WHEN COPRODUCTS ARE BIPRODUCTS

RICHARD GARNER AND DANIEL SCHÄPPI

Abstract. Among right-closed monoidal categories with finite coproducts, we characterise those with finite *biproducts* as being precisely those in which the initial object and the coproduct of the unit with itself admit right duals. This generalises Houston's result that any compact closed category with finite coproducts admits biproducts.

1. Background and statement of results

Recall that a monoidal category is *compact closed* (also *autonomous*) when every object has both a left and right dual; key examples include the categories of finite dimensional vector spaces, and of sets and relations. In [2], Houston proves that in a compact closed category, finite products and coproducts coincide; more precisely, they are *biproducts*:

Definition 1. Let $\mathscr{C}$ be a category with a *zero object*: an object $0 \in \mathscr{C}$ which is both initial and terminal. A coproduct $\Sigma_{i \in I} A_i$ in $\mathscr{C}$ is called a *biproduct* if the cone

$$(\pi_k \colon \Sigma_{i \in I} A_i \to A_k)_{k \in I} \tag{1}$$

is a product cone, where π_k is the unique morphism with $\pi_k \iota_k = 1_{A_k}$ and with $\pi_k \iota_i = 0$ for $i \neq k$; here, for any $X, Y \in \mathscr{C}$, $0 \colon X \to Y$ is the composite of unique maps $X \to 0 \to Y$.

Houston's proof does not adapt to give a characterisation of categories with biproducts among the (different) class of symmetric monoidal closed categories; in a question on MathOverflow [1], Barton asked whether such a characterisation could be given as the two requirements that finite coproducts exist, and that the initial object and the coproduct of the unit with itself have duals. After helpful conversations with Mike Shulman, the second author was able to answer this question affirmatively; some time later, the first author, inspired by discussions with James Dolan, found a simpler version of the proof which generalises to the non-symmetric monoidal case (and thus recovers Houston's result). The goal of this note, then, is to give a streamlined proof of:

Theorem 2. *If $\mathscr{C} = (\mathscr{C}, \otimes, I)$ is a monoidal category possessing finite coproducts preserved by each $A \otimes (–)$, then $\mathscr{C}$ has a zero object and finite biproducts if and only if the initial object 0 and coproduct $I + I$ have right duals.*

In fact, we prove something slightly more general. When $\mathscr{C}$ has finite coproducts, the existence of biproducts is equivalent to *semi-additivity*: the existence of commutative monoid structures on the hom-sets which are preserved by composition in each variable.

Theorem 3. *If $\mathscr{C}$ is a monoidal category with nullary and binary coproducts of I, preserved by each $A \otimes (–)$, then $\mathscr{C}$ is semi-additive if and only if 0 and $I + I$ have right duals.*

The first author gratefully acknowledges the support of Australian Research Council Discovery Projects DP110102360 and DP130101969.

The second author gratefully acknowledges support through the Swiss National Foundation Fellowship P2SKP2_148479.

A key ingredient in the proof of these theorems is the "terminal object lemma":

Lemma 4. *An object T of a category $\mathscr{C}$ is terminal if and only if there is a cocone $\varepsilon\colon \mathrm{id}_{\mathscr{C}} \Rightarrow \Delta T$ under the identity functor on $\mathscr{C}$ for which $\varepsilon_T = 1_T\colon T \to T$.*

Proof. The "only if" follows as the unique morphisms $C \to T$ are natural in C. Conversely, given ε, there is the map $\varepsilon_C\colon C \to T$ from each $C \in \mathscr{C}$; to show unicity, we use $\varepsilon_T = 1_T$ and naturality of ε to conclude that for any $f\colon C \to T$ we have $f = \varepsilon_T f = \varepsilon_C$. □

From this we recover the following well-known result, which provides the link between Theorems 2 and 3.

Proposition 5. *If $\mathscr{C}$ is semi-additive with an initial object 0, then the initial object is a zero object and any finite coproduct that exists in $\mathscr{C}$ is a biproduct.*

Proof. The neutral elements of the monoids $\mathscr{C}(C, 0)$ give a cocone $\mathrm{id}_{\mathscr{C}} \Rightarrow \Delta 0$, so that 0 is terminal by Lemma 4. Suppose now that $(\iota_i\colon A_i \to A)_{i\in I}$ is a finite coproduct cocone. We will use Lemma 4 to show that the cone $\pi = (\pi_k\colon A \to A_k)_{k\in I}$ of (1) is terminal among all such cones. Given a cone $f = (f_k\colon B \to A_k)_{k\in I}$, let $\varepsilon_f = \Sigma_{i\in I}\iota_i f_i\colon B \to A$. Then $\pi_j \varepsilon_f = \Sigma_i \pi_j \iota_i f_i = f_j$, so $\varepsilon_f\colon f \to \pi$ is a map of cones; moreover, $\varepsilon_{(-)}$ is natural as composition in $\mathscr{C}$ is bilinear. Finally, we have $\varepsilon_\pi \iota_j = \Sigma_i \iota_i \pi_i \iota_j = \iota_j$ and so $\varepsilon_\pi = 1_A$. □

2. Proofs and examples

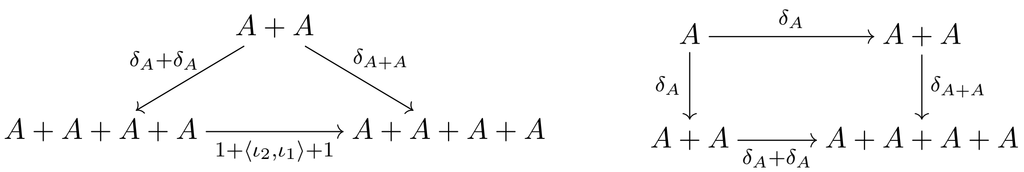
Our main result is a consequence of the following necessary and sufficient condition for a reasonable category $\mathscr{C}$ to be semi-additive. We say that $\mathscr{C}$ has *2-fold copowers* if all coproducts $A + A$ exist, and that it has *binary copowers* if, for each $n \in \mathbb{N}$, all 2^n-fold coproducts $A + \cdots + A$ exist; which is so just when $\mathscr{C}$ has 2-fold copowers and an initial object 0. In a category with binary copowers, the coproducts $0 + A$ and $A + 0$ always exist (since they can be taken to be A), and so we can talk about *counital comagmas*: objects A endowed with a comultiplication $\delta\colon A \to A + A$ and a counit $\varepsilon\colon A \to 0$ such that $(\varepsilon + 1_A)\delta = 1_A = (1_A + \varepsilon)\delta$.

Proposition 6. *Let $\mathscr{C}$ be a category with binary copowers. The following are equivalent:*

(i) The category $\mathscr{C}$ is semi-additive;
(ii) The initial object 0 is a zero object and the 2-fold copowers $A + A$ are biproducts;
(iii) The identity functor has a counital comagma structure in the functor category $[\mathscr{C}, \mathscr{C}]$.

Proof. The implication (i) ⇒ (ii) follows from Proposition 5; while (ii) ⇒ (iii) follows since the diagonal morphism and the zero morphism are natural.

For (iii) ⇒ (i), suppose that $\varepsilon_A\colon A \to 0$ and $\delta_A\colon A \to A + A$ are natural families with $(1_A + \varepsilon_A)\delta_A = 1_A = (\varepsilon_A + 1_A)\delta_A$. Naturality at coproduct injections $\iota_i\colon A \to A + A$ implies that any triangle as on the left in



commutes in $\mathscr{C}$, while naturality at δ_A says that each square as to the right commutes. By combining these two diagrams we find that the unital magma $(A, \delta_A, \varepsilon_A)$ in $\mathscr{C}^{\mathrm{op}}$ is medial; here, a magma $m\colon M \times M \to M$ is said to be *medial* if for all generalised elements $a, b, c, d\colon X \to M$ we have $m(m(a,b), m(c,d)) = m(m(a,c), m(b,d))$.

The Eckmann–Hilton argument shows that any unital medial magma in a category with finite products[1] is a commutative monoid. Thus for each pair of objects $A, B \in \mathscr{C}$ we get a commutative monoid structure $\big(\mathscr{C}(A,B), \mathscr{C}(\delta_A, B), \mathscr{C}(\varepsilon_A, B)\big)$, which by naturality of δ and ε gives the desired semi-additive structure on $\mathscr{C}$. □

We now prove our main theorems. First we recall:

Definition 7. Let $(\mathscr{C}, \otimes, I)$ be a monoidal category. A *right dual* for $X \in \mathscr{C}$ comprises an object $Y \in \mathscr{C}$ and a map $\eta\colon I \to Y \otimes X$ such that each map $f\colon A \to B \otimes X$ is of the form

$$A \xrightarrow{A \otimes \eta} A \otimes Y \otimes X \xrightarrow{g \otimes X} B \otimes X$$

for a unique $g\colon A \otimes Y \to B$. Note that Y is right dual to X if and only if there are natural isomorphisms $\theta_{A,B}\colon \mathscr{C}(A, B \otimes X) \to \mathscr{C}(A \otimes Y, B)$ which are stable under tensor, in the sense that $(A' \otimes {-}) \circ \theta_{A,B} = \theta_{A' \otimes A, A' \otimes B} \circ (A' \otimes {-})\colon \mathscr{C}(A, B \otimes X) \to \mathscr{C}(A' \otimes A \otimes Y, A' \otimes B)$.

Proof of Theorem 3. If $\mathscr{C}$ is semi-additive, then by Proposition 5, the initial object is terminal and the coproduct $B \otimes (I + I) \cong B + B$ is a product. Thus there are natural bijections between maps $A \otimes 0 \to B$ and $A \to B \otimes 0$ on the one hand and maps $A \otimes (I + I) \to B$ and $A \to B \otimes (I + I)$ on the other; these isomorphisms are easily seen to be stable under tensor, whence both 0 and $I + I$ are self-dual.

In the converse direction, the assumption that $A \otimes (-)$ preserves the 2-fold copower $I + I$ implies that $\mathscr{C}$ has binary copowers, and so Proposition 6 is applicable. Since there are natural isomorphisms $A \otimes (I + I) \cong A + A$ and $A \otimes 0 \cong 0$, we may verify Proposition 6(iii) by constructing a counital comagma structure on the object $I \in \mathscr{C}$.

Suppose, then, that $\varepsilon\colon I \to Z \otimes 0 \cong 0$ exhibits Z as right dual to 0, and that $\eta\colon I \to D \otimes (I + I)$ exhibits D as right dual to $I + I$. Lemma 4 applied to the natural family $\varepsilon_A = A \otimes \varepsilon\colon A \to 0$ shows that 0 is a terminal object; in particular, $\varepsilon f = \varepsilon g$ for any two maps $f, g\colon X \rightrightarrows I$. Now let $\pi_1, \pi_2\colon D \to I$ be the unique maps such that

$$\iota_i = I \xrightarrow{\eta} D \otimes (I + I) \xrightarrow{\pi_i \otimes 1} I + I\ .$$

Writing $\bar\eta$ for the composite $I \xrightarrow{\eta} D \otimes (I + I) \cong D + D$, it follows that

$$\iota_i = I \xrightarrow{\bar\eta} D + D \xrightarrow{\pi_i + \pi_i} I + I\ .$$

Now take $\delta\colon I \to I + I$ to be the composite $(\pi_1 + \pi_2)\bar\eta\colon I \to D + D \to I + I$ and observe that $(1_I + \varepsilon)\delta = (1_I + \varepsilon)(\pi_1 + \pi_2)\bar\eta = (\pi_1 + \varepsilon\pi_2)\bar\eta = (\pi_1 + \varepsilon\pi_1)\bar\eta = (1_I + \varepsilon)\iota_1 = 1_I$ and dually $(\varepsilon + 1_I)\delta = 1_I$. It follows that the identity functor on $\mathscr{C}$ bears a counital comagma structure, and so by Proposition 6 that $\mathscr{C}$ is semi-additive. □

Proof of Theorem 2. Propositions 5 and 6 show that a category with finite coproducts is semi-additive if and only if it has finite biproducts. The claim therefore follows from Theorem 3. □

We conclude the paper with some examples showing that our result is, in a certain sense, the best possible: the assumptions in Theorem 3 that $A \otimes (-)$ preserves the initial object 0 and the coproduct $I + I$ cannot be relaxed.

---

[1] It suffices to assume that the 3-fold product $M \times M \times M$ exists; but this is necessary in order to express the associativity axiom of a monoid internally.

Example 8. Let $\mathscr{C}$ be the category of endofunctors of the category $\mathbf{2} = \{0 < 1\}$, with tensor product given by composition. The category $\mathscr{C}$ is isomorphic to the ordered set $\{0 < \mathrm{id} < 1\}$, so the coproduct $\mathrm{id} + \mathrm{id}$ is equal to id and is therefore both self-dual and also preserved by each $A \otimes (-)$. The initial object 0 has as right dual the terminal object 1, but 0 is not preserved by $1 \otimes (-)$. The category $\mathscr{C}$ is not semi-additive since 0 is not isomorphic to 1.

Example 9. Let $\mathbf{FinSet}_*$ be the category of finite pointed sets, and $\mathscr{C}$ the category of zero-object-preserving endofunctors of $\mathbf{FinSet}_*$, with tensor product given by composition. The constant functor at 0 is self-dual, and $\mathrm{id} + \mathrm{id}$ has $\mathrm{id} \times \mathrm{id}$ as right dual (here: right adjoint). Every $A \otimes (-)$ preserves the initial object by assumption, but need not preserve the coproduct $\mathrm{id} + \mathrm{id}$. The category $\mathscr{C}$ is not semi-additive since the canonical morphism $\mathrm{id} + \mathrm{id} \to \mathrm{id} \times \mathrm{id}$ is not invertible.

Note that in the above two examples, the functors $(-) \otimes A$ preserve all finite coproducts because these are computed pointwise in functor categories. By passing to the monoidal categories with the reverse tensor product, we obtain examples satisfying all the conditions of Theorem 3 except that one of 0 and $I + I$ merely has a *left* dual. This shows that, in the situation of Theorem 3, such left duals need not imply the semi-additivity of the underlying category.

References

[1] Barton, R. Semiadditivity and dualizability of **2**. Question on MathOverflow, February 2010. Available at http://mathoverflow.net/questions/14402.

[2] Houston, R. Finite products are biproducts in a compact closed category. *Journal of Pure and Applied Algebra 212*, 2 (2008), 394–400.

Department of Mathematics, Macquarie University, NSW 2109, Australia
E-mail address: richard.garner@mq.edu.au

School of Mathematics and Statistics, University of Sheffield, Sheffield, S3 7RH, UK
E-mail address: d.schaeppi@sheffield.ac.uk