).

Explicitly:

- For each $B \in \underline{\mathbf{T}}(\Gamma.A)$, an object $\Sigma_A(B) \in \underline{\mathbf{T}}(\Gamma)$;
- For each $B \in \underline{\mathbf{T}}(\Gamma.A)$, a unit map $\eta_{A,B} \colon B \to \Sigma_A(B) \times_\Gamma A$ in $\underline{\mathbf{T}}(\Gamma.A)$; which is a map $i_{A,B}$ as in:

$$\begin{array}{ccc}
\Gamma.A.B & \xrightarrow{i_{A,B}} & \Gamma.\Sigma_A B \\
{\scriptstyle \pi_{\Gamma.A}}\downarrow & & \downarrow{\scriptstyle \pi_\Gamma} \\
\Gamma.A & \xrightarrow[\pi_\Gamma]{} & \Gamma
\end{array}$$

- For each $D \in \underline{\mathbf{T}}(\Gamma)$, a right strict pseudoinverse for the functor

$$\underline{\mathbf{C}}/\Gamma\big(\Gamma.\Sigma_A(B),\ D\big) \to \underline{\mathbf{C}}/\Gamma\big(\Gamma.A.B,\ D\big)$$

  induced by precomposition with $i_{A,B}$.

Say we have *strong sums* just when each $i_{A,B}$ has a chosen left strict pseudoinverse which induces the right strict pseudoinverses above.

# Example: 2-dimensional syntactic model

This has strong sums:

- For each $B \in \underline{\textbf{Type}}(\Gamma.A)$, we take $\Sigma_A(B) = (\Gamma \vdash \Sigma x : A.\, B(x))$;
- For each $B \in \underline{\textbf{Type}}(\Gamma.A)$, we take $i_{A,B} \colon \Gamma.A.B. \to \Gamma.\Sigma_A B$ to be

$$\Gamma,\ x : A,\ y : B(x) \ \vdash\ \langle x, y \rangle : \Sigma x : A.\, B(x)$$

- By $\Sigma$-elimination we can provide judgements

$$\Gamma,\ z : \Sigma x : A.\, B(x) \ \vdash\ \pi_1(z) : A$$
$$\Gamma,\ z : \Sigma x : A.\, B(x) \ \vdash\ \pi_2(z) : B(\pi_1(z))$$

with $\pi_1(\langle x, y \rangle) = x$ and $\pi_2(\langle x, y \rangle) = y$, and judgements

$$\Gamma,\ z : \Sigma x : A.\, B(x) \ \vdash\ \phi(z) : \mathrm{Id}_{\Sigma_A B}\left(z,\ \langle \pi_1(z), \pi_2(z) \rangle\right);$$

these provide a left strict pseudoinverse for $i_{A,B}$.

# Product types

A split comprehension 2-category has *products* if:

- For each $\Gamma \in \underline{\mathbf{C}}$ and $A \in \underline{\mathbf{T}}(\Gamma)$, the reindexing functor

$$(-) \times_{\Gamma} A := \underline{\mathbf{T}}(\pi_{\Gamma}) \colon \underline{\mathbf{T}}(\Gamma) \to \underline{\mathbf{T}}(\Gamma.A)$$

  has a right strict right biadjoint $\Pi_A$.

- These right adjoints satisfy some kind of *Beck-Chevalley condition*.

Explicitly:

- For each $B \in \underline{\mathbf{T}}(\Gamma.A)$, an object $\Pi_A(B) \in \underline{\mathbf{T}}(\Gamma)$;
- For each $B \in \underline{\mathbf{T}}(\Gamma.A)$, a map $\mathsf{ev}_{A,B} \colon \Pi_A(B) \times_\Gamma A \to B$ in $\underline{\mathbf{T}}(\Gamma.A)$;
- For each map $f \colon D \times_\Gamma A \to B$ in $\underline{\mathbf{T}}(\Gamma.A)$, a map $\lambda_f \colon D \to \Pi_A(B)$ in $\underline{\mathbf{T}}(\Gamma)$ fitting into:

$$\begin{array}{ccc} D \times_\Gamma A & \xrightarrow{\lambda_f \times_\Gamma A} & \Pi_A(B) \times_\Gamma A \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle \mathsf{ev}_{A,B}} \\ & & B \end{array}$$

Such that:

- For each $D \in \underline{\mathbf{T}}(\Gamma)$ and $B \in \underline{\mathbf{T}}(\Gamma.A)$, the functor

$$\underline{\mathbf{T}}(\Gamma)\big(D,\ \Pi_A(B)\big) \to \underline{\mathbf{T}}(\Gamma.A)\big(D \times_\Gamma A,\ B\big)$$

induced by composition with $\mathsf{ev}_{A,B}$ is full and faithful.

# Example: 2-dimensional syntactic model

- For each $B \in \underline{\mathbf{Type}}(\Gamma.A)$, we take $\Pi_A(B) = (\Gamma \;\vdash\; \Pi x : A.\, B(x))$;
- For each $B \in \underline{\mathbf{Type}}(\Gamma.A)$, we take $\mathsf{ev}_{A,B} \colon \Pi_A(B) \times_\Gamma A \to B$ to be

$$\Gamma,\; M : \Pi x : A.\, B(x),\; x : A \;\vdash\; M \cdot x : B(x);$$

- For each map $f \colon D \times_\Gamma A \to B$ in $\underline{\mathbf{Type}}(\Gamma.A)$, i.e.

$$\Gamma,\; w : D,\; x : A \;\vdash\; f(w, x) : B(x),$$

we take $\lambda_f \colon D \to \Pi_A(B)$ in $\underline{\mathbf{Type}}(\Gamma)$ to be

$$\Gamma,\; w : D \;\vdash\; \lambda x : A.\, f(w, x) : \Pi x : A.\, B(x);$$

However, that the functor

$$\underline{\mathbf{Type}}(\Gamma)\big(D,\ \Pi_A(B)\big) \to \underline{\mathbf{Type}}(\Gamma.A)\big(D \times_\Gamma A,\ B\big)$$

induced by composition with $\mathsf{ev}_{A,B}$ to be full and faithful says that given

$$\Gamma,\ w : D \vdash\ M(w) : \Pi x : A.\, B(x) \quad \text{and} \quad \Gamma,\ w : D \vdash\ N(w) : \Pi x : A.\, B(x)$$

we have a bijection between judgements

$$\Gamma,\ w : D\ \vdash\ \alpha(w) : \mathrm{Id}_{\Pi x:A.\, B(x)}(M(w), N(w))$$

and judgements

$$\Gamma,\ w : D,\ x : A\ \vdash\ \beta(x, w) : \mathrm{Id}_{B(x)}(M(w) \cdot x, N(w) \cdot x)$$

which is the principle of *function extensionality*.

Thus we have shown that

| | | | | |
|---|---|---|---|---|
| Martin-Löf type theory | + | 2-dimensionality | + | function extensionality |

may be modelled by:

| | | | | |
|---|---|---|---|---|
| Comprehension 2-category | + | strong sums | + | products |

(Still need to formulate equality in a satisfactory way)

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