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The Vietoris Monad and Weak Distributive Laws

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Abstract
The Vietoris monad on the category of compact Hausdorff spaces is a topological analogue of the power-set monad on the category of sets. Exploiting Manes' characterisation of the compact Hausdorff spaces as algebras for the ultrafilter monad on sets, we give precise form to the above analogy by exhibiting the Vietoris monad as induced by a weak distributive law, in the sense of Böhm, of the power-set monad over the ultrafilter monad.

Keywords Vietoris hyperspace · Monads · Distributive laws · Weak distributive laws · Continuous lattices

1 Introduction

In his 1922 paper [27], Vietoris described how the set of closed subspaces of a compact Hausdorff space X can itself be made into a compact Hausdorff space, now often referred to as the *Vietoris hyperspace* VX. The Vietoris construction is important not just in topology, but also in theoretical computer science, where its various generalisations provide different notions of *power domain* [22], and in general algebra, where its restriction to zero-dimensional spaces links up under Stone duality with the theory of *Boolean algebras with operators* [14].

The assignation $X \mapsto VX$ in fact underlies a monad $\mathbb{V}$ on the category $\mathcal{KH}$aus of compact Hausdorff spaces. This monad structure was sketched briefly by Manes in [19, Exercise I.5.23], but received its first detailed treatment by Wyler in [28]; in particular, Wyler identified the $\mathbb{V}$-algebras as Scott's *continuous lattices* [21].

Clearly, the Vietoris monad is related to the *power-set* monad $\mathbb{P}$ on the category of sets. In both cases, the monad unit and multiplication are given by inclusion of singletons and by set-theoretic union; and both underlying functors are "power-object" constructions—differing in

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the distinction between *closed* subspaces and *arbitrary* subsets, and in the need to topologise in the former case.

In this article, we give a new account of the Vietoris monad on $\mathcal{KH}$aus which explains its similarities with the power-set monad on $\mathcal{S}$et by deriving it from it in a canonical way; for good measure, this account also renders the slightly delicate topological aspects of the Vietoris construction entirely automatic.

The starting point is Manes' result [18] identifying compact Hausdorff spaces as the algebras for the ultrafilter monad $\mathbb{\beta}$ on $\mathcal{S}$et. In light of this, we recognise our situation as the following one: we have a monad—namely, the power-set monad $\mathbb{P}$ on sets—which we would like to "lift" appropriately to the category of algebras for another monad on the same base—namely, the ultrafilter monad $\mathbb{\beta}$.

At this point, the categorically-minded reader will doubtless think of Beck's theory [2] of distributive laws. For monads $\mathbb{S}$, $\mathbb{T}$ on $\mathcal{C}$, a *distributive law* of $\mathbb{S}$ over $\mathbb{T}$ is a natural transformation $\delta\colon TS \Rightarrow ST$ satisfying four axioms expressing compatibility with the monad structures of $\mathbb{S}$ and $\mathbb{T}$. As we will recall in Sect. 3.1 below, distributive laws correspond to *liftings* of $\mathbb{S}$ to a monad on the category of $\mathbb{T}$-algebras, and also to *extensions* of $\mathbb{T}$ to a monad on the Kleisli category of $\mathbb{S}$.

In particular, we can ask: is there a distributive law of $\mathbb{P}$ over $\mathbb{\beta}$ for which the associated lifting of $\mathbb{P}$ to $\mathcal{KH}$aus, the category of $\mathbb{\beta}$-algebras, is the Vietoris monad? Unfortunately, the answer to this question is *no*, since the kind of lifting mandated by the theory of distributive laws is too strong; if the Vietoris monad did lift the power-set monad in this sense, then the underlying set of VX would comprise the full power-set of X, rather than just the closed subsets.

However, we are clearly very close to having a lifting of $\mathbb{P}$ to $\mathbb{\beta}$-algebras; and, in fact, we are also very close to having a distributive law of $\mathbb{P}$ over $\mathbb{\beta}$. For indeed, such a distributive law would be the same as an extension of $\mathbb{\beta}$ to the Kleisli category of $\mathbb{P}$, which is the category $\mathcal{R}$el of sets and relations; and the extension of structure from $\mathcal{S}$et to $\mathcal{R}$el was analysed in detail by Barr [1]. As observed in [26, § 2.11], it is a direct consequence of Barr's results that:

- A functor $F\colon \mathcal{S}\text{et} \to \mathcal{S}\text{et}$ has *at most* one extension to a locally monotone functor $\tilde{F}\colon \mathcal{R}\text{el} \to \mathcal{R}\text{el}$, which exists just when F is weakly cartesian;
- If F, G are weakly cartesian, then $\alpha\colon F \Rightarrow G$ has *at most* one extension to a natural transformation $\tilde{\alpha}\colon \tilde{F} \Rightarrow \tilde{G}$, existing just when α is weakly cartesian.

(The definition of weak cartesianness is recalled in Sect. 4.1 below.) In the case of the ultrafilter monad $\mathbb{\beta}$ on $\mathcal{S}$et, it is well known that the underlying endofunctor and the monad multiplication are weakly cartesian, and so extend; while the unit is not, and so does not. This not-quite extension of $\mathbb{\beta}$ to $\mathcal{R}$el turns out to correspond to a not-quite distributive law $\delta\colon \beta P \Rightarrow P\beta$, which is compatible with both monad multiplications and the unit of $\mathbb{P}$, but not with the unit of $\mathbb{\beta}$.

One perspective on this situation can be found in [5,10,24,25]. As was already essentially observed in [1], the not-quite extension of $\mathbb{\beta}$ to $\mathcal{R}$el is an example of a *lax* monad extension in the sense of [5]. It was noted in [10, Exercise 1.I], and confirmed in [25], that such lax monad extensions correspond to suitably-defined lax distributive laws, and further explained in [24] that these correspond, in turn, to suitable lax liftings. These facts are important for the area of *monoidal topology*; see, for example, [11].

However, for the ends we wish to pursue here, a different point of view is relevant. In 2009, with motivation from quantum algebra, Street [23] and Böhm [3] introduced various notions of *weak distributive law* of a monad $\mathbb{S}$ over a monad $\mathbb{T}$, involving a natural transformation $\delta\colon TS \Rightarrow ST$ satisfying Beck's original axioms relating to the monad multiplications, but weakening in different ways those relating to the monad units. Each of these kinds of weak distributive law of $\mathbb{S}$ over $\mathbb{T}$ was shown to correspond to a kind of "weak lifting" of $\mathbb{S}$ to $\mathbb{T}$-algebras.

In particular, one of the kinds of weak distributive law involves simply dropping from Beck's original notion the axiom relating to the unit of $\mathbb{T}$. Thus, the not-quite distributive law $\delta\colon \beta P \Rightarrow P\beta$ we described above is a weak distributive law, in this sense, of $\mathbb{P}$ over $\mathbb{\beta}$; and so there is a corresponding weak lifting of $\mathbb{P}$ to $\mathbb{\beta}$-algebras. Our main result identifies this weak lifting by proving:

Theorem *The Vietoris monad on the category of compact Hausdorff spaces is the weak lifting of the power-set monad associated to the canonical weak distributive law of the power-set monad over the ultrafilter monad.*

As an application of this result, we obtain a simple new proof of Wyler's characterisation of the $\mathbb{V}$-algebras as the continuous lattices; and we conclude the paper with remarks on possible variations and generalisations of our main result.

2 The Monads

2.1 The Power-Set Monad

We begin by recalling the various monads of interest and their categories of algebras. Most straightforwardly, we have:

Definition 1 The power-set monad $\mathbb{P}$ on $\mathcal{S}\mathrm{et}$ has PX given by the set of all subsets of X, and $Pf\colon PX \to PY$ given by direct image. The unit $\eta_X\colon X \to PX$ and multiplication $\mu_X\colon PPX \to PX$ are given by $\eta_X(x) = \{x\}$ and $\mu_X(\mathcal{A}) = \bigcup \mathcal{A}$.

The $\mathbb{P}$-algebras can be identified as complete lattices in two different ways, depending on whether we view the $\mathbb{P}$-algebra structure as providing the sup operation or the inf operation; the maps of the category of $\mathbb{P}$-algebras are then respectively the sup-preserving maps and the inf-preserving maps.

2.2 The Ultrafilter Monad

Recall that a *filter* on a set X is a non-empty subset $\mathcal{F} \subseteq PX$ such that, for all $A, B \subseteq X$, we have $A, B \in \mathcal{F}$ if and only if $A \cap B \in \mathcal{F}$. A filter is an *ultrafilter* if it contains exactly one of A and $X \backslash A$ for each $A \subseteq X$.

Definition 2 The ultrafilter monad $\mathbb{\beta}$ on $\mathcal{S}\mathrm{et}$ has βX given by the set of all ultrafilters on X, and $\beta f\colon \beta X \to \beta Y$ the function taking *pushforward* along f:

$$\mathcal{F} \mapsto f_!(\mathcal{F}) = \{B \subseteq Y : f^{-1}(B) \in \mathcal{F}\} = \{B \subseteq Y : f(A) \subseteq B \text{ for some } A \in \mathcal{F}\}.$$

The unit $\eta_X\colon X \to \beta X$ and multiplication $\mu_X\colon \beta\beta X \to \beta X$ are defined by $\eta_X(x) = \{A \subseteq X : x \in A\}$ and $\mu_X(\mathbf{F}) = \{A \subseteq X : A^\# \in \mathbf{F}\}$, where for any $A \subseteq X$ we define $A^\# = \{\mathcal{F} \in \beta X : A \in \mathcal{F}\}$.

The algebras for the ultrafilter monad were identified by Manes [18] as the compact Hausdorff spaces. Recall that, for a topological space X, an ultrafilter $\mathcal{F} \in \beta X$ is said to *converge* to $x \in X$ if each neighbourhood of x is in $\mathcal{F}$; and that, when X is compact Hausdorff, each $\mathcal{F} \in \beta X$ converges to a unique point $\xi(\mathcal{F})$. Manes showed that the function $\xi \colon \beta X \to X$ so determined endows the compact Hausdorff X with $\mathbb{\beta}$-algebra structure, and that *every* $\mathbb{\beta}$-algebra arises thus. Under this identification, the $\mathbb{\beta}$-algebra maps are the continuous ones.

2.3 The Vietoris Monad

The *Vietoris hyperspace* [27] VX of a compact Hausdorff space X is the set of all closed subspaces of X, endowed with the topology (sometimes called the "hit-and-miss" topology) generated by the following subbasic open sets for each $C \in VX$:

$$C^+ = \{A \in VX : A \cap C = \emptyset\} \quad \text{and} \quad C^- = \{A \in VX : A \nsubseteq C\}.$$

Definition 3 [28] The Vietoris monad $\mathbb{V}$ on $\mathcal{KH}aus$ has VX given as above, and action on maps $Vf \colon VX \to VY$ given by direct image. The unit $\eta_X \colon X \to VX$ and multiplication $\mu_X \colon VVX \to VX$ are given by $\eta_X(x) = \{x\}$ and $\mu_X(\mathcal{A}) = \bigcup \mathcal{A}$.

It was shown in [28] that the $\mathbb{V}$-algebras are the *continuous lattices* of [21]. Recall that, for elements x, y of a poset L, we write $x \ll y$ if, whenever $D \subseteq L$ is directed and $y \leqslant \sup D$, there exists some $d \in D$ with $x \leqslant d$. A *continuous lattice* is a complete lattice L such that every $x \in L$ satisfies $x = \sup\{s : s \ll x\}$. Under Wyler's identification, a continuous lattice L becomes a compact Hausdorff space under its *Lawson topology*, which is generated by the subbasic open sets

$$s^+ = \{x \in L : s \ll x\} \quad \text{and} \quad s^- = \{x \in L : s \nleqslant x\} \quad \text{for } s \in L,$$

and a $\mathbb{V}$-algebra via the function $VL \to L$ taking infima of closed sets.

3 Distributive Laws and Weak Distributive Laws

3.1 Distributive Laws

We now recall Beck's classical theory [2] of *distributive laws* and their associated liftings and extensions, and the generalisation of this theory to *weak distributive laws* [3] which will be necessary for our main result. We begin with Beck's original notion.

Definition 4 Let $\mathbb{S} = (S, \nu, \omega)$ and $\mathbb{T} = (T, \eta, \mu)$ be monads on a category $\mathcal{C}$. A *distributive law of* $\mathbb{S}$ *over* $\mathbb{T}$ is a natural transformation $\delta \colon TS \Rightarrow ST$ rendering commutative the four diagrams:

$$\begin{array}{ccccc} TSS & \xrightarrow{\delta S} & STS & \xrightarrow{S\delta} & SST \\ {\scriptstyle T\omega}\downarrow & & & & \downarrow{\scriptstyle \omega T} \\ TS & & \xrightarrow{\delta} & & ST \end{array} \qquad \begin{array}{ccccc} TTS & \xrightarrow{T\delta} & TST & \xrightarrow{\delta T} & STT \\ {\scriptstyle \mu S}\downarrow & & & & \downarrow{\scriptstyle S\mu} \\ TS & & \xrightarrow{\delta} & & ST \end{array}$$

$$\begin{array}{ccc} & T & \\ {\scriptstyle T\nu}\swarrow & & \searrow{\scriptstyle \nu T} \\ TS & \xrightarrow{\delta} & ST \end{array} \qquad \begin{array}{ccc} & S & \\ {\scriptstyle \eta S}\swarrow & & \searrow{\scriptstyle S\eta} \\ TS & \xrightarrow{\delta} & ST. \end{array}$$

The basic result about distributive laws is that they correspond both to liftings and to extensions, in the sense of the following definition.

Definition 5 Let $\mathbb{S} = (S, \nu, \omega)$ and $\mathbb{T} = (T, \eta, \mu)$ be monads on a category $\mathcal{C}$. If we write $U^{\mathbb{T}}: \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ for the forgetful functor from the category of $\mathbb{T}$-algebras, then a *lifting* of $\mathbb{S}$ to $\mathcal{C}^{\mathbb{T}}$ is a monad $\tilde{\mathbb{S}}$ on $\mathcal{C}^{\mathbb{T}}$ such that

$$U^{\mathbb{T}} \circ \tilde{S} = S \circ U^{\mathbb{T}} \qquad U^{\mathbb{T}} \circ \tilde{\nu} = \nu \circ U^{\mathbb{T}} \quad \text{and} \quad U^{\mathbb{T}} \circ \tilde{\omega} = \omega \circ U^{\mathbb{T}}.$$

On the other hand, if we write $F_{\mathbb{S}}: \mathcal{C} \to \mathcal{C}_{\mathbb{S}}$ for the free functor into the Kleisli category of $\mathbb{S}$, then an *extension* of $\mathbb{T}$ to $\mathcal{C}_{\mathbb{S}}$ is a monad $\tilde{\mathbb{T}}$ on $\mathcal{C}_{\mathbb{S}}$ such that

$$\tilde{T} \circ F_{\mathbb{S}} = F_{\mathbb{S}} \circ T \qquad \tilde{\eta} \circ F_{\mathbb{S}} = F_{\mathbb{S}} \circ \eta \quad \text{and} \quad \tilde{\mu} \circ F_{\mathbb{S}} = F_{\mathbb{S}} \circ \mu.$$

Proposition 6 [2, § 1], [20, Theorem 2.5]. *For monads $\mathbb{S}$, $\mathbb{T}$ on $\mathcal{C}$, there are bijections between distributive laws of $\mathbb{S}$ over $\mathbb{T}$, liftings of $\mathbb{S}$ to $\mathcal{C}^{\mathbb{T}}$ and extensions of $\mathbb{T}$ to $\mathcal{C}_{\mathbb{S}}$.*

Proof. Given a distributive law $\delta: TS \Rightarrow ST$, we define the corresponding lifting of $\mathbb{S}$ to $\mathbb{T}$-algebras to have action on objects given by

$$\tilde{S}(X, TX \xrightarrow{x} X) = (SX, TSX \xrightarrow{\delta_X} STX \xrightarrow{Sx} SX)$$

and remaining data inherited from $\mathbb{S}$: thus $\tilde{S}(f) = Sf$, $\tilde{\nu}_X = \nu_X$ and $\tilde{\omega}_X = \omega_X$. Conversely, for a lifting of $\mathbb{S}$ to $\mathbb{T}$-algebras with action $\tilde{S}(X, x) = (SX, \sigma_{X,x})$, the corresponding distributive law $\delta: TS \Rightarrow ST$ is given by:

$$\delta_X = TSX \xrightarrow{TS\eta_X} TSTX \xrightarrow{\sigma_{F^{\mathbb{T}}X}} STX. \tag{3.1}$$

Next, for a distributive law $\delta: TS \Rightarrow ST$, the corresponding extension of $\mathbb{T}$ to $\mathcal{C}_{\mathbb{S}}$ is given on objects by $\tilde{T}X = TX$ and on a Kleisli map from X to Y by

$$\tilde{T}(X \xrightarrow{f} SY) = TX \xrightarrow{Tf} TSY \xrightarrow{\delta_Y} STY,$$

while the unit and multiplication have components

$$X \xrightarrow{\eta_X} TX \xrightarrow{\nu_{TX}} STX \quad \text{and} \quad TTX \xrightarrow{\mu_X} TX \xrightarrow{\nu_{TX}} STX.$$

Conversely, given an extension $\tilde{\mathbb{T}}$ of $\mathbb{T}$, we may view each map $1_{SX}: SX \to SX$ as a Kleisli map from SX to X, and applying $\tilde{T}$ yields a Kleisli map from TSX to TX, which provides the X-component of the corresponding distributive law:

$$\tilde{T}(SX \xrightarrow{1_{SX}} SX) = TSX \xrightarrow{\delta_X} STX. \qquad \square$$

We can describe the algebras for the lifted monad $\tilde{\mathbb{S}}$ associated to a distributive law in various other ways. One is in terms of the *composite monad* $\mathbb{ST}$ on $\mathcal{C}$, which is the monad induced by the composite adjunction $(\mathcal{C}^{\mathbb{T}})^{\tilde{\mathbb{S}}} \leftrightarrows \mathcal{C}^{\mathbb{T}} \leftrightarrows \mathcal{C}$; its underlying endofunctor is ST, its unit is $\nu\eta: 1 \Rightarrow ST$ and its multiplication is $\omega\mu \circ S\delta T: STST \Rightarrow ST$. Another is in terms of "δ-algebras":

Definition 7 Let $\delta: TS \Rightarrow ST$ be a distributive law of $\mathbb{S}$ over $\mathbb{T}$. A *δ-algebra* is an object $X \in \mathcal{C}$ endowed with $\mathbb{T}$-algebra structure $t: TX \to X$ and $\mathbb{S}$-algebra structure $s: SX \to X$ and rendering commutative the diagram below. The category $\mathcal{C}^{\delta}$ of δ-algebras is the full subcategory of $\mathcal{C}^{\mathbb{S}} \times_{\mathcal{C}} \mathcal{C}^{\mathbb{T}}$ on the δ-algebras.

$$\begin{array}{ccccc} TSX & \xrightarrow{\delta_X} & STX & \xrightarrow{St} & SX \\ \downarrow{\scriptstyle Ts} & & & & \downarrow{\scriptstyle s} \\ TX & & \xrightarrow{\quad t \quad} & & X \end{array} \tag{3.2}$$

The basic result relating these notions is the following; for the proof, see [2].

Lemma 8 *For any distributive law $\delta\colon TS \Rightarrow ST$ of $\mathbb{S}$ over $\mathbb{T}$, there are canonical isomorphisms between the category of $\tilde{\mathbb{S}}$-algebras in $\mathcal{C}^{\mathbb{T}}$, the category of $\mathbb{S}\mathbb{T}$-algebras in $\mathcal{C}$, and the category of δ-algebras in $\mathcal{C}$.*

3.2 Weak Distributive Laws

As explained in the introduction, weak distributive laws generalise distributive laws by relaxing the axioms relating to the monad units. There are various ways of doing this, studied in Street [23] and Böhm [3], but we will need only one, which we henceforth refer to with the unadorned name "weak distributive law". In the terminology of [3], our notion is that of a monad in $\mathcal{EM}^w(\mathcal{C}\mathrm{at})$ whose 2-cell data satisfy the conditions of Lemma 1.2(3) of *ibid*.

Definition 9 Let $\mathbb{S} = (S, \nu, \omega)$ and $\mathbb{T} = (T, \eta, \mu)$ be monads on a category $\mathcal{C}$. A *weak distributive law of* $\mathbb{S}$ *over* $\mathbb{T}$ is a natural transformation $\delta\colon TS \Rightarrow ST$ rendering commutative the three diagrams:

$$\begin{array}{ccccc} TSS & \xrightarrow{\delta S} & STS & \xrightarrow{S\delta} & SST \\ {\scriptstyle T\omega}\downarrow & & & & \downarrow{\scriptstyle \omega T} \\ TS & & \xrightarrow{\quad\delta\quad} & & ST \end{array} \qquad \begin{array}{ccccc} TTS & \xrightarrow{T\delta} & TST & \xrightarrow{\delta T} & STT \\ {\scriptstyle \mu S}\downarrow & & & & \downarrow{\scriptstyle S\mu} \\ TS & & \xrightarrow{\quad\delta\quad} & & ST \end{array} \qquad \begin{array}{ccc} & T & \\ {\scriptstyle T\nu}\swarrow & & \searrow{\scriptstyle \nu T} \\ TS & \xrightarrow{\quad\delta\quad} & ST . \end{array}$$

Thus, a weak distributive law in our sense simply drops from Beck's definition the axiom relating to the unit of $\mathbb{T}$. Such weak distributive laws correspond to *weak* liftings and to *weak* extensions, where the definitions of these are a bit more subtle.

Definition 10 Let $\mathbb{S} = (S, \nu, \omega)$ and $\mathbb{T} = (T, \eta, \mu)$ be monads on a category $\mathcal{C}$. A *weak lifting* of $\mathbb{S}$ to $\mathcal{C}^{\mathbb{T}}$ comprises a monad $\tilde{\mathbb{S}}$ on $\mathcal{C}^{\mathbb{T}}$ and natural transformations

$$U^{\mathbb{T}}\tilde{S} \xRightarrow{\;\iota\;} SU^{\mathbb{T}} \xRightarrow{\;\pi\;} U^{\mathbb{T}}\tilde{S} \tag{3.3}$$

such that $\pi\iota = 1$, and such that each of the following diagrams commutes:

$$\begin{array}{ccccc} U^{\mathbb{T}}\tilde{S}\tilde{S} & \xrightarrow{\iota\tilde{S}} & SU^{\mathbb{T}}\tilde{S} & \xrightarrow{S\iota} & SSU^{\mathbb{T}} \\ {\scriptstyle U^{\mathbb{T}}\tilde{\omega}}\downarrow & & & & \downarrow{\scriptstyle \omega U^{\mathbb{T}}} \\ U^{\mathbb{T}}\tilde{S} & & \xrightarrow{\quad\iota\quad} & & SU^{\mathbb{T}} \end{array} \qquad \begin{array}{ccc} & U^{\mathbb{T}} & \\ {\scriptstyle U^{\mathbb{T}}\tilde{\nu}}\swarrow & & \searrow{\scriptstyle \nu U^{\mathbb{T}}} \\ U^{\mathbb{T}}\tilde{S} & \xrightarrow{\quad\iota\quad} & SU^{\mathbb{T}} \end{array} \tag{3.4}$$

$$\begin{array}{ccccc} SSU^{\mathbb{T}} & \xrightarrow{S\pi} & SU^{\mathbb{T}}\tilde{S} & \xrightarrow{\pi\tilde{S}} & U^{\mathbb{T}}\tilde{S}\tilde{S} \\ {\scriptstyle \omega U^{\mathbb{T}}}\downarrow & & & & \downarrow{\scriptstyle U^{\mathbb{T}}\tilde{\omega}} \\ SU^{\mathbb{T}} & & \xrightarrow{\quad\pi\quad} & & U^{\mathbb{T}}\tilde{S} \end{array} \qquad \begin{array}{ccc} & U^{\mathbb{T}} & \\ {\scriptstyle \nu U^{\mathbb{T}}}\swarrow & & \searrow{\scriptstyle U^{\mathbb{T}}\tilde{\nu}} \\ SU^{\mathbb{T}} & \xrightarrow{\quad\pi\quad} & U^{\mathbb{T}}\tilde{S}; \end{array} \tag{3.5}$$

while a *weak extension* of $\mathbb{T}$ to $\mathcal{C}_{\mathbb{S}}$ comprises a functor $\tilde{T}\colon \mathcal{C}_{\mathbb{S}} \to \mathcal{C}_{\mathbb{S}}$ and natural transformation $\tilde{\mu}\colon \tilde{T}\tilde{T} \Rightarrow \tilde{T}$ such that $\tilde{T} \circ F_{\mathbb{S}} = F_{\mathbb{S}} \circ T$ and $\tilde{\mu} \circ F_{\mathbb{S}} = F_{\mathbb{S}} \circ \mu$.

Note that our "weak liftings" are exactly the simultaneous weak ι- and π-liftings of [3]. By exactly the same constructions as in Proposition 6, we have:

Proposition 11 *For monads $\mathbb{S}$, $\mathbb{T}$ on $\mathcal{C}$, there is a bijective correspondence between weak distributive laws of $\mathbb{S}$ over $\mathbb{T}$ and weak extensions of $\mathbb{T}$ to $\mathcal{C}_{\mathbb{S}}$.*

The correspondence between weak distributive laws and weak liftings is more interesting. It is proved by Proposition 4.4 and Theorem 4.5 of [3] in a more general context; however, for the particular kind of weakness we are interested in, the following more direct proof is possible.

To begin with, we define a *semialgebra* for a monad $\mathbb{T} = (T, \eta, \mu)$ to be given by a pair $(X \in \mathcal{C}, x : TX \to X)$ satisfying the associativity axiom $x.Tx = x.\mu_X$ but not necessarily the unit axiom $x.\eta_X = 1_X$. The $\mathbb{T}$-semialgebras form a category $\mathcal{C}^{\mathbb{T}}_s$, wherein a map from (X, x) to (Y, y) is a map $f : X \to Y$ with $y.Tf = f.x$.

Lemma 12 *If idempotents split in $\mathcal{C}$, then the full inclusion $I : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}^{\mathbb{T}}_s$ has a simultaneous left and right adjoint $K : \mathcal{C}^{\mathbb{T}}_s \to \mathcal{C}^{\mathbb{T}}$.*

Proof. For any $(X, x) \in \mathcal{C}^{\mathbb{T}}_s$ we have $x.\eta_X.x = x.Tx.\eta_{TX} = x.\mu_X.\eta_{TX} = x = x.\mu_X.T\eta_X = x.Tx.T\eta_X$ so that $x.\eta_X : (X, x) \to (X, x)$ is an idempotent of $\mathbb{T}$-semialgebras. Splitting this idempotent yields a diagram

$$(X, x) \xrightarrow{p} (Y, y) \xrightarrow{i} (X, x) \tag{3.6}$$

in $\mathcal{C}^{\mathbb{T}}_s$ with $pi = 1_Y$ and $ip = x.\eta_X$. The semialgebra (Y, y) is in fact a $\mathbb{T}$-algebra since $y\eta_Y = piy\eta_Y = px.Ti.\eta_Y = px.\eta_X i = pipi = 1_Y$. Moreover, if (Z, z) is a $\mathbb{T}$-algebra and $f : (X, x) \to (Z, z)$, then $f = z\eta_Z.f = z.Tf.\eta_X = f.x.\eta_X = fip$ so that f factors through p. On the other hand, if $g : (Z, z) \to (X, x)$, then $g = gz\eta_Z = x.Tg.\eta_Z = x.\eta_X.g = ipg$ so that g factors through i. Thus i and p exhibit (Y, y) as the value at (X, x) of the desired left and right adjoint K. □

Proposition 13 *If idempotents split in $\mathcal{C}$, then for any monads $\mathbb{S}$, $\mathbb{T}$ on $\mathcal{C}$, there is a bijective correspondence between weak distributive laws of $\mathbb{S}$ over $\mathbb{T}$ and weak liftings of $\mathbb{S}$ to $\mathcal{C}^{\mathbb{T}}$.*

Proof. Given a weak distributive law $\delta : TS \Rightarrow ST$, we may define a *strict* lifting $\check{\mathbb{S}}$ of $\mathbb{S}$ to $\mathbb{T}$-*semi*algebras by taking, as in Proposition 6, $\check{S}(X, x) = (SX, Sx.\delta_X)$ and with the remaining data inherited from $\mathbb{S}$. We now obtain the desired *weak* lifting $\tilde{\mathbb{S}}$ of $\mathbb{S}$ to $\mathcal{C}^{\mathbb{T}}$ as the monad generated by the composite adjunction:

$$(\mathcal{C}^{\mathbb{T}}_s)^{\check{\mathbb{S}}} \underset{U^{\check{\mathbb{S}}}}{\overset{F^{\check{\mathbb{S}}}}{\leftrightarrows}} \mathcal{C}^{\mathbb{T}}_s \underset{I}{\overset{K}{\leftrightarrows}} \mathcal{C}^{\mathbb{T}}.$$

In particular, $\tilde{S}$ sends a $\mathbb{T}$-algebra (X, x) to the $\mathbb{T}$-algebra obtained as the splitting

$$(SX, Sx.\delta_X) \xrightarrow{\pi_{X,x}} \tilde{S}(X, x) \xrightarrow{\iota_{X,x}} (SX, Sx.\delta_X) \tag{3.7}$$

of the idempotent $Sx.\delta_X.\eta_{SX} : (SX, Sx.\delta_X) \to (SX, Sx.\delta_X)$ in the category of $\mathbb{T}$-semialgebras. Applying the forgetful functor $\mathcal{C}^{\mathbb{T}}_s \to \mathcal{C}$ to (3.7) yields the components of the ι and π required in (3.3), and it is clear from the manner of definition that the lifted unit $\tilde{\nu}$ is the *unique* map rendering the triangles in (3.4) and (3.5) commutative. As for the lifted multiplication $\tilde{\omega}$, a short calculation shows that, for any $\mathbb{T}$-semialgebra (X, x) with $\mathbb{T}$-algebra splitting (3.6), the maps

$$(SX, Sx.\delta_X) \xrightarrow{Sp} (SY, Sy.\delta_Y) \xrightarrow{\pi_{Y,y}} \tilde{S}(Y, y) \xrightarrow{\iota_{Y,y}} (SY, Sy.\delta_Y) \xrightarrow{Si} (SX, Sx.\delta_X)$$

compose to the idempotent $Sx.\delta_X.\eta_{SX}$, and so exhibit $\tilde{S}(Y, y)$ as the $\mathbb{T}$-algebra splitting of $\check{S}(X, x) = (SX, Sx.\delta_X)$. Thus, for any $\mathbb{T}$-algebra (X, x), the maps

$$\check{S}\check{S}(X,x) \xrightarrow{\check{S}\pi_{X,x}} \check{S}\tilde{S}(X,x) \xrightarrow{\pi_{\tilde{S}(X,x)}} \tilde{S}\tilde{S}(X,x) \xrightarrow{\iota_{\tilde{S}(X,x)}} \check{S}\tilde{S}(X,x) \xrightarrow{\check{S}\iota_{X,x}} \check{S}\check{S}(X,x)$$

exhibit $\tilde{S}\tilde{S}(X, x)$ as the $\mathbb{T}$-algebra splitting of $\check{S}\check{S}(X, x)$; whence $\tilde{\omega}$ is the *unique* map rendering commutative the rectangles in (3.4) and (3.5), as required.

This concludes the construction of a weak lifting from a weak distributive law. Suppose conversely we have a weak lifting of $\mathbb{S}$ to $\mathbb{T}$-algebras. For each $\mathbb{T}$-algebra (X, x) with $\tilde{S}(X, x) = (Y, y)$, define the map $\sigma_{X,x} \colon TSX \to SX$ as the composite

$$TSX \xrightarrow{T\pi_{X,x}} TY \xrightarrow{y} Y \xrightarrow{\iota_{X,x}} SX$$

and now define $\delta \colon TS \Rightarrow ST$ to have components (3.1). Direct calculation shows this to be a weak distributive law. $\square$

Just as before, there are various ways of describing the algebras for the weakly lifted monad $\tilde{\mathbb{S}}$ associated to a weak distributive law. We can consider the composite monad $\widetilde{\mathbb{S}\mathbb{T}}$ induced by the adjunction $(\mathcal{C}^{\mathbb{T}})^{\tilde{\mathbb{S}}} \leftrightarrows \mathcal{C}^{\mathbb{T}} \leftrightarrows \mathcal{C}$, whose underlying endofunctor $\widetilde{ST}$ is obtained by splitting the idempotent

$$ST \xrightarrow{\eta ST} TST \xrightarrow{\delta T} STT \xrightarrow{S\mu} ST,$$

or we can consider the category of δ-algebras defined exactly as in Definition 7. The relation between these notions is the same as before, and we record it as follows; for the proof, we refer the reader to [3, Proposition 3.7].

Lemma 14 *For any weak distributive law $\delta \colon TS \Rightarrow ST$ of $\mathbb{S}$ over $\mathbb{T}$, there are canonical isomorphisms between the category of $\tilde{\mathbb{S}}$-algebras in $\mathcal{C}^{\mathbb{T}}$, the category of $\widetilde{\mathbb{S}\mathbb{T}}$-algebras in $\mathcal{C}$, and the category of δ-algebras in $\mathcal{C}$.*

4 Weakly Lifting the Power-Set Monad

If, in the results of the previous section, we take $\mathcal{C}$ to be $\mathcal{S}\mathrm{et}$, $\mathbb{S}$ to be the power-set monad, and $\mathbb{T}$ to be any $\mathcal{S}\mathrm{et}$-monad, then we establish a bijection between (weak) liftings of the power-set monad to $\mathbb{T}$-algebras and (weak) extensions of $\mathbb{T}$ to $\mathcal{S}\mathrm{et}_{\mathbb{P}}$. Now $\mathcal{S}\mathrm{et}_{\mathbb{P}}$ is the category $\mathcal{R}\mathrm{el}$ of *sets and relations*, and the possibility of extending structure from $\mathcal{S}\mathrm{et}$ to $\mathcal{R}\mathrm{el}$ was analysed by [1], as we now recall.

4.1 Extending Structure from Sets to Relations

The category $\mathcal{R}\mathrm{el}$ has sets as objects, and as morphisms $R \colon X \nrightarrow Y$, relations $R \subseteq X \times Y$; we write $x \; R \; y$ to indicate that $(x, y) \in R$. Identity maps are equality relations, and the composite of $R \colon X \nrightarrow Y$ and $S \colon Y \nrightarrow Z$ is given by:

$$x \; SR \; z \qquad \iff \qquad (\exists y \in Y)\,(x \; R \; y) \wedge (y \; S \; z).$$

Under the identification of $\mathcal{R}\mathrm{el}$ as $\mathcal{S}\mathrm{et}_{\mathbb{P}}$, the free functor $F_{\mathbb{P}} \colon \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}_{\mathbb{P}}$ corresponds to the identity-on-objects embedding $(-)_* \colon \mathcal{S}\mathrm{et} \to \mathcal{R}\mathrm{el}$ which sends a function $f \colon X \to Y$ to its graph $f_* = \{(x, fx) : x \in X\} \subseteq X \times Y$. We also have the reverse relation $f^* = \{(fx, x) :$

$x \in X\} \subseteq Y \times X$, and in fact, relations of these two forms generate $\mathcal{R}\mathrm{el}$ under composition, since every $R\colon X \twoheadrightarrow Y$ in $\mathcal{R}\mathrm{el}$ can be written as $q_* \circ p^*$ where $p\colon X \leftarrow R \to Y : q$ are the two projections.

Importantly, $\mathcal{R}\mathrm{el}$ is not just a category; each hom-set is partially ordered by inclusion, and composition preserves the order on each side, so making $\mathcal{R}\mathrm{el}$ a *locally partially ordered 2-category*. With respect to this structure, it is easy to see for any function $f\colon X \to Y$ that f_* is *left adjoint* to f^* in $\mathcal{R}\mathrm{el}$. This observation is key to proving the following result, essentially due to Barr [1]; for a detailed proof, see [17].

Proposition 15 *Any $F\colon \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ has at most one extension to a 2-functor $\tilde{F}\colon \mathcal{R}\mathrm{el} \to \mathcal{R}\mathrm{el}$. This exists just when F is weakly cartesian, and is then defined on a relation $R\colon X \twoheadrightarrow Y$ with projections $p\colon X \leftarrow R \to Y : q$ by*

$$\tilde{F}(R) = (Fq)_*(Fp)^*\colon FX \twoheadrightarrow FY. \tag{4.1}$$

Any $\alpha\colon F \Rightarrow G\colon \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ has at most one extension to a 2-natural $\tilde{\alpha}\colon \tilde{F} \Rightarrow \tilde{G}$. This exists just when α is weakly cartesian, and has components $(\tilde{\alpha})_X = (\alpha_X)_*$.

Here, a functor $F\colon \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ is *weakly cartesian* if it preserves weak pullback squares, and a natural transformation $\alpha\colon F \Rightarrow G$ is *weakly cartesian* if its naturality squares are weak pullbacks; recall that a weak pullback square is one for which the induced comparison map into the pullback is an epimorphism.

Corollary 16 *For any monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathcal{S}\mathrm{et}$:*

(i) *If T, η and μ are all weakly cartesian, then there is a canonical extension of $\mathbb{T}$ to $\mathcal{R}\mathrm{el}$, and so by Proposition 6, a canonical lifting of $\mathbb{P}$ to $\mathbb{T}$-algebras;*
(ii) *If only T and μ are weakly cartesian, then there is still a canonical* weak *extension of $\mathbb{T}$ to $\mathcal{R}\mathrm{el}$, and so a canonical* weak *lifting of $\mathbb{P}$ to $\mathbb{T}$-algebras.*

The intended application of this takes $\mathbb{T}$ to be the ultrafilter monad, but before turning to this, we consider two simpler examples.

4.2 First Example

Let $\mathbb{T} = (T, \eta, \mu)$ be the commutative monoid monad. This is an *analytic monad* in the sense of [15]—in fact, the terminal one—so that each of T, η and μ is weakly cartesian: thus $\mathbb{T}$ extends *strictly* from $\mathcal{S}\mathrm{et}$ to $\mathcal{R}\mathrm{el}$. Using the formula (4.1), we see that the action of this extension on a relation $R\colon X \twoheadrightarrow Y$ is the relation $\tilde{T}R\colon TX \twoheadrightarrow TY$ with

$$x_1 \cdots x_n \; \tilde{T}R \; y_1 \cdots y_m \iff (\exists\, \sigma : n \cong m)(x_1 \; R \; y_{\sigma(1)}) \wedge \cdots \wedge (x_n \; R \; y_{\sigma(n)}).$$

Plugging this in to the proof of Proposition 6, we see that the distributive law corresponding to this extension has components $\delta_X\colon TPX \to PTX$ given by

$$A_1 \cdots A_n \quad \mapsto \quad \{a_1 \cdots a_n : \text{each } a_i \in A_i\}. \tag{4.2}$$

and so that, under the identification of $\mathbb{T}$-algebras with commutative monoids, the lifted monad $\tilde{\mathbb{P}}$ takes a commutative monoid $(X, \cdot, 1)$ to the commutative monoid with underlying set PX, unit $\{1\}$ and multiplication $A \cdot B = \{a \cdot b : a \in A, b \in B\}$. The algebras for the lifted monad $\tilde{\mathbb{P}}$ are precisely the *commutative unital quantales*: complete lattices X endowed with a commutative monoid structure $(X, \cdot, 1)$ whose binary multiplication preserves sups separately in each variable.

4.3 Second Example

We now consider the finite power-set monad $\mathbb{P}_f$ on $\mathcal{S}\mathrm{et}$, whose algebras are idempotent commutative monoids. Unlike the commutative monoid monad, this does not extend strictly from $\mathcal{S}\mathrm{et}$ to $\mathcal{R}\mathrm{el}$, due to:

Lemma 17 *The endofunctor and the multiplication of the finite power-set monad $\mathbb{P}_f$ on $\mathcal{S}\mathrm{et}$ are weakly cartesian, but the unit is not.*

(In fact the same is true on replacing $\mathbb{P}_f$ by the full power-set monad $\mathbb{P}$. In other words, $\mathbb{P}$ does not distribute over itself; see [16].)

Proof. For the endofunctor part, see, for instance, [13, Proposition 1.4]. For the multiplication, we must show that, given a function $f\colon X \to Y$, a finite subset $A \subseteq X$ and a finite subset $B = \{B_1, \ldots, B_n\} \subseteq P_f Y$ with $f(A) = B_1 \cup \cdots \cup B_n$, there exists a finite subset $\{C_1, \ldots, C_m\} \subseteq P_f X$ with $\{f(C_1), \ldots, f(C_m)\} = B$ and $A = C_1 \cup \cdots \cup C_m$. We have such on taking $m = n$ and $C_i = A \cap f^{-1}(B_i)$. Finally, to see the unit is not weakly cartesian, note that under the function $\{0, 1\} \to \{0\}$, the finite $\{0, 1\} \subseteq \{0, 1\}$ maps to the singleton $\{0\}$, but is not itself a singleton. □

However, we still have a weak extension of $\mathbb{P}_f$ to $\mathcal{R}\mathrm{el}$; this observation is apparently due to Ehrhard, and is discussed in detail in [12]. Calculating explicitly using (4.1), we see that the action of $\tilde{P}_f$ on a relation $R\colon X \nrightarrow Y$ is given by the “Egli–Milner relation”:

$$A\ \tilde{P}_f R\ B \iff (\forall a \in A. \exists b \in B. a\ R\ b) \wedge (\forall b \in B. \exists a \in A. a\ R\ b).$$

Thus, by Proposition 11, the weak distributive law corresponding to this weak extension has components $\delta_X\colon P_f P X \to P P_f X$ given by

$$\mathcal{A} \mapsto \left\{B \subseteq X \text{ finite} : B \subseteq \bigcup \mathcal{A} \text{ and } A \cap B \neq \emptyset \text{ for all } A \in \mathcal{A}\right\}.$$

We now calculate the corresponding weak lifting of the power-set monad to the category of $\mathbb{P}_f$-algebras. Given such an algebra (X, x), we first form the associated semialgebra $(PX, Px.\delta_X)$, whose action map $P_f PX \to PX$ is

$$\{A_1, \ldots, A_n\} \mapsto \{a_1 \cdots a_m : \text{each } a_i \text{ is in some } A_j, \text{ and some } a_i \text{ is in each } A_j\}.$$

In particular, the idempotent $Px.\delta_X.\eta_X\colon PX \to PX$ takes $A \subseteq X$ to the set of non-empty finite products of elements of A. Clearly A is fixed by this idempotent just when it is a *subsemigroup*—i.e., closed under binary multiplication.

It follows that the lifted monad $\tilde{\mathbb{P}}$ acts on (X, x) to yield the set $P_\bullet(X)$ of all subsemigroups of X, under the P_f-algebra structure given as in the previous display. Reading off the monoid structure from this, we see that the unit of $P_\bullet X$ is given by $\{1\}$, while the binary multiplication is given by

$$A \cdot B = \{a_1 \cdots a_n \cdot b_1 \cdots b_m : n, m \geqslant 1, \text{each } a_i \in A \text{ and each } b_j \in B\}.$$

In this expression, since A and B are already subsemigroups, we have that $a = a_1 \cdots a_n$ is itself in A and $b = b_1 \cdots b_m$ is itself in B; so, more succinctly,

$$A \cdot B = \{a \cdot b : a \in A, b \in B\}, \tag{4.3}$$

i.e., the same formula that we derived in Sect. 4.2 for the commutative monoid monad. It now follows from Lemma 14 that the algebras for the lifted monad $\tilde{\mathbb{P}}$ are exactly the commutative (unital) quantales whose multiplication is idempotent.

This example can be extended in various directions. Firstly, recall that a *normal band* is an idempotent semigroup satisfying the axiom $xyzw = xzyw$. The free normal band on a set X is the set $P_f^{**}X$ of *bipointed* finite subsets of X under the multiplication $(A, a, b)\cdot(B, c, d) = (A \cup B, a, d)$. The induced monad $\mathbb{P}_f^{**}$ does not have weakly cartesian unit, but has weakly cartesian endofunctor and multiplication; so we have a weak lifting of $\mathbb{P}$ to the category of normal bands. Like before, $\tilde{\mathbb{P}}$ takes a normal band X to the normal band $P_\bullet X$ of sub-semigroups under the binary operation (4.3). This construction is also given in [30], but without the monadic context.

A second direction of extension is to consider the monad $\mathbb{T}_S$ on $\mathcal{S}\mathrm{et}$ whose algebras are semimodules over a given commutative semiring S. In [4, Theorem 8.10] conditions are given on S which characterise *precisely* when the associated $\mathbb{T}_S$ has weakly cartesian endofunctor and multiplication; under these conditions, then, we obtain a weak lifting of the power-set monad to the category of S-modules. Our two preceding examples fit into this framework on taking $S = (\mathbb{N}, \times, +)$ and $S = (\{0, 1\}, \wedge, \vee)$; as was shown in [4, Example 9.5], other legitimate choices of S include $(\mathbb{Q}_+, \times, +)$ and $(\mathbb{R}_+, \times, +)$.

5 The Vietoris Monad and Weak Distributive Laws

5.1 Recovering the Vietoris Monad

We now prove our main theorem, recovering the Vietoris monad as the weak lifting of the power-set monad associated to the canonical weak distributive law of $\mathbb{P}$ over β. We begin with the following well known result; see, for example, [10, Examples III.1.12.3 and Proposition III.1.12.4].

Lemma 18 *The endofunctor and multiplication of the monad β are weakly cartesian, but the unit is not.*

As such, we have a canonical weak extension of β to $\mathcal{R}\mathrm{el}$. The action of $\tilde{\beta}\colon \mathcal{R}\mathrm{el} \to \mathcal{R}\mathrm{el}$ on a relation $R\colon X \to\!\!\!\!\!\to Y$ is given by

$$\mathcal{F}\ \tilde{\beta}R\ \mathcal{G} \iff (A \in \mathcal{F} \implies R(A) \in \mathcal{G}) \tag{5.1}$$

where we write $R(A)$ for $\{y \in Y : (\exists a \in A)(a\ R\ y)\}$; see, for example, [10, Examples III.1.10.3(3)]. Corresponding to this weak extension, we have a weak distributive law of $\mathbb{P}$ over β; calculating from the above expression, we see that its components $\delta_X\colon \beta PX \to P\beta X$ are given by

$$\delta_X(\mathbf{F}) = \left\{ \mathcal{F} \in \beta X : \bigcup \mathcal{A} \in \mathcal{F} \text{ for all } \mathcal{A} \in \mathbf{F} \right\}. \tag{5.2}$$

We now wish to calculate the associated weak lifting of $\mathbb{P}$ to β-algebras, i.e., to compact Hausdorff spaces. We begin with:

Lemma 19 *Let $(X, \xi\colon \beta X \to X)$ be a β-algebra. The action map $\beta PX \to PX$ of the semialgebra $(PX, P\xi.\delta_X)$ is given by*

$$\mathbf{F} \mapsto \bigcap_{\mathcal{A}\in\mathbf{F}} \overline{\bigcup \mathcal{A}} \tag{5.3}$$

where $\overline{(\ )}$ denotes closure in the topology on X. It follows that the idempotent function $P\xi.\delta_X.\eta_{PX}\colon (PX, P\xi.\delta_X) \to (PX, P\xi.\delta_X)$ sends each $B \in PX$ to its closure.

Proof. Given $x \in X$, we have $x \in \bigcap_{\mathcal{A} \in \mathbf{F}} \overline{\bigcup \mathcal{A}}$ if and only if each open neighbourhood of x meets each $\bigcup \mathcal{A}$, if and only if there exists an ultrafilter containing each $\bigcup \mathcal{A}$ and converging to x. But by (5.2), this happens just when $x \in P\xi(\delta_X(\mathbf{F}))$. Finally, since $\eta_{PX} \colon PX \to \beta PX$ sends B to the ultrafilter $\{\mathcal{A} \subseteq PX : B \in \mathcal{A}\}$, the idempotent $P\xi.\delta_X.\eta_{PX}$ sends each $B \in PX$ to $\bigcap_{B \in \mathcal{A}} \overline{\bigcup \mathcal{A}} = \overline{B}$. □

Theorem 20 *The Vietoris monad on the category of compact Hausdorff spaces is the weak lifting of the power-set monad associated to the canonical weak distributive law of the power-set monad over the ultrafilter monad.*

Proof. Let X be a compact Hausdorff space seen as a $\mathbb{\beta}$-algebra $(X, \xi \colon \beta X \to X)$. The $\mathbb{\beta}$-algebra $\tilde{P}(X, \xi)$ is obtained by splitting the idempotent $P\xi.\delta_X.\eta_{PX}$ on $(PX, P\xi.\delta_X)$; so by the previous lemma, its underlying set is the set VX of *closed* subsets of X, and its $\mathbb{\beta}$-algebra structure is given by the same formula as in (5.3). By naturality in (3.3), the action of $\tilde{P}$ on maps is given by direct image, while by the formulae in (3.4), the unit and multiplication of $\tilde{\mathbb{P}}$ are once again given by inclusion of singletons and set-theoretic union.

As such, it remains only to show that the $\mathbb{\beta}$-algebra structure on $\tilde{P}(X, \xi)$ describes the Vietoris topology; in light of Lemma 19, we must thus show that any $\mathcal{F} \in \beta VX$ converges in the Vietoris topology to the unique point $L = \bigcap_{A \in \mathcal{F}} \overline{\bigcup A}$. This follows from Lemma 22 below, since the Vietoris topology on VX is the Lawson topology on the continuous lattice $(VX, \supseteq)$. □

5.2 Vietoris Algebras

The composite monad associated to the weak distributive law of $\mathbb{P}$ over $\mathbb{V}$ is easily seen to be the well known *filter monad* $\mathbb{F}$; as such, Lemma 14 asserts a canonical isomorphism between the categories of $\mathbb{V}$-algebras in $\mathcal{KH}$aus and of $\mathbb{F}$-algebras in $\mathcal{S}$et. This was originally proven as [28, Theorem 6.3] and is, in fact, how Wyler identified the $\mathbb{V}$-algebras as the continuous lattices—by first identifying the $\mathbb{F}$-algebras as such (a result originally proved by Day [6]).

Now Lemma 14 also identifies $\mathbb{V}$-algebras with δ-algebras for the weak distributive law $\delta \colon \beta P \Rightarrow P\beta$, i.e., as sets X endowed with $\mathbb{\beta}$-algebra structure $\xi \colon \beta X \to X$ and $\mathbb{P}$-algebra structure $i \colon PX \to X$ subject to commutativity in

$$\begin{array}{ccccc} \beta PX & \xrightarrow{\delta_X} & P\beta X & \xrightarrow{P\xi} & PX \\ \downarrow{\scriptstyle \beta i} & & & & \downarrow{\scriptstyle i} \\ \beta X & & \xrightarrow{\xi} & & X. \end{array} \tag{5.4}$$

In [28], Wyler does note that a $\mathbb{V}$-algebra is a $\mathbb{\beta}$-algebra and a $\mathbb{P}$-algebra subject to some compatibility—see, for example, Proposition 6.4 of *ibid.*—but does not express this in terms of the square (5.4). In fact, by using (5.4) it is easy to give a *direct* proof that Vietoris algebras are continuous lattices, as we will now do.

In what follows, given a filter $\mathcal{F}$ on a topological space, we write adh $\mathcal{F}$ for $\bigcap_{A \in \mathcal{F}} \overline{A}$; recall that, for an ultrafilter $\mathcal{F}$, the points in adh $\mathcal{F}$ are precisely those to which $\mathcal{F}$ converges. On the other hand, for a filter $\mathcal{F}$ on a complete lattice, we write lim inf $\mathcal{F}$ for $\sup\{\inf A : A \in \mathcal{F}\}$.

Proposition 21 *Let X be a complete lattice and a compact Hausdorff space, seen as a $\mathbb{\beta}$-algebra $\xi \colon \beta X \to X$ via ultrafilter convergence and as a $\mathbb{P}$-algebra $i \colon PX \to X$ by taking infima. The following are equivalent:*

(i) (X, ξ, i) *is a* δ*-algebra, i.e., renders* (5.4) *commutative.*
(ii) $\lim\inf \mathcal{F} = \inf \operatorname{adh} \mathcal{F}$ *for any filter* $\mathcal{F}$ *on* X.

Proof. (ii) $\Rightarrow$ (i). Let $\mathbf{F} \in \beta\mathcal{P}X$. By Lemma 19, the upper path around (5.4) takes $\mathbf{F}$ to $\inf \operatorname{adh} \bigcup \mathbf{F}$ where $\bigcup \mathbf{F}$ is the filter generated by $\bigcup \mathcal{A}$ for each $\mathcal{A} \in \mathbf{F}$. The lower path takes $\mathbf{F}$ to the limit point $\inf \operatorname{adh} \mathcal{F}$ of the ultrafilter $\mathcal{F}$ generated by the sets $\mathcal{A}^i = \{\inf A : A \in \mathcal{A}\}$ for all $\mathcal{A} \in \mathbf{F}$. So given (ii), it suffices to show that $\lim\inf \bigcup \mathbf{F} = \lim\inf \mathcal{F}$, which follows since $\inf \mathcal{A}^i = \inf(\bigcup \mathcal{A})$ for all $\mathcal{A} \in \mathbf{F}$.

(i) $\Rightarrow$ (ii) We first show that for all $a \in X$, the principal upset $\uparrow a$ and downset $\downarrow a$ are closed. Consider for any ultrafilter $\mathcal{F}$ on X the ultrafilter $\mathbf{F}$ on $\mathcal{P}X$ generated by the sets $\{a, B\} = \{\{a, b\} : b \in B\}$ for all $B \in \mathcal{F}$. Note that:

- If $\downarrow a \in \mathcal{F}$, then for each $B \in \mathcal{F}$ also $B_a = B \cap \downarrow a \in \mathcal{F}$. Thus $B \supseteq B_a = i(\{a, B_a\}) \in \beta i(\mathbf{F})$, so that $\beta i(\mathbf{F}) = \mathcal{F}$ and $\xi(\beta i(\mathbf{F})) = \xi(\mathcal{F})$.
- If $\uparrow a \in \mathcal{F}$, then $\{a\} \in \beta i(\mathbf{F})$ whence $\xi(\beta i(\mathbf{F})) = a$.

In either case, $P\xi(\delta_X(\mathbf{F})) = \bigcap_{B \in \mathcal{F}} \overline{\{a\} \cup B} = \{a\} \cup \bigcap_{B \in \mathcal{F}} \overline{B} = \{a, \xi(\mathcal{F})\}$ by Lemma 19, and so $i(P\xi(\delta_X(\mathbf{F}))) = a \wedge \xi(\mathcal{F})$. So by the assumption, if $\downarrow a \in \mathcal{F}$ then $\xi(\mathcal{F}) = a \wedge \xi(\mathcal{F})$ so that $\xi(\mathcal{F}) \in \downarrow a$; while if $\uparrow a \in \mathcal{F}$ then $a = a \wedge \xi(\mathcal{F})$ so that $\xi(\mathcal{F}) \in \uparrow a$. This proves that both $\downarrow a$ and $\uparrow a$ are closed.

We now prove (ii). Given a filter $\mathcal{F}$ on X, the family of subsets of $\mathcal{P}X$ given by $\mathcal{F}$ together with $\downarrow A$ for each $A \in \mathcal{F}$ has the finite intersection property; let $\mathbf{F}$ be any ultrafilter on $\mathcal{P}X$ which extends it. Now, for each $A \in \mathcal{F}$ we have $\downarrow A \in \mathbf{F}$ and so $A = \bigcup \downarrow A \in \bigcup \mathbf{F}$. On the other hand, each $\mathcal{A} \in \mathbf{F}$ meets $\mathcal{F}$, say in A and so $\bigcup \mathcal{A} \supseteq A$ is in $\mathcal{F}$. So $\bigcup \mathbf{F} = \mathcal{F}$, and so by Lemma 19 the upper path around (5.4) takes $\mathbf{F}$ to $\inf \operatorname{adh} \mathcal{F}$. As for the lower path, $\beta i(\mathbf{F})$ contains $\{\inf A : A \in \mathcal{F}\}$ and $\uparrow(\inf A)$ for each $A \in \mathcal{F}$; so $\xi(\beta i(\mathbf{F}))$ is contained in the intersection of closed sets $\bigcap \uparrow(\inf A) = \uparrow(\lim\inf \mathcal{F})$, but also in the closed set $\downarrow(\lim\inf \mathcal{F}) \supseteq \{\inf A : A \in \mathcal{F}\}$ and so must equal $\lim\inf \mathcal{F}$. Thus $\inf \operatorname{adh} \mathcal{F} = \lim\inf \mathcal{F}$ as desired. ☐

The remainder of the argument is standard continuous lattice theory, contained in, say, [8]; we include it here for the sake of a self-contained presentation.

Lemma 22 *An ultrafilter* $\mathcal{F}$ *on a continuous lattice converges in the Lawson topology to the unique point* $\ell = \lim\inf \mathcal{F}$.

Proof. We first show $\mathcal{F}$ contains every subbasic open neighbourhood of ℓ. First, if $\ell \in s^+$, i.e., $s \ll \ell$, then $s \ll \inf A$ for some $A \in \mathcal{F}$, and so $s \ll a$ for all $a \in A$; whence $A \subseteq s^+$ and so $s^+ \in \mathcal{F}$. Next, if $\ell \in s^-$, i.e., $s \not\leqslant \ell$ then $s \not\leqslant \inf A$ for all $A \in \mathcal{F}$. So for each $A \in \mathcal{F}$, we have $s \not\leqslant a$ for some $a \in A$, i.e., each $A \in \mathcal{F}$ meets s^-, and so, since $\mathcal{F}$ is an ultrafilter, we have $s^- \in \mathcal{F}$. Thus $\mathcal{F}$ converges to ℓ; suppose it also converges to y. Then for each $s \ll y$ we have $s^+ \in \mathcal{F}$ and so $s \leqslant \inf s^+ \leqslant \ell$. Since $y = \bigvee\{s : s \ll y\}$ we must have $y \leqslant \ell$. We claim $\ell \leqslant y$, i.e., $\inf A \leqslant y$ for each $A \in \mathcal{F}$. If not, then $\inf A \not\leqslant y$ for some $A \in \mathcal{F}$, so that $(\inf A)^-$ is in $\mathcal{F}$. So $(\inf A)^-$ and A are disjoint sets in $\mathcal{F}$, a contradiction. ☐

Proposition 23 *Let* X *be a complete lattice and a compact Hausdorff space. The following are equivalent:*

(i) $\lim\inf \mathcal{F} = \inf \operatorname{adh} \mathcal{F}$ *for any filter* $\mathcal{F}$ *on* X*;*
(ii) X *is a continuous lattice and its topology is the Lawson topology.*

Proof. (i) $\Rightarrow$ (ii). By the proof of Proposition 21, each principal upset $\uparrow x$ is closed in X. We now show that, if U is up-closed and open and $x \in U$, then $\inf U \ll x$. Indeed, suppose $x \leqslant \sup D$ for some directed $D \subseteq X$. Then $\uparrow(\sup D) \subseteq U$ since U is up-closed, and so $\emptyset = (X\backslash U) \cap \uparrow(\sup D) = (X\backslash U) \cap \bigcap\{\uparrow d : d \in D\}$. By compactness of X and downward-directedness of $\{\uparrow d : d \in D\}$, it follows that $\emptyset = (X\backslash U) \cap \uparrow d$ for some $d \in D$; thus $d \in U$ and so $\inf U \leqslant d$ as required.

We now show that X is continuous. Let $x \in X$ and let $\mathcal{F}$ be the neighbourhood filter of x. Since X is Hausdorff, we have $\operatorname{adh} \mathcal{F} = \{x\}$ and so by (i) that $\liminf \mathcal{F} = x$. Clearly $\liminf \mathcal{F}$ is the supremum of $\{\inf U : U \text{ up-closed in } \mathcal{F}\}$, and by above $\inf U \ll x$ for each such U. It follows that $x = \sup\{s : s \ll x\}$ so that X is continuous. Finally, the condition $\liminf \mathcal{F} = \inf \operatorname{adh} \mathcal{F}$ applied to an ultrafilter implies by Lemma 22 that the topology on X is the Lawson topology.

(ii) $\Rightarrow$ (i). We first show $\inf A = \inf \overline{A}$ for any $A \subseteq X$. Clearly $\inf \overline{A} \leqslant \inf A$; while if $x \in \overline{A}$, then x is the convergence point of some ultrafilter $\mathcal{F}$ containing A, whence $x = \liminf \mathcal{F} \geqslant \inf A$, so that $\inf A \leqslant \inf \overline{A}$. We now prove (i). Given a filter $\mathcal{F}$, we have for each $A \in \mathcal{F}$ that $\inf A = \inf \overline{A} \leqslant \inf \operatorname{adh} \mathcal{F}$ and so $\liminf \mathcal{F} \leqslant \inf \operatorname{adh} \mathcal{F}$. It remains to show $\inf \operatorname{adh} \mathcal{F} \leqslant \liminf \mathcal{F}$. By continuity, we can write $\inf \operatorname{adh} \mathcal{F}$ as $\bigvee\{s : s \ll \inf \operatorname{adh} \mathcal{F}\}$, so it suffices to show $s \ll \inf \operatorname{adh} \mathcal{F}$ implies $s \leqslant \liminf \mathcal{F}$. We prove the contrapositive: if $s \nleqslant \liminf \mathcal{F}$ then $s \not\ll \inf \operatorname{adh} \mathcal{F}$. Now if $s \nleqslant \liminf \mathcal{F}$, then $s \nleqslant \inf A = \inf \overline{A}$ for each $A \in \mathcal{F}$. Thus, for each $A \in \mathcal{F}$ there is some $a \in \overline{A}$ with $s \nleqslant a$ and hence $s \not\ll a$. This says that the closed set $X\backslash s^+$ meets $\bar{A}$ for each $A \in \mathcal{F}$; whence $\{\overline{A} : A \in \mathcal{F}\} \cup \{X\backslash s^+\}$ has the finite intersection property, so that by compactness, $X\backslash s^+$ meets $\bigcap_{A \in \mathcal{F}} \overline{A} = \operatorname{adh} \mathcal{F}$. This means $s \not\ll a$ for some $a \in \operatorname{adh} \mathcal{F}$, and so $s \not\ll \inf \operatorname{adh} \mathcal{F}$ as desired. □

So δ-algebras are continuous lattices, and it is easy to identify the corresponding δ-algebra maps as the inf- and directed-sup preserving functions. We thus recover:

Theorem 24 [28] *The category of $\mathbb{V}$-algebras is isomorphic to the category of continuous lattices with inf- and directed sup-preserving maps.*

5.3 Variations

It is natural to consider variations on Theorem 20, involving different weak distributive laws on possibly different categories. Treating these in detail must await further work, but it is worth sketching a couple of possibilities.

On the one hand, we may replace $\mathbb{P}$ by the *non-empty* power-set monad $\mathbb{P}_+$ on $\mathcal{S}$et, while keeping β the same. In this case, we expect to obtain a weak distributive law of $\mathbb{P}_+$ over β whose corresponding weak lifting to the category $\mathcal{KH}$aus of β-algebras is the *proper* Vietoris monad $\mathbb{V}_+$. This monad, considered in [29], sends a compact Hausdorff space X to its set of non-empty closed subsets under the Vietoris topology, and has as its algebras the continuous *semilattices*.

On the other hand, we can replace $\mathbb{P}$ by the *upper-set monad* $\mathbb{P}^{\uparrow}$ on the category of posets, and β by the *prime filter monad* $\mathbb{P}\mathbb{f}$. As in [7], this latter monad has as algebras the *compact pospaces*—compact spaces X with a partial order $\leqslant$ which is closed in $X \times X$. In this case, via the partially ordered version of Barr's relation lifting [17, Section 3.3], we expect to find a weak distributive law of $\mathbb{P}^{\uparrow}$ over $\mathbb{P}\mathbb{f}$, with corresponding weak lifting the "ordered Vietoris monad" $\mathbb{V}^{\uparrow}$ on compact pospaces. This takes a compact pospace X to its space $V^{\uparrow}X$ of closed upper-sets ordered by reverse inclusion, with a modified version of the Vietoris topology; see [8, Example VI-3.10]. As explained in [9], the $\mathbb{V}^{\uparrow}$-algebras are, once again, the continuous lattices. In fact, [9] describes other Vietoris-like monads, and it is natural to hope that these may arise in a similar manner; but again, we leave this to future work.

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