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Ionads

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ABSTRACT

The notion of Grothendieck topos may be considered as a generalisation of that of topological space, one in which the points of the space may have non-trivial automorphisms. However, the analogy is not precise, since in a topological space, it is the points which have conceptual priority over the open sets, whereas in a topos it is the other way around. Hence a topos is more correctly regarded as a generalised locale than as a generalised space. In this article we introduce the notion of *ionad*, which stands in the same relationship to a topological space as a (Grothendieck) topos does to a locale. We develop basic aspects of their theory and discuss their relationship with toposes.

1. Introduction

Grothendieck introduced *toposes* in [6] in order to describe a more general kind of "space" than that given by general topology, one whose "points" could possess non-trivial automorphisms. However, as Grothendieck himself immediately points out, the notion of topos is not a faithful generalisation of that of topological space; for though each space gives rise to a topos—namely the topos of sheaves on that space—we may only reconstruct the space from the topos if the space satisfies a separability axiom ("sobriety"). This is a reflection of a more general fact concerning the continuous maps between spaces. Every such map induces a geometric morphism between the corresponding sheaf toposes, and so we have a functor $\mathbf{Sp} \to \mathbf{GTop}$ from the category of spaces to the category of Grothendieck toposes; but this functor is neither full nor faithful (not even in the bicategorical sense). As is well known, the reason for these discrepancies is that a Grothendieck topos is not really a more general kind of *space*, but rather a more general kind of *locale* [3,9]. Indeed, to every locale we may assign a topos—again, by the sheaf construction—but, by contrast with the case for spaces, it is always possible to reconstruct the locale from the topos. We obtain similar good behaviour with respect to continuous maps of locales: the sheaf construction extends to a functor $\mathbf{Loc} \to \mathbf{GTop}$ which is (bicategorically) full and faithful and has a (bicategorical) left adjoint.

The reason that we bring this up is to point out a gap in our conceptual framework: there is no established structure which generalises the notion of topological space in a manner corresponding to that in which a Grothendieck topos generalises a locale. The purpose of this paper is to fill this gap by introducing the notion of *ionad*[1]. Like a topological space, an ionad comprises a set of points together with a "topology". The notion of topology employed is not the classical one, but is at least a generalisation of it: which in particular means that there is a canonical way of viewing a topological space as an ionad, giving rise to a full, reflective embedding of the category of spaces into the category of ionads. Thus an ionad is indeed a "generalised space", and the tightness of the correspondence is confirmed by many further correlations between the theory of topological spaces and that of ionads: for instance, a continuous map between ionads is a function on the underlying sets which commutes with the topologies in an appropriate sense; (co)limits in the category of ionads are constructed by equipping the (co)limit of the underlying diagram of sets with a suitably universal topology; and the most common way of constructing an ionad is in terms of a set of points together with a (suitably-generalised) basis of opens. Moreover, any ionad

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1 "ionad" is a Gaelic word meaning "place".

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has a collection of "opens", and just as the opens of a topological space form a locale, so the opens of an ionad form a topos. For those ionads arising from topological spaces, this "topos of opens" is just the topos of sheaves on the space; which is to say that the passage from spaces to ionads to toposes coincides with that from spaces to locales to toposes.

If we do take a topos-theoretic perspective, then the notion of ionad turns out to have a succinct and familiar expression—it is nothing other than a topos equipped with a separating set of points. Many other aspects of ionad theory admit similarly familiar topos-theoretic interpretations. However, the view we take here is that it should be perfectly possible to develop the basic theory of ionads without presupposing the corresponding aspects of topos theory, and we have arranged our account accordingly.

The paper is structured as follows. In Section 2, we introduce the notion of ionad, and give such examples as we may construct with our bare hands. We see that the "generalised opens" of an ionad are always a topos, and that, as mentioned above, the notion of ionad is essentially the same as that of spatial topos. Then in Section 3 we describe how ionads may be generated from a *basis*; where this notion naturally generalises the corresponding one for spaces. Using this we are able to give several more examples of ionads, which correspond to some familiar examples of toposes. We investigate the connection between the ionad generated from a basis, and the Grothendieck topos generated from a site, and finally characterise the ionads which may be generated from a basis (which we call *bounded*) as being those whose category of opens is a Grothendieck topos.

In Section 4 we define *continuous maps* between ionads and give a number of examples. We shall see that both the category of topological spaces and the category of small categories embed fully and faithfully into the category of bounded ionads, with the former embedding having a left adjoint; later, we show that the latter one has a right adjoint. In Sections 5 and 6, we briefly consider further aspects of the theory. Section 5 shows that the category of ionads may be enriched to a 2-category, whose 2-cells are *specialisations* between continuous maps, generalising the (pointwise) specialisation order on maps of topological spaces; and Section 6 describes the limits and colimits possessed by the 2-category of ionads: we see that the 2-category of *all* ionads has rather few limits and colimits, but that the 2-category of bounded ionads is complete and cocomplete. The paper is concluded in Section 7 by a short discussion on the comparative advantages and disadvantages of the notions of topos and ionad.

2. Ionads

We usually define a topological space to be a set X of points together with a *topology*: a collection of subsets of X closed under finite intersections and arbitrary unions. However, we may equally well give the set X together with an *interior operator*: an order-preserving map $i\colon \mathcal{P}X \to \mathcal{P}X$ which is a coclosure operator (i.e., deflationary and idempotent) and preserves finite intersections. The passage between the two definitions is straightforward: given a topology on X, there is an interior operator sending $A \subseteq X$ to the largest open set contained in it; and given an interior operator $i\colon \mathcal{P}X \to \mathcal{P}X$, there is a topology on X consisting of all those $A \subseteq X$ for which $A = i(A)$. Now by taking this second definition of topological space and replacing every poset-theoretic device which appears in it with a corresponding category-theoretic one, we obtain the notion of ionad.

Definition 2.1. An *ionad* is given by a set X of points together with a cartesian (i.e., finite limit preserving) comonad $I_X\colon \mathbf{Set}^X \to \mathbf{Set}^X$.

Remark 2.2. It will be convenient to carry over from the topological case the abuse of notation which names a space by its set of points: thus we may refer to an ionad (X, I_X) simply as X, with the interior comonad being left implicit.

If we are given a topological space presented in terms of its interior operator i, then we can reconstruct its open sets as the collection of i-fixpoints. In the case of an ionad, the only presentation we have is in terms of a generalised interior operator: but we can use the category-theoretic analogue of the fixpoint construction in order to *define* its "generalised opens".

Definition 2.3. The *category of opens* $\mathbf{O}(X)$ of an ionad X is the category of I_X-coalgebras.

Remark 2.4. In the definition of ionad, we have chosen to have a mere *set* of points, rather than a category of them. We do so for a number of reasons. The first is that this choice mirrors most closely the definition of topological space, where we have a set, and not a poset, of points. The second is that we would in fact obtain no extra generality by allowing a category of points. We may see this by analogy with the topological case, where to give an interior operator on a poset of points $(X, \leqslant)$ is equally well to give a topology $\mathcal{O}(X)$ on X such that every open set is upwards-closed with respect to $\leqslant$. Similarly, to equip a small category $\mathbf{C}$ with an interior comonad is equally well to give an interior comonad on $X := \mathrm{ob}\,\mathbf{C}$ together with a factorisation of the forgetful functor $\mathbf{O}(X) \to \mathbf{Set}^X$ through the presheaf category $\mathbf{Set}^{\mathbf{C}}$; this is an easy consequence of Example 2.7 below. However, the most compelling reason for not admitting a category of points is that, if we were to do so, then adjunctions such as that between the category of ionads and the category of topological spaces would no longer exist. Note that, although we do not allow a category of points, the points of any (well-behaved) ionad bear nonetheless a canonical category structure—described in Definition 5.7 and Remark 5.9 below—which may be understood as an analogue of the specialisation ordering on the points of a space.

Remark 2.5. As stated in the Introduction, the category of opens of an ionad is always a (cocomplete, elementary) topos: this because $\mathbf{Set}^X$ is a topos for any set X, and the category of coalgebras for a cartesian comonad on a topos is again a topos. Moreover, the cofree/forgetful adjunction between $\mathbf{Set}^X$ and $\mathbf{O}(X)$ yields a surjective geometric morphism $\mathbf{Set}^X \to \mathbf{O}(X)$. To give such a geometric morphism is to give an X-indexed family of points of the topos $\mathbf{O}(X)$; to say that it is surjective is to say that these points separate the generalised opens in $\mathbf{O}(X)$, in the sense that their inverse image functors jointly reflect isomorphisms. In particular, this makes $\mathbf{O}(X)$ a topos with *enough points* [11, §C2.2].

In fact, given any surjective geometric morphism $f : \mathbf{Set}^X \to \mathcal{E}$, we obtain an ionad (X, f^*f_*) whose category of open sets is equivalent (by surjectivity of f) to $\mathcal{E}$. Thus ionads are essentially the same things as toposes equipped with a separating set of points. There is a parallel here with the theory of topological spaces: where a space may be identified with a surjective locale morphism out of a discrete locale—one of the form $\mathcal{P}X$ for some set X, an ionad may be identified with a surjective geometric morphism out of a discrete topos—one of the form $\mathbf{Set}^X$ for some set X.

Remark 2.6. We shall see in Examples 3.5.2 below that every topological space A gives rise to an ionad ΣA, and that the topos of opens $\mathbf{O}(\Sigma A)$ is equivalent to the topos of sheaves $\mathbf{Sh}(A)$. With this in mind, we could have chosen to refer to the topos $\mathbf{O}(X)$ as the *topos of sheaves* on the "generalised space" X. We will not do so here, for the following two reasons. The first is that, in generalising further concepts from topological spaces to ionads, we often need do nothing more than replace $\mathcal{O}(X)$ everywhere by $\mathbf{O}(X)$, and the inevitability of this replacement would be diminished if we were to refer to this latter topos as $\mathbf{Sh}(X)$. The second reason is that, whilst it is indeed true that objects of $\mathcal{O}(X)$ look very much like sheaves on a topological space—as evidenced by Proposition 3.4, for instance—it is equally true that they look very much like generalised open sets. We will expand on this point in Remark 3.2 below.

Example 2.7. If $(X, \leqslant)$ is a partially ordered set, then there is a topology on X—the *Alexandroff topology*—whose open sets are the upwards-closed subsets of X with respect to $\leqslant$. In a similar way, if $\mathbf{C}$ is a small category, then there is an ionad $A(\mathbf{C})$ on the set of objects of $\mathbf{C}$ whose generalised opens are the "generalised upsets" in $\mathbf{C}$: that is, the covariant presheaves on $\mathbf{C}$. The interior comonad of this ionad is induced by the adjunction

$$\mathbf{Set}^{\mathrm{ob}\,\mathbf{C}} \underset{\mathrm{Ran}_J}{\overset{\mathbf{Set}^J}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Set}^{\mathbf{C}} \qquad (1)$$

obtained by restriction and right Kan extension along the inclusion functor $J : \mathrm{ob}\,\mathbf{C} \to \mathbf{C}$. Observe that the functor $\mathbf{Set}^J$, since it strictly creates equalisers, is strictly comonadic; and so the category of open sets for $A(\mathbf{C})$ is isomorphic to $\mathbf{Set}^{\mathbf{C}}$.

Recall that the lattice of open sets of an Alexandroff topology is closed under *arbitrary* intersections, and that this property serves to completely characterise the Alexandroff topologies. Likewise, for an Alexandroff ionad, the forgetful functor $U : \mathbf{O}(A(\mathbf{C})) \to \mathbf{Set}^{\mathrm{ob}\,\mathbf{C}}$ creates limits (since $\mathbf{Set}^J$, and hence the interior comonad, preserve them); and this property completely characterises the Alexandroff ionads. Indeed, if for some ionad X the forgetful functor $\mathbf{O}(X) \to \mathbf{Set}^X$ creates limits, then the interior operator $I_X : \mathbf{Set}^X \to \mathbf{Set}^X$ will preserve them; and so have a colimit-preserving left adjoint K. The comonad structure of I_X transposes across the adjunction to give a monad structure on K, which is equivalently a monoid structure on the functor

$$M := X \xrightarrow{y} \mathbf{Set}^X \xrightarrow{K} \mathbf{Set}^X$$

with respect to profunctor composition; and this in turn amounts to specifying a category $\mathbf{C}$ with object set X and homsets $\mathbf{C}(x, y) := M(x)(y)$. Moreover, it follows ("adjoint triples" [4]) that there is an isomorphism between the category of I_X-coalgebras—which is $\mathbf{O}(X)$—and the category of K-algebras, which, by an easy calculation, is $\mathbf{Set}^{\mathbf{C}}$.

3. Generalised bases

In order to produce more sophisticated examples of ionads, we will need a way of generating topologies from bases. Recall that a *basis* for an ordinary topology on a set X is a collection $\mathcal{B} \subseteq \mathcal{P}X$ satisfying the following properties:

- For every $x \in X$, there is some $B \in \mathcal{B}$ with $x \in \mathcal{B}$;
- If $x \in X$ and $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, then there is some $B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$ and $x \in B_3$.

The open sets of the topology this generates are arbitrary unions of elements of $\mathcal{B}$. However, since our aim is to generalise this definition from spaces to ionads, we will be more interested in describing the interior operator generated by $\mathcal{B}$. To this end, we regard $\mathcal{B}$ as a poset under inclusion, and write $m : \mathcal{B} \to \mathcal{P}X$ for the (order-preserving) inclusion map. Now the basis axioms for $\mathcal{B}$ correspond to the requirement that m should be *flat*, in the sense that, for each $x \in X$, the set $\{B \in \mathcal{B} \mid x \in m(B)\}$ should be a downwards directed poset. In fact, we can drop the requirement that m should be injective entirely; this gives a more "intensional" notion of basis, where the same open set may be named by more than one basis element. (Observe that this happens quite frequently in practice: think, for example, of the Zariski topology on the prime spectrum of a ring; or of the logical topology on the set of complete theories extending a first-order theory $\mathbb{T}$.)

Now given any flat morphism $m : \mathcal{B} \to \mathcal{P}X$, we may define an interior operator on X in the following manner. We write $\downarrow\mathcal{B}$ for the poset of downsets in $\mathcal{B}$, and $y : \mathcal{B} \to \downarrow\mathcal{B}$ for the order-preserving map sending $B \in \mathcal{B}$ to the downset of all elements below B. This map exhibits $\downarrow\mathcal{B}$ as the free join-completion of $\mathcal{B}$, and so there is a unique way of extending m along y to yield a join-preserving map $m \otimes (-)\colon \downarrow\mathcal{B} \to \mathcal{P}X$:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{m} & \mathcal{P}X \\ {\scriptstyle y}\downarrow & \nearrow_{m\otimes(-)} & \\ \downarrow\mathcal{B} & & \end{array} .$$

Since this $m \otimes (-)$ preserves joins, it has a right adjoint $[m, -]\colon \mathcal{P}X \to \downarrow\mathcal{B}$, and composing these yields a coclosure operator $i := m \otimes [m, -]$ on $\mathcal{P}X$. In order to show that this i preserves finite meets, it suffices to show that $m \otimes -$ does so; but a standard piece of lattice theory tells us that this is equivalent to the flatness of m. It remains to check that this interior operator i is the one associated to the topology generated by $\mathcal{B}$. For this, we calculate that

$$m \otimes \mathcal{X} = \bigcup_{B \in \mathcal{X}} m(B) \qquad \text{and} \qquad [m, A] = \{B \in \mathcal{B} \,|\, m(B) \subseteq A\}$$

so that the composite $i\colon \mathcal{P}X \to \mathcal{P}X$ sends A to the union of all those $m(B)$'s with $m(B) \subseteq A$, as required. Consideration of the above now leads us to propose:

Definition 3.1. A *basis* for an ionad with set of points X is given by a small category $\mathbf{B}$ together with a functor $M\colon \mathbf{B} \to \mathbf{Set}^X$ which is *flat*, in the sense that for each $x \in X$, the category of elements of the functor $M(-)(x)\colon \mathbf{B} \to \mathbf{Set}$ is cofiltered.

The construction of an ionad from a basis mirrors that of a space from a basis. The Yoneda embedding $y\colon \mathbf{B} \to \mathbf{Set}^{\mathbf{B}^{\mathrm{op}}}$ exhibits $\mathbf{Set}^{\mathbf{B}^{\mathrm{op}}}$ as the free colimit-completion of $\mathbf{B}$, and so we may extend M along it to yield a colimit-preserving functor $M \otimes (-)\colon \mathbf{Set}^{\mathbf{B}^{\mathrm{op}}} \to \mathbf{Set}^X$:

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{M} & \mathbf{Set}^X \\ {\scriptstyle y}\downarrow & \cong \quad \nearrow_{M\otimes(-)} & \\ \mathbf{Set}^{\mathbf{B}^{\mathrm{op}}} & & \end{array} .$$

Since this $M \otimes (-)$ preserves colimits, it has a right adjoint $[M, -]$, and composing these together yields a comonad on $\mathbf{Set}^X$. Again, to ensure that this comonad preserves finite limits, it suffices to show that $M \otimes (-)$ does; and a standard piece of category theory says that this is equivalent to the flatness of M.

Remark 3.2. If $\mathcal{B}$ is a basis for an ordinary topology on a set X, then a subset $A \subset X$ is open in that topology just when every $x \in A$ is contained in some $B \in \mathcal{B}$ with $B \subset A$. If $M\colon \mathbf{B} \to \mathbf{Set}^X$ is a basis for an ionad X, then we may view objects of the category of opens $\mathbf{O}(X)$ in a corresponding manner. Unravelling the definitions, we see that the interior comonad I generated by the basis M has its value at $A \in \mathbf{Set}^X$ given by

$$(IA)(x) = \int^{B \in \mathbf{B}} (MB)(x) \times \prod_{y \in X} A(y)^{(MB)(y)} .$$

Thus, if we think of a typical $A \in \mathbf{Set}^X$ as specifying, for each $x \in X$, a set Ax of proofs that x lies in A, then to give an I-coalgebra structure on A is to give a mapping which, to each proof that x lies in A, coherently assigns an element $B \in \mathbf{B}$, together with proofs that x lies in MB and that MB is contained in A.

Remark 3.3. Having motivated the preceding construction purely from topological considerations, we now see that it is a familiar one in topos-theory. A basis for an ionad is a flat functor $\mathbf{B} \to \mathbf{Set}^X$, which corresponds to a colimit- and finite-limit-preserving functor $\mathbf{Set}^{\mathbf{B}^{\mathrm{op}}} \to \mathbf{Set}^X$, and hence to a geometric morphism $\mathbf{Set}^X \to \mathbf{Set}^{\mathbf{B}^{\mathrm{op}}}$. Any such geometric morphism factors as

$$\mathbf{Set}^X \xrightarrow{p} \mathcal{E} \xrightarrow{i} \mathbf{Set}^{\mathbf{B}^{\mathrm{op}}} \tag{2}$$

where p is a surjection, and i an inclusion (see [10, Theorem A4.2.10] for example). By Remark 2.5, the map p determines an ionad on X, which is by inspection precisely the ionad generated by the basis $\mathbf{B} \to \mathbf{Set}^X$. The fact that i is a geometric inclusion tells us that $\mathcal{E}$, the category of opens of this ionad, is a subtopos of $\mathbf{Set}^{\mathbf{B}^{\mathrm{op}}}$. In fact, we have:

Proposition 3.4. *If $M\colon \mathbf{B} \to \mathbf{Set}^X$ is a basis for an ionad X, then the category $\mathbf{O}(X)$ is equivalent to the category of sheaves on the site whose underlying category is $\mathbf{B}$ and whose covering sieves are those $(f_i\colon U_i \to U \mid i \in I)$ in $\mathbf{B}$ which M sends to jointly epimorphic families in $\mathbf{Set}^X$.*

Proof. Since $\mathbf{O}(X)$ is the category of coalgebras for the comonad generated by the adjunction

$$\mathbf{Set}^X \underset{[M,-]}{\overset{M\otimes(-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Set}^{\mathbf{B}^{\mathrm{op}}},$$

there is a canonical comparison functor $L\colon \mathbf{Set}^{\mathbf{B}^{\mathrm{op}}} \to \mathbf{O}(X)$, which preserves finite limits since $M \otimes (-)$ does. Because $\mathbf{Set}^{\mathbf{B}^{\mathrm{op}}}$ has equalisers, this functor has a right adjoint R; because $M \otimes (-)$ preserves them, this R is fully faithful, and so exhibits $\mathbf{O}(X)$, up-to-equivalence, as the category of sheaves for a Grothendieck topology on $\mathbf{B}$. A family of morphisms $(f_i\colon U_i \to U \mid i \in I)$ is covering for this topology just when the sieve $\varphi\colon A \rightarrowtail \mathbf{B}(-, U)$ they generate is inverted by L. Since the forgetful functor $U_X\colon \mathbf{O}(X) \to \mathbf{Set}^X$ is comonadic, hence conservative, $L\varphi$ is invertible if and only if $U_X L\varphi = M \otimes \varphi$ is so in $\mathbf{Set}^X$. But φ is the image of the map $[\mathcal{C}(-, f_i)]_{i\in I}\colon \sum_{i\in I} \mathcal{C}(-, U_i) \to \mathcal{C}(-, U)$; and since $M \otimes (-)$ preserves finite limits and colimits, it preserves image factorisations, whence $M \otimes \varphi$ is an isomorphism if and only if $(M \otimes \mathcal{C}(-, f_i) \mid i \in I) = (Mf_i \mid i \in I)$ is a jointly epimorphic family, as required. □

Examples 3.5. (1) If $\mathbf{C}$ is a small category, then we obtain a basis for the Alexandroff ionad $A(\mathbf{C})$ of Example 2.7 by taking $\mathbf{B} := \mathbf{C}^{\mathrm{op}}$ and $M := [J, -]\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathrm{ob}\,\mathbf{C}}$, where $J\colon \mathrm{ob}\,\mathbf{C} \to \mathbf{C}^{\mathrm{op}}$ is the canonical inclusion. We see that the ionad this basis generates is $A(\mathbf{C})$ by noting that the adjunction $M \otimes (-) \dashv [M, -]$ induced by M is precisely the adjunction of (1).

(2) Every topological space X gives rise to an ionad ΣX on the same set of points, generated by the following basis. We take $\mathbf{B} := \mathcal{O}(X)$, the lattice of open sets of the topology (though we could equally well take it to be any basis, in the classical sense, for the topology on X), and $M\colon \mathcal{O}(X) \to \mathbf{Set}^X$ the composite of $\mathcal{O}(X) \rightarrowtail \mathcal{P}X$ with the obvious inclusion $\mathcal{P}X \rightarrowtail \mathbf{Set}^X$. Thus we have

$$M(U)(x) = \begin{cases} 1 & \text{if } x \in U; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly M preserves finite limits, and so is flat; and it is easy to see that the covering families of the induced site structure on $\mathbf{B}$ are of the form $(U_i \subseteq U \mid i \in I)$ where $\bigcup U_i = U$. Thus by Proposition 3.4, the category of opens $\mathbf{O}(\Sigma X)$ is equivalent to the category of sheaves on the space X.

(3) Let X be a topological space equipped with an action by a discrete group G. We define the *G-equivariant ionad* $\Sigma_G X$ to have set of points X, and topology generated by the following basis. We take $\mathbf{B} := \mathcal{O}_G(X)$, the category whose objects are open sets of X, and whose morphisms $U \to V$ are elements $g \in G$ for which $g(U) \subseteq V$; and define $M\colon \mathcal{O}_G(X) \to \mathbf{Set}^X$ by

$$M(U)(x) = \{h \in G \mid hx \in U\} \quad \text{and} \quad M(g)(x)\colon h \mapsto gh\,.$$

It is easy to show that M is flat, and so defines a basis for an ionad $\Sigma_G X$. The induced site structure on $\mathcal{O}_G(X)$ has as covering families all those $(g_i\colon U_i \to V \mid i \in I)$ such that (Mg_i) is jointly epimorphic; that is, such that for every $x \in X$ and $h \in G$ with $hx \in V$, there exists $i \in I$ such that $g_i^{-1}hx \in U_i$; that is, such that the family of maps $g_i|_{U_i}\colon U_i \to X$ jointly cover V. Thus by Proposition 3.4 and [10, Examples A.2.1.11(c)], we conclude that $\mathbf{O}(\Sigma_G X)$ is equivalent to the topos of G-equivariant sheaves on X.

(4) If A is a commutative ring, we define its *étale ionad* as follows. Its set of points is $\mathrm{Spec}(A)$, the set of prime ideals of A, whilst its topology is generated by the following basis. The category $\mathbf{B}$ is (a skeleton of) $\mathbf{Et}_A^{\mathrm{op}}$, the opposite of the category of étale A-algebras, whilst $M\colon \mathbf{Et}_A^{\mathrm{op}} \to \mathbf{Set}^{\mathrm{Spec}(A)}$ sends an étale A-algebra $f\colon A \to B$ and a prime ideal $P \lhd A$ to the set of all prime ideals $Q \lhd B$ for which $f^{-1}(Q) = P$. The induced site structure on $\mathbf{Et}_A^{\mathrm{op}}$ has as covering families those $(f_i\colon B_i \leftarrow B \mid i \in I)$ such that $(Mf_i \mid i \in I)$ is jointly epimorphic; but since the disjoint union of the sets $(MB)(P)$, as P ranges over the prime ideals of A, is clearly the set of all prime ideals of B, to say that the family $(Mf_i \mid i \in I)$ is jointly epimorphic is equally well to say that every prime ideal of B is the inverse image of a prime ideal of some B_i. Thus by Proposition 3.4 and [8, Exercise 0.11], the topos of opens of the étale ionad is the little étale topos of A.

(5) Let $\mathbb{T}$ be a coherent first-order theory over a language of cardinality κ. For any regular cardinal $\lambda > \kappa$, we define the *λ-small classifying ionad* as follows. Let X be a set of representatives of isomorphism classes of models of $\mathbb{T}$ of cardinality $< \lambda$, and let $\mathbf{B}$ be the *syntactic category* [11, §D1.4] of the theory $\mathbb{T}$; it has as objects, coherent formulae-in-context $\{\vec{x}.\phi\}$, and as morphisms, equivalence classes of provably functional relations between them. We define a functor $M\colon \mathbf{B} \to \mathbf{Set}^X$ which takes a formula-in-context $\{\vec{x}.\phi\}$ and a model A and returns the interpretation $[\![\phi]\!]_A$ of ϕ in A. We may show that $\mathbf{B}$ has, and M preserves, all finite limits, so that M is a basis for an ionad. Moreover, if $\big(\theta_i\colon \{\vec{y}_i.\phi_i\} \to \{\vec{x}.\psi\} \mid 1 \leqslant i \leqslant n\big)$ is a family of maps in $\mathbf{B}$, then it is easy to see that the sequent $\psi \vdash_{\vec{x}} \bigvee_{i=1}^n (\exists \vec{y}_i)\theta_i$ is validated in a given model A of $\mathbb{T}$ if and only if the family of functions $[\![\theta_i]\!]_A\colon [\![\phi_i]\!]_A \to [\![\psi]\!]_A$ is jointly epimorphic. Since the collection of models of cardinality $< \lambda$ is complete for $\mathbb{T}$, it follows that the family $(\theta_i \mid i \in I)$ is sent to a jointly epimorphic family in $\mathbf{Set}^X$ if and only if the sequent $\psi \vdash_{\vec{x}} \bigvee_{i=1}^n (\exists \vec{y}_i)\theta_i$ is provable in $\mathbb{T}$: so that the induced site of this ionad is the syntactic site of $\mathbb{T}$, and the topos of opens, the classifying topos of $\mathbb{T}$.

Although every ionad we meet in practice will be generated from a basis, it is not *a priori* the case that every ionad need arise in this way. Indeed, if an ionad is generated by a basis $M\colon \mathbf{B} \to \mathbf{Set}^X$, then its category of opens is equivalent to a category of sheaves, and hence a Grothendieck topos. On the other hand, the category of coalgebras for an *inaccessible* cartesian comonad on $\mathbf{Set}^X$ is not locally presentable, and hence not a Grothendieck topos. In fact, we have:

Proposition 3.6. *The following conditions on an ionad X are equivalent:*

(1) *It may be generated (up to isomorphism) from a basis;*
(2) *Its interior comonad is accessible;*
(3) *Its category of opens is a Grothendieck topos.*

Proof. First we show (1) ⇒ (2). Given a basis $M: \mathbf{B} \to \mathbf{Set}^X$, we will show that $[M, -]: \mathbf{Set}^X \to \mathbf{Set}^{\mathbf{B}^{op}}$ preserves λ-filtered colimits for some regular cardinal λ; it then follows that also its composite with the left adjoint $M \otimes (-)$ will do so. So let λ be such that the set of objects $\{MB \mid B \in \mathbf{B}\}$ are all λ-presentable in $\mathbf{Set}^X$ (such a λ exists since $\mathbf{Set}^X$ is locally presentable). Now given a λ-filtered diagram $A: \mathbf{I} \to \mathbf{Set}^X$ we calculate that $[M, \mathrm{colim}_i A_i](B) = \mathbf{Set}^X(MB, \mathrm{colim}_i A_i) \cong \mathrm{colim}_i \mathbf{Set}^X(MB, A_i) = \mathrm{colim}_i [M, A_i](B)$ for every $B \in \mathbf{B}$ as required. For (2) ⇒ (3), we simply observe that if I_X is an accessible comonad, then its category of coalgebras $\mathbf{O}(X)$ is locally presentable, and hence a Grothendieck topos. Finally we show that (3) ⇒ (1). If $\mathbf{O}(X)$ is a Grothendieck topos, then it is in particular locally presentable; and so we may find a small full dense subcategory $N: \mathbf{B} \rightarrowtail \mathbf{O}(X)$ which, without loss of generality, we may take to be closed under finite limits. Then the composite $M := \mathbf{B} \rightarrowtail \mathbf{O}(X) \to \mathbf{Set}^X$ is a cartesian functor on $\mathbf{B}$, and so gives a basis for an ionad. The interior comonad of this ionad is isomorphic to that generated by the string of adjunctions

$$\mathbf{Set}^X \underset{\text{cofree}}{\overset{\text{forget}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{O}(X) \underset{[N,-]}{\overset{N\otimes(-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Set}^{\mathbf{B}^{op}} ,$$

but since $\mathbf{B}$ is dense in $\mathbf{O}(X)$, the counit of the right-hand adjunction is an isomorphism, and it follows that the interior comonad of the resultant ionad is isomorphic to I_X. □

Definition 3.7. We call an ionad *bounded* if it satisfies any one of the three equivalent conditions of Proposition 3.6.

Remark 3.8. Every ionad that we meet in mathematical practice is bounded: moreover, it is quite probable that the existence or otherwise of unbounded ionads is a problem that is independent of the usual axioms of set theory. A construction given in [10, Example B3.1.12] shows that any inaccessible cartesian endofunctor of **Set** gives rise to an unbounded ionad on the two-element set; but the only known construction of an inaccessible, cartesian endofunctor of **Set**, given in [2], requires the existence of a proper class of measurable cardinals.

Remark 3.9. In Remark 2.5, we noted that the category of opens of an ionad is always a topos with enough points. For bounded ionads, we can say more: the toposes arising as their categories of opens are *precisely* the Grothendieck toposes with enough points. Indeed, for a topos $\mathcal{E}$ to have enough points is for the class of all inverse image functors $\mathcal{E} \to \mathbf{Set}$ to be jointly conservative. In the case of a Grothendieck topos, this implies the existence of a mere *set* X of inverse image functors with this property (as is shown in [11, Proposition C2.2.12]), and hence of a geometric surjection $\mathbf{Set}^X \to \mathcal{E}$ exhibiting $\mathcal{E}$ as the category of opens of an ionad.

4. Maps of ionads

We now consider the appropriate notion of morphism between ionads. For ordinary topological spaces X and Y, a continuous map is a morphism of underlying sets $f: X \to Y$ such that the induced inverse image mapping $f^{-1}: \mathcal{P}Y \to \mathcal{P}X$ maps open sets to open sets; which is to say that there exists a (necessarily unique) lifting

$$\begin{array}{ccc} \mathcal{O}(Y) & \overset{f^*}{\dashrightarrow} & \mathcal{O}(X) \\ \downarrow & & \downarrow \\ \mathcal{P}Y & \underset{f^{-1}}{\longrightarrow} & \mathcal{P}X \end{array}$$

of f^{-1} as indicated. We are therefore led to propose:

Definition 4.1. A *continuous map* of ionads $X \to Y$ is a function $f: X \to Y$ of the underlying sets together with a lifting

$$\begin{array}{ccc} \mathbf{O}(Y) & \overset{f^*}{\longrightarrow} & \mathbf{O}(X) \\ {\scriptstyle U_Y}\downarrow & & \downarrow{\scriptstyle U_X} \\ \mathbf{Set}^Y & \underset{f^{-1}}{\longrightarrow} & \mathbf{Set}^X \end{array} \tag{3}$$

of $f^{-1} := \mathbf{Set}^f$ through the corresponding categories of open sets. We write **Ion** for the category of ionads and continuous maps, and **BIon** for the full subcategory determined by the bounded ionads.

Remark 4.2. In accordance with the notational convention established in Remark 2.2, we may choose to denote a continuous map of ionads by naming its underlying map of sets $f\colon X \to Y$ whilst leaving the corresponding lifting $f^*\colon \mathbf{O}(Y) \to \mathbf{O}(X)$ implicit.

Remark 4.3. Observe that if f is an ionad morphism, then $f^*\colon \mathbf{O}(Y) \to \mathbf{O}(X)$ is cartesian, because finite limits are preserved by $f^{-1}.U_Y$ and reflected by U_X. Moreover, since U_Y and U_X are comonadic and $\mathbf{O}(Y)$ has equalisers, the adjoint lifting theorem [7] permits us to lift the right adjoint Π_f of f^{-1} to a right adjoint f_* for f^*, and so to make f^* into the inverse image part of a geometric morphism $\mathbf{O}(X) \to \mathbf{O}(Y)$. This construction yields a functor[2] $\mathbf{O}(\text{–})\colon \mathbf{Ion} \to \mathbf{Top}$, which is analogous to the functor $\mathcal{O}(\text{–})\colon \mathbf{Sp} \to \mathbf{Loc}$ assigning to every topological space its locale of open sets.

Example 4.4. If $\mathbf{C}$ and $\mathbf{D}$ are small categories, then to give a continuous map $A(\mathbf{C}) \to A(\mathbf{D})$ between the corresponding Alexandroff ionads is to give a function $f\colon \operatorname{ob}\mathbf{C} \to \operatorname{ob}\mathbf{D}$ together with a lifting

$$\begin{array}{ccc} \mathbf{Set}^{\mathbf{D}} & \xrightarrow{f^*} & \mathbf{Set}^{\mathbf{C}} \\ \downarrow & & \downarrow \\ \mathbf{Set}^{\operatorname{ob}\mathbf{D}} & \xrightarrow[f^{-1}]{} & \mathbf{Set}^{\operatorname{ob}\mathbf{C}} . \end{array} \tag{4}$$

In particular, any extension of f to a functor $F\colon \mathbf{C} \to \mathbf{D}$ determines such a lifting by taking $f^* = \mathbf{Set}^F$, and so the assignation $\mathbf{C} \mapsto A(\mathbf{C})$ extends to a functor $A\colon \mathbf{Cat} \to \mathbf{BIon}$. In fact, this functor is fully faithful. To see this, we must prove that every lifting in (4) is induced by a unique extension of f to a functor. So given such a lifting f^*, we must construct for each $g\colon x \to x'$ in $\mathbf{C}$ a morphism $F(g)\colon fx \to fx'$. Note first that commutativity in (4) forces $f^*(H)(x) = H(fx)$ and $f^*(H)(x') = H(fx')$ for every $H \in \mathbf{Set}^{\mathbf{D}}$. In particular, taking $H = y_{fx}$, we obtain a map of sets

$$f^*(y_{fx})(g)\colon \mathbf{D}(fx, fx) \to \mathbf{D}(fx, fx')$$

and evaluating this at 1_{fx} yields the required morphism $fx \to fx'$. Straightforward diagram chasing shows this assignation to be functorial, and that the resultant functor uniquely induces the lifting f^*. Thus we have a full embedding $A\colon \mathbf{Cat} \to \mathbf{BIon}$; in Remark 6.5 below, we will see that this embedding is in fact coreflective.

Remark 4.5. To give a continuous map of ionads $(X, I) \to (Y, J)$ is equally well to give a function $f\colon X \to Y$ and a natural transformation $\delta\colon f^{-1}J \Rightarrow If^{-1}$ such that the diagrams

$$\begin{array}{ccc} f^{-1}J & \xrightarrow{f^{-1}\epsilon} & f^{-1} \\ {\scriptstyle\delta}\downarrow & \nearrow_{\epsilon f^{-1}} & \\ If^{-1} & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} f^{-1}J & \xrightarrow{f^{-1}\Delta} & f^{-1}JJ & \xrightarrow{\delta J} & If^{-1}J \\ {\scriptstyle\delta}\downarrow & & & & \downarrow{\scriptstyle I\delta} \\ If^{-1} & & \xrightarrow[\Delta f^{-1}]{} & & IIf^{-1} \end{array} \tag{5}$$

commute; that is, such that the pair (f^{-1}, δ) is a comonad morphism $(\mathbf{Set}^Y, J) \to (\mathbf{Set}^X, I)$ in the sense of [14]. The passage between the two descriptions is as follows: given $\delta\colon f^{-1}J \Rightarrow If^{-1}$, we define the corresponding $f^*\colon \mathbf{O}(Y) \Rightarrow \mathbf{O}(X)$ by $f^*(a\colon A \to JA) = \delta_A.f^{-1}a\colon f^{-1}A \to If^{-1}A$. Conversely, given f^*, we obtain the A-component of the corresponding δ by applying f^* to the cofree J-coalgebra $\Delta_A\colon JA \to JJA$—yielding an I-coalgebra $f^*(\Delta_A)\colon f^{-1}JA \to If^{-1}JA$—and then postcomposing with $If^{-1}\epsilon_A\colon If^{-1}JA \to If^{-1}A$.

Example 4.6. Let Y be the ionad generated by a basis $M\colon \mathbf{B} \to \mathbf{Set}^Y$. By the preceding Remark, to give a continuous map $X \to Y$ is to give a function $f\colon X \to Y$ together with a natural transformation $f^{-1}.M \otimes [M, \text{–}] \Rightarrow I_X.f^{-1}$ satisfying the axioms of (5). Now by virtue of the adjunction $M \otimes (\text{–}) \dashv [M, \text{–}]$, to give this natural transformation is equally well to give a natural transformation $f^{-1}.M \otimes (\text{–}) \Rightarrow I_X.f^{-1}.M \otimes (\text{–})\colon \mathbf{Set}^{\mathbf{B}^{\mathrm{op}}} \to \mathbf{Set}^X$; and since both f^{-1} and $I_X.f^{-1}$ preserve colimits, such a natural transformation is uniquely determined by a natural transformation $\alpha\colon f^{-1}.M \Rightarrow I_X.f^{-1}.M\colon \mathbf{B} \to \mathbf{Set}^X$. Under these correspondences, the two axioms of (5) become two axioms on α which say exactly that it equips $f^{-1}.M$ with the structure of a coalgebra for the comonad $(I_X)^{\mathbf{B}}$ on $(\mathbf{Set}^X)^{\mathbf{B}}$. But to give such a coalgebra structure on $f^{-1}.M$ is equally well to give a lifting

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{f'} & \mathbf{O}(X) \\ {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle U_X} \\ \mathbf{Set}^Y & \xrightarrow[f^{-1}]{} & \mathbf{Set}^X \end{array} \tag{6}$$

of $f^{-1}.M$ through $\mathbf{O}(X)$, the category of I_X-coalgebras.

2 With the usual definition of geometric morphism, this is really only a *pseudo*functor, because we must choose a right adjoint f_* for each f^*, and in general cannot expect these choices to satisfy $g_*f_* = (gf)_*$, but only $g_*f_* \cong (gf)_*$. However, we shall take the slightly non-standard definition of a geometric morphism $\mathcal{E} \to \mathcal{F}$ as a left adjoint cartesian functor $\mathcal{F} \to \mathcal{E}$; whereupon we do indeed obtain a genuine functor $\mathbf{Ion} \to \mathbf{Top}$.

A consequence of this is that the collection of ionad morphisms $X \to Y$ will be a set whenever Y is bounded. For in this case, continuous maps are given by diagrams as in (6); there are only a set of functions $f : X \to Y$; and for every such f, there are only a set of liftings f', since $\mathbf{B}$ is small and each $H \in \mathbf{Set}^X$ admits only a set of I_X-coalgebra structures. In particular, we deduce that the category **BIon** of bounded ionads is locally small.

Remark 4.7. Writing $\mathbf{Set}^{(-)} : \mathbf{Set} \to \mathbf{CAT}^{\mathrm{op}}$ for the functor sending X to $\mathbf{Set}^X$ and $f : X \to Y$ to $f^{-1} : \mathbf{Set}^Y \to \mathbf{Set}^X$, we may regard **Ion** as a full subcategory of the comma category $\mathbf{CAT}^{\mathrm{op}} \downarrow \mathbf{Set}^{(-)}$. The preceding example shows that, for any basis $M : \mathbf{B} \to \mathbf{Set}^Y$, the corresponding ionad Y is a coreflection of M into this full subcategory. The counit of this coreflection is a map

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\overline{M}} & \mathbf{O}(Y) \\ & {\scriptstyle M}\searrow \quad \swarrow{\scriptstyle U_Y} & \\ & \mathbf{Set}^Y & \end{array}$$

composition with which induces the bijection between squares of the form (3) and of the form (6).

Example 4.8. Let $f : X \to Y$ be a continuous map of topological spaces. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(Y) & \xrightarrow{f^*} & \mathcal{O}(X) \\ \downarrow & & \downarrow \\ \mathcal{P}Y & & \mathcal{P}X \\ \downarrow & & \downarrow \\ \mathbf{Set}^Y & \xrightarrow[f^{-1}]{} & \mathbf{Set}^X \end{array}$$

and so applying the coreflection of the preceding remark, we obtain a continuous map of ionads $\Sigma X \to \Sigma Y$. Thus the assignation $X \mapsto \Sigma X$ extends to a functor $\Sigma : \mathbf{Sp} \to \mathbf{Ion}$, which in fact exhibits **Sp** as a full reflective subcategory of **Ion** (and indeed also of **BIon**). Let us describe the left adjoint Λ of Σ. Given an ionad X, we observe that $\mathcal{P}X$ is the lattice of subobjects of 1 inside $\mathbf{Set}^X$; and that since I_X is cartesian, it restricts and corestricts to this lattice, thus yielding an interior operator $i : \mathcal{P}X \to \mathcal{P}X$. We claim that the space ΛX with this interior operator gives a reflection of X along the functor Σ. To see this, first observe that the open sets of ΛX, which are the fixpoints of i, are precisely the subobjects of 1 inside $\mathbf{O}(X)$; and so we have a pullback diagram

$$\begin{array}{ccc} \mathcal{O}(\Lambda X) & \rightarrowtail & \mathbf{O}(X) \\ \downarrow & & \downarrow \\ \mathcal{P}X & \rightarrowtail & \mathbf{Set}^X . \end{array} \qquad (7)$$

Now if Y is a topological space, then by Example 4.6, to give an ionad map $X \to \Sigma Y$ is to give a function $f : X \to Y$ and a functor $f' : \mathcal{O}(Y) \to \mathbf{O}(X)$ making the following square commute:

$$\begin{array}{ccc} \mathcal{O}(Y) & \xrightarrow{f'} & \mathbf{O}(X) \\ {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle U_X} \\ \mathbf{Set}^Y & \xrightarrow[f^{-1}]{} & \mathbf{Set}^X . \end{array}$$

But the lower composite $f^{-1}.M$ factors through the inclusion $\mathcal{P}X \rightarrowtail \mathbf{Set}^X$ (since f^{-1} preserves finite limits, and hence subobjects of 1); and so, since (7) is a pullback, f' must factor through the inclusion $\mathcal{O}(\Lambda X) \rightarrowtail \mathbf{O}(X)$. But this implies that there is at most one f' lifting f^{-1}, and that such a lifting will exist precisely when f is continuous as a map of spaces $\Lambda X \to Y$. Thus for each X, we have established a bijection $\mathbf{Ion}(X, \Sigma Y) \cong \mathbf{Sp}(\Lambda X, Y)$—whose naturality in Y is easily checked—so that the assignation $X \mapsto \Lambda X$ extends to a functor $\Lambda : \mathbf{Ion} \to \mathbf{Sp}$ left adjoint to Σ. It remains to observe that for any space Z, we have $\Lambda \Sigma Z = Z$, so that the adjunction $\Lambda \dashv \Sigma$ is a reflection as claimed.

Remark 4.9. The reflection constructed in the previous example again has a familiar topos-theoretic interpretation. Given an ionad X, we may factorise the unique geometric morphism $\mathbf{O}(X) \to \mathbf{Set}$ (note that this exists since $\mathbf{O}(X)$ is cocomplete) as

$$\mathbf{O}(X) \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathbf{Set}$$

where i is hyperconnected and p is localic (cf. [10, §A4.6]). That p is localic means that $\mathcal{E}$ is equivalent to $\mathbf{Sh}(K)$ for some locale K; whilst that i is hyperconnected means in particular that it is a surjection. Hence so also is the composite geometric morphism

$$\mathbf{Set}^X \to \mathbf{O}(X) \xrightarrow{i} \mathcal{E}\,. \tag{8}$$

But since $\mathbf{Set}^X$ is itself equivalent to the category of sheaves on the discrete locale $\mathcal{P}X$, the geometric morphism (8) is induced by a surjective locale morphism $\mathcal{P}X \to K$; and, as we noted in Example 2.5, to give this is to give a topological space with set of points X: which is the reflection of X into $\mathbf{Sp}$ described above.

5. The specialisation enrichment

Recall that if X is a space, then its set of points may be preordered by the *specialisation order*, in which $x \leqslant y$ whenever every open set of X that contains x also contains y. The specialisation order induces a preordering on each hom-set $\mathbf{Sp}(X, Y)$ in which $f \leqslant g$ iff $fx \leqslant gx$ for all $x \in X$, and with respect to these preorderings, the composition functions $\mathbf{Sp}(Y, Z) \times \mathbf{Sp}(X, Y) \to \mathbf{Sp}(X, Z)$ become order-preserving maps. Consequently, this structure enriches $\mathbf{Sp}$ to a locally preordered 2-category. We now wish to describe a corresponding enrichment of $\mathbf{Ion}$ to a (no longer locally preordered) 2-category. In order to do so, we first recast the definition of the 2-cells of $\mathbf{Sp}$ in a manner which makes the correct generalisation obvious. Observe that $f \leqslant g : X \to Y$ just when $fx \in U$ implies $gx \in U$ for every $x \in X$ and open $U \subseteq Y$. This is equivalently to ask that $f^{-1}(U) \subseteq g^{-1}(U)$ for every open set $U \subseteq Y$: or in other words, that there should be an inequality

$$\mathcal{O}(Y) \underset{g^*}{\overset{f^*}{\rightrightarrows}} \mathcal{O}(X) \qquad (f^* \leqslant g^*)$$

between the (unique) liftings of f^{-1} and g^{-1} through the corresponding open set lattices. We are therefore led to propose:

Definition 5.1. A *specialisation* between ionad morphisms $f, g : X \to Y$ is a natural transformation

$$\alpha : f^* \Rightarrow g^* : \mathbf{O}(Y) \to \mathbf{O}(X) \tag{9}$$

subject to no further conditions (in particular, this means no compatibility conditions with f^{-1} or g^{-1}). The ionads, continuous maps and specialisations form a 2-category, for which we reuse the notation $\mathbf{Ion}$; similarly, we write $\mathbf{BIon}$ to denote the full and locally full sub-2-category spanned by the bounded ionads.

Remark 5.2. We saw in Remark 4.3 that the assignation $(X, I) \mapsto \mathbf{O}(X)$ yields a functor $\mathbf{O}(\text{–}) : \mathbf{Ion} \to \mathbf{Top}$, with a continuous map (f, f^*) of ionads being sent to the geometric morphism whose inverse image part is f^*. Consequently, the specialisations between continuous maps are in bijection with the geometric transformations between the corresponding geometric morphisms, so that the functor $\mathbf{O}(\text{–})$ extends to a locally fully faithful 2-functor.

Example 5.3. If $F, G : \mathbf{C} \to \mathbf{D}$ are functors between small categories, then the specialisations $A(F) \Rightarrow A(G) : A(\mathbf{C}) \Rightarrow A(\mathbf{D})$ are given by natural transformations $\mathbf{Set}^F \Rightarrow \mathbf{Set}^G : \mathbf{Set}^{\mathbf{D}} \to \mathbf{Set}^{\mathbf{C}}$; and these are in bijection with natural transformations $F \Rightarrow G : \mathbf{C} \to \mathbf{D}$. Thus the ordinary functor $A : \mathbf{Cat} \to \mathbf{BIon}$ extends to a 2-fully faithful 2-functor.

Remark 5.4. Recall from Remark 4.5 that a map of ionads $(X, I) \to (Y, J)$ corresponds to a pair (f, δ) where $f : X \to Y$ and $\delta : f^{-1}J \Rightarrow If^{-1}$ is a natural transformation satisfying two axioms. If (f, δ) and (g, γ) are a parallel pair of ionad morphisms given in this manner, then the specialisations between them are in correspondence with natural transformations $\rho : f^{-1}J \Rightarrow g^{-1}$ making the diagram

$$\begin{array}{ccccc} f^{-1}J & \xrightarrow{f^{-1}\Delta} & f^{-1}JJ & \xrightarrow{\rho J} & g^{-1}J \\ \downarrow{\scriptstyle f^{-1}\Delta} & & & & \downarrow{\scriptstyle \gamma} \\ f^{-1}JJ & \xrightarrow[\delta J]{} & If^{-1}J & \xrightarrow[I\rho]{} & Ig^{-1} \end{array} \tag{10}$$

commute. This is a consequence of [12, Section 2.1]; we summarise the argument as follows. Given a natural transformation $\alpha : f^* \Rightarrow g^* : \mathbf{O}(Y) \to \mathbf{O}(X)$, then the component at A of the corresponding $\rho : f^{-1}J \Rightarrow g^{-1}$ is obtained by first evaluating $U_X.\alpha$ at the cofree coalgebra $\Delta_A : JA \to JJA$—which yields a map $f^{-1}JA \to g^{-1}JA$—and then postcomposing this with $g^{-1}\epsilon_A : g^{-1}JA \to g^{-1}A$. Conversely, if given $\rho : f^{-1}J \Rightarrow g^{-1}$ making (10) commute, then the corresponding $\alpha : f^* \Rightarrow g^*$ has its component at a coalgebra $a : A \to JA$ given by $\rho_A.f^{-1}a : f^*A \to g^*A$.

Example 5.5. Recall from Example 4.6 that if the ionad Y is generated by the basis $M: \mathbf{B} \to \mathbf{Set}^Y$, then ionad morphisms $X \to Y$ are in bijection with pairs (f, f') where $f: X \to Y$ is a function and $f': \mathbf{B} \to \mathbf{O}(X)$ a functor making (6) commute. Suppose now that (f, f') and (g, g') are two ionad morphisms specified in this way. By the preceding Remark, the specialisations between them correspond with natural transformations $\rho: f^{-1}J \Rightarrow g^{-1}$ making (10) commute, where J is the comonad $M \otimes [M, –]$ generated by the basis M. But to give such a ρ is equally well to give a natural transformation $f^{-1}.M \otimes (–) \Rightarrow g^{-1}.M \otimes (–)$ satisfying one axiom; and since both f^{-1} and g^{-1} preserve colimits, such a natural transformation is determined uniquely by a natural transformation $\phi: f^{-1}.M \Rightarrow g^{-1}.M$ satisfying one axiom, which is easily shown to amount to the requirement that ϕ should lift through the forgetful functor $U_X: \mathbf{O}(X) \to \mathbf{Set}^X$. Thus we have shown that to give a specialisation from (f, f') to (g, g') is equally well to give a natural transformation $\alpha: f' \Rightarrow g': \mathbf{B} \to \mathbf{O}(X)$.

We may deduce from this that the category $\mathbf{Ion}(X, Y)$ is small whenever Y is a bounded ionad. Indeed, we know from Example 4.6 that this category has only a set of objects; moreover, the collection of morphisms between any two such objects may be identified with the collection of natural transformations $f' \Rightarrow g'$ for some $f', g': \mathbf{B} \to \mathbf{O}(X)$, and this is a set because $\mathbf{B}$ is small. In particular, we may conclude that $\mathbf{BIon}$ is a locally small 2-category.

Example 5.6. Taking the preceding example together with Example 4.8, we see that if $f, g: X \to Y$ are continuous maps of topological spaces, then the specialisations $\Sigma f \Rightarrow \Sigma g: \Sigma X \to \Sigma Y$ are given by natural transformations

$$\begin{array}{ccccc} & & \mathcal{O}(X) & & \\ & \overset{f^*}{\nearrow} & & \searrow & \\ \mathcal{O}(Y) & & \Downarrow \alpha & & \mathbf{O}(\Sigma X) \\ & \underset{g^*}{\searrow} & & \nearrow & \\ & & \mathcal{O}(X) & & \end{array}$$

as indicated. But as the embedding $\mathcal{O}(X) \rightarrowtail \mathbf{O}(\Sigma X)$ is full and faithful, any such α is induced by a unique natural transformation $f^* \Rightarrow g^*$; and as $\mathcal{O}(X)$ is a poset, there can be at most one such, which exists precisely when $f \leqslant g$ in the 2-category $\mathbf{Sp}$. Thus we have shown that the embedding functor $\Sigma: \mathbf{Sp} \to \mathbf{BIon}$ extends to a 2-fully faithful 2-functor; and much the same argument shows that the left adjoint of Σ extends to a left 2-adjoint.

When at the start of this Section we described the two-dimensional structure of $\mathbf{Sp}$, we did so by constructing it from the specialisation order on each space. By contrast, when we defined the two-dimensional structure of $\mathbf{Ion}$ we did so directly; and this raises the question of what the appropriate ionad-theoretic analogue of the specialisation order should be. In order to answer this, let us observe that in $\mathbf{Sp}$, the specialisation order on a space X is encoded in the two-dimensional structure as the hom-category $\mathbf{Sp}(1, X)$. This immediately suggests the following:

Definition 5.7. The *specialisation functor* $V: \mathbf{BIon} \to \mathbf{Cat}$ is the representable functor $\mathbf{BIon}(1, –)$, and its value at a bounded ionad X is the *specialisation category* of X.

Observe that we are forced to define V only on the bounded ionads, since if X is an unbounded ionad, then there is no reason to expect that $\mathbf{BIon}(1, X)$ should be a small category (though it will always have a mere set of objects).

Remark 5.8. In the topological case, the 2-functor $\mathbf{Sp}(1, –): \mathbf{Sp} \to \mathbf{Poset}$ is right 2-adjoint to the 2-functor $\mathbf{Poset} \to \mathbf{Sp}$ sending a poset to the corresponding Alexandroff space. We shall see in Remark 6.5 below that the corresponding result holds for ionads: the 2-functor $V: \mathbf{BIon} \to \mathbf{Cat}$ of the previous definition is right 2-adjoint to the embedding $A: \mathbf{Cat} \to \mathbf{BIon}$.

Remark 5.9. We may extract the following explicit description of the specialisation category VX of a bounded ionad X. An object of VX is given by a function $1 \to X$, which is a point $x \in X$, together with a lifting x^* making

$$\begin{array}{ccc} \mathbf{O}(X) & \xrightarrow{x^*} & \mathbf{Set} \\ {\scriptstyle U_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbf{Set}^X & \xrightarrow[x^{-1}]{} & \mathbf{Set} \end{array}$$

commute. Obviously, there is exactly one such lifting, namely $x^* = x^{-1}.U_X$, and so we may identify objects of VX with elements of X. Now the morphisms $x \to y$ in VX are the specialisations $(x, x^*) \Rightarrow (y, y^*)$, and these are given by natural transformations $x^{-1}.U_X \Rightarrow y^{-1}.U_X: \mathbf{O}(X) \to \mathbf{Set}$.

A priori this is as much as we can say; however, if we suppose given some basis $M: \mathbf{B} \to \mathbf{Set}^X$ which generates the ionad X then we can simplify this description further. For then, by Remark 5.5, we may identify specialisations $(x, x^*) \Rightarrow (y, y^*)$ with natural transformations $x^{-1}.U_X.\overline{M} \Rightarrow y^{-1}.U_X.\overline{M}$, where $\overline{M}: \mathbf{B} \to \mathbf{O}(X)$ is the coreflection map of Remark 4.7; and since $x^{-1}.U_X.\overline{M} = x^{-1}.M = M(–)(x)$ and likewise $y^{-1}.U_X.\overline{M} = M(–)(y)$, we may identify these in turn with natural transformations $M(–)(x) \Rightarrow M(–)(y): \mathbf{B} \to \mathbf{Set}$. Thus we arrive at the following simple description of VX. Starting from the basis $M: \mathbf{B} \to \mathbf{Set}^X$ we may transpose it to a functor $M': X \to \mathbf{Set}^{\mathbf{B}}$, and VX is now obtained by factorising M' as a functor bijective on objects, followed by one that is fully faithful.

6. Limits and colimits of ionads

In this section, we describe, with sketches of proofs, the limits and colimits that exist in the 2-category of ionads. It turns out that in the category of *all* ionads, rather few of these exist:

Proposition 6.1. *The* 2*-category* **Ion** *has coproducts, tensors by small categories, and a terminal object.*

Proof. The terminal object of **Ion** is given by $(1, \mathrm{id}_{\mathbf{Set}})$. Given a family $(X_k, I_k)_{k \in K}$ of ionads, their coproduct has as its underlying set $\sum_k X_k$ and as its interior comonad the composite

$$\mathbf{Set}^{\sum_k X_k} \xrightarrow{\cong} \prod_k \mathbf{Set}^{X_k} \xrightarrow{\prod_k I_k} \prod_k \mathbf{Set}^{X_k} \xrightarrow{\cong} \mathbf{Set}^{\sum_k X_k} .$$

As regards tensor products, suppose given an ionad X and a small category $\mathbf{C}$. We define the ionad $\mathbf{C} \otimes X$ to have underlying set $\mathrm{ob}\,\mathbf{C} \times X$, and interior comonad generated by the composite adjunction

$$\mathbf{Set}^{\mathrm{ob}\,\mathbf{C} \times X} \cong (\mathbf{Set}^X)^{\mathrm{ob}\,\mathbf{C}} \underset{(\mathrm{cofree})^{\mathrm{ob}\,\mathbf{C}}}{\overset{(\mathrm{forget})^{\mathrm{ob}\,\mathbf{C}}}{\rightleftarrows}} \mathbf{O}(X)^{\mathrm{ob}\,\mathbf{C}} \underset{\mathrm{Ran}_J}{\overset{\mathbf{O}(X)^J}{\rightleftarrows}} \mathbf{O}(X)^{\mathbf{C}} , \tag{11}$$

where $J\colon\ \mathrm{ob}\,\mathbf{C} \to \mathbf{C}$ is the canonical inclusion. Observe that in order for Ran_J to exist here we must know $\mathbf{O}(X)$ to be complete: but being a topos, it is complete if and only if cocomplete, and it is certainly the latter by virtue of being comonadic over $\mathbf{Set}^X$. Since both left adjoint functors in (11) strictly create equalisers, the adjunction they generate is strictly comonadic, so that $\mathbf{O}(\mathbf{C} \otimes X) \cong \mathbf{O}(X)^{\mathbf{C}}$: from which the universal property of the tensor product follows easily. □

However, on restricting to the 2-category of bounded ionads, the situation is much more satisfying:

Theorem 6.2. **BIon** *is cocomplete as a* 2*-category.*

Proof (*Sketch*)**.** It is easy to see that the constructions of coproducts and tensor products in **Ion** restrict to **BIon**. It therefore suffices to prove that **BIon** has coequalisers. Given a parallel pair $f, g\colon X \rightrightarrows Y$, we first form the coequaliser $q\colon Y \to Z$ of the functions between the underlying sets of points, and then the equaliser $E\colon \mathbf{E} \to \mathbf{O}(Y)$ of $f^*, g^*\colon \mathbf{O}(Y) \rightrightarrows \mathbf{O}(X)$ in **CAT**. Observing that q^{-1} is the equaliser of f^{-1} and g^{-1}, we thereby induce a morphism $V\colon \mathbf{E} \to \mathbf{Set}^Z$ fitting into a commutative diagram:

$$\begin{array}{ccccc}
\mathbf{E} & \xrightarrow{E} & \mathbf{O}(Y) & \overset{f^*}{\underset{g^*}{\rightrightarrows}} & \mathbf{O}(X) \\
{\scriptstyle V}\downarrow & & \downarrow{\scriptstyle U_Y} & & \downarrow{\scriptstyle U_X} \\
\mathbf{Set}^Z & \xrightarrow[q^{-1}]{} & \mathbf{Set}^Y & \overset{f^{-1}}{\underset{g^{-1}}{\rightrightarrows}} & \mathbf{Set}^X .
\end{array}$$

We shall show that $\mathbf{E}$ is isomorphic to the category of opens of a bounded ionad structure on the set Z; it is then easy to see that this ionad must be the coequaliser of f and g in **BIon**. The key step in the proof will be to show that $\mathbf{E}$ is an accessible category, which we will do using the fact that the 2-category **ACC** of accessible categories is closed under the formation of inserters and equifiers in **CAT** (see [13, Theorem 5.1.6]). So let $\gamma\colon f^{-1}.J \Rightarrow I.f^{-1}$ and $\delta\colon g^{-1}.J \Rightarrow I.g^{-1}$ be the natural transformations corresponding to the functors f^* and g^*. Now an object of $\mathbf{E}$ consists of a coalgebra $a\colon A \to JA$ in $\mathbf{Set}^Y$ such that the equality

$$f^{-1}A \xrightarrow{f^{-1}a} f^{-1}JA \xrightarrow{\gamma_A} If^{-1}A = g^{-1}A \xrightarrow{g^{-1}a} g^{-1}JA \xrightarrow{\delta_A} Ig^{-1}A$$

holds. In particular, this means that $f^{-1}A = g^{-1}A$, so that to give such an object is equally well to give an object $W \in \mathbf{Set}^Z$ and a coalgebra $a\colon q^{-1}W \to Jq^{-1}W$ such that $\gamma_X.f^{-1}q^{-1}u = \delta_X.g^{-1}q^{-1}u$. Using this explicit description of $\mathbf{E}$, it is easy to give a construction of it from inserters and equifiers in **ACC**: first we form the inserter of the two functors $q^{-1}, Jq^{-1}\colon \mathbf{Set}^Z \rightrightarrows \mathbf{Set}^Y$, and then equify three pairs of 2-cells, imposing the coalgebra axioms and the additional compatibility with γ_X and δ_X. It follows that $\mathbf{E}$ is accessible as claimed.

We may now show by a straightforward diagram chase that, because U_Y and U_X strictly create colimits and finite limits, so too does V. Since $\mathbf{Set}^Z$ has all colimits, it follows that $\mathbf{E}$ does too, and that V preserves them. Hence $\mathbf{E}$ is locally presentable, and by the special adjoint functor theorem, V has a right adjoint G. Moreover, since V creates finite limits, it in particular preserves them, so that the composite VG is the interior comonad of an ionad on Z. Since V strictly creates equalisers, it is strictly comonadic, so that $\mathbf{E}$ is isomorphic to the category of opens of this ionad; and it remains only to show the ionad to be bounded. But since $\mathbf{E}$ is locally presentable, it is in particular a Grothendieck topos, so we are done by Proposition 3.6. □

We have similarly good behaviour with respect to limits:

Theorem 6.3. **BIon** *is complete as a* 2*-category.*

Proof (*Sketch*)**.** We begin by showing that **BIon** is complete as a 1-category. First we prove that the forgetful functor $U\colon \mathbf{BIon} \to \mathbf{Set}$ is a fibration; then we show that all the fibres of U are complete; and then we show that reindexing between those fibres preserves limits. These three conditions together imply that **BIon**, the total category of this fibration, is complete as a 1-category.

To show that $U\colon \mathbf{BIon} \to \mathbf{Set}$ is a fibration, we must, given a bounded ionad (Y, J) and a map of sets $f : X \to Y$, produce a cartesian lift $(f, f^*)\colon (X, I) \to (Y, J)$ in **BIon**. We take $I\colon \mathbf{Set}^X \to \mathbf{Set}^X$ to be the comonad generated by the string of adjunctions

$$\mathbf{Set}^X \underset{\Pi_f}{\overset{f^{-1}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Set}^Y \underset{\text{cofree}}{\overset{\text{forget}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{O}(Y)\,,$$

and take $(f, f^*)\colon (X, I) \to (Y, J)$ to be the map corresponding, under Remark 4.5, to the natural transformation $f^{-1}.J.\eta\colon f^{-1}.J \Rightarrow f^{-1}.J.\Pi_f.f^{-1} = I.f^{-1}$. It is easy to check that this map is cartesian; and so U is a fibration.

Secondly, we show that each of the fibres of U is complete. For a given set X, the fibre category U_X is the opposite of the category of accessible, cartesian comonads on $\mathbf{Set}^X$, and to show this complete, it suffices to show that the category $\mathbf{AC}(\mathbf{Set}^X, \mathbf{Set}^X)$ of accessible, cartesian endofunctors of $\mathbf{Set}^X$ is cocomplete. But this category is isomorphic to $\mathbf{AC}(\mathbf{Set}^X, \mathbf{Set})^X$, so it is enough to show that $\mathbf{AC}(\mathbf{Set}^X, \mathbf{Set})$ is cocomplete; which we do by proving it reflective in the cocomplete $\mathbf{Acc}(\mathbf{Set}^X, \mathbf{Set})$. So given a functor $A\colon \mathbf{Set}^X \to \mathbf{Set}$ which preserves κ-filtered colimits, we let $\mathbf{C}$ denote a skeleton of the full subcategory of $\mathbf{Set}^X$ on the κ-presentable objects; by elementary cardinal arithmetic, $\mathbf{C}$ has finite limits and the inclusion $V\colon \mathbf{C} \to \mathbf{Set}^X$ preserves them. The category $\mathbf{Cart}(\mathbf{C}, \mathbf{Set})$ is reflective in $[\mathbf{C}, \mathbf{Set}]$; let B denote the reflection of $A \circ V$ into it. Now $\mathrm{Lan}_V B$ is clearly accessible, but we claim it is also cartesian: whereupon it easily provides the required reflection of A into $\mathbf{AC}(\mathbf{Set}^X, \mathbf{Set})$. To prove the claim, note that, since V is dense, $\mathrm{Lan}_V B$ is the composite of $B \otimes (-)\colon \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{Set}$ with $[V, -]\colon \mathbf{Set}^X \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$. The former is cartesian because B is, and the latter because it is a right adjoint; so $\mathrm{Lan}_V B$ is cartesian as desired.

Thirdly, we show that for every map of sets $f : X \to Y$, the reindexing functor $U_f\colon U_Y \to U_X$ preserves limits. This is equivalent to showing that U_f^{op} is cocontinuous, for which it is enough to show that

$$f^{-1}.(-).\Pi_f\colon \mathbf{AC}(\mathbf{Set}^Y, \mathbf{Set}^Y) \to \mathbf{AC}(\mathbf{Set}^X, \mathbf{Set}^X)$$

is cocontinuous; but this is immediate from the fact that it has a right adjoint $\Pi_f.(-).f^{-1}$. This completes the proof that **BIon** is complete as a 1-category.

To show that **BIon** is complete as a 2-category, it now suffices to show that it admits cotensors products with the arrow category **2**. If X is the ionad generated by a basis $M\colon \mathbf{B} \to \mathbf{Set}^X$, then the cotensor product $\mathbf{2} \pitchfork X$ will be the ionad whose set of points Z is the set of triples (x, y, α), where $x, y \in X$ and $\alpha\colon M(-)(x) \Rightarrow M(-)(y)$; observe that Z is the set of morphisms of the category VX of Definition 5.7. The topology on this ionad is generated by the following basis $N\colon \mathbf{B}^{\mathbf{2}} \to \mathbf{Set}^Z$. For an object $k\colon c \to d$ of $\mathbf{B}^{\mathbf{2}}$ and element $\alpha\colon M(-)(x) \Rightarrow M(-)(y)$ of Z, the set $N(k)(\alpha)$ is obtained as the pullback

$$\begin{array}{ccc} N(k)(\alpha) & \longrightarrow & M(d)(x) \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \alpha_d} \\ M(c)(y) & \xrightarrow[M(k)(y)]{} & M(d)(y)\,. \end{array} \qquad (12)$$

With some effort, we may check that N is flat; and with considerable further effort, may verify the ionad it induces does indeed possess the universal property required of the cotensor $\mathbf{2} \pitchfork X$. The argument is similar to, but more elaborate than, the one given in the following Remark in regard of finite products; it is closely related to the corresponding topos-theoretic argument, as given in [10, Proposition B4.1.2], for example. □

Remark 6.4. We may describe the product of two bounded ionads more concretely, in terms of a basis generated by open rectangles. More precisely, if the ionads X and Y are generated by bases $M\colon \mathbf{B} \to \mathbf{Set}^X$ and $N\colon \mathbf{C} \to \mathbf{Set}^Y$, then their product is the ionad with set of points $X \times Y$ and topology generated by the basis

$$M \otimes N\colon \mathbf{B} \times \mathbf{C} \to \mathbf{Set}^{X \times Y}$$
$$(M \otimes N)(b, c)(x, y) = M(b)(x) \times N(c)(y)\,.$$

Flatness of $M \otimes N$ follows easily from that of M and N. To show that the ionad it generates is a product of X and Y, we must show that, for any pair of functions $f : Z \to X$, $g : Z \to Y$, we have a bijection between squares of the form

$$\begin{array}{ccc} \mathbf{B} \times \mathbf{C} & \xrightarrow{h'} & \mathbf{O}(Z) \\ {\scriptstyle M\otimes N}\big\downarrow & & \big\downarrow{\scriptstyle U_Z} \\ \mathbf{Set}^{X \times Y} & \xrightarrow[(f,g)^{-1}]{} & \mathbf{Set}^Z \end{array} \qquad (13)$$

and pairs of squares of the form

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{f'} & \mathbf{O}(Z) \\ {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle U_Z} \\ \mathbf{Set}^X & \xrightarrow[f^{-1}]{} & \mathbf{Set}^Z \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{C} & \xrightarrow{g'} & \mathbf{O}(Z) \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle U_Z} \\ \mathbf{Set}^Y & \xrightarrow[g^{-1}]{} & \mathbf{Set}^Z \end{array} . \tag{14}$$

On the one hand, if given f' and g' as in (14), then we define the corresponding h' by $h'(b, c) = f'(b) \times g'(c)$; since U_Z strictly creates finite limits, we may always choose this product in such a way as to make (13) commute. Conversely, if given h' as in (13), we define the corresponding f' and g' by $f'(b) = \int^{c \in \mathbf{C}} h'(b, c)$ and $g'(c) = \int^{b \in \mathbf{B}} h'(b, c)$; again, since U_Z strictly creates colimits, we may choose the colimits in question so as to render the squares in (14) commutative.

Remark 6.5. We may also describe the tensor product of a bounded ionad by a small category in terms of bases: given a basis $M : \mathbf{B} \to \mathbf{Set}^X$ for an ionad X, easy calculation shows that $\mathbf{C} \otimes X$ may be generated by the basis

$$N : \mathbf{C}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{Set}^{\mathrm{ob}\,\mathbf{C} \times X}$$
$$N(c, b)(c', x) = \mathbf{C}(c', c) \times M(b)(x) .$$

Now by comparing this description with Examples 3.5.1 and Remark 6.4, we conclude that for a bounded ionad X, the tensor product $\mathbf{C} \otimes X$ is equally well the product $A(\mathbf{C}) \times X$; in particular, there is a 2-natural isomorphism $A \cong (–) \otimes 1 : \mathbf{Cat} \to \mathbf{BIon}$, and so by virtue of the 2-adjunction

$$(–) \otimes 1 \dashv \mathbf{BIon}(1, –) : \mathbf{BIon} \to \mathbf{Cat}$$

we deduce, as promised in Remark 5.8, that the Alexandroff embedding $A : \mathbf{Cat} \to \mathbf{BIon}$ is left 2-adjoint to the specialisation 2-functor $V : \mathbf{BIon} \to \mathbf{Cat}$.

7. Conclusions

In this final section, we make a few comments on the advantages and disadvantages of the notion of ionad as compared with the notion of topos. The obvious starting point for such a discussion is a consideration of the analogous relationship between the notions of topological space and locale.

One of the major advantages that locales have over spaces is the ease with which their theory may be *relativised*. Maps of locales $X \to Y$ may be identified with internal locales in the sheaf topos $\mathbf{Sh}(Y)$, and so properties of, and constructions on, locales—so long as these are expressed in the logic common to any topos—may without effort be transferred to properties of, and constructions on, maps of locales. For instance, as soon as we know how to form the product of locales, we also know how to form the fibre product over X: it is simply the product of locales internal to $\mathbf{Sh}(X)$. The theory of topological spaces does not relativise in the same way, since many parts of its development makes essential use of classical logic, and so do not internalise well to an arbitrary topos.

It seems likely that this advantage of locales over spaces propagates upwards to a corresponding advantage of toposes over ionads. Certainly, the theory of toposes relativises very satisfactorily: for example, bounded geometric morphisms into a topos $\mathcal{F}$ may be represented by sites internal to that topos; and this means that, for example, constructing the pullback of bounded geometric morphisms is scarcely more problematic than constructing the product of two Grothendieck toposes. Yet it seems unlikely that the theory of ionads relativises in the same manner: so, for instance, we should not expect our concrete description of the product of two bounded ionads to yield a corresponding concrete description of pullbacks of bounded ionads. This, then, is one reason for preferring toposes over ionads.

A second reason is that many toposes of interest do not have a natural expression as an ionad. Most obviously, this could be because the topos we are interested in does not have enough points: which mirrors the corresponding fact that a non-spatial locale will not admit a natural expression as a topological space. More subtly, it could be that the topos we are interested in has *too many* points: namely, a proper class of them. Such a topos, if spatial, will admit any number of different representations as an ionad, but each such representation will require the selection of a mere *set* of separating points: and since maps of ionads are required to preserve these selected sets of points, none of the ionads representing the topos will be able to capture the full range of geometric morphisms into it. This means, amongst other things, that the theory of classifying toposes has no ionad-theoretic analogue. For instance, there can be no bounded ionad Y which "classifies groups" in the sense that ionad morphisms $X \to Y$ correspond with group objects in $\mathbf{O}(X)$. The best we can do is to construct, as in Examples 3.5.5, the ionad Y which "classifies λ-small groups"—in the sense that ionad morphisms $X \to Y$ correspond with group objects $G \in \mathbf{O}(X)$ whose "stalks are λ-small"; in other words, such that $(U_X G)(x)$ is a set of cardinality $< \lambda$ for each $x \in X$. Clearly this is nowhere near as useful a notion, which is something of a pity: the classifying topos of groups really should be considered as "the generalised space of all groups equipped with the Scott topology", and the language of ionads would appear ideal for the expression of this idea. It is conceivable that this problem could be overcome with a sufficiently clever definition of "large ionad"—one endowed with a proper class of points—but whilst there are a few obvious candidates for such a notion, none seems to be wholly satisfactory.

These, then, are two quite general grounds for preferring toposes over ionads; yet there remain good reasons for having the notion of ionad available to us. The first is that some particular applications of topos theory may be more perspicuously expressed in the language of ionads than of toposes: two examples that come to mind are the sheaf-theoretic semantics for first-order modal logic given in [1], and the generalised Stone duality of [5]. The second reason is pedagogical. Many aspects of topos theory are abstractions of corresponding aspects of general topology, but the abstraction is twice removed: first one must pass from spaces to locales, and then from locales to toposes. At the first step, one loses the points, which to many, is already to enter a quite unfamiliar world, and the second step can only compound this unfamiliarity. With the notion of ionad available, one may arrive at these same abstractions by a different route, passing first from spaces to ionads, and then from ionads to toposes. The advantage of doing so is that one retains the tangibility afforded by the presence of points for as long as possible. It seems to me that it is in this pedagogical aspect that ionads are likely to make their most useful contribution.

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