

CARTESIAN CLOSED VARIETIES I: THE CLASSIFICATION THEOREM

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ABSTRACT. In 1990, Johnstone gave a syntactic characterisation of the equational theories whose associated varieties are cartesian closed. Among such theories are all *unary* theories—whose models are sets equipped with an action by a monoid M —and all *hyperaffine* theories—whose models are sets with an action by a Boolean algebra B . We improve on Johnstone’s result by showing that an equational theory is cartesian closed just when its operations have a unique hyperaffine–unary decomposition. It follows that any non-degenerate cartesian closed variety is a variety of sets equipped with compatible actions by a monoid M and a Boolean algebra B ; this is the classification theorem of the title.

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1. INTRODUCTION

In [10], Johnstone considered the following very natural question: when is a variety—by which we mean the category of models of a single-sorted equational algebraic theory—a cartesian closed category? This was, in fact, a follow-up to an earlier question—“when is a variety a topos?”—asked by Johnstone in [9], with the answers in the two cases turning out to be surprisingly similar. The solutions Johnstone provides are *syntactic recognition theorems*, giving necessary and sufficient conditions on the operations of an equational theory for the variety it presents to be cartesian closed or a topos. We recall the cartesian closed result as Theorem 5.2 below, and the reader will readily observe that, while a little delicate, the conditions involved are straightforward enough to be practically useful; and indeed, a very similar set of conditions finds computational application in [13].

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Be this as it may, Johnstone’s conditions do little to help us delineate the scope of the cartesian closed varieties. Much as we can say that every Grothendieck topos is the topos of sheaves on a site, we would like to say that every cartesian closed variety is . . . and filling this gap would amount to providing a *semantic classification theorem* for cartesian closed varieties. This is one of the main objectives of this paper: we will show that every cartesian closed variety is the variety of sets endowed with two actions, one by a monoid M and one by a Boolean algebra B , which interact in a suitable way. Thus, our classification shows that any cartesian closed variety is a kind of “bicrossed product” of the variety of M -sets, which as a presheaf category, is well known to be cartesian closed; and the variety of B -sets, as introduced in [1] and recalled in Section 3, which was shown to be cartesian closed in [10, Example 8.8].

Our semantic classification theorem will be obtained by way of a *syntactic classification theorem* derived from Johnstone’s recognition theorem. To motivate this result, observe first that the cartesian closed varieties of M -sets are precisely those which can be presented by *unary* algebraic theories, that is, theories whose operations and equations are all of arity 1. On the other hand, as shown in [10] and recalled in Section 4, the cartesian closed varieties of B -sets are precisely those which are presented by *hyperaffine* algebraic theories, that is ones whose every operation is hyperaffine (Definition 4.1); here, for (say) a ternary operation f , hyperaffineness asserts that $f(x, x, x) = x$, i.e., f is affine, and moreover that:

$$f(f(x_{11}, x_{12}, x_{13}), f(x_{21}, x_{22}, x_{23}), f(x_{31}, x_{32}, x_{33})) = f(x_{11}, x_{22}, x_{33}) .$$

Our syntactic classification theorem (Theorem 5.5) now states that:

Theorem. *An equational theory presents a cartesian closed variety if, and only if, every operation f has a unique decomposition as a hyperaffine operation h applied to a unary one m , i.e., $f(x_1, x_2, \dots, x_n) = h(m(x_1), m(x_2), \dots, m(x_n))$.*

The proof of this result is simply Johnstone’s recognition theorem together with a little calculation, but we should note that our result does not supplant Johnstone’s, but rather complements it: for while our condition may be simpler to state, it is harder to check if one wants to determine if a given variety is cartesian closed.

Where our formulation comes into its own is in deriving our semantic classification theorem. If \mathbb{T} presents a cartesian closed variety, then the syntactic classification theorem tell us that its operations are completely determined by the monoid of unary operations M together with the subtheory $\mathbb{H} \subseteq \mathbb{T}$ of hyperaffine operations. However, just knowing these is not enough to recover the *substitution* of our equational theory, and so we must also record the manner in which \mathbb{H} and M act on each other by substitution. This leads to what we term a *matched pair of theories* $[\mathbb{H} \mid M]$ (Definition 6.3), and our second main result (Theorem 6.9):

Theorem. *The category of non-degenerate cartesian closed varieties is equivalent to the category of non-degenerate matched pairs of theories.*

Applying the correspondence between hyperaffine algebraic theories \mathbb{H} and theories of B -sets over Boolean algebras B now transforms each matched pair of theories into what we term a *matched pair of algebras* $[B \mid M]$ (Definition 7.1). This involves a Boolean algebra B and a monoid M such that M is a B -set, B is an

M -set, and various further equational axioms hold. In fact, this structure has been studied in the literature: in the nomenclature of [6], we would say that M is a B -monoid. Concomitantly, we have a notion of $[B | M]$ -set (Definition 7.3), which is a set equipped with B -action and M -action in a manner which is compatible with the B -action on M and the M -action on B . In terms of this, we finally obtain our semantic classification result (Theorem 7.10):

Theorem. *The category of non-degenerate cartesian closed varieties is equivalent to the category of non-degenerate matched pairs of algebras via an equivalence which identifies the matched pair $[B | M]$ with the cartesian closed variety of $[B | M]$ -sets.*

There is one point we should clarify about the preceding result. As stated, it is only valid for varieties and equational theories which are *finitary*, i.e., generated by operations of finite arity. However, in the paper proper, it will also be valid in the *infinitary* case; and the adjustments needed to account for this are entirely confined to the Boolean algebra side of things. Indeed, whereas *finitary* hyperaffine theories correspond to Boolean algebras B , arbitrary hyperaffine theories correspond to *strongly zero-dimensional locales*; these are locales (= complete Heyting algebras) in which every cover can be refined to a partition. Two different proofs of this correspondence can be found in [10, Example 8.8] and in [7, §2]; we in fact provide a third proof (Theorem 4.9), but with respect to a slightly different presentation of strongly zero-dimensional locales, inspired by [14]: we consider Boolean algebras B equipped with a collection \mathcal{J} of well-behaved partitions of B , under axioms which make them correspond to strongly zero-dimensional topologies on B . We refer to such a pair (B, \mathcal{J}) as a *Grothendieck Boolean algebra* $B_{\mathcal{J}}$; and we now have notions of Grothendieck matched pair of algebras $[B_{\mathcal{J}} | M]$ and of $[B_{\mathcal{J}} | M]$ -set which, when deployed in the theorem above, make it valid for *infinitary* cartesian closed varieties.

This concludes our overview of the paper, and the reader will notice that we give scarcely any examples. The justification for this is the companion paper [5], which begins a programme to develop the theory of $[B_{\mathcal{J}} | M]$ -sets and link them to structures from operator algebra; in particular, we will see that any matched pair $[B | M]$ has an associated topological category, and that for suitable, and natural, choices of $[B | M]$, we can recover the étale topological groupoids which give rise to structures such as Cuntz C^* -algebras, Leavitt path algebras, and C^* -algebras associated to self-similar groups, and so on.

2. BACKGROUND

2.1. Conventions. Given sets I and J we write J^I for the set of functions from I to J . If $u \in J^I$, we write u_i for the value of the function u at $i \in I$; on the other hand, given a family of elements $(t_i \in J : i \in I)$, we write $\lambda i. t_i$ for the corresponding element of J^I . Given $t \in J^I$, $i \in I$ and $j \in J$, we may write $t[j/t_i]$ for the function which agrees with t except that its value at i is given by j . We may identify a natural number n with the set $\{1, \dots, n\} \subseteq \mathbb{N}$.

A category \mathcal{C} is *concrete* if it comes equipped with a faithful functor U to the category of sets. This U is often an obvious “forgetful” functor, in which case we suppress it from our notation. A *concrete functor* $(\mathcal{C}, U) \rightarrow (\mathcal{C}', U')$ is a functor $H: \mathcal{C} \rightarrow \mathcal{C}'$ with $U'H = U$. Such an H associates to each \mathcal{C} -structure on a set X a corresponding \mathcal{C}' -structure, in such a way that each \mathcal{C} -homomorphism $f: X \rightarrow Y$

is also a homomorphism of the associated \mathcal{C}' -structures. A *concrete isomorphism* is an invertible concrete functor; this amounts to a bijection between \mathcal{C} -structures and \mathcal{C}' -structures on each set X for which the homomorphisms match up.

If \mathcal{C} is a concrete category and X a set, then a *free \mathcal{C} -object on X* is a \mathcal{C} -object $F(X)$ endowed with a function $\eta_X: X \rightarrow F(X)$, the *unit*, such that, for any \mathcal{C} -object Y , each function $f: X \rightarrow Y$ has a unique factorisation through η_X via a \mathcal{C} -homomorphism $f^\dagger: F(X) \rightarrow Y$. We say that *free \mathcal{C} -structures exist* if they exist for every set X ; this is equivalent to the faithful functor U having a left adjoint.

2.2. Varieties and algebraic theories. By a *variety* \mathcal{V} , we mean the concrete category of (possibly empty) models of a (possibly infinitary) single-sorted equational theory. For theories with a mere *set* of function symbols, free \mathcal{V} -structures always exist; we will relax this by allowing a proper *class* of function symbols, but still assuming that free \mathcal{V} -structures exist. So the category of complete join-lattices is a variety in our sense, but not the category of complete Boolean algebras. We write $\mathcal{V}\text{ar}$ for the category of varieties and concrete functors between them.

A variety is *non-degenerate* if it contains a structure with at least two elements. To within concrete isomorphism, there are two degenerate varieties: \mathcal{V}_1 is the full subcategory of Set on the one-element sets, while \mathcal{V}_2 is the full subcategory on the zero- and one-element sets. The former is the category of models of the equational theory with no operation symbols and the axiom $x = y$; while the latter is the category of models of the theory with a single constant c and the axiom $c = x$.

A given variety may be axiomatised by operations and equations in many ways; however, there is always a maximal choice, which is captured by the following notion of *algebraic theory*. This is what a universal algebraist would call an (infinitary) abstract clone, and what a category theorist would call a monad relative to the identity functor $\text{Set} \rightarrow \text{Set}$.

Definition 2.1 (Algebraic theories). An *algebraic theory* \mathbb{T} comprises:

- For each set I , a set $T(I)$ of \mathbb{T} -operations of arity I ;
- For each set I and $i \in I$, an element $\pi_i \in T(I)$ (the *i th projection*);
- For all sets I, J a *substitution* function $T(I) \times T(J)^I \mapsto T(J)$, written as $(t, u) \mapsto t(u)$, or when $I = n$ as $(t, u) \mapsto t(u_1, \dots, u_n)$;

all subject to the axioms:

- $t(\lambda i. \pi_i) = t$ for all $t \in T(I)$;
- $\pi_i(u) = u_i$ for all $u \in T(J)^I$ and $i \in I$;
- $(t(u))(v) = t(\lambda i. u_i(v))$ for all $t \in T(I)$, $u \in T(J)^I$ and $v \in T(K)^J$.

If \mathbb{S} and \mathbb{T} are algebraic theories, then a *homomorphism of algebraic theories* $\varphi: \mathbb{S} \rightarrow \mathbb{T}$ comprises functions $\varphi_I: S(I) \rightarrow T(I)$ for each set I , such that:

- $\varphi_I(\pi_i) = \pi_i$ for all $i \in I$;
- $\varphi_J(t(u)) = \varphi_I(t)(\lambda i. \varphi_J(u_i))$ for all $t \in T(I)$ and $u \in T(J)^I$.

We write $\mathcal{T}\text{hy}$ for the category of algebraic theories and homomomorphisms.

An algebraic theory is said to be *non-degenerate* if $\pi_1 \neq \pi_2 \in T(2)$, or equivalently, if $i \neq j \in I$ implies $\pi_i \neq \pi_j \in T(I)$. To within isomorphism, there are exactly two degenerate algebraic theories: \mathbb{T}_1 , in which $T_1(I) = 1$ for all I ; and \mathbb{T}_2 , in which $T_2(0) = 0$ and $T_2(I) = 1$ otherwise.

When working with an algebraic theory \mathbb{T} we will deploy *variable notation*. For example, in the algebraic theory of semigroups, the defining axiom is expressed by the equality left below in $T(3)$; however, we would prefer to write it as to the right.

$$m(m(\pi_1, \pi_2), \pi_3) = m(\pi_1, m(\pi_2, \pi_3)) \quad m(m(x, y), z) = m(x, m(y, z)) .$$

We may do so if we view this right-hand equality as universally quantified over all sets I and all elements $x, y, z \in T(I)$. It then implies the left-hand equality on taking $(x, y, z) = (\pi_1, \pi_2, \pi_3)$, and conversely, is implied by the left equality via substitution. Our convention throughout will be that any x -, y - or z -symbol (possibly subscripted) appearing in an equality is to be interpreted in this way.

2.3. Semantics and realisation. We now draw the link between algebraic theories and varieties via the *semantics* of an algebraic theory.

Definition 2.2 (Category of models of a theory). A *model* \mathbf{X} for an algebraic theory \mathbb{T} is a set X together with for each set I an *interpretation function* $T(I) \times X^I \rightarrow M$, written as $(t, a) \mapsto \llbracket t \rrbracket(a)$ satisfying the following axioms:

- $\llbracket \pi_i \rrbracket(x) = x_i$ for all $x \in X^I$ and $i \in I$;
- $\llbracket t(u) \rrbracket(x) = \llbracket t \rrbracket(\lambda i. \llbracket u_i \rrbracket(x))$ for all $t \in T(I)$, $u \in T(J)^I$ and $x \in A^J$.

A \mathbb{T} -*model homomorphism* $\mathbf{X} \rightarrow \mathbf{Y}$ is a function $f: X \rightarrow Y$ with $f(\llbracket t \rrbracket_{\mathbf{X}}(x)) = \llbracket t \rrbracket_{\mathbf{Y}}(f(x))$ for all $t \in T(I)$ and $a \in X^I$. We write $\mathbb{T}\text{-Mod}$ for the concrete category of \mathbb{T} -models and homomorphisms.

The category $\mathbb{T}\text{-Mod}$ can be presented as the models of an equational first-order theory, whose proper class of function-symbols is given by the disjoint union of the $T(I)$'s. Moreover, free \mathbb{T} -models exist; for indeed, given a set X , the set $T(X)$ becomes a \mathbb{T} -model $\mathbf{T}(X)$ on defining $\llbracket t \rrbracket_{\mathbf{T}(X)}(u) = t(u)$, and now the map $\eta_X: X \rightarrow T(X)$ sending x to π_x exhibits $\mathbf{T}(X)$ as free on X . Thus, the concrete category $\mathbb{T}\text{-Mod}$ is a variety for any theory \mathbb{T} . In fact, this process is functorial:

Definition 2.3 (Semantics of algebraic theories). For any homomorphism $\varphi: \mathbb{S} \rightarrow \mathbb{T}$ of algebraic theories, we write $\varphi^*: \mathbb{T}\text{-Mod} \rightarrow \mathbb{S}\text{-Mod}$ for the concrete functor which to each \mathbb{T} -model \mathbf{X} associates the \mathbb{S} -model structure $\varphi^* \mathbf{X}$ on X with $\llbracket t \rrbracket_{\varphi^* \mathbf{X}}(x) = \llbracket \varphi(t) \rrbracket_{\mathbf{X}}(x)$. We write $(-)\text{-Mod}: \mathcal{T}\text{hy}^{\text{op}} \rightarrow \mathcal{V}\text{ar}$ for the functor sending each algebraic theory to its concrete category of models, and each homomorphism φ to φ^* .

A basic result in the functorial semantics of algebraic theories is that $(-)\text{-Mod}$ is an equivalence of categories. In particular, it is essentially surjective, which is to say that every variety is realised by some algebraic theory; here, we say that an algebraic theory \mathbb{T} *realises* a variety \mathcal{V} if $\mathbb{T}\text{-Mod}$ and \mathcal{V} are concretely isomorphic. For example, the degenerate varieties $\mathcal{V}_1, \mathcal{V}_2$ are realised by the degenerate algebraic theories $\mathbb{T}_1, \mathbb{T}_2$. In general, we can find a \mathbb{T} which realises a variety \mathcal{V} using free objects in \mathcal{V} . Writing $\mathbf{T}(I)$ for the free \mathcal{V} -object on X , with unit $\eta_I: I \rightarrow T(I)$, the desired theory \mathbb{T} has sets of operations $T(I)$; projection elements $\pi_i = \eta_X(i) \in T(I)$; and substitution given by $t(u) = u^\dagger(t)$.

2.4. Cartesian closed varieties. Any variety \mathcal{V} has finite products, with the product of $\mathbf{X}, \mathbf{Y} \in \mathcal{V}$ being $X \times Y$ with the componentwise \mathcal{V} -structure. We say \mathcal{V} is *cartesian closed* if for every $\mathbf{Y} \in \mathcal{V}$, the functor $(-) \times \mathbf{Y}: \mathcal{V} \rightarrow \mathcal{V}$ has a right adjoint. More elementarily, this means that for every \mathbf{Y}, \mathbf{Z} in \mathcal{V} , there is a “function-space”

$Z^Y \in \mathcal{V}$ and a homomorphism $\text{ev}: Z^Y \times Y \rightarrow Z$ (“evaluation”), such that for all $f: X \times Y \rightarrow Z$, there is a unique $\bar{f}: X \rightarrow Z^Y$ with $\text{ev} \circ (\bar{f} \times 1) = f$. Note that, in particular, the degenerate varieties \mathcal{V}_1 and \mathcal{V}_2 are cartesian closed, since they are equivalent to the one- and two-element Heyting algebras respectively.

The simplest possible class of non-degenerate cartesian closed varieties are the varieties of M -sets for a monoid M : sets equipped with an associative, unital left M -action. It is well known that the variety of M -sets, being a presheaf category, is cartesian closed; we record the structure here for future reference.

Proposition 2.4. *The variety of M -sets is cartesian closed.*

Proof. For M -sets Y and Z , the function-space Z^Y is the set of M -set maps $\varphi: M \times Y \rightarrow Z$ (where M acts on itself by multiplication) under the action

$$m, f \quad \mapsto \quad m^* f = (\lambda n, y. f(nm, y)) . \quad (2.1)$$

Evaluation $\text{ev}: Z^Y \times Y \rightarrow Z$ is given by $\text{ev}(f, y) = f(1, y)$; and given a homomorphism $f: X \times Y \rightarrow Z$, its transpose $\bar{f}: X \rightarrow Z^Y$ is given by $\bar{f}(x)(m, y) = f(mx, y)$. \square

3. BOOLEAN ALGEBRAS AND B -SETS

3.1. Varieties of B -sets. In this section, we discuss another important class of non-degenerate cartesian closed varieties, namely the varieties of B -sets for a Boolean algebra B , as introduced by Bergman in [1]. In what follows, we write $(\vee, \wedge, 0, 1)$ for the distributive lattice structure of a Boolean algebra B , and $(-)'$ for its negation; we say that B is *non-degenerate* if $0 \neq 1$.

Definition 3.1 (Variety of B -sets). Let B be a non-degenerate Boolean algebra. A B -set is a set X endowed with an action $B \times X \times X \rightarrow X$, written $(b, x, y) \mapsto b(x, y)$, satisfying the axioms

$$\begin{aligned} b(x, x) = x & & b(b(x, y), z) = b(x, z) & & b(x, b(y, z)) = b(x, z) \\ 1(x, y) = x & & b'(x, y) = b(y, x) & & (b \wedge c)(x, y) = b(c(x, y), y) . \end{aligned} \quad (3.1)$$

We write $B\text{-Set}$ for the variety of B -sets.

As explained in [1], the first three axioms make each $b(-, -): X \times X \rightarrow X$ into a *decomposition operation* [12, Definition 4.32], meaning that it induces a direct product decomposition $X \cong X_1 \times X_2$ where X_1 and X_2 are quotients of X by suitable equivalence relations. The first equivalence relation \equiv_b is defined by

$$x \equiv_b y \iff b(x, y) = y ; \quad (3.2)$$

the second dually relates x and y just when $b(x, y) = x$ but, in light of the fifth B -set axiom, can equally be described as $\equiv_{b'}$. In fact, as in [12, Theorem 4.33], we can recover $b(-, -)$ from \equiv_b and $\equiv_{b'}$, since $b(x, y)$ is the unique element of X with

$$b(x, y) \equiv_b x \quad \text{and} \quad b(x, y) \equiv_{b'} y . \quad (3.3)$$

Thus, we can recast the notion of B -set in terms of a set equipped with a suitable family of equivalence relations:

Proposition 3.2. *Let B be a non-degenerate Boolean algebra. Each B -set structure on a set X induces equivalence relations $(\equiv_b : b \in B)$ as in (3.2) which satisfy:*

- (i) *If $x \equiv_b y$ and $c \leq b$ then $x \equiv_c y$;*

- (ii) $x \equiv_1 y$ if and only if $x = y$, and $x \equiv_0 y$ always;
 (iii) If $x \equiv_b y$ and $x \equiv_c y$ then $x \equiv_{b \vee c} y$;
 (iv) For any $x, y \in X$ and $b \in B$, there is $z \in X$ such that $z \equiv_b x$ and $z \equiv_{b'} y$.

Any family of equivalence relations $(\equiv_b : b \in B)$ satisfying (i)–(iv) arises in this way from a unique B -set structure on X whose operations are characterised by (3.3). Furthermore, under this correspondence, a function $X \rightarrow Y$ between B -sets is a homomorphism if and only if it preserves each equivalence relation \equiv_b .

Proof. Given B -set structure on X , each \equiv_b as in (3.2) is an equivalence relation by [12, Lemma on p.162]. To verify (i), if $x \equiv_b y$ and $c \leq b$, then $c(x, y) = (c \wedge b)(x, y) = c(b(x, y), y) = c(y, y) = y$, so $x \equiv_c y$. Next, (ii) follows immediately from $1(x, y) = x$ and $0(x, y) = y$. For (iii), if $b(x, y) = y$ and $c(x, y) = y$, then $(b \vee c)(x, y) = b(x, c(x, y)) = b(x, y) = y$. Finally, for (iv), we take $z = b(x, y)$; then $b(z, x) = b(b(x, y), x) = b(x, x) = x$ and $b'(z, y) = b(y, z) = b(y, b(x, y)) = b(y, y) = y$ as desired. We argued above that we can reconstruct the B -set operations from the \equiv_b 's, so this gives an injective map from B -set structures on X to families of equivalence relations satisfying (i)–(iv).

To show surjectivity, consider a family $(\equiv_b : b \in B)$ satisfying (i)–(iv). For any $x, y \in X$ and $b \in B$, the element whose existence is asserted by (iv) is, by (ii) and (iii), *unique*. If we write it as $b(x, y)$ as in (3.3), then we claim this assignment endows X with B -set structure. Indeed:

- Since $x \equiv_b x$ and $x \equiv_{b'} x$, we have $b(x, x) = x$;
- Since $b(b(x, y), z) \equiv_b b(x, y) \equiv_b x$ and $b(b(x, y), z) \equiv_{b'} z$, we have $b(b(x, y), z) = b(x, z)$, and likewise we have $b(x, b(y, z)) = b(x, z)$;
- Since x is the *only* z with $x \equiv_1 z$, we have $1(x, y) = x$;
- Since $b(x, y) \equiv_{b'} y$ and $b(x, y) \equiv_b y$ we have $b'(y, x) = b(x, y)$;
- By (i) we have $b(c(x, y), y) \equiv_{b \wedge c} c(x, y) \equiv_{b \wedge c} x$. Similarly $b(c(x, y), y) \equiv_{b \wedge c'} c(x, y) \equiv_{b \wedge c'} y$, and also $b(c(x, y), y) \equiv_{b'} y$, whence by (iii), $b(c(x, y), y) \equiv_{(b \wedge c)'} y$. Thus $b(c(x, y), y) = (b \wedge c)(x, y)$.

Moreover, this B -set structure induces the given equivalence relations \equiv_b ; indeed, since $b(x, y) \equiv_{b'} y$ and $b(x, y) \equiv_b x$ we have by (i)–(iii) that $b(x, y) = y$ if and only if $x \equiv_b y$. Finally, any B -set homomorphism $f : X \rightarrow Y$ clearly preserves each \equiv_b ; conversely, if f preserves each \equiv_b , then from (3.3) in X we have $f(b(x, y)) \equiv_b f(x)$ and $f(b(x, y)) \equiv_{b'} f(y)$, and so $f(b(x, y)) = b(f(x), f(y))$ by (3.3) in Y . \square

Remark 3.3. Conditions (i)–(iii) above say that, for any elements x, y of a B -set X , the set $\llbracket x=y \rrbracket = \{b \in B : x \equiv_b y\}$ is an *ideal* of the Boolean algebra B ; and since each \equiv_b is an equivalence relation, the function $\llbracket = \rrbracket : X \times X \rightarrow \text{Idl}(B)$ is an $\text{Idl}(B)$ -valued *equivalence relation*, in the sense that $\llbracket x=x \rrbracket = 1$, $\llbracket x=y \rrbracket = \llbracket y=x \rrbracket$ and $\llbracket x=y \rrbracket \wedge \llbracket y=z \rrbracket \leq \llbracket x=z \rrbracket$. So X becomes an $\text{Idl}(B)$ -valued *set* in the sense of [4]—but one of a rather special kind, since in a general $\text{Idl}(B)$ -valued set the equality $\llbracket = \rrbracket$ need only be a *partial* equivalence relation. As explained in [4], $\text{Idl}(B)$ -valued sets are a way of presenting sheaves on B , and so the preceding observations draw the link between B -sets and sheaves that was central to [1]. In this context, the totality of our $\llbracket = \rrbracket$ reflects the fact that the elements of a B -set X correspond to *total* elements of the corresponding sheaf.

By exploiting Proposition 3.2 we can now prove easily that:

Proposition 3.4. *The variety of B -sets is cartesian closed.*

Proof. Given B -sets Y and Z , we consider the set Z^Y of B -set homomorphisms $Y \rightarrow Z$. We claim this is a B -set under the pointwise equivalence relations \equiv_b . Only axiom (iv) is non-trivial. So suppose $f, g \in Z^Y$ and $b \in B$. For each $y \in Y$, we have $h(y) \in Z$ such that $h(y) \equiv_b f(y)$ and $h(y) \equiv_{b'} g(y)$, and so $h: Y \rightarrow Z$ will satisfy $h \equiv_b f$ and $h \equiv_{b'} g$ so long as it is in fact a homomorphism. So suppose that $y_1 \equiv_c y_2$ in Y ; we must show $f(y_1) \equiv_c f(y_2)$. Since $h(y_i) \equiv_b f(y_i)$ and $f(y_1) \equiv_c f(y_2)$ (as f is a homomorphism) we have by (i) that $h(y_1) \equiv_{b \wedge c} f(y_1) \equiv_{b \wedge c} f(y_2) \equiv_{b \wedge c} h(y_2)$; and similarly $h(y_1) \equiv_{b' \wedge c} h(y_2)$. Thus $h(y_1) \equiv_c h(y_2)$ by (iii) and so h is a homomorphism as desired. So Z^Y is a B -set under the pointwise structure; whereupon it is clear that the usual evaluation map $\text{ev}: Z^Y \times Y \rightarrow Z$ is a homomorphism, and that for any homomorphism $f: X \times Y \rightarrow Z$, the usual transpose $\bar{f}: X \rightarrow Z^Y$ is a homomorphism: so Z^Y is a function-space as desired. \square

3.2. Varieties of $B_{\mathcal{J}}$ -sets. If n is a finite set, then as in [10, Proposition 4.3], a $\mathcal{P}(n)$ -set structure on a set X determines and is determined by equivalence relations $\equiv_{\{1\}}, \dots, \equiv_{\{n\}}$ on X , for which the quotient maps exhibit X as the product of the sets $X/\equiv_{\{i\}}$. Thus the category of $\mathcal{P}(n)$ -sets is equivalent to the category of n -fold cartesian products of sets. However, this does not carry over to infinite sets I , for which a $\mathcal{P}(I)$ -set is more general than an I -fold cartesian product of sets. The reason is that the notion of $\mathcal{P}(I)$ -set does not pay regard to the *infinite* joins needed to construct each $A \subseteq I$ from atoms $\{i\}$. This can be rectified by equipping $\mathcal{P}(I)$ with a suitable collection of “well-behaved” joins.

Definition 3.5 (Partition). Let B be a Boolean algebra and $b \in B$. A *partition* of b is a subset $P \subseteq B \setminus \{0\}$ such that $\bigvee P = b$, and $c \wedge d = 0$ whenever $c \neq d \in P$. An *extended partition* of b is a subset $P \subseteq B$ (possibly containing 0) satisfying the same conditions. If P is an extended partition of b , then we write $P^- = P \setminus \{0\}$ for the corresponding partition. We say merely “partition” to mean “partition of 1”.

Definition 3.6 (Zero-dimensional topology, Grothendieck Boolean algebra). A *zero-dimensional topology* on a Boolean algebra B is a collection \mathcal{J} of partitions of B which contains every finite partition, and satisfies:

- (i) If $P \in \mathcal{J}$, and $Q_b \in \mathcal{J}$ for each $b \in P$, then $P(Q) = \{b \wedge c : b \in P, c \in Q_b\}^- \in \mathcal{J}$;
- (ii) If $P \in \mathcal{J}$ and $\alpha: P \rightarrow I$ is a surjective map, then each join $\bigvee \alpha^{-1}(i)$ exists and $\alpha!(P) = \{\bigvee \alpha^{-1}(i) : i \in I\} \in \mathcal{J}$.

A *Grothendieck Boolean algebra* $B_{\mathcal{J}}$ is a Boolean algebra B with a zero-dimensional topology \mathcal{J} . A *homomorphism* of Grothendieck Boolean algebras $f: B_{\mathcal{J}} \rightarrow C_{\mathcal{K}}$ is a Boolean homomorphism $f: B \rightarrow C$ such that $P \in \mathcal{J}$ implies $f(P)^- \in \mathcal{K}$.

A zero-dimensional topology on B is a special kind of Grothendieck topology on B in the sense of [8, §II.2.11], wherein the covers of $1 \in B$ are the elements of \mathcal{J} , and the covers of an arbitrary $b \in B$ are given by:

Definition 3.7 (Local partitions). Let $B_{\mathcal{J}}$ be a Grothendieck Boolean algebra and $b \in B$. We write \mathcal{J}_b for the set of partitions of b characterised by:

$$P \in \mathcal{J}_b \iff P \cup \{b'\} \in \mathcal{J} \iff P \subseteq Q \in \mathcal{J} \text{ and } \bigvee P = b.$$

However, our presentation follows not [8] but rather [14]—according to which, our Grothendieck Boolean algebras are the “subcomplete, locally refinable Boolean partition algebras”. Via the general theory of [8, §II.2.11], any Grothendieck Boolean algebra generates a *locale* (= complete Heyting algebra) given by the set $\text{Idl}_{\mathcal{J}}(B)$ of ideals $I \subseteq B$ which are *\mathcal{J} -closed*, meaning that $b \in I$ as soon as $P \subseteq I$ for some $P \in \mathcal{J}_b$. The locales so arising are the *strongly zero-dimensional locales* considered in [7], and in fact, our category of Grothendieck Boolean algebras is dually equivalent to the category of strongly zero-dimensional locales [15, Theorem 24].

Definition 3.8 (Variety of $B_{\mathcal{J}}$ -sets). Let $B_{\mathcal{J}}$ be a non-degenerate Grothendieck Boolean algebra. A $B_{\mathcal{J}}$ -set is a B -set X endowed with a function $P: X^P \rightarrow X$ for each infinite $P \in \mathcal{J}$, satisfying:

$$P(\lambda b. x) = x \quad P(\lambda b. b(x_b, y_b)) = P(\lambda b. x_b) \quad b(P(x), x_b) = x_b \quad \forall b \in P. \quad (3.4)$$

We write $B_{\mathcal{J}}\text{-Set}$ for the variety of $B_{\mathcal{J}}$ -sets.

Note that any non-degenerate Boolean algebra B has a least zero-dimensional topology given by the collection of all finite partitions of B . In this case, $B_{\mathcal{J}}$ -sets are just B -sets, so that Definition 3.8 includes Definition 3.1 as a special case.

While the existence of functions like P above looks like extra structure on a B -set, it is in fact a property, rather like the existence of inverses in a monoid:

Proposition 3.9. *Let B be a non-degenerate Boolean algebra and P a partition of B .*

- (i) *An operation P on a B -set X satisfying the axioms (3.4) is unique if it exists.*
- (ii) *If X and Y are B -sets admitting the operation P , then any homomorphism of B -sets $f: X \rightarrow Y$ will preserve it.*

Proof. For (i), suppose P and P' both satisfy the axioms of (3.4). For any $x \in X^P$ we have $P(x) = P(\lambda b. b(P'(x), x_b)) = P(\lambda b. P'(x)) = P'(x)$ and so $P = P'$. For (ii), let $x \in X^P$ again; since $b(P(x), x_b) = x_b$ and f preserves b , we have $b(f(P(x)), f(x_b)) = f(x_b)$, and so

$$P(\lambda b. f(x_b)) = P(\lambda b. b(f(P(x)), f(x_b))) = P(\lambda b. f(P(x))) = f(P(x)). \quad \square$$

It follows that $B_{\mathcal{J}}\text{-Set}$ is a full subcategory of $B\text{-Set}$. We can also characterise this subcategory in terms of the induced equivalence relations of Proposition 3.2.

Proposition 3.10. *Let $B_{\mathcal{J}}$ be a non-degenerate Grothendieck Boolean algebra. A B -set X is a $B_{\mathcal{J}}$ -set if, and only if, for each infinite $P \in \mathcal{J}$ and $x \in X^P$, there is a unique element $z \in X$ with $z \equiv_b x_b$ for all $b \in P$.*

Proof. First, suppose X is a $B_{\mathcal{J}}$ -set. Given $P \in \mathcal{J}$ infinite and $x \in X^P$, we define $z = P(x)$; we now have $z \equiv_b x_b$ by the right-hand axiom in (3.4), and if $z' \equiv_b x_b$ for each $b \in P$, then $z = P(x) = P(\lambda b. b(x_b, z')) = P(\lambda b. z') = z'$ by the other two axioms, so that z is *unique* with the desired property.

Suppose conversely that the stated condition holds; we endow X with $B_{\mathcal{J}}$ -set structure. Given $P \in \mathcal{J}$ infinite and $x \in X^P$, we define $P(x)$ as the unique element such that $P(x) \equiv_b x_b$ for all $b \in P$. Since $y \equiv_b z$ just when $b(y, z) = z$ we have $b(P(x), x_b) = x_b$ for all $b \in P$; we also have $P(\lambda b. x) = x$ as $x \equiv_b x$ for each x . Finally, for the second axiom in (3.4), given $b \in P$ we have $P(\lambda b. b(x_b, y_b)) \equiv_b b(x_b, y_b) \equiv_b x_b \equiv_b P(\lambda b. x_b)$ and so $P(\lambda b. b(x_b, y_b)) = P(\lambda b. x_b)$ by unicity. \square

There is some awkwardness above in the distinction between infinite and non-infinite partitions; but in fact, this can be avoided.

Proposition 3.11. *Let $B_{\mathcal{J}}$ be a non-degenerate Grothendieck Boolean algebra. A family of equivalence relations $(\equiv_b : b \in B)$ on a set X determines $B_{\mathcal{J}}$ -set structure on X if, and only if, it satisfies axiom (i) of Proposition 3.2 together with (ii)' For any $P \in \mathcal{J}$ and $x \in X^P$, there is a unique $z \in X$ with $z \equiv_b x_b$ for all $b \in P$.*

Proof. Suppose first that (i) and (ii)' hold. It suffices to show that axioms (ii)–(iv) of Proposition 3.2 also hold. (ii) and (iv) follow easily on applying (ii)' to the partitions $\{1\}$ and $\{b, b'\}$ respectively. As for (iii), given its hypotheses, let z be unique by (ii)' such that $z \equiv_{b \vee c} x$ and $z \equiv_{b' \wedge c'} y$. Then using (i) and the hypotheses, we have $z \equiv_b x \equiv_b y$ and $z \equiv_{b' \wedge c} x \equiv_{b' \wedge c} y$. So z agrees with y on each part of $\{b, b' \wedge c, b' \wedge c'\}$ and so $z = y$; whence $y = z \equiv_{b \vee c} x$ as desired.

For the converse, we must show that (i)–(iv) of Proposition 3.2 imply (ii)' for any finite partition $P = \{b_1, \dots, b_k\}$. The unicity in (ii)' holds by repeatedly applying axiom (iii) then axiom (i). For existence, the base case $k = 1$ is trivial; so let us assume the result for $k - 1$, and prove it for k . By induction, we find y such that $y \equiv_{b_1 \vee b_2} x_2$ and $y \equiv_{b_i} x_i$ for $i > 2$; now by (iv) we can find z such that $z \equiv_{b_1} x_1$ and $z \equiv_{b'_1} y$. Since $b_i \leq b'_1$ for $i \geq 2$ also $z \equiv_{b_i} y \equiv_{b_i} x_i$ for all $2 \leq i \leq k$, as desired. \square

Again, using the equalities \equiv_b makes it easy to show that $B_{\mathcal{J}}$ -sets are a cartesian closed variety. First we need a preparatory lemma.

Lemma 3.12. *Let X be a $B_{\mathcal{J}}$ -set and let $x, y \in X$.*

- (i) *For any $P \in \mathcal{J}$ and $b \in B$, if $x \equiv_{b \wedge c} y$ for all $c \in P$, then $x \equiv_b y$.*
- (ii) *For any $P \in \mathcal{J}_b$, if $x \equiv_c y$ for all $c \in P$, then $x \equiv_b y$.*

Proof. For (i), first note $b \wedge P = (\{b \wedge c : c \in P\} \cup \{b'\})^- \in \mathcal{J}$ by applying condition (i) for a zero-dimensional topology with $P = \{b, b'\}$, $Q_b = P$ and $Q_{b'} = \{1\}$. Now let $z \in X$ be unique such that $z \equiv_b y$ and $z \equiv_{b'} x$. For each $c \in P$ we have $z \equiv_{b \wedge c} y \equiv_{b \wedge c} x$, and so z agrees with x on each part of the partition $b \wedge P$ in \mathcal{J} . Thus $z = x$ and so $x = z \equiv_b y$ as desired. For (ii), apply (i) with $b = \bigvee P$. \square

Remark 3.13. Note that part (ii) of this Lemma asserts that, for all elements x, y in a $B_{\mathcal{J}}$ -set X , the ideal $\llbracket x=y \rrbracket = \{b \in B : x \equiv_b y\}$ of Remark 3.3 is a \mathcal{J} -closed ideal.

Proposition 3.14. *The variety of $B_{\mathcal{J}}$ -sets is cartesian closed.*

Proof. It suffices to show that for $B_{\mathcal{J}}$ -sets Y and Z , the B -set exponential Z^Y of Proposition 3.14 is itself a $B_{\mathcal{J}}$ -set. So suppose given a partition $P \in \mathcal{J}$ and a family of homomorphisms $f_b : Y \rightarrow Z$ for each $b \in P$, and define $g : Y \rightarrow Z$ by the property that $g(y) \equiv_b f_b(y)$ for each $b \in P$; this g will be unique such that $g \equiv_b f_b$ for each $b \in P$, so long as it is a B -set map. So suppose that $y_1 \equiv_c y_2$; we must show that $g(y_1) \equiv_c g(y_2)$, for which by the preceding lemma it suffices to show that $g(y_1) \equiv_{b \wedge c} g(y_2)$ for all $b \in P$: and this follows exactly as in Proposition 3.4. \square

3.3. Theories of B -sets and $B_{\mathcal{J}}$ -sets. To conclude this section, we describe algebraic theories which realise the varieties of B -sets and $B_{\mathcal{J}}$ -sets.

Definition 3.15. Let $B_{\mathcal{J}}$ be a non-degenerate Grothendieck Boolean algebra. A $B_{\mathcal{J}}$ -valued distribution on a set I is a function $\omega : I \rightarrow B$ whose restriction to

$\text{supp}(\omega) = \{i \in I : \omega(i) \neq 0\}$ is an injection onto a partition in \mathcal{J} . The *theory of $B_{\mathcal{J}}$ -sets* $\mathbb{T}_{B_{\mathcal{J}}}$ has $T_{B_{\mathcal{J}}}(I)$ given by the set of $B_{\mathcal{J}}$ -valued distributions on I ; the projection element $\pi_i \in T_{B_{\mathcal{J}}}(I)$ given by $\pi_i(j) = 1$ if $i = j$ and $\pi_i(j) = 0$ otherwise; and the composition $\omega(\gamma) \in T_{B_{\mathcal{J}}}(J)$ of $\omega \in T_{B_{\mathcal{J}}}(I)$ and $\gamma \in T_{B_{\mathcal{J}}}(J)^I$ given by $\omega(\gamma)(j) = \bigvee_{i \in I} \omega(i) \wedge \gamma_i(j)$, where this join exists using axioms (ii) and (iii) for a zero-dimensional topology.

When \mathcal{J} is the topology of finite partitions, we will write \mathbb{T}_B in place of $\mathbb{T}_{B_{\mathcal{J}}}$ and call it the *theory of B -sets*. In this case, a B -valued distribution is simply an $\omega: I \rightarrow B$ whose support injects onto a *finite* partition of B .

Proposition 3.16. *For any non-degenerate Grothendieck Boolean algebra $B_{\mathcal{J}}$ the theory of $B_{\mathcal{J}}$ -sets realises the variety of $B_{\mathcal{J}}$ -sets. In particular, for any non-degenerate Boolean algebra, the theory of B -sets realises the variety of B -sets.*

Proof. Suppose first that X is a $\mathbb{T}_{B_{\mathcal{J}}}$ -model. For each $b \in B$, we have the element $\omega_b \in T_{B_{\mathcal{J}}}(2)$ with $\omega_b(1) = b$ and $\omega_b(2) = b'$, while for each infinite partition $P \in \mathcal{J}$, we have the element $\omega_P \in T_{B_{\mathcal{J}}}(P)$ given by the inclusion map $P \hookrightarrow B$. It is straightforward to verify that these elements satisfy the axioms of (3.1) and (3.4) in $\mathbb{T}_{B_{\mathcal{J}}}$, so their interpretations equip any $\mathbb{T}_{B_{\mathcal{J}}}$ -model with the structure of a $B_{\mathcal{J}}$ -set.

Suppose conversely that X is a $B_{\mathcal{J}}$ -set, and let $\omega \in T_{B_{\mathcal{J}}}(I)$ and $x \in X^I$. Since $(\text{im } \omega)^-$ is a partition in \mathcal{J} , we may use the $B_{\mathcal{J}}$ -set structure of X to define $\llbracket \omega \rrbracket(x)$ as the unique element with $\llbracket \omega \rrbracket(x) \equiv_{\omega(i)} x_i$ for all $i \in \text{supp}(\omega)$. We now check the two $\mathbb{T}_{B_{\mathcal{J}}}$ -model axioms. First, we have $\llbracket \pi_i \rrbracket(x) \equiv_1 x_i$, i.e., $\llbracket \pi_i \rrbracket(x) = x_i$. Second, given $\omega \in T_B(I)$ and $\gamma \in T_B(J)^I$ and $x \in X^J$, we have $\llbracket \omega \rrbracket(\lambda i. \llbracket \gamma_i \rrbracket(x)) \equiv_{\omega(i)} \llbracket \gamma_i \rrbracket(x)$ and $\llbracket \gamma_i \rrbracket(x) \equiv_{\gamma_i(j)} x_j$ for all $i \in I, j \in J$, and so $\llbracket \omega \rrbracket(\lambda i. \llbracket \gamma_i \rrbracket(x)) \equiv_{\omega(i) \wedge \gamma_i(j)} x_j$. Now Lemma 3.12(ii) yields

$$\llbracket \omega \rrbracket(\lambda i. \llbracket \gamma_i \rrbracket(x)) \equiv_{\bigvee_i \omega(i) \wedge \gamma_i(j)} x_j \quad \text{for all } j \in J;$$

but $\llbracket \omega(\gamma) \rrbracket(x)$ is unique with this property, whence $\llbracket \omega \rrbracket(\lambda i. \llbracket \gamma_i \rrbracket(x)) = \llbracket \omega(\gamma) \rrbracket(x)$.

Starting from a $B_{\mathcal{J}}$ -set structure on X , the induced $\mathbb{T}_{B_{\mathcal{J}}}$ -model on X satisfies $\llbracket \omega_b \rrbracket(x, y) \equiv_b x$ and $\llbracket \omega_b \rrbracket(x, y) \equiv_{b'} y$, i.e., $\llbracket \omega_b \rrbracket(x, y) = b(x, y)$, and also satisfies $\llbracket \omega_P \rrbracket(x) \equiv_b x_b$ for all $b \in P$, i.e., $\llbracket \omega_P \rrbracket(x) = P(x)$, and so yields the original B -set structure back. Conversely, given a $\mathbb{T}_{B_{\mathcal{J}}}$ -model structure $\llbracket - \rrbracket$, the model structure $\llbracket - \rrbracket'$ induced from the associated $B_{\mathcal{J}}$ -set satisfies $\llbracket \omega \rrbracket'(x) \equiv_{\omega(i)} x_i$, i.e., $\llbracket \omega_{\omega(i)} \rrbracket(\llbracket \omega \rrbracket'(x), x_i) = x_i$ for each i . But by an easy calculation, we have $\omega(\lambda i. \omega_{\omega(i)}(x_i, y_i)) = \omega(\lambda i. x_i)$ in $\mathbb{T}_B(2 \times I)$, and so

$$\llbracket \omega \rrbracket(x) = \llbracket \omega \rrbracket(\lambda i. \llbracket \omega_{\omega(i)} \rrbracket(\llbracket \omega \rrbracket'(x), x_i)) = \llbracket \omega \rrbracket(\lambda i. \llbracket \omega \rrbracket'(x), x_i) = \llbracket \omega \rrbracket'(x) . \quad \square$$

Remark 3.17. We can read off from this proof that the free $B_{\mathcal{J}}$ -set on a set X is given by $T_{B_{\mathcal{J}}}(X)$, endowed with the $B_{\mathcal{J}}$ -set structure in which $\omega \equiv_b \gamma$ just when $b \wedge \omega(x) = b \wedge \gamma(x)$ for all $x \in X$. Given a partition $P \in \mathcal{J}$ and family of elements $\omega \in T_{B_{\mathcal{J}}}(X)^P$, the element $P(\omega) \in T_{B_{\mathcal{J}}}(X)$ is given by $P(\omega)(x) = \bigvee_{b \in P} b \wedge \omega_b(x)$. The function $X \rightarrow T_{B_{\mathcal{J}}}(X)$ exhibiting $T_{B_{\mathcal{J}}}(X)$ as free on X is given by $x \mapsto \pi_x$.

As a special case, the free $B_{\mathcal{J}}$ -set on two generators can be identified with B itself (with generators 0 and 1) under the $B_{\mathcal{J}}$ -set structure of ‘‘conditioned disjunction’’:

$$b(c, d) = (b \wedge c) \vee (b' \wedge d) \quad \text{and} \quad P(\lambda b. c_b) = \bigvee_{b \in P} b \wedge c_b .$$

4. HYPERAFFINE THEORIES

In this section, we describe, following [10, 7], the syntactic characterisation of theories of B_j -sets as (non-degenerate) *hyperaffine* algebraic theories, with the B -sets matching under this correspondence with the *finitary* hyperaffine theories.

As is well known, an algebraic theory is *finitary* if it corresponds to a finitary variety: which is to say that for each $t \in T(I)$, there exist $i_1, \dots, i_n \in I$ and $u \in T(n)$ such that $t(x) = u(x_{i_1}, \dots, x_{i_n})$. The notion of hyperaffine algebraic theory is perhaps slightly less familiar:

Definition 4.1 (Hyperaffine operation, hyperaffine algebraic theory). Let \mathbb{T} be an algebraic theory. We say that $t \in T(I)$ is *affine* if $t(\lambda i. x) = x$ in $T(1)$, and *hyperaffine* if also $t(\lambda i. t(\lambda j. x_{ij})) = t(\lambda i. x_{ii})$ in $T(I \times I)$. We say that \mathbb{T} is *hyperaffine* if each of its operations is so.

Our objective now is to prove:

Proposition 4.2. *A non-degenerate algebraic theory \mathbb{T} is hyperaffine if, and only if, it is isomorphic to \mathbb{T}_{B_j} for some non-degenerate Grothendieck Boolean algebra B_j ; and it is finitary and hyperaffine if, and only if, it is isomorphic to some \mathbb{T}_B .*

For the “if” direction, we need only show that each \mathbb{T}_{B_j} is hyperaffine, and that each \mathbb{T}_B is finitary. For the first claim, since $T_{B_j}(1)$ is a singleton, every operation of \mathbb{T}_{B_j} must be affine. To see that each $\omega \in T_{B_j}(I)$ is hyperaffine, we observe that $\alpha = \omega(\lambda i. \omega(\lambda j. x_{ij}))$ and $\beta = \omega(\lambda i. x_{ii})$ correspond to the elements of $T_B(I \times I)$ given by, respectively,

$$\alpha(i, j) = \omega(i) \wedge \omega(j) \quad \text{and} \quad \beta(i, j) = \begin{cases} \omega(i) & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

and these are equal since ω is an injection of $\text{supp}(\omega)$ onto a partition of B . To show finitariness of each \mathbb{T}_B , note that any $\omega : I \rightarrow B$ in $T_B(I)$ has *finite* support i_1, \dots, i_n ; so we have the element $\gamma \in T(n)$ given by $\gamma(k) = \omega(i_k)$ and see easily that $\omega = \gamma(\pi_{i_1}, \dots, \pi_{i_n})$. So \mathbb{T}_B is finitary.

The “only if” direction of Proposition 4.2 is harder, and we will attack it in stages. We begin by establishing an important property of hyperaffine theories:

Lemma 4.3. *If \mathbb{T} is a hyperaffine algebraic theory, then every pair of operations $t \in T(I)$ and $u \in T(J)$ commute, i.e., we have $t(\lambda i. u(\lambda j. x_{ij})) = u(\lambda j. t(\lambda i. x_{ij}))$.*

Proof. The operation $t(\lambda i. u(\lambda j. x_{ij}))$ is hyperaffine, which says that

$$t(\lambda i. u(\lambda j. x_{ijij})) = t(\lambda i. u(\lambda j. t(\lambda k. u(\lambda \ell. x_{ijk\ell})))) .$$

Now taking $x_{ijk\ell} = y_{jk}$ gives the desired result:

$$t(\lambda i. u(\lambda j. y_{ji})) = t(\lambda i. u(\lambda j. t(\lambda k. u(\lambda \ell. y_{jk})))) = u(\lambda j. t(\lambda k. y_{jk})) . \quad \square$$

Now, we observed in Remark 3.17 that, in the theory of B_j -sets, the Boolean algebra B appears as the free B_j -set on two generators. This indicates how we should reconstruct a Boolean algebra from a hyperaffine theory.

Proposition 4.4. *If \mathbb{T} is non-degenerate hyperaffine, then $T(2)$ underlies a non-degenerate Boolean algebra with $1 = \pi_1$, $0 = \pi_2$, and \wedge , \vee and $(-)'$ determined by*

$$(b \wedge c)(x, y) = b(c(x, y), y) \quad (b \vee c)(x, y) = b(x, c(x, y)) \quad b'(x, y) = b(y, x) .$$

Proof. According to [3, Theorem 1], to give Boolean algebra structure on $T(2)$ is to give constants $1, 0$ and a ternary operation $a, b, c \mapsto a(b, c)$ (thought of as encoding the Boolean operation “if a then b else c ”) satisfying the five axioms:

$$\begin{aligned} a(b, c)(d, e) &= a(b(d, e), c(d, e)) & 0(b, c) &= c & 1(b, c) &= b \\ a(0, a) &= 0 & a(b, 0) &= b(a, 0) ; \end{aligned} \quad (4.1)$$

and in this presentation, the usual Boolean operations \wedge, \vee and $(-)'$ are re-found as $b \wedge c = b(c, 0)$ and $b \vee c = b(b, c)$ and $b' = b(0, 1)$. In the case of $T(2)$, if we take $1 = \pi_1$ and $0 = \pi_2$ and $a(b, c)$ to be the substitution operation in \mathbb{T} , then all but the last-displayed axiom are trivial. For this last axiom, we compute that $a(b(x, y), y) = a(b(x, y), b(y, y)) = b(a(x, y), a(y, y)) = b(a(x, y), y)$ using affineness of b ; commutativity of a and b via Lemma 4.3; and affineness of a . So $T(2)$ is a Boolean algebra structure, with operations \wedge, \vee and $(-)'$ as displayed above; it is non-degenerate by the assumption that $\pi_1 \neq \pi_2 \in T(2)$. \square

We now explain how to endow the Boolean algebra of this proposition with a zero-dimensional topology. In this we follow [7, §2] by first introducing *binary reducts* and proving some important facts about them.

Definition 4.5 (Binary reducts). Let \mathbb{T} be an algebraic theory and $t \in T(I)$. For each subset $U \subseteq I$, we write $t^{(U)} \in T(2)$ for the binary operation with

$$t^{(U)}(x, y) = t\left(\lambda i. \begin{cases} x & \text{if } i \in U; \\ y & \text{otherwise.} \end{cases}\right)$$

When U is a singleton $\{i\}$, we may write $t^{(i)}$ rather than $t^{\{i\}}$.

Lemma 4.6. *Let \mathbb{T} be a non-degenerate hyperaffine algebraic theory, let $B = T(2)$ be the Boolean algebra of Proposition 4.4, and let $t, u \in T(I)$.*

- (i) *If $t^{(i)} = u^{(i)}$ for all $i \in I$ then $t = u$;*
- (ii) *If $i \neq j$ then $t^{(i)} \wedge t^{(j)} = 0$ in B ;*
- (iii) *For all $U \subseteq I$, we have $t^{(U)} = \bigvee_{i \in U} t^{(i)}$ in B .*

Proof. The elements of $B = T(2)$ easily satisfy the axioms of (3.1) in \mathbb{T} and so via their action by substitution endow each $T(J)$ with a B -set structure. We claim that, with respect to this structure, we have for each $h \in T(I)$ that:

$$h(x) \text{ is unique such that } h(x) \equiv_{h^{(i)}} x_i \text{ for all } i \in I. \quad (4.2)$$

To see that h satisfies the displayed condition, fix $i \in I$ and define $w_{jk} = x_k$ if $j = i$ and $w_{jk} = x_i$ otherwise; then we have $h^{(i)}(h(x), x_i) = h^{(i)}(h(\lambda k. x_k), h(\lambda k. x_i)) = h(\lambda j. h(\lambda k. w_{jk})) = h(\lambda j. w_{jj}) = x_i$ as required. To show the unicity, suppose $h^{(i)}(k(x), x_i) = x_i$ for each i , and set $z_{ij} = k(x)$ if $j = i$ and $z_{ij} = x_i$ otherwise; then $h(x) = h(\lambda i. h^{(i)}(k(x), x_i)) = h(\lambda i. h(\lambda j. z_{ij})) = h(\lambda i. z_{ii}) = k(x)$ as desired.

We now prove (i)–(iii). For (i), if $t^{(i)} = u^{(i)}$ for each i , then $u(x) \equiv_{t^{(i)}} x_i$ for each i , whence $u(x) = t(x)$ by unicity in (4.2). For (ii), (4.2) yields $h^{(j)}(x, y) \equiv_{h^{(i)}} y$ for $i \neq j$, i.e., $y = h^{(i)}(h^{(j)}(x, y), y)$ which says $h^{(i)} \wedge h^{(j)} = 0$. Finally for (iii), observe that (4.2) implies that $h(x) = h(y)$ if and only if $x_i \equiv_{h^{(i)}} y_i$ for all $i \in I$. Thus, for any $t \in T(I)$, $U \subseteq I$ and $u \in T(2)$, we have $t^{(U)} \leq u$ in $T(2)$ just when $t^{(U)}(u(x, y), y) = t^{(U)}(x, y)$, just when $u(x, y) \equiv_{t^{(i)}} x$ for all $i \in U$. In particular, $t^{(U)} \leq u$ if and only if $t^{(i)} \leq u$ for all $i \in U$, so $t^{(U)} = \bigvee_{i \in U} t^{(i)}$ as desired. \square

This lemma implies, in particular, that if \mathbb{T} is hyperaffine and $h \in T(I)$, then the set $P = \{h^{(i)} : i \in I\}^-$ is a partition of $B = T(2)$. In this situation, we will say that the operation h realises the partition P .

Proposition 4.7. *Let \mathbb{T} be a non-degenerate hyperaffine theory and $B = T(2)$ the Boolean algebra of Proposition 4.4. The set \mathcal{J} of all partitions realised by operations of \mathbb{T} constitutes a zero-dimensional topology on B . If \mathbb{T} is finitary, then \mathcal{J} is necessarily the topology of finite partitions.*

Proof. We first show that any $P \in \mathcal{J}$ has a *canonical realisation* by a (necessarily unique) $h \in T(P)$ with $h^{(b)} = b$ for all $b \in P$. To this end, let $k \in T(I)$ be any realiser for P , pick an arbitrary element $b \in P$, and define a function $f: I \rightarrow P$ by taking $f(i) = k^{(i)}$ if $k^{(i)} \neq 0$ and $f(i) = b$ otherwise; we now easily see that $h(x) = k(\lambda i. x_{f(i)})$ is the desired canonical realisation for P .

We now verify the axioms for a zero-dimensional topology. First observe that the trivial partition $\{1\}$ is (canonically) realised by $\pi_1 \in T(1)$; and that if all $(n-1)$ -fold partitions are realised, then so is every n -fold partition $\{b_1, \dots, b_n\}$: for indeed, if $h \in T(n-1)$ realises $\{b_1 \vee b_2, \dots, b_n\}$, then $k(x_1, \dots, x_n) = b_1(x_1, h(x_2, \dots, x_n))$ is easily seen to realise $\{b_1, \dots, b_n\}$. So all finite partitions are in \mathcal{J} .

Now for (i), suppose given $P \in \mathcal{J}$ and $Q_b \in \mathcal{J}$ for each $b \in B$. Let $h \in T(P)$ and $k_b \in T(Q_b)$ be their canonical realisers, and consider the term $\ell \in T(\sum_b Q_b)$ with $\ell(x) = h(\lambda b. k_b(\lambda c. x_{bc}))$. Easily we have $\ell^{(b,c)}(x, y) = h^{(b)}(k_b^{(c)}(x, y), y) = b(c(x, y), y) = (b \wedge c)(x, y)$ so that ℓ realises the partition $P(Q)$ as desired. Finally for (ii), let $P \in \mathcal{J}$ with canonical realiser $h \in T(P)$, and let $\alpha: P \rightarrow I$ be a surjection. Let $k(x) = h(\lambda i. x_{\alpha(i)})$ in $T(I)$; then by Lemma 4.6(iii), we have $k^{(i)} = h^{(\alpha^{-1}(i))} = \bigvee_{b \in \alpha^{-1}(i)} h^{(b)} = \bigvee \alpha^{-1}(i)$, so that k realises the partition $\alpha!(P)$.

Finally, if \mathbb{T} is finitary, then we can write any $h \in T(I)$ as $k(x_{i_1}, \dots, x_{i_k})$ for some finite list $i_1, \dots, i_k \in I$. It follows that $h^{(i)} = 0$ unless $i \in \{i_1, \dots, i_k\}$, so that partitions which \mathbb{T} realises are precisely the finite ones. \square

The following result now completes the proof of Proposition 4.2.

Proposition 4.8. *Let \mathbb{T} be a non-degenerate hyperaffine theory, and $B_{\mathcal{J}}$ the non-degenerate Grothendieck Boolean algebra of Propositions 4.4 and 4.7. The maps*

$$\omega_{(-)}: T(I) \rightarrow T_{B_{\mathcal{J}}}(I) \quad t \mapsto \lambda i. t^{(i)} \quad (4.3)$$

are the components of an isomorphism of algebraic theories $\mathbb{T} \cong \mathbb{T}_{B_{\mathcal{J}}}$. In particular, if \mathbb{T} is finitary, then we have an isomorphism $\mathbb{T} \cong \mathbb{T}_B$.

Proof. By Lemma 4.6 and definition of \mathcal{J} , each $\omega_t: I \rightarrow B$ is injective from its support onto a partition in \mathcal{J} , so that the maps in (4.3) are well-defined.

For any $i \in I$ it is clear that $\omega_{\pi_i} = \pi_i \in T_{B_{\mathcal{J}}}(I)$. As for preservation of composition, let $t \in T(I)$ and $u \in T(J)^I$; we must show $t(u)^{(j)} = \bigvee_{i \in I} t^{(i)} \wedge u_i^{(j)}$ for each $j \in J$. To this end, note that the term $v = t(\lambda i. u_i(\lambda j. x_{ij}))$ satisfies

$$v^{(i,j)}(x, y) = t^{(i)}(u^{(j)}(x, y), y) = (t^{(i)} \wedge u_i^{(j)})(x, y);$$

whence, by Lemma 4.6(iii), $t(u)^{(j)} = v^{(I \times \{j\})} = \bigvee_{i \in I} \omega(i) \wedge \gamma_i(j)$ as desired.

So we have a theory morphism $\omega_{(-)}: \mathbb{T} \rightarrow \mathbb{T}_{B_{\mathcal{J}}}$, whose components are injective by Lemma 4.6(i). It remains to show they are also surjective. To this end, let $\omega: I \rightarrow B$ be a distribution. By assumption, $\omega|_{\text{supp}(\omega)}$ is an injection onto some

$P \in \mathcal{J}$. So let $\iota: P \rightarrow I$ be the injective function sending $\omega(i)$ to i , let $h \in T(P)$ be the canonical realiser of P , and define $t \in T(I)$ to be $t(x) = h(\lambda b. x_{\iota(b)})$. It is now clear that $t^{(i)} = h^{\omega(i)} = \omega(i)$ for all $i \in \text{supp}(\omega)$, and that $t^{(i)} = 0$ for all $i \notin \text{supp}(\omega)$. Thus $\omega_t = \omega$ as desired. \square

We now make the correspondences between non-degenerate Grothendieck Boolean algebras, varieties of $B_{\mathcal{J}}$ -sets, and non-degenerate hyperaffine algebraic theories—and their finitary variants—into functorial equivalences. Let us write:

- $\mathcal{B}\text{Alg}$ (resp., $\text{gr}\mathcal{B}\text{Alg}$) for the category of non-degenerate Boolean (resp., Grothendieck Boolean) algebras and their homomorphisms.
- $\mathcal{H}\text{Aff}$ (resp., $\mathcal{H}\text{Aff}^{\omega}$) for the full subcategory of $\mathcal{T}\text{hy}$ on the non-degenerate hyperaffine (resp., finitary hyperaffine) algebraic theories.
- $B\text{-Var}$ (resp., $B_{\mathcal{J}}\text{-Var}$) for the full subcategory of $\mathcal{V}\text{ar}$ on the varieties isomorphic to some $B\text{-Set}$ (resp. $B_{\mathcal{J}}\text{-Set}$).

The assignments $B_{\mathcal{J}} \mapsto B_{\mathcal{J}}\text{-Set}$ and $B_{\mathcal{J}} \mapsto \mathbb{T}_{B_{\mathcal{J}}}$ can now be made functorial. A homomorphism of non-degenerate Grothendieck Boolean algebras $f: B_{\mathcal{J}} \rightarrow B'_{\mathcal{J}}$ induces, on the one hand, a concrete functor $f^*: B'_{\mathcal{J}}\text{-Set} \rightarrow B_{\mathcal{J}}\text{-Set}$, where f^* assigns to a $B'_{\mathcal{J}}$ -set structure on X the $B_{\mathcal{J}}$ -set structure with $b, x, y \mapsto (fb)(x, y)$; and on the other hand, a theory homomorphism $\mathbb{T}_f: \mathbb{T}_{B_{\mathcal{J}}} \rightarrow \mathbb{T}_{B'_{\mathcal{J}}}$ with components

$$(\mathbb{T}_f)_I: T_{B_{\mathcal{J}}}(I) \rightarrow T_{B'_{\mathcal{J}}}(I) \quad \omega \mapsto f \circ \omega . \quad (4.4)$$

In this way, we obtain functors $\mathbb{T}_{(-)}$ and $(-)\text{-Set}$ as in the statement of:

Theorem 4.9. *We have a triangle of equivalences, commuting to within natural isomorphism, as to the left in:*

$$\begin{array}{ccc} \text{gr}\mathcal{B}\text{Alg} & \xrightarrow{\mathbb{T}_{(-)}} & \mathcal{H}\text{Aff} \\ & \searrow & \swarrow \\ & (-)\text{-Set} & (-)\text{-Mod} \\ & & (B_{\mathcal{J}}\text{-Var})^{\text{op}} \end{array} \quad \begin{array}{ccc} \mathcal{B}\text{Alg} & \xrightarrow{\mathbb{T}_{(-)}} & \mathcal{H}\text{Aff}^{\omega} \\ & \searrow & \swarrow \\ & (-)\text{-Set} & (-)\text{-Mod} \\ & & (B\text{-Var})^{\text{op}} \end{array} \quad (4.5)$$

which restricts back to a triangle of equivalences as to the right.

Proof. By Propositions 3.16 and 4.2, the equivalence $(-)\text{-Mod}: \mathcal{T}\text{hy} \rightarrow \mathcal{V}\text{ar}^{\text{op}}$ restricts to one $\mathcal{H}\text{Aff} \rightarrow (B_{\mathcal{J}}\text{-Var})^{\text{op}}$ and further back to one $\mathcal{H}\text{Aff}^{\omega} \rightarrow (B\text{-Var})^{\text{op}}$. Again because of Proposition 3.16, the triangles commute to within isomorphism. So to complete the proof, it suffices to show that $\mathbb{T}_{(-)}$ is an equivalence both to the left and the right. We already know by Proposition 4.2 that in both cases it is essentially surjective, so we just need to check it is also full and faithful.

As in the proof of Proposition 3.16, given $b \in B$ we write $\omega_b \in T_{B_{\mathcal{J}}}(2)$ for the element with $\omega_b(1) = b$ and $\omega_b(2) = b'$, and given $P \in \mathcal{J}$ we write $\omega_P \in T_{B_{\mathcal{J}}}(P)$ for the element given by the inclusion $P \hookrightarrow B$. Note that every element of $T_B(2)$ is of the form ω_b for a unique $b \in B$; so given a theory homomorphism $\varphi: \mathbb{T}_B \rightarrow \mathbb{T}_C$ there is a unique map $f: B \rightarrow C$ such that $\varphi(\omega_b) = \omega_{f(b)}$ for each $b \in B$. Since $\omega_1 = \pi_1$, $\omega_{b'} = \omega_b(\pi_2, \pi_1)$ and $\omega_{b \wedge c} = \omega_b(\omega_c(\pi_1, \pi_2)\pi_2)$ in $T_B(2)$, and φ preserves these identities, this f is a Boolean homomorphism. Moreover, for each $\omega \in T_B(I)$ and $i \in I$, we have $\omega_{\omega(i)} = \omega^{(i)} \in T_B(2)$, and so $\varphi(\omega)^{(i)} = \varphi(\omega^{(i)}) = \varphi(\omega_{\omega(i)}) = \omega_{f(\omega(i))}$, so that $\varphi(\omega) = \lambda i. f(\omega(i))$. In particular, f must be

a homomorphism of Grothendieck Boolean algebras: for indeed, if $P \in \mathcal{J}$, then $(\text{im } \varphi(\omega_P))^- = f(P)^- \in \mathcal{K}$. Moreover, we have that $\varphi = \mathbb{T}_f$; now if also $\varphi = \mathbb{T}_g$, then $\omega_{f(b)} = \varphi(\omega_b) = \omega_{g(b)}$ for all $b \in B$ and so $f = g$. So $\mathbb{T}_{(-)}$ is full and faithful as claimed, and this completes the proof. \square

5. HYPERAFFINE–UNARY THEORIES

In this section, we prove our first main result, which gives a syntactic characterisation of the algebraic theories which correspond to cartesian closed varieties. This simplifies an existing characterisation due to Johnstone in [10]; as such, we begin by recalling Johnstone’s result, and then use it to deduce ours.

Notation 5.1 (Placed equality, dependency; [10]). Let \mathbb{T} be an algebraic theory, let $q \in T(I)$ and let $i \in I$. Given $t, u \in T(J)$, we write $t \equiv_{q,i} u$ (read as “ t and u are equal in the i th place of q ”) as an abbreviation for the assertion that

$$q(x[t(y)/x_i]) = q(x[u(y)/x_i]) .$$

We say that $q \in T(I)$ *does not depend on* $i \in I$ if $x \equiv_{q,i} y$.

Theorem 5.2 ([10]). *A non-degenerate algebraic theory \mathbb{T} presents a cartesian closed variety if, and only if, the following two conditions hold:*

(i) *For every $p \in T(A)$, there exist $q \in T(B)$, families $u, v \in T(1)^B$, and a function $\alpha: B \rightarrow A$ such that*

$$q(\lambda b. u_b(x)) = x \quad \text{and} \quad u_b(p(\lambda a. x_a)) \equiv_{q,b} v_b(x_{\alpha(b)}) \quad \text{for all } b \in B. \quad (5.1)$$

(ii) *For any $q \in T(B)$, $u \in T(1)^B$ and $\alpha: B \rightarrow A$, if $q(\lambda b. u_b(x_{\alpha(b)})) \in T(A)$ does not depend on i , then q does not depend on any $j \in \alpha^{-1}(i)$.*

Our improved characterisation says that \mathbb{T} presents a cartesian closed variety if, and only if, each operation decomposes uniquely into hyperaffine and unary parts.

Definition 5.3 (Hyperaffine–unary decomposition, hyperaffine–unary theory). Let \mathbb{T} be an algebraic theory. Given a hyperaffine $h \in T(I)$ and a unary $m \in T(1)$, we write $[h \mid m]$ for the operation $h(\lambda i. m) \in T(I)$. A *hyperaffine–unary decomposition* of $t \in T(I)$ is a choice of h and m as above such that $t = [h \mid m]$. We say that the theory \mathbb{T} is *hyperaffine–unary* if every operation $t \in T(I)$ admits a unique hyperaffine–unary decomposition.

The crucial lemma which will enable us to prove this is:

Lemma 5.4. *If the algebraic theory \mathbb{T} satisfies condition (i) of Theorem 5.2, then any affine operation of \mathbb{T} is hyperaffine.*

Proof. Let $p \in T(A)$ be affine, and let q, u, v, α be as in Theorem 5.2(i). Note first that, since p is affine, on substituting x for each x_a in the right-hand equation of (5.1), we have that $u_b \equiv_{q,b} v_b$ for all $b \in B$. To show p is hyperaffine, it suffices, by the left equation of (5.1), to prove that $u_b(p(\lambda a. p(\lambda a'. x_{aa'}))) \equiv_{q,b} u_b(p(\lambda a. x_{aa}))$ for all $b \in B$. But we calculate that

$$\begin{aligned} u_b(p(\lambda a. p(\lambda a'. x_{aa'}))) &\equiv_{q,b} v_b(p(\lambda a'. x_{\alpha(b), a'})) \equiv_{q,b} u_b(p(\lambda a'. x_{\alpha(b), a'})) \\ &\equiv_{q,b} v_b(x_{\alpha(b), \alpha(b)}) \equiv_{q,b} u_b(p(\lambda a. x_{aa})) \end{aligned}$$

using the right equation of (5.1) three times and the fact that $u_b \equiv_{q,b} v_b$ once. \square

Theorem 5.5. *An algebraic theory \mathbb{T} presents a cartesian closed variety if, and only if, it is hyperaffine–unary.*

Proof. Firstly, if \mathbb{T} is degenerate then it is both cartesian closed and hyperaffine, so *a fortiori* hyperaffine–unary. Thus, we may assume henceforth that \mathbb{T} is non-degenerate and so apply Johnstone’s characterisation theorem.

We first prove the **only if** direction. To begin with, we show that each $p \in T(A)$ has some hyperaffine–unary decomposition. To this end, let q, u, v, α be as in Theorem 5.2(i). Let $h \in T(A)$ be given by $h(x) = q(\lambda b. u_b(x_{\alpha(b)}))$. By the left condition of (5.1) h is affine, and so it is hyperaffine by Lemma 5.4. Let $m \in T(1)$ be given by $m(x) = p(\lambda a. x)$. We now calculate that

$$\begin{aligned} p(x) &= q(\lambda b. u_b(p(x))) = q(\lambda b. v_b(x_{\alpha(b)})) = q(\lambda b. u_b(p(\lambda a. x_{\alpha(b)}))) \\ &= q(\lambda b. u_b(m(x_{\alpha(b)}))) = h(\lambda a. m(x_a)) \end{aligned}$$

using in succession, the left equality in (5.1); the right equality twice; the definition of m ; and the definition of h .

We now show this decomposition of p is unique. Suppose we have $h, h' \in T(A)$ hyperaffine and $m, m' \in T(1)$ with $[h \mid m] = [h' \mid m']$. As h and h' are affine, we have $m(x) = h(\lambda a. m(x)) = h'(\lambda a. m'(x)) = m'(x)$. We must show also that $h = h'$. Note that $h(\lambda a. m(x_{aa})) = h(\lambda a. h(\lambda a'. m(x_{aa'}))) = h(\lambda a. h'(\lambda a'. m(x_{aa'})))$, so that the right-hand side does not depend on $x_{aa'}$ whenever $a \neq a'$. Thus by Theorem 5.2(ii), we conclude that $h(\lambda a. h'(\lambda a'. x_{aa'}))$ does not depend on $x_{aa'}$ for any $a \neq a'$. It follows that $h(\lambda a. h'(\lambda a'. x_{aa})) = h(\lambda a. h'(\lambda a'. x_{aa})) = h(\lambda a. x_{aa})$ and so taking $x_{aa'} = y_a$, we conclude that $h'(\lambda a. y_a) = h(\lambda a. h'(\lambda a. y_a)) = h(\lambda a. y_a)$ so that $h = h'$ as desired.

We now prove the **if** direction. Supposing that every operation of \mathbb{T} has a unique hyperaffine–unary decomposition, we prove (i) and (ii) of Theorem 5.2. For (i), given $p \in T(A)$ with decomposition $p = [h \mid m]$, we take $q = h$, $u_a = \text{id}$, $v_a = m$ and $\alpha = \text{id}$ to obtain the required data satisfying (5.1). It remains to verify condition (ii). So let $q \in T(B)$, $u \in T(1)^B$ and $\alpha: B \rightarrow A$ be such that $p \in T(A)$ given by $p(x) = q(\lambda b. u_b(x_{\alpha(b)}))$ does not depend on x_a . Writing $q = [h \mid m]$, we have

$$\begin{aligned} p(x) &= [h \mid m](\lambda b. u_b(x_{\alpha(b)})) = h(\lambda b. m(u_b(x_{\alpha(b)}))) \\ &= h(\lambda b. h(\lambda b'. m(u_{b'}(x_{\alpha(b)})))) = h(\lambda b. n(x_{\alpha(b)})) = k(\lambda a. n(x_a)) \end{aligned}$$

where we define $n(x) = h(\lambda b'. m(u_{b'}(x)))$ and $k(x) = h(\lambda b. x_{\alpha(b)})$. Now consider the hyperaffine operations $k', k'' \in T(A+1)$ with

$$k'(x, y) = k(x) \quad \text{and} \quad k''(x, y) = k(x[y/x_a]) .$$

Because $p = [k \mid n]$ does not depend on x_a , the operations $[k' \mid n]$ and $[k'' \mid n]$ are equal. By unicity of decompositions, we have $k' = k''$ in $T(A+1)$, so that $k(x)$ does not depend on x_a . We claim it follows that $h(y)$ does not depend on y_b for any $b \in \alpha^{-1}(a)$. For indeed, we have that $x = k(\lambda i. x) = k^{(a)}(y, x) = h^{(\alpha^{-1}(a))}(y, x)$ in $T(2)$, since $k(x)$ does not depend on x_a . Since, by hyperaffineness of h , we have $h(x) \equiv_{h,b} x_b$ for each $b \in B$, we conclude that, for any $b \in \alpha^{-1}(a)$, we have $x = h^{(\alpha^{-1}(a))}(y, x) \equiv_{h,b} y$ so that h does not depend on b . \square

The proof of Theorem 5.5 just given provides an intellectually honest account of the genesis of the ideas in this paper; but for good measure, we will also give

a direct proof of the theorem which does not rely on [10]. For now we only show that every cartesian closed variety is hyperaffine-unary; we will close the loop in Section 7 once we have a better handle on what hyperaffine-unary theories are.

It will in fact be clearer if we prove something more general. Recall that the *copower* $I \cdot X$ of some $X \in \mathcal{C}$ by a set I is a coproduct of I copies of X . We write $\iota_i: X \rightarrow I \cdot X$ for the coproduct coprojections, and given maps $(f_i: X \rightarrow Y)_{i \in I}$, write $\langle f_i \rangle: I \cdot X \rightarrow Y$ for the unique map with $\langle f_i \rangle \circ \iota_i = f_i$ for each i .

Definition 5.6 (Complete theory of dual operations). Let \mathcal{C} be a category with all set-indexed copowers and let $X \in \mathcal{C}$. The *complete theory of dual operations of X* \mathbb{T}_X is the algebraic theory with:

- $T_X(I) = \mathcal{C}(X, I \cdot X)$;
- Projection elements $\iota_i \in T(I)$;
- Substitution $\mathcal{C}(X, I \cdot X) \times \mathcal{C}(X, J \cdot X)^I \rightarrow \mathcal{C}(X, J \cdot X)$ given by

$$(t, u) \mapsto X \xrightarrow{t} I \cdot X \xrightarrow{\langle u_i \rangle} J \cdot X .$$

To a universal algebraist, this is the (infinitary) *clone of co-operations* of X [2]; to a category theorist, it is the *structure* of the hom-functor $\mathcal{C}(X, -): \mathcal{C} \rightarrow \text{Set}$ [11]. The following standard result is now [11, Theorem III.2]:

Proposition 5.7. *Let \mathcal{C} be a category with set-indexed copowers and let $X \in \mathcal{C}$. For any $Y \in \mathcal{C}$, the set $\mathcal{C}(X, Y)$ is a model for \mathbb{T}_X with $\llbracket t \rrbracket(f) = \langle f_i \rangle \circ t: X \rightarrow I \cdot X \rightarrow Y$ for each $t \in T(I)$. For any $g: Y \rightarrow Z$, the function $g \circ (-): \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$ is a \mathbb{T}_X -homomorphism, and so we induce a factorisation*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{K} & \mathbb{T}_X\text{-Mod} , \\ \mathcal{C}(X, -) \searrow & & \swarrow U \\ & \text{Set} & \end{array} \quad (5.2)$$

which is universal among factorisations of $\mathcal{C}(X, -)$ through a variety.

When, in the above proposition, we take \mathcal{C} itself to be a variety $\mathbb{T}\text{-Mod}$, and take $X = \mathbf{T}(1)$, the free model on one generator, it is visibly the case that $\mathbb{T}_X \cong \mathbb{T}$ and that K is an *isomorphism*. Thus, the fact that any cartesian closed variety is hyperaffine-unary follows from:

Proposition 5.8. *Let \mathcal{C} be a category with finite products and set-indexed copowers, and suppose that each $(-) \times X: \mathcal{C} \rightarrow \mathcal{C}$ preserves copowers; in particular, this will be so if \mathcal{C} is cartesian closed. For any $X \in \mathcal{C}$, the complete theory of dual operations \mathbb{T}_X is hyperaffine-unary.*

Proof. Since $(-) \times X$ preserves copowers, we may realise the copower $I \cdot X$ as the product $(I \cdot 1) \times X$ via the coprojection maps

$$X \xrightarrow{\cong} 1 \times X \xrightarrow{\iota_i \times X} (I \cdot 1) \times X .$$

Thus, we may write operations in $T(I)$ as pairs $(h, m): X \rightarrow (I \cdot 1) \times X$ where $h: X \rightarrow I \cdot 1$ and $m: X \rightarrow X$. From the definition of substitution in \mathbb{T}_X , such an operation is *affine* just when

$$X \xrightarrow{(h, m)} (I \cdot 1) \times X \xrightarrow{\pi_2} X = X \xrightarrow{\text{id}} X ,$$

i.e., just when $m = \text{id}$. We claim that any such $t = (h, \text{id})$ is in fact *hyperaffine*. This follows from commutativity in:

$$\begin{array}{ccccc}
 & & (I \cdot 1) \times X & & \\
 & \nearrow^{(h, \text{id})} & & \searrow^{(I \cdot 1) \times (h, \text{id})} & \\
 X & \xrightarrow{(\Delta, \text{id})} & X \times X \times X & \xrightarrow{h \times h \times X} & (I \cdot 1) \times (I \cdot 1) \times X \\
 \downarrow^{(h, \text{id})} & & \nearrow^{\Delta \times X} & & \downarrow^{\theta \times X} \\
 (I \cdot 1) \times X & \xrightarrow{(\Delta \cdot 1) \times X} & & \xrightarrow{} & ((I \times I) \cdot 1) \times X
 \end{array}$$

whose uppermost composite is the interpretation of $t(\lambda i. t(\lambda j. x_{ij}))$ and whose lower composite interprets $t(\lambda i. x_{ii})$; here $\theta: (I \cdot 1) \times (I \cdot 1) \rightarrow (I \times I) \cdot 1$ is the canonical isomorphism characterised by $\theta \cdot (\iota_i, \iota_j) = \iota_{(i,j)}$.

So the hyperaffine operations of \mathbb{T}_X are those of the form $(h, \text{id}): X \rightarrow (1 \cdot I) \times X$; and, of course, the unary operations are those of the form $m: X \rightarrow X$. Moreover, if h is hyperaffine and m is unary, then the operation $[h \mid m] = h(\lambda i. m(i))$ is interpreted by the composite

$$X \xrightarrow{(h, \text{id})} (1 \cdot I) \times X \xrightarrow{\text{id} \times m} (1 \cdot I) \times X = X \xrightarrow{(h, m)} (1 \cdot I) \times X .$$

So each operation $t = (h, m) \in T(I)$ has a unique hyperaffine–unary decomposition into the hyperaffine $(h, \text{id}) \in T(I)$ and the unary $m \in T(1)$, as desired. \square

6. MATCHED PAIRS OF THEORIES

As the name suggests, a hyperaffine–unary theory has a hyperaffine part and a unary part. In this section, we show that these two parts, together with their actions on each other, provide an entirely equivalent description of the notion of hyperaffine–unary theory. To begin with, let us record how we extract out the two parts of a hyperaffine–unary theory.

Proposition 6.1. *Let \mathbb{T} be a hyperaffine–unary theory. The hyperaffine operations of \mathbb{T} form a subtheory \mathbb{H} , called the hyperaffine part; while $T(1)$ forms a monoid M under substitution, called the unary part.*

Proof. The only non-trivial point is that hyperaffine operations are closed under substitution. But every affine operation is hyperaffine by Lemma 5.4, and the affine operations in any theory are easily closed under substitution. \square

Clearly, if we know \mathbb{H} and M then we know every operation of \mathbb{T} . However, the monoid structure of M and the substitution structure of \mathbb{H} do *not* determine the substitution structure of \mathbb{T} . For this, we also need to record how \mathbb{H} and M act on each other via substitution in \mathbb{T} .

Proposition 6.2. *Let \mathbb{T} be a hyperaffine–unary theory with hyperaffine and unary parts \mathbb{H} and M . We may determine operations*

$$\begin{array}{ll}
 M \times H(I) \rightarrow H(I) & H(I) \times M^I \rightarrow M \\
 (m, h) \mapsto m^* h & (h, n) \mapsto h \triangleright n
 \end{array}$$

as follows: if $h \in H(I)$ and $m \in M$, then $m^*h \in H(I)$ is unique such that

$$m(h(\lambda i. x_i)) = (m^*h)(\lambda i. m(x_i)) ; \quad (6.1)$$

while if $h \in H(I)$ and $n \in M^I$, then $h \triangleright n \in M$ is unique such that

$$h(\lambda i. n_i(x)) = (h \triangleright n)(x) . \quad (6.2)$$

These operations uniquely determine the substitution structure of \mathbb{T} via the formulae:

- $\pi_i = [\pi_i \mid 1]$ in $T(I)$, where $1 = \pi_1$ is the identity element of M ;
- If $[h \mid m] \in T(I)$ and $[k \mid n] \in T(J)^I$ (i.e., $[k_i \mid n_i] \in T(J)$ for each i), then

$$[h \mid m] ([k \mid n]) = [h(\lambda i. m^*k_i) \mid h \triangleright (\lambda i. mn_i)] \text{ in } T(J) . \quad (6.3)$$

Proof. For the unique existence of m^*h , the composite operation $m(h(\lambda i. x_i)) \in T(I)$ admits a unique hyperaffine–unary decomposition; but since $m(h(\lambda i. x)) = m(x)$, the unary part of this must be m . The corresponding hyperaffine part $m^*h \in H(I)$ is thus the unique operation making (6.1) hold. The unique existence of $h \triangleright n$ is trivial: take it as the substitution $h(n)$ in \mathbb{T} , and then (6.2) follows from associativity of substitution. It remains to prove that these operations determine the projections and substitutions for \mathbb{T} . That $\pi_i = [\pi_i \mid 1]$ is trivial; as for (6.3), we calculate that:

$$\begin{aligned} [h \mid m] ([k \mid n])(x) &= h(\lambda i. m(k_i(\lambda j. n_j(x_j)))) && \text{definition of } [\mid] \\ &= h(\lambda i. (m^*k_i)(\lambda j. m(n_j(x_j)))) && \text{definition of } m^*k_i \\ &= h(\lambda i. h(\lambda i'. (m^*k_i)(\lambda j. m(n_{i'}(x_j))))) && \text{hyperaffineness of } h \\ &= h(\lambda i. (m^*k_i)(\lambda j. h(\lambda i'. m(n_{i'}(x_j))))) && \text{commutativity in } \mathbb{H} \\ &= (h(\lambda i. m^*k_i)(\lambda j. h(\lambda i. mn_i)(x_j))) && \text{associativity in } \mathbb{H} \\ &= [h(\lambda i. m^*k_i) \mid h \triangleright (\lambda i. mn_i)](x) && \text{definition of } [\mid] . \quad \square \end{aligned}$$

So any hyperaffine–unary algebraic theory is determined by its hyperaffine and unary parts, together with the operations of (6.1) and (6.2). However, if given a hyperaffine theory \mathbb{H} and a monoid M , together with operations of the same form, we should *not* expect to obtain a structure of hyperaffine–unary theory on the sets $T(I) = H(I) \times M$: for although the preceding proposition indicates how to define substitution from these operations, it does not ensure that the axioms of a theory are satisfied. For this, we must impose axioms on the operations relating \mathbb{H} and M .

Definition 6.3 (Matched pairs of theories). A *matched pair of theories* $[\mathbb{H} \mid M]$ comprises a hyperaffine theory \mathbb{H} and a monoid M together with operations:

$$\begin{aligned} M \times H(I) &\rightarrow H(I) & H(I) \times M^I &\rightarrow M \\ (m, h) &\mapsto m^*h & (h, n) &\mapsto h \triangleright n \end{aligned} \quad (6.4)$$

satisfying the following axioms:

- (i) For $m \in M$, the maps $m^*(-): H(I) \rightarrow H(I)$ give a homomorphism of algebraic theories $m^*: \mathbb{H} \rightarrow \mathbb{H}$:

$$m^*(\pi_i) = \pi_i \quad (6.5)$$

$$m^*(h(k)) = (m^*h)(\lambda i. m^*k_i) ; \quad (6.6)$$

(ii) The maps in (i) constitute an M -action on \mathbb{H} :

$$1^*h = h \quad (6.7)$$

$$m^*n^*h = (mn)^*(h) ; \quad (6.8)$$

(iii) The operations $\triangleright : H(I) \times M^I \rightarrow M$ make M into a \mathbb{H} -model \mathbf{M} :

$$\pi_i \triangleright m = m_i \quad (6.9)$$

$$h(k) \triangleright (m) = h \triangleright (\lambda i. k_i \triangleright m) ; \quad (6.10)$$

(iv) Right multiplication by $n \in M$ is a \mathbb{H} -model homomorphism $(-)n : \mathbf{M} \rightarrow \mathbf{M}$:

$$(h \triangleright m)n = h \triangleright (\lambda i. m_i n) ; \quad (6.11)$$

(v) Left multiplication by $n \in M$ is a \mathbb{H} -model homomorphism $n(-) : \mathbf{M} \rightarrow n^*\mathbf{M}$, where $n^*\mathbf{M}$ is the \mathbb{H} -model obtained by pulling back \mathbf{M} along the theory homomorphism $n^* : \mathbb{H} \rightarrow \mathbb{H}$:

$$n(h \triangleright m) = n^*h \triangleright (\lambda i. nm_i) ; \quad (6.12)$$

(vi) For $h \in H(I)$, the map $(-)^*h : M \rightarrow H(I)$ is a \mathbb{H} -model homomorphism $\mathbf{M} \rightarrow \mathbf{H}(I)$, where $\mathbf{H}(I)$ is the free \mathbb{H} -model structure on $H(I)$:

$$(k \triangleright m)^*(h) = k(\lambda j. m_j^*h) . \quad (6.13)$$

We call a matched pair of theories *finitary* or *non-degenerate* when \mathbb{H} is so.

A *homomorphism* $[\varphi | f] : [\mathbb{H} | M] \rightarrow [\mathbb{H}' | M']$ of matched pairs of theories is a homomorphism of theories $\varphi : \mathbb{H} \rightarrow \mathbb{H}'$ and a monoid homomorphism $f : M \rightarrow M'$ such that for all $h \in H(I)$, $m \in M$ and $n \in M^I$, we have:

$$\varphi(m^*h) = f(m)^*(\varphi(h)) \quad \text{and} \quad f(h \triangleright n) = \varphi(h) \triangleright (\lambda i. f(n_i)) . \quad (6.14)$$

We now show soundness and completeness of this axiomatisation.

Proposition 6.4. *If \mathbb{T} is a hyperaffine-unary theory, then its hyperaffine and unary parts \mathbb{H} and M constitute a matched pair of theories $\mathbb{T}^\downarrow = [\mathbb{H} | M]$ under the operations of Proposition 6.2; and \mathbb{T}^\downarrow is finitary or non-degenerate just when \mathbb{T} is so.*

Proof. For (6.5)–(6.8), we calculate using (6.1) and the theory axioms of \mathbb{T} and conclude using unicity of hyperaffine-unary decompositions. For (6.5), the calculation is $(m^*\pi_i)(\lambda j. m(x_j)) = m(\pi_i(\lambda j. x_j)) = m(x_i) = \pi_i(\lambda j. m(x_j))$. For (6.6):

$$\begin{aligned} (m^*(t(u)))(\lambda j. m(x_j)) &= m(t(\lambda i. u_i(x))) = (m^*t)(\lambda i. m(u_i(x))) \\ &= (m^*t)(\lambda i. (m^*u_i)(\lambda j. m(x_j))) \\ &= ((m^*t)(\lambda i. m^*u_i))(\lambda j. m(x_j)) . \end{aligned}$$

For (6.7), we have $(1^*h)(x) = (1^*h)(\lambda i. 1(x_i)) = 1(h(x)) = h(x)$, and for (6.8),

$$\begin{aligned} (m^*n^*h)(\lambda i. m(n(x_i))) &= m((n^*h)(\lambda i. n(x_i))) = m(n(h(x))) \\ &= (mn)(h(x)) = ((mn)^*h)(\lambda i. m(n(x_i))) . \end{aligned}$$

Next, (6.9) and (6.10) follow directly from the theory axioms for \mathbb{T} . Finally, for (6.11)–(6.13), we calculate using (6.1) and (6.2) and conclude using unicity of decompositions. For (6.11) the calculation is that $(h \triangleright m)(n)(x) = h(\lambda i. m_i(n(x))) = (h \triangleright (\lambda i. m_i n))(x)$. For (6.12) we have:

$$(n(h \triangleright m))(x) = n(h(\lambda i. m_i(x))) = (n^*h)(\lambda i. n(m_i(x))) = (n^*h \triangleright (\lambda i. nm_i))(x) ;$$

and finally, for (6.13) we have:

$$\begin{aligned}
((k \triangleright m)^* h)(\lambda i. (k \triangleright m)(x_i)) &= (k \triangleright m)(h(x)) = k(\lambda j. m_j(h(x))) \\
&= k(\lambda j. (m_j^* h)(\lambda i. m_j(x_i))) \\
&= k(\lambda j. k(\lambda j'. (m_j^* h)(\lambda i. m_{j'}(x_i)))) \\
&= k(\lambda j. (m_j^* h)(\lambda i. k(\lambda j'. m_{j'}(x_i)))) \\
&= (k(\lambda j. (m_j^* h)))(\lambda i. (k \triangleright m)(x_i)) . \quad \square
\end{aligned}$$

Proposition 6.5. *For any matched pair of theories $[\mathbb{H} \mid M]$, there is a hyperaffine-
unary theory $\mathbb{H} \bowtie M$, the bicrossed product of \mathbb{H} and M , with $(\mathbb{H} \bowtie M)^\downarrow \cong [\mathbb{H} \mid M]$.*

Proof. For each set I , we take $(\mathbb{H} \bowtie M)(I) = H(I) \times M$, and write a typical element like before as $[h \mid m]$. We define projection elements $\pi_i = [\pi_i \mid 1]$ and substitution operations by the formula (6.3), and claim that, upon doing so, we obtain an algebraic theory $\mathbb{H} \bowtie M$. For this, we must check the three theory axioms. Firstly: $[h \mid m](\lambda i. [\pi_i \mid 1]) = [h(\lambda i. m^* \pi_i) \mid h \triangleright (\lambda i. m1)] = [h(\lambda i. \pi_i) \mid h \triangleright (\lambda i. m)] = [h \mid m]$ by the definition, axiom (6.5), and axiom (6.10). Secondly,

$$[\pi_i \mid 1]([k \mid n]) = [\pi_i(\lambda j. 1^* k_j) \mid \pi_i \triangleright (\lambda j. 1n_j)] = [1^* k_i \mid 1n_i] = [k_i \mid n_i] .$$

by the definition, axiom (6.9) and axiom (6.7). Finally, for associativity of substitution, we first compute that $([h \mid m]([k \mid n]))([\ell \mid p])$ is given by

$$\begin{aligned}
&[h(\lambda i. m^* k_i) \mid h \triangleright (\lambda i. mn_i)]([\ell \mid p]) \\
&= [(h(\lambda i. m^* k_i))(\lambda j. (h \triangleright (\lambda i. mn_i))^* \ell_j) \mid h(\lambda i. m^* k_i) \triangleright (\lambda j. (h \triangleright (\lambda i. mn_i))p_j)]
\end{aligned}$$

while $[h \mid m](\lambda i. [k_i \mid n_i]([\ell \mid p]))$ is given by

$$\begin{aligned}
&[h \mid m](\lambda i. [k_i(\lambda j. n_i^* \ell_j) \mid k_i \triangleright (\lambda j. n_i p_j)]) \\
&= [h(\lambda i. m^*(k_i(\lambda j. n_i^* \ell_j))) \mid h \triangleright (\lambda i. m(k_i \triangleright (\lambda j. n_i p_j)))] .
\end{aligned}$$

Comparing first terms we have:

$$\begin{aligned}
&(h(\lambda i. m^* k_i))(\lambda j. h \triangleright (\lambda i. mn_i)^* \ell_j) \\
&= h(\lambda i. m^* k_i(\lambda j. (h \triangleright (\lambda i'. mn_{i'}))^* \ell_j)) && \text{associativity in } \mathbb{H} \\
&= h(\lambda i. m^* k_i(\lambda j. h(\lambda i'. (mn_{i'}^* \ell_j)))) && (6.13) \\
&= h(\lambda i. h(\lambda i'. m^* k_i(\lambda j. (mn_{i'}^* \ell_j)))) && \text{commutativity in } \mathbb{H} \\
&= h(\lambda i. m^* k_i(\lambda j. (mn_i)^* \ell_j)) && \text{hyperaffinness} \\
&= h(\lambda i. m^* k_i(\lambda j. m^* n_i^* \ell_j)) && (6.8) \\
&= h(\lambda i. m^*(k_i(\lambda j. n_i^* \ell_j))) ; && (6.6)
\end{aligned}$$

while comparing second terms, we have

$$\begin{aligned}
&h(\lambda i. m^* k_i) \triangleright (\lambda j. (h \triangleright (\lambda i. mn_i))p_j) \\
&= h(\lambda i. m^* k_i) \triangleright (\lambda j. h \triangleright (\lambda i. mn_i p_j)) && (6.11) \\
&= h \triangleright (\lambda i. m^* k_i \triangleright (\lambda j. h \triangleright (\lambda i'. mn_{i'} p_j))) && (6.10) \\
&= h \triangleright (\lambda i. h \triangleright (\lambda i'. m^* k_i \triangleright (\lambda j. mn_{i'} p_j))) && \text{commutativity in } \mathbb{H} \\
&= h \triangleright (\lambda i. m^* k_i \triangleright (\lambda j. mn_i p_j)) && \text{hyperaffinness} \\
&= h \triangleright (\lambda i. m(k_i \triangleright (\lambda j. n_i p_j))) && (6.12)
\end{aligned}$$

as desired. So $\mathbb{H} \bowtie M$ is an algebraic theory.

We next characterise the unary and hyperaffine operations of $\mathbb{H} \bowtie M$. Clearly the unary operations are those of the form $[1 \mid m]$. As for the hyperaffines, note that $[h \mid m] \in (\mathbb{H} \bowtie M)(I)$ will be hyperaffine when $[h \mid m](\lambda i. x) = x$, i.e., when $[h(\lambda i. x) \mid m] = [x \mid 1]$. In particular, we must have $m = 1$; and such an element will be hyperaffine just when also $[h \mid 1](\lambda i. [h \mid 1](\lambda j. x_{ij})) = [h \mid 1](\lambda i. x_{ii})$, i.e., $[h(\lambda i. h(\lambda j. x_{ij})) \mid 1] = [h(\lambda i. x_{ii}) \mid 1]$. Since each $h \in H(I)$ is hyperaffine, we conclude that the hyperaffines in $(\mathbb{H} \bowtie M)(I)$ are all elements of the form $[h \mid 1]$. It follows from this characterisation that each $[h \mid m] \in (\mathbb{H} \bowtie M)(I)$ has the unique hyperaffine–unary decomposition $[h \mid m](x) = [h \mid 1](\lambda i. [1 \mid m](x_i))$, whence it follows that $\mathbb{H} \bowtie M$ is a hyperaffine–unary theory.

It remains to show that $(\mathbb{H} \bowtie M)^\downarrow \cong [\mathbb{H} \mid M]$. Writing \mathbb{H}' and M' for the hyperaffine and unary parts of $\mathbb{H} \bowtie M$, we have isomorphisms $\varphi_I: H(I) \rightarrow H'(I)$ and $f: M \rightarrow M'$ given by $h \mapsto [h \mid 1]$ and $m \mapsto [1 \mid m]$. Because $[h \mid 1](\lambda i. [k_i \mid 1]) = [h(k) \mid 1]$ and $\pi_i = [\pi_i \mid 1]$ in $\mathbb{H} \bowtie M$, the maps φ_I constitute an isomorphism of theories $\mathbb{H} \rightarrow \mathbb{H}'$; and because $[1 \mid m]([1 \mid n]) = [1 \mid mn]$ and $1 = \pi_1 = [1 \mid 1]$ in $T(1)$, the map f is a monoid isomorphism $M \rightarrow M'$.

We now verify the two axioms in (6.14). For the first, observe that the operation $(-)^*$ on $(\mathbb{H} \bowtie M)^\downarrow$ has $([1 \mid m])^*([h \mid 1])$ given by the unique element $[k \mid 1] \in H'(I)$ for which $[k \mid 1](\lambda i. [1 \mid m](x_i)) = [1 \mid m]([h \mid 1])$. But $[k \mid 1](\lambda i. [1 \mid m](x_i)) = [k \mid m]$ and $[1 \mid m]([h \mid 1]) = [m^*h \mid 1]$, whence $([1 \mid m])^*([h \mid 1]) = [m^*h \mid 1]$, i.e., $f(m)^*(\varphi(h)) = \varphi(m^*h)$ as required. For the second axiom in (6.14), note that the operation \triangleright on $(\mathbb{H} \bowtie M)^\downarrow$ is given by $[h \mid 1] \triangleright (\lambda i. [1 \mid m_i]) = [h \mid 1](\lambda i. [1 \mid m_i]) = [1 \mid h \triangleright m]$, which says that $\varphi(h) \triangleright (\lambda i. f(m_i)) = f(h \triangleright m)$ as required.

Finally, it is trivial to observe that \mathbb{T} is finitary or non-degenerate if and only if \mathbb{H} is so, i.e., if and only if \mathbb{T}^\downarrow is so. \square

So we have a correspondence between hyperaffine–unary theories and matched pairs of theories; we now describe how this correspondence interacts with semantics.

Definition 6.6 (Models of matched pairs of theories). Let $[\mathbb{H} \mid M]$ be a matched pair of theories. A $[\mathbb{H} \mid M]$ -model \mathbf{X} is a set X endowed with both \mathbb{H} -model structure $h, x \mapsto \llbracket h \rrbracket(x)$ and M -set structure $m, x \rightarrow m \cdot x$ in such a way that

$$(h \triangleright m) \cdot x = \llbracket h \rrbracket(\lambda i. m_i \cdot x) \quad \text{and} \quad n \cdot \llbracket h \rrbracket(x) = \llbracket n^*h \rrbracket(\lambda i. n \cdot x) \quad (6.15)$$

for all $h \in H(I)$, $x \in X^I$, $m \in M^I$ and $n \in M$; while a homomorphism of $[\mathbb{H} \mid M]$ -models is a function preserving both \mathbb{H} -model and M -set structure. We write $[\mathbb{H} \mid M]\text{-Mod}$ for the variety of $[\mathbb{H} \mid M]$ -models.

Proposition 6.7. *Let \mathbb{T} be a hyperaffine–unary theory with $\mathbb{T}^\downarrow = [\mathbb{H} \mid M]$. The variety of \mathbb{T} -models is concretely isomorphic to the variety of $[\mathbb{H} \mid M]$ -models.*

Proof. Firstly, restricting back a \mathbb{T} -model structure on a set X to the hyperaffine and unary parts yields \mathbb{H} -model and M -set structure which satisfy the axioms in (6.15) due to the definitions of the operations $(-)^*$ and \triangleright in \mathbb{T}^\downarrow . Conversely, \mathbb{H} -model and M -set structure on X yields \mathbb{T} -model structure with

$$\llbracket [h \mid m] \rrbracket(x) = \llbracket h \rrbracket(\lambda i. m \cdot x_i) . \quad (6.16)$$

The projection axioms hold as every projection is in \mathbb{H} . As for substitution:

$$\begin{aligned}
& \llbracket [h \mid m] \rrbracket (\lambda i. \llbracket [k_i \mid n_i] \rrbracket (x)) \\
&= \llbracket [h] (\lambda i. m \cdot \llbracket [k_i] (\lambda j. n_i \cdot x_j)) \rrbracket && \text{definition} \\
&= \llbracket [h] (\lambda i. \llbracket [m^* k_i] (\lambda j. mn_i \cdot x_j)) \rrbracket && (6.15) \\
&= \llbracket [h] (\lambda i. \llbracket [h] (\lambda i'. \llbracket [m^* k_i] (\lambda j. mn_{i'} \cdot x_j)) \rrbracket) \rrbracket && \text{hyperaffinness} \\
&= \llbracket [h] (\lambda i. \llbracket [m^* k_i] (\lambda j. \llbracket [h] (\lambda i'. mn_{i'} \cdot x_j)) \rrbracket) \rrbracket && \text{commutativity} \\
&= \llbracket [h] (\lambda i. \llbracket [m^* k_i] (\lambda j. (h \triangleright (\lambda i'. mn_{i'})) \cdot x_j)) \rrbracket && (6.15) \\
&= \llbracket [h (\lambda i. m^* k_i)] (\lambda j. (h \triangleright (\lambda i. mn_i)) \cdot x_j) \rrbracket && \mathbb{H}\text{-model axiom} \\
&= \llbracket [h (\lambda i. m^* k_i) \mid h \triangleright (\lambda i. mn_i)] \rrbracket (x) && \text{definition} \\
&= \llbracket [h \mid m] ([k \mid n]) \rrbracket (x) && (6.3).
\end{aligned}$$

It is easy to see that these assignments are mutually inverse; and in light of (6.16), the homomorphisms match up under the correspondence. \square

To conclude this section, we make the correspondence between hyperaffine–unary theories and matched pairs of theories functorial. We write:

- $[\mathcal{H}\text{Aff} \mid \mathcal{U}\text{n}]$ (resp., $[\mathcal{H}\text{Aff}^\omega \mid \mathcal{U}\text{n}]$) for the category of non-degenerate matched pairs (resp., finitary matched pairs) of theories and their homomorphisms;
- $\mathcal{H}\text{Aff}\text{--}\mathcal{U}\text{n}$ (resp., $\mathcal{H}\text{Aff}\text{--}\mathcal{U}\text{n}^\omega$) for the full subcategory of $\mathcal{T}\text{hy}$ on the non-degenerate hyperaffine–unary (resp. finitary hyperaffine–unary) theories;
- $\text{cc}\mathcal{V}\text{ar}$ (resp., $\text{cc}\mathcal{V}\text{ar}^\omega$) for the full subcategory of $\mathcal{V}\text{ar}$ on the non-degenerate cartesian closed (resp., cartesian closed finitary) varieties.

Now the assignment $[\mathbb{H} \mid M] \mapsto [\mathbb{H} \mid M]\text{-Mod}$ can be made functorial. Indeed, given a homomorphism $[\varphi \mid f] : [\mathbb{H} \mid M] \rightarrow [\mathbb{H}' \mid M']$ of non-degenerate matched pairs of theories, we have a concrete functor $[\varphi \mid f]^* : [\mathbb{H}' \mid M']\text{-Mod} \rightarrow [\mathbb{H} \mid M]\text{-Mod}$ which acts by φ^* and f^* on the \mathbb{H}' -model and M' -set structures. In this way, we obtain a functor $(-)\text{-Mod} : [\mathcal{H}\text{Aff} \mid \mathcal{U}\text{n}]^{\text{op}} \rightarrow \mathcal{V}\text{ar}$ which, in light of Proposition 6.7, the final clause of Proposition 6.4, and Theorem 5.5, must land inside $\text{cc}\mathcal{V}\text{ar}$.

The assignment $(-)^{\downarrow}$ of Proposition 6.4 can also be made functorial:

Proposition 6.8. *The assignment $\mathbb{T} \mapsto \mathbb{T}^{\downarrow}$ is the action on objects of a functor $(-)^{\downarrow} : \mathcal{H}\text{Aff}\text{--}\mathcal{U}\text{n} \rightarrow [\mathcal{H}\text{Aff} \mid \mathcal{U}\text{n}]$, which on morphisms takes $\varphi : \mathbb{S} \rightarrow \mathbb{T}$ to the homomorphism $[\varphi|_{\mathbb{H}} \mid \varphi|_M] : \mathbb{S}^{\downarrow} \rightarrow \mathbb{T}^{\downarrow}$.*

Proof. $(-)^{\downarrow}$ is clearly functorial so long as it is well-defined on morphisms. To show this, let $\varphi : \mathbb{T} \rightarrow \mathbb{T}'$ be a homomorphism between hyperaffine–unary theories. Clearly, φ preserves both hyperaffine operations and unary operations, and so restricts back to $\varphi|_{\mathbb{H}} : \mathbb{H} \rightarrow \mathbb{H}'$ and $\varphi|_M : M \rightarrow M'$. We must verify that these restrictions satisfy the axioms in (6.14). The second axiom is simply an instance of the homomorphism axiom for φ ; as for the first, we have:

$$\begin{aligned}
\varphi(m^* h)(\lambda i. \varphi(m)(x_i)) &= \varphi(m^* h(\lambda i. m(x_i))) = \varphi(m(h(x))) \\
&= \varphi(m)(\varphi(h)(x)) = (\varphi(m)^* \varphi(h))(\lambda i. \varphi(m)(x_i))
\end{aligned}$$

whence $\varphi|_{\mathbb{H}}(m^* h) = (\varphi|_M(m))^*(\varphi|_{\mathbb{H}}(h))$ by unicity of decompositions. \square

Given the above, we are now ready to state the main result of this section:

Theorem 6.9. *We have a triangle of equivalences, commuting to within natural isomorphism, as to the left in:*

$$\begin{array}{ccc}
 \mathcal{H}\text{Aff}\text{-}\mathcal{U}\mathfrak{n} & \xrightarrow{(-)^\downarrow} & [\mathcal{H}\text{Aff} \mid \mathcal{U}\mathfrak{n}] \\
 \searrow^{(-)\text{-Mod}} & & \swarrow^{(-)\text{-Mod}} \\
 & & (\text{cc}\mathcal{V}\text{ar})^{\text{op}}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{H}\text{Aff}\text{-}\mathcal{U}\mathfrak{n}^\omega & \xrightarrow{(-)^\downarrow} & [\mathcal{H}\text{Aff}^\omega \mid \mathcal{U}\mathfrak{n}] \\
 \searrow^{(-)\text{-Mod}} & & \swarrow^{(-)\text{-Mod}} \\
 & & (\text{cc}\mathcal{V}\text{ar}^\omega)^{\text{op}}
 \end{array}$$

which restricts back to a triangle of equivalences as to the right.

Proof. The triangles commute to within isomorphism by Proposition 6.7, and by Theorem 5.5, their left edges are equivalences. So to complete the proof it suffices to show that $(-)^{\downarrow}$ is an equivalence to the left and the right. We know that in both cases it is essentially surjective by Proposition 6.5, and so it remains only to show it is also full and faithful. For fidelity, note that any homomorphism $\varphi: \mathbb{S} \rightarrow \mathbb{T} \in \mathcal{H}\text{Aff}\text{-}\mathcal{U}\mathfrak{n}$ must, by unicity of decompositions, send $[h \mid m]$ to $[\varphi(h) \mid \varphi(m)]$, and so is determined by its hyperaffine and unary restrictions. For fullness, let $\mathbb{S}, \mathbb{T} \in \mathcal{H}\text{Aff}\text{-}\mathcal{U}\mathfrak{n}$ and let $[\varphi \mid f]: \mathbb{S}^\downarrow \rightarrow \mathbb{T}^\downarrow$. We must show that the functions

$$\psi: S(I) \rightarrow T(I) \quad [h \mid m] \mapsto [\varphi(h) \mid f(m)]$$

preserve projections and substitution. For projections, we have $\psi(\pi_i) = \varphi([\pi_i \mid 1]) = [\varphi(\pi_i) \mid f(1)] = [\pi_i \mid 1] = \pi_i$, while for substitution, we have:

$$\begin{aligned}
 \psi([h \mid m]([k \mid n])) &= \psi([h(\lambda i. m^* k_i) \mid h \triangleright (\lambda i. mn_i)]) \\
 &= [\varphi(h(\lambda i. m^* k_i)) \mid f(h \triangleright (\lambda i. mn_i))] \\
 &= [(\varphi(h))(\lambda i. \varphi(m^* k_i)) \mid \varphi(h) \triangleright (\lambda i. f(mn_i))] \\
 &= [(\varphi(h))(\lambda i. f(m)^*(\varphi(k_i))) \mid \varphi(h) \triangleright (\lambda i. f(m)f(n_i))] \\
 &= [\varphi(h) \mid f(m)](\lambda i. [\varphi(k) \mid f(n)]) \\
 &= \psi([h \mid m])(\lambda i. \psi([k_i \mid n_i])) . \quad \square
 \end{aligned}$$

7. MATCHED ALGEBRAS AND $[B \mid M]$ -SETS

In this section we use Theorem 4.9 to recast the notion of matched pair of theories in terms of what we will call a *matched pair of algebras*. This yields a reformulation of Theorem 6.9 giving a functorial equivalence between (non-degenerate) hyperaffine–unary theories, matched pairs of algebras, and cartesian closed varieties. We begin in the finitary case, where a matched pair $[B \mid M]$ involves a Boolean algebra B and a monoid M which act on each in a suitable way—a structure which was already considered in [6, §4], in a related, though different, context.

Definition 7.1 (Matched pair of algebras). A non-degenerate *matched pair of algebras* $[B \mid M]$ comprises a non-degenerate Boolean algebra B , a monoid M and:

- B -set structure on M , which we write as $b, m, n \mapsto b(m, n)$;
- M -set structure on B , which we write as $m, b \mapsto m^*b$;

such that M acts on B by Boolean homomorphisms, and such that:

- $b(m, n)p = b(mp, np)$;
- $m(b(n, p)) = (m^*b)(mn, mp)$; and
- $b(m, n)^*(c) = b(m^*c, n^*c)$,

for all $m, n, p \in M$ and $b, c \in B$. Here, in the final axiom, we recall from Remark 3.17 that B itself is a B -set under the operation of conditioned disjunction $b(c, d) = (b \wedge c) \vee (b' \wedge d)$. These axioms are equivalently the conditions that:

- $m \equiv_b n \implies mp \equiv_b np$;
- $n \equiv_b p \implies mn \equiv_{m^*b} mp$;
- $m \equiv_b n \implies m^*c \equiv_b n^*c$, i.e., $b \wedge m^*c = b \wedge n^*c$.

A *homomorphism of matched pairs of algebras* $[\varphi | f] : [B | M] \rightarrow [B' | M']$ comprises a Boolean homomorphism $\varphi : B \rightarrow B'$ and a monoid homomorphism $f : M \rightarrow M'$ such that, for all $m, n \in M$ and $b \in B$ we have:

$$\varphi(b)(f(m), f(n)) = f(b(m, n)) \quad \text{and} \quad f(m)^*(\varphi(b)) = \varphi(m^*b), \quad (7.1)$$

or equivalently, such that

$$m \equiv_b n \implies f(m) \equiv_{\varphi(b)} f(n) \quad \text{and} \quad f(m)^*(\varphi(b)) = \varphi(m^*b). \quad (7.2)$$

We write $[\mathcal{B}Alg | \text{Mon}]$ for the category of non-degenerate matched pairs of algebras.

We now establish the desired equivalence between finitary matched pairs of theories, and matched pairs of algebras.

Proposition 7.2. *The assignment sending a non-degenerate finitary hyperaffine theory \mathbb{H} to the Boolean algebra $B = H(2)$ of Proposition 4.4 induces an equivalence of categories $\Theta : [\mathcal{H}Aff^\omega | \text{Un}] \rightarrow [\mathcal{B}Alg | \text{Mon}]$.*

Proof. The assignment $\mathbb{H} \mapsto H(2)$ is, by Theorem 4.9, the action on objects of an equivalence $\mathcal{H}Aff^\omega \rightarrow \mathcal{B}Alg$. Under this equivalence, the data and axioms of a matched pair of theories $[\mathbb{H} | M]$ as in Definition 6.3 transform as follows:

- \mathbb{H} and M correspond to the Boolean algebra $B = H(2)$ and monoid M ;
- The maps to the left of (6.4), satisfying the bicrossed pair axioms (i) and (ii), correspond to a monoid action $m, b \mapsto m^*b$ of M on B by Boolean homomorphisms;
- The maps to the right of (6.4), satisfying the axiom (iii), correspond to a B -set structure $b, m, n \mapsto b(m, n)$ on M ;
- The axioms (iv) and (v) correspond directly to the first two axioms for a matched pair of algebras.

As for axiom (vi), we claim that this corresponds to the final axiom for a matched pair of algebras. This is not completely immediate: we must first observe that (vi) can be replaced by the apparently weaker special case which takes $I = 2$:

(vi)' For $h \in H(2)$, the map $(-)^*h : M \rightarrow H(2)$ is an \mathbb{H} -model map $\mathbf{M} \rightarrow \mathbf{H}(2)$.

To see that this special case implies the general one, we must show that for any $h \in H(I)$, $k \in H(J)$ and $m \in M^J$ we have: $(k \triangleright m)^*(h) = k(\lambda j. m_j^*h)$. By Lemma 4.6(i), it suffices to verify for each $i \in I$ the equality of the binary reducts $(-)^{(i)}$ of each side. Since the operations $(k \triangleright m)^*$ and m_j^* are theory homomorphisms $\mathbb{H} \rightarrow \mathbb{H}$, we have

$$((k \triangleright m)^*h)^{(i)} = (k \triangleright m)^*(h^{(i)}) \quad \text{and} \quad k(\lambda j. m_j^*h)^{(i)} = k(\lambda j. m_j^*(h^{(i)}))$$

and since the $h^{(i)}$ are binary, these terms are equal by the special case (vi)'. Observing that the \mathbb{H} -model structure on $\mathbf{H}(2)$ corresponds to the B -action on B

by conditioned disjunction, we thus conclude that (vi)', and hence also (vi), are equivalent to the final axiom for a matched pair of theories.

It remains to show that homomorphisms match up under the above correspondences: for which we must show that the two conditions of (6.14) correspond to the two conditions of (7.1). For the first condition in (6.14), this is achieved by exploiting Lemma 4.6(i) like before to reduce to then case $I = 2$. As for the second condition, we may re-express it as saying that f is a homomorphism of \mathbb{H} -models $\mathbf{M} \rightarrow \varphi^*(\mathbf{M}')$, from which the correspondence with (7.1) is immediate. \square

So each finitary matched pair of theories $[\mathbb{H} \mid M]$ has a more concrete expression as a matched pair of algebras $[B \mid M]$; we now show that, correspondingly, the variety of $[\mathbb{H} \mid M]$ -models has a more concrete expression as a variety of $[B \mid M]$ -sets:

Definition 7.3 (Variety of $[B \mid M]$ -sets). Let $[B \mid M]$ be a non-degenerate matched pair of algebras. A $[B \mid M]$ -set is a set X endowed with B -set structure and M -set structure, such that in addition we have:

$$b(m, n) \cdot x = b(m \cdot x, n \cdot x) \quad \text{and} \quad m \cdot b(x, y) = (m^*b)(m \cdot x, m \cdot y) \quad (7.3)$$

for all $b \in B$, $m, n \in M$ and $x, y \in X$; or equivalently, such that:

$$m \equiv_b n \implies m \cdot x \equiv_b n \cdot x \quad \text{and} \quad x \equiv_b y \implies m \cdot x \equiv_{m^*b} m \cdot y. \quad (7.4)$$

A homomorphism of $[B \mid M]$ -sets is a function which is at once a B -set and an M -set homomorphism. We write $[B \mid M]$ -Set for the variety of $[B \mid M]$ -sets.

We noted in the preceding sections that any non-degenerate Boolean algebra B is always a B -set over itself, and that any monoid M is always an M -set over itself. If $[B \mid M]$ is a matched pair, then by definition we also have that B is an M -set, and M is a B -set; it should therefore be no surprise that, when endowed with these actions, both B and M become $[B \mid M]$ -sets. In fact, M is the free $[B \mid M]$ -set on one generator; while B is the coproduct of two copies of the terminal $[B \mid M]$ -set.

Proposition 7.4. *The equivalence Θ of Proposition 7.2 fits into a triangle of equivalences, commuting to within natural isomorphism:*

$$\begin{array}{ccc} [\mathcal{H}\text{Aff}^\omega \mid \mathcal{U}\mathfrak{n}] & \xrightarrow{\Theta} & [\mathcal{B}\text{Alg} \mid \text{Mon}] \\ & \searrow \text{(-)-Mod} & \swarrow \text{(-)-Set} \\ & & (\text{ccVar}^\omega)^{\text{op}}. \end{array}$$

Proof. Given a matched pair of theories $[\mathbb{H} \mid M] \in [\mathcal{H}\text{Aff}^\omega \mid \mathcal{U}\mathfrak{n}]$ with associated matched pair of algebras $[B \mid M]$, we know by Theorem 4.9 that the data of an $[\mathbb{H} \mid M]$ -model structure on X , as in Definition 6.6, will transform as follows:

- The \mathbb{H} -model and M -set structure on X correspond to a B -set structure $b, x, y \mapsto b(x, y)$ and an M -set structure;
- The left-hand axiom in (6.15), after reducing to the case $I = 2$ as in the proof of Proposition 7.2, becomes the left-hand axiom in (7.3).
- The right-hand axiom in (6.15) states that $n \cdot (-)$ is an \mathbb{H} -model homomorphism $\mathbf{X} \rightarrow n^*\mathbf{X}$, and thus becomes the right-hand axiom in (7.3).

It is clear that the homomorphisms match up under this correspondence, and so the variety of $[\mathbb{H} \mid M]$ -models and the variety of $[B \mid M]$ -sets are concretely isomorphic. It is easy to check that these isomorphisms are natural in $[\mathbb{H} \mid M]$ as required. \square

Now extending the preceding arguments to the non-finitary case is straightforward. First we generalise the notion of matched pair of algebras.

Definition 7.5. A non-degenerate *Grothendieck matched pair of algebras* $[B_{\mathcal{J}} \mid M]$ comprises a non-degenerate matched pair of algebras $[B \mid M]$ and a zero-dimensional topology \mathcal{J} on B , such that:

- The B -set M is a $B_{\mathcal{J}}$ -set;
- The M -action on B is by Grothendieck Boolean homomorphisms $B_{\mathcal{J}} \rightarrow B_{\mathcal{J}}$.

A *homomorphism of Grothendieck matched pairs of algebras* $[\varphi \mid f] : [B_{\mathcal{J}} \mid M] \rightarrow [B'_{\mathcal{J}'} \mid M']$ is a homomorphism of matched pairs of algebras for which $\varphi : B_{\mathcal{J}} \rightarrow B'_{\mathcal{J}'}$ is a Grothendieck Boolean homomorphism. We write $[\text{grBA} \mid \text{Mon}]$ for the category of non-degenerate Grothendieck matched pairs of algebras.

We next generalise the correspondence between finitary hyperaffine theories and matched pairs of algebras; the proof of this result is *mutatis mutandis* the same as Proposition 7.2.

Proposition 7.6. *The assignment sending a non-degenerate hyperaffine theory \mathbb{H} to the Grothendieck Boolean algebra $B_{\mathcal{J}}$ of Proposition 4.7 induces an equivalence of categories $\Theta : [\mathcal{H}\text{Aff} \mid \mathcal{Un}] \rightarrow [\text{grBA} \mid \text{Mon}]$.* \square

Finally, we introduce the varieties associated to Grothendieck matched pairs of algebras $[B_{\mathcal{J}} \mid M]$, and show that they match up with the models of the corresponding matched pairs of theories.

Definition 7.7 (Variety of $[B_{\mathcal{J}} \mid M]$ -sets). Let $[B_{\mathcal{J}} \mid M]$ be a non-degenerate Grothendieck matched pair of algebras. A $[B_{\mathcal{J}} \mid M]$ -set is a $[B \mid M]$ -set X whose underlying B -set is in fact a $B_{\mathcal{J}}$ -set; a homomorphism of $[B_{\mathcal{J}} \mid M]$ -sets is just a homomorphism of $[B \mid M]$ -sets. We write $[B_{\mathcal{J}} \mid M]\text{-Set}$ for the variety of $[B_{\mathcal{J}} \mid M]$ -sets.

Like before, both B and M are $[B_{\mathcal{J}} \mid M]$ -sets via their canonical actions on themselves and each other.

Proposition 7.8. *The equivalence Θ of Proposition 7.6 fits into a triangle of equivalences, commuting to within natural isomorphism:*

$$\begin{array}{ccc}
 [\mathcal{H}\text{Aff} \mid \mathcal{Un}] & \xrightarrow{\Theta} & [\text{grBA} \mid \text{Mon}] \\
 \searrow (-)\text{-Mod} & & \swarrow (-)\text{-Set} \\
 & (\text{ccVar})^{\text{op}} &
 \end{array}$$

\square

Remark 7.9. We can extract from the above development a description of the free $[B_{\mathcal{J}} \mid M]$ -set on a set X as given by the product of $[B_{\mathcal{J}} \mid M]$ -sets $M \times T_{B_{\mathcal{J}}} X$. Here, M is seen as a $[B_{\mathcal{J}} \mid M]$ -set via its canonical structures of $B_{\mathcal{J}}$ - and M -set, while $T_{B_{\mathcal{J}}}(X)$ is seen as a $B_{\mathcal{J}}$ -set as in Remark 3.17 and as an M -set via the action $n \cdot (m, \omega) = (nm, n^* \circ \omega)$. The function $\eta : X \rightarrow M \times T_{B_{\mathcal{J}}}(X)$ exhibiting $M \times T_{B_{\mathcal{J}}}(X)$ as free on X is given by $x \mapsto (1, \pi_x)$.

Combining Propositions 7.6 and 7.8 with Theorem 6.9, we obtain the main theorem of this section, relating (non-degenerate) Grothendieck matched pairs, hyperaffine–unary theories and cartesian closed varieties.

Theorem 7.10. *We have a triangle of equivalences, commuting to within natural isomorphism, as to the left in:*

$$\begin{array}{ccc}
 \mathcal{H}\text{Aff-}\mathcal{U}\text{n} & \xrightarrow{(-)^\downarrow} & [\text{gr}\mathcal{B}\mathcal{A}\text{lg} \mid \text{Mon}] \\
 \searrow & & \swarrow \\
 (-)\text{-Mod} & & (-)\text{-Set} \\
 & \searrow & \swarrow \\
 & (\text{cc}\mathcal{V}\text{ar})^{\text{op}} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{H}\text{Aff-}\mathcal{U}\text{n}^\omega & \xrightarrow{(-)^\downarrow} & [\mathcal{B}\mathcal{A}\text{lg} \mid \text{Mon}] \\
 \searrow & & \swarrow \\
 (-)\text{-Mod} & & (-)\text{-Set} \\
 & \searrow & \swarrow \\
 & (\text{cc}\mathcal{V}\text{ar}^\omega)^{\text{op}} &
 \end{array}$$

which restricts back to a triangle of equivalences as to the right.

Another way to say this is that every non-degenerate cartesian closed variety is a variety of $[B_{\mathcal{J}} \mid M]$ -sets for some Grothendieck matched pair of algebras $[B_{\mathcal{J}} \mid M]$. We now make explicit the cartesian closed structure of $[B_{\mathcal{J}} \mid M]$ -Set.

Proposition 7.11. *The variety of $[B_{\mathcal{J}} \mid M]$ -sets is cartesian closed. In particular, the variety of $[B \mid M]$ -sets is cartesian closed.*

Proof. Given $[B_{\mathcal{J}} \mid M]$ -sets Y and Z , we define the function space Z^Y to be the set of $[B_{\mathcal{J}} \mid M]$ -set homomorphisms $f: M \times Y \rightarrow Z$. We make this into an M -set under the same action as in Proposition 2.4:

$$m, f \quad \mapsto \quad m^*f = (\lambda n, y. f(nm, y)) .$$

We must check m^*f is a $[B_{\mathcal{J}} \mid M]$ -set homomorphism if f is one. For the M -set aspect this is just as in Proposition 2.4; for the $B_{\mathcal{J}}$ -set aspect, if $n \equiv_b p$ and $y \equiv_b z$ then $nm \equiv_b pm$ and so $f(nm, y) \equiv_b f(pm, y)$, i.e., $(m^*f)(n, y) \equiv_b (m^*f)(p, z)$ as desired. We now make Z^Y into a $B_{\mathcal{J}}$ -set via the equivalence relations:

$$f \equiv_b g \quad \iff \quad f(m, y) \equiv_{m^*b} g(m, y) \text{ for all } m, y \in M \times Y .$$

Axiom (i) of Proposition 3.2 is straightforward, given that each m^* is a Grothendieck Boolean homomorphism; so it now suffices to check axiom (ii)' of Proposition 3.11. Thus, given a partition $P \in \mathcal{J}$ and homomorphisms $f_b: M \times Y \rightarrow Z$ for each $b \in P$, we must show there is a unique $g: M \times Y \rightarrow Z$ with $g \equiv_b f_b$ for all $b \in P$, i.e.,

$$g(m, y) \equiv_{m^*b} f_b(m, y) \quad \text{for all } b \in B. \quad (7.5)$$

As m^* is a Grothendieck Boolean homomorphism, the set $m^*P = \{m^*b : b \in B\}^-$ is in \mathcal{J} , and so for each (m, y) there is a *unique* element $g(m, y)$ satisfying (7.5). It remains to show that the $g: M \times Y \rightarrow Z$ so defined is a $[B_{\mathcal{J}} \mid M]$ -set homomorphism.

To see g preserves the $B_{\mathcal{J}}$ -set structure, suppose that $m \equiv_c n$ in M and $y \equiv_c z$ in Y ; we must show $g(m, y) \equiv_c g(n, z)$ in Z . Since $m \equiv_c n$ we have for each $b \in P$ that $c \wedge m^*b = c \wedge n^*b$, and so $g(m, y) \equiv_{c \wedge m^*b} f_b(m, y) \equiv_{c \wedge m^*b} f_b(n, z) \equiv_{c \wedge m^*b} g(n, z)$, using that $g(m, y) \equiv_{m^*b} f_b(m, y)$ and $f_b(m, y) \equiv_c f_b(n, z)$ and $f_b(n, z) \equiv_{n^*b} g(n, z)$. Thus, for all $m^*b \in m^*P$ we have $g(m, y) \equiv_{c \wedge m^*b} g(n, z)$ and so by Lemma 3.12(i) that $g(m, y) \equiv_c g(n, z)$ as required. To see g preserves the M -set structure, we must show $m \cdot g(n, y) = g(mn, my)$. But for each $b \in P$ we have $g(n, y) \equiv_{n^*b} f_b(n, y)$, and so $m \cdot g(n, y) \equiv_{(mn)^*b} m \cdot f_b(n, y) = f_b(mn, my) \equiv_{(mn)^*b} g(mn, my)$. Thus $m \cdot g(n, y) = g(mn, my)$ by Lemma 3.12(i).

So Z^Y is a well-defined $[B_{\mathcal{J}} | M]$ -set. We now define the evaluation homomorphism $\text{ev}: Z^Y \times Y \rightarrow Z$ as in Proposition 2.4 by $\text{ev}(f, y) = f(1, y)$. This preserves M -set structure as there; while for the $B_{\mathcal{J}}$ -set structure, if $f \equiv_b g$ in Z^Y and $y \equiv_b y'$ in Y , then $\text{ev}(f, y) = f(1, y) \equiv_b g(1, y) \equiv_b g(1, y') = \text{ev}(g, y')$ as desired.

Finally, given a $[B_{\mathcal{J}} | M]$ -set homomorphism $f: X \times Y \rightarrow Z$, its transpose $\bar{f}: X \rightarrow Z^Y$ is given by $\bar{f}(x)(m, y) = f(mx, y)$. As in Proposition 2.4, this is an M -set homomorphism, and is the unique such with $\text{ev}(\bar{f}(x), y) = f(x, y)$ for all x, y . It remains to show that \bar{f} preserves $B_{\mathcal{J}}$ -set structure. But if $x \equiv_b x'$, then $mx \equiv_{m^*b} mx'$ for all $m \in M$, and so $\bar{f}(x)(m, y) = f(mx, y) \equiv_{m^*b} f(mx', y) = \bar{f}(x')(m, y)$ for all $(m, y) \in M \times Y$, i.e., $\bar{f}(x) \equiv_b \bar{f}(x')$. \square

Note that the results of the preceding sections have shown, without reference to [10], that every non-degenerate hyperaffine-unary theory is a theory of $[B_{\mathcal{J}} | M]$ -sets. Since every degenerate hyperaffine-unary theory clearly presents a cartesian closed variety, the preceding result thus completes a proof of the “if” direction of Theorem 5.5 that does not rely on [10]. Taken together with Proposition 5.8, we thus obtain our desired independent proof of Theorem 5.5.

As mentioned in the introduction, we defer substantive examples of varieties of $[B_{\mathcal{J}} | M]$ -sets to the companion paper [5], where we establish links with topics in operator algebra. However, Proposition 5.8 assures us that there is a plentiful supply of such varieties: we have one for any object X of a category with finite products and distributive set-indexed copowers. We can now be more explicit about the $[B_{\mathcal{J}} | M]$ associated to such an X .

Proposition 7.12. *Let \mathcal{C} be a category with finite products and set-based copowers for which each functor $C \times (-)$ preserves copowers, and let $X \in \mathcal{C}$.*

(a) *The set $\mathcal{M} = C(X, X)$ is a monoid with unit id_X under the operation of composition in diagrammatic order, i.e.:*

$$mn = X \xrightarrow{m} X \xrightarrow{n} X ;$$

(b) *Writing $\iota_{\top}, \iota_{\perp}: 1 \rightarrow 1 + 1$ for the first and second copower coprojections, the set $B = \mathcal{C}(X, 1 + 1)$ is a Boolean algebra under the operations*

$$1 = X \xrightarrow{\iota_{\top}} 1 \xrightarrow{\iota_{\top}} 1 + 1 \quad b' = X \xrightarrow{b} 1 + 1 \xrightarrow{\langle \iota_2, \iota_1 \rangle} 1 + 1$$

$$\text{and } b \wedge c = X \xrightarrow{(b,c)} (1 + 1) \times (1 + 1) \xrightarrow{\wedge} 1 + 1$$

where $\wedge: (1 + 1) \times (1 + 1) \rightarrow 1 + 1$ satisfies $\wedge \circ (\iota_i \times \iota_j) = \iota_{i \wedge j}$ for $i, j \in \{\top, \perp\}$;

(c) *There is a zero-dimensional coverage \mathcal{J} on B in which $P \subseteq B$ is in \mathcal{J} just when there exists a map $f: X \rightarrow P \cdot 1$ with $\langle \delta_{bc} \rangle_{b \in B} \circ f = c$ for all $c \in P$, where here $\delta_{bc}: 1 \rightarrow 1 + 1$ is given by $\delta_{bc} = \iota_{\top}$ when $b = c$ and $\delta_{bc} = \iota_{\perp}$ otherwise;*

(d) *M acts on B via precomposition;*

$$m^*b = X \xrightarrow{m} X \xrightarrow{b} 1 + 1 ;$$

(e) *B acts on M via:*

$$(b, m, n) \mapsto X \xrightarrow{(b, \text{id})} (1 + 1) \times X \xrightarrow{\cong} X + X \xrightarrow{\langle m, n \rangle} X .$$

So long as B is non-degenerate, the above operations make $[B_{\mathcal{J}} | M]$ into a non-degenerate Grothendieck matched pair of algebras. Moreover, for all $Y \in \mathcal{C}$, the set

$\mathcal{C}(X, Y)$ becomes a $[B_{\mathcal{J}} \mid M]$ -set, where M acts on $\mathcal{C}(X, Y)$ via precomposition, and B acts on $\mathcal{C}(X, Y)$ via

$$(b, x, y) \mapsto X \xrightarrow{(b, \text{id})} (1 + 1) \times X \xrightarrow{\cong} X + X \xrightarrow{\langle x, y \rangle} Y .$$

In this manner, we obtain a factorisation of the hom-functor $\mathcal{C}(X, -)$ as

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{-K-} & [B_{\mathcal{J}} \mid M]\text{-Set} \\ & \searrow & \swarrow U \\ \mathcal{C}(X, -) & & \text{Set} \end{array} \tag{7.6}$$

which is universal among factorisations of $\mathcal{C}(X, -)$ through a variety. In particular, if \mathcal{C} is a non-degenerate cartesian closed variety and X is the free model $F1$ on one generator, then K is an isomorphism.

Proof. By Proposition 5.8, the complete theory \mathbb{T}_X of dual operations of $X \in \mathcal{C}$ is hyperaffine–unary. If \mathbb{T}_X is degenerate, then so is the Boolean algebra B described above, and there is nothing to do; otherwise, we know that the non-degenerate hyperaffine–unary \mathbb{T}_X corresponds to a Grothendieck matched pair $[B_{\mathcal{J}} \mid M]$ for which $\mathbb{T}\text{-Mod}$ is concretely isomorphic to $[B_{\mathcal{J}} \mid M]\text{-Set}$. Using that the hyperaffine operations of \mathbb{T}_X are, as in the proof of Proposition 5.8, those of the form $(h, 1): X \rightarrow (I \cdot 1) \times X \cong I \cdot X$, and following through the construction of $[B_{\mathcal{J}} \mid M]$ from the hyperaffine–unary theory \mathbb{T}_X as in Sections 6 and 7, yields the description above. Finally, the factorisation (7.6) is simply the factorisation (5.2) after transporting across the isomorphism $\mathbb{T}\text{-Mod} \cong [B_{\mathcal{J}} \mid M]\text{-Set}$. \square

We encourage the reader to apply this result in any category satisfying its rather mild hypotheses. For example, when \mathcal{C} is the category of topological spaces, the monoid M associated to a space X comprises all continuous endomorphisms of X , while the Grothendieck Boolean algebra $B_{\mathcal{J}}$ comprises all clopen subsets of X , with the infinite partitions in \mathcal{J} being all infinite clopen partitions of X . Now M acts on B by inverse image, $\varphi, U \mapsto \varphi^{-1}(U)$, while B acts on M by restriction and glueing: $U, f, g \mapsto \langle f|_U, g|_{U^c} \rangle$. In this example, $[B_{\mathcal{J}} \mid M]$ is rather large, in much the same way that full automorphism groups of objects tend to be rather large, and a key aspect of [5] will be to apply this result in carefully chosen situations where $[B_{\mathcal{J}} \mid M]$ comes out as something combinatorially tractable and of independent interest.

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