ALGEBRAIC WEAK FACTORISATION SYSTEMS I: ACCESSIBLE AWFS

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Abstract. Algebraic weak factorisation systems (awfs) refine weak factorisation systems by requiring that the assignations sending a map to its first and second factors should underlie an interacting comonad–monad pair on the arrow category. We provide a comprehensive treatment of the basic theory of awfs—drawing on work of previous authors—and complete the theory with two main new results. The first provides a characterisation of awfs and their morphisms in terms of their double categories of left or right maps. The second concerns a notion of *cofibrant generation* of an awfs by a small double category; it states that, over a locally presentable base, any small double category cofibrantly generates an awfs, and that the awfs so arising are precisely those with accessible monad and comonad. Besides the general theory, numerous applications of awfs are developed, emphasising particularly those aspects which go beyond the non-algebraic situation.

1. Introduction

A *weak factorisation system* on a category $\mathcal{C}$ comprises two classes of maps $\mathcal{L}$ and $\mathcal{R}$, each closed under retracts in the arrow category, and obeying two axioms: firstly, that each map $f \in \mathcal{C}$ admit a factorisation $f = Rf \cdot Lf$ with $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$; and secondly, that each $r \in \mathcal{R}$ have the right lifting property with respect to each $\ell \in \mathcal{L}$—meaning that, for every square as in the solid part of:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "\ell"'] & C \arrow[d, "r"] \\
B \arrow[r] \arrow[ur, dotted] & D
\end{tikzcd}
\tag{1.1}
$$

there should exist a commuting diagonal filler as indicated. Weak factorisation systems play a key role in *Quillen model structures* [42], which comprise two intertwined weak factorisation systems on a category; but they also arise elsewhere, for example in the categorical semantics of intensional type theory [4, 20].

Algebraic weak factorisation systems were introduced in [25]; they refine the basic notion by requiring that the factorisation process $f \mapsto (Lf, Rf)$ yield a compatible comonad L and monad R on the arrow category of $\mathcal{C}$. Given (L, R), we re-find $\mathcal{L}$ and $\mathcal{R}$ as the retract-closures of the classes of maps admitting

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L-coalgebra or R-algebra structure, so that one may define an algebraic weak factorisation system (henceforth AWFS) purely in terms of a comonad–monad pair (L, R) satisfying suitable axioms; we recall these in Section 2 below.

As shown in [22], any cofibrantly generated weak factorisation system on a well-behaved category may be realised as an AWFS, so that the algebraic notions are entirely appropriate for doing homotopy theory; this point of view has been pushed by Riehl, who in [43, 44] gives definitions of algebraic model category and algebraic monoidal model category, and in subsequent collaboration has used these notions to obtain non-trivial homotopical results [6, 12, 15].

Yet AWFS can do more than just serve as well-behaved realisations of their underlying weak factorisation systems; by making serious use of the monad R and comonad L, we may capture phenomena which are invisible in the non-algebraic setting. For example, each AWFS on $\mathcal{C}$ induces a *cofibrant replacement comonad* on $\mathcal{C}$ by factorising the unique maps out of 0; and if we choose our AWFS carefully, then the Kleisli category of this comonad Q—whose maps $A \rightsquigarrow B$ are maps $QA \to B$ in the original category—will equip $\mathcal{C}$ with a usable notion of *weak map*. For instance, there is an AWFS on the category of tricategories [24] and strict morphisms (preserving all structure on the nose) for which $\mathbf{Kl}(\mathsf{Q})$ comprises the tricategories and their trihomomorphisms (preserving all structure up to coherent equivalence); this example and others were described in [23], and will be revisited in the companion paper [13].

Probably the most important expressive advantage of AWFS is that their left and right classes can delineate kinds of map which mere weak factorisation systems cannot—the reason being that we interpret the classes of an AWFS as being composed of the L-coalgebras and R-algebras, rather than the underlying $\mathcal{L}$-maps and $\mathcal{R}$-maps. For example, here are some classes of map in **Cat** which are not the $\mathcal{R}$-maps of any weak factorisation system, but which—as shown in Examples 28 below—may be described in terms of the possession of R-algebra structure for a suitable AWFS:

- The Grothendieck fibrations;
- The Grothendieck fibrations whose fibres are groupoids;
- The Grothendieck fibrations whose fibres have finite limits, and whose reindexing functors preserve them;
- The left adjoint left inverse functors.

At a crude level, the reason that these kinds of map cannot be expressed as classes of $\mathcal{R}$-maps is that they are not retract-closed. The deeper explanation is that being an $\mathcal{R}$-map is a mere *property*, while being an R-algebra is a *structure* involving choices of basic lifting operations—and this choice allows for necessary equational axioms to be imposed between derived operations. As a further demonstration of the power this affords, we mention the result of [27] that for any monad T on a category $\mathcal{C}$ with finite products, there is an AWFS on $\mathcal{C}$ whose "algebraically fibrant objects"—R-algebras $X \to 1$—are precisely the T-algebras.

The existence of AWFS that reach beyond the scope of the non-algebraic theory opens up intriguing vistas. In a projected sequel to this paper, we will consider the theory of enrichment over monoidal AWFS [44] and use it to develop an abstract "homotopy coherent enriched category theory". In fact, we touch on this already in the current paper; Section 8 describes the *enriched small object argument* and sketches some of its applications to two-dimensional category theory, and to notions from the theory of quasicategories [32, 33, 40] such as *limits*, *Grothendieck fibrations* and *Kan extensions*.

The role of these examples, and others like them, will be to illuminate and justify the main contribution of this paper—that of giving a comprehensive account of the theory of unenriched AWFS. Parts of this theory can be found developed across the papers [6, 22, 25, 43, 44]; our objective is to draw the most important of these results together, and to complete them with two new theorems that clarify and simplify both the theory and the practice of AWFS.

In order to explain our two main theorems, we must first recall some *double-categorical* aspects of AWFS. A double category $\mathbb{A}$ is an internal category in **CAT**, as on the left below; we refer to objects and arrows of $\mathcal{A}_0$ as *objects* and *horizontal arrows*, and to objects and arrow of $\mathcal{A}_1$ as *vertical arrows* and *squares*. Internal functors and internal natural transformations between internal categories in **CAT** will be called *double functors* and *horizontal natural transformations*; they comprise a 2-category **DBL**.

$$\mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \xrightarrow{\ \circ\ } \mathcal{A}_1 \underset{\text{cod}}{\overset{\text{dom}}{\underset{\longleftarrow \text{id} \longrightarrow}{\longrightarrow}}} \mathcal{A}_0 \qquad\qquad \mathsf{R}\text{-}\mathbf{Alg} \times_{\mathcal{C}} \mathsf{R}\text{-}\mathbf{Alg} \xrightarrow{\ \circ\ } \mathsf{R}\text{-}\mathbf{Alg} \underset{\text{cod}}{\overset{\text{dom}}{\underset{\longleftarrow \text{id} \longrightarrow}{\longrightarrow}}} \mathcal{C}\ .$$

To each AWFS (L, R) on a category $\mathcal{C}$ we may associate a double category $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$, as on the right above, whose objects and horizontal arrows are the objects and arrows of $\mathcal{C}$, whose vertical arrows are the R-algebras, and whose squares are maps of R-algebras. The functor $\circ\colon \mathsf{R}\text{-}\mathbf{Alg} \times_{\mathcal{C}} \mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{R}\text{-}\mathbf{Alg}$ encodes a canonical *composition law* on R-algebras—recalled in Section 2.8 below—which is an algebraic analogue of the fact that $\mathcal{R}$-maps in a weak factorisation system are closed under composition.

There is a forgetful double functor $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ into the double category of *squares* in $\mathcal{C}$—wherein objects are those of $\mathcal{C}$, vertical and horizontal arrows are arrows of $\mathcal{C}$, and squares are commuting squares in $\mathcal{C}$. In [43, Lemma 6.9], it was shown that the assignation sending an AWFS (L, R) on $\mathcal{C}$ to the double functor $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ constitutes the action on objects of a 2-fully faithful 2-functor

$$(–)\text{-}\mathbb{A}\mathbf{lg}\colon \mathbf{AWFS}_{\text{lax}} \to \mathbf{DBL}^{\mathbf{2}} \tag{1.2}$$

from the 2-category of AWFS, lax AWFS morphisms and AWFS 2-cells—whose definition we recall in Section 2.9 below—to the arrow 2-category $\mathbf{DBL}^{\mathbf{2}}$.

Our first main result, Theorem 6 below, gives an elementary characterisation of the essential image of (1.2) and of its dual, the fully faithful coalgebra 2-functor $(–)\text{-}\mathbb{C}\mathbf{oalg}\colon \mathbf{AWFS}_{\text{oplax}} \to \mathbf{DBL}^{\mathbf{2}}$. In the theory of monads, a corresponding characterisation of the *strictly monadic* functors—those in the essential image of the 2-functor sending a monad T to the forgetful functor $U^{\mathsf{T}}\colon \mathsf{T}\text{-}\mathbf{Alg} \to \mathcal{C}$—is given by Beck's monadicity theorem; and so we term our result a *Beck theorem* for AWFS. Various aspects of this result are already in the literature—see [22, Appendix], [23, Proposition 2.8] or [43, Theorem 2.24]—and during the preparation of this paper, we became aware that Athorne had independently arrived at a similar result as [3, Theorem 2.5.3]. We nonetheless provide a complete treatment here, as this theorem is crucial to a smooth handling of AWFS, in particular allowing them to be constructed simply by giving a double category of the correct form to be one's double category of R-algebras or L-coalgebras.

The second main result of this paper deals with the appropriate notion of cofibrant generation for AWFS. Recall that a weak factorisation system $(\mathcal{L}, \mathcal{R})$ is *cofibrantly generated* if there is a mere *set* of maps J such that $\mathcal{R}$ is the precisely the class of maps with the right lifting property against each $j \in J$. Cofibrantly generated weak factorisation systems are commonplace due to Quillen's *small object argument* [42, §II.3.2], which ensures that for any set of maps J in a locally presentable category [19], the AWFS $(\mathcal{L}, \mathcal{R})$ cofibrantly generated by J exists.

In [22, Definition 3.9] is described a notion of cofibrant generation for AWFS: given a small category $U \colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ over $\mathcal{C}^{\mathbf{2}}$, the AWFS cofibrantly generated by $\mathcal{J}$, if it exists, has as R-algebras, maps g equipped with choices of right lifting against each map Uj, naturally with respect to maps of $\mathcal{J}$. This notion is already more permissive than the usual one—as witnessed by [45, Example 13.4.5], for example—but we argue that it is still insufficient, since it excludes important examples on AWFS, such as the ones on **Cat** listed above, whose R-algebras are not retract-closed.

To rectify this, we introduce in Section 6.2 the notion of an AWFS being cofibrantly generated by a small *double category* $U \colon \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$. This condition specifies the R-algebras as being maps equipped with liftings against Uj for each vertical map $j \in \mathbb{J}$, naturally with respect to squares of $\mathbb{J}$, but now with the extra requirement that, for each pair of composable vertical maps $j \colon x \to y$ and $k \colon y \to z$ of $\mathbb{J}$, the specified lifting against kj should be obtained by taking specified lifts first against k and then against j.

This broader notion of cofibrant generation is permissive enough to capture all our leading examples, including the ones on **Cat** listed above. Our second main theorem justifies it at a theoretical level, by characterising the cofibrantly generated AWFS on locally presentable categories as being exactly the *accessible* AWFS—those (L, R) whose comonad L and monad R preserve κ-filtered colimits for some regular cardinal κ. More precisely, we show for a locally presentable $\mathcal{C}$ that every small $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ cofibrantly generates an accessible AWFS on $\mathcal{C}$; and that every accessible AWFS on $\mathcal{C}$ is cofibrantly generated by a small $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$.

We now give a summary of the contents of the paper. We begin in Section 2 with a revision of the basic theory of AWFS: the definition, the relation with weak factorisation systems, double categories of algebras and coalgebras, morphisms of AWFS, and the fully faithful 2-functors $(-)$-$\mathbb{A}\mathbf{lg}$ and $(-)$-$\mathbb{C}\mathbf{oalg}$. In Section 3, we give our first main result, the Beck theorem for AWFS described above, together with two useful variants. Section 4 then uses the Beck theorem to give constructions of a wide range of AWFS; in particular, we discuss the AWFS for *split epis* in any category with binary coproducts; the AWFS for *lalis* (left adjoint left inverse functors) in any 2-category with oplax limits of arrows; and the construction of *injective* and *projective* liftings of AWFS.

Section 5 revisits the notion of cofibrant generation of AWFS by small categories, as introduced in [22], and uses the Beck theorem to give a simplified proof that such AWFS always exist in a locally presentable $\mathcal{C}$. This prepares the way for our second main result; in Section 6 we introduce cofibrant generation by small double categories, and in Section 33, show that over a locally presentable base $\mathcal{C}$, such AWFS always exist and are precisely the accessible AWFS on $\mathcal{C}$.

Finally in Section 8 we say a few words about *enriched cofibrant generation*. As mentioned above, a sequel to this paper will deal with this in greater detail; here, we content ourselves with giving the basic construction and a range of applications. In particular, we will see how to express notions such as *Grothendieck fibrations*, *categories with limits*, and *Kan extensions* in terms of algebras for suitable AWFS, and explain how to extend these constructions to the quasicategorical context.

2. Revision of algebraic weak factorisation systems

Algebraic weak factorisation systems were introduced in [25]—there called *natural* weak factorisation systems—and their theory developed further in [2, 6, 22, 43, 44]. They are highly structured objects, and an undisciplined approach runs the risk of foundering in a morass of calculations. The above papers, taken together, show that a smoother presentation is possible; in this introductory section, we draw together the parts of this presentation so as to give a concise account of the basic aspects of the theory.

Before beginning, let us state our foundational assumptions. κ will be a Grothendieck universe, and sets in κ will be called *small*, while general sets will be called *large*. **Set** and **SET** are the categories of small and large sets; **Cat** and **CAT** are the 2-categories of small categories (ones internal to **Set**) and of locally small categories (ones enriched in **Set**). Throughout the paper, all categories will be assumed to be locally small and all 2-categories will be assumed to be locally small (=**Cat**-enriched) except for ones whose names, like **SET** or **CAT**, are in capital letters.

2.1. **Functorial factorisations.** By a *functorial factorisation* on a category $\mathcal{C}$, we mean a functor $F\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{3}}$ from the category of arrows to that of composable pairs which is a section of the composition functor $\mathcal{C}^{\mathbf{3}} \to \mathcal{C}^{\mathbf{2}}$. We write $F = (L, E, R)$, to indicate that the value of F at an object f or morphism $(h, k)\colon f \to g$ of $\mathcal{C}^{\mathbf{2}}$ is given as on the left and the right of:

$$X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y \qquad\qquad \begin{tikzcd} X \arrow[r, "Lf"] \arrow[d, "h"'] & Ef \arrow[r, "Rf"] \arrow[d, "E(h,k)"] & Y \arrow[d, "k"] \\ W \arrow[r, "Lg"] & Eg \arrow[r, "Rg"] & Z \end{tikzcd} .$$

Associated to (L, E, R) are the functors $L, R\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ with actions on objects $f \mapsto Lf$ and $f \mapsto Rf$, and the natural transformations $\epsilon\colon L \Rightarrow 1$ and $\eta\colon 1 \Rightarrow R$ with components at f given by the commuting squares:

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "Lf"'] & A \arrow[d, "f"] \\ Ef \arrow[r, "Rf"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\ B \arrow[r, "1"] & B \end{tikzcd} . \tag{2.1}$$

Note that (L, E, R) is completely determined by either (L, ϵ) or (R, η).

2.2. **Algebraic weak factorisation systems.** An *algebraic weak factorisation system* (AWFS) on $\mathcal{C}$ comprises:

(i) A functorial factorisation (L, E, R) on $\mathcal{C}$;
(ii) An extension of (L, ϵ) to a comonad $\mathsf{L} = (L, \epsilon, \Delta)$;
(iii) An extension of (R, η) to a monad $\mathsf{R} = (R, \eta, \mu)$;
(iv) A distributive law [9] of the comonad L over the monad R, whose underlying transformation $\delta \colon LR \Rightarrow RL$ satisfies $\mathrm{dom} \cdot \delta = \mathrm{cod} \cdot \Delta$ and $\mathrm{cod} \cdot \delta = \mathrm{dom} \cdot \mu$.

This definition involves less data than may be immediately apparent. The counit and unit axioms $\epsilon L \cdot \Delta = 1$ and $\mu \cdot \eta R = 1$ force the components of $\Delta \colon L \to LL$ and $\mu \colon RR \to R$ to be identities in their respective domain and codomain components, so given by commuting squares as on the left and right of:

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "Lf"'] & A \arrow[d, "LLf"] \\ Ef \arrow[r, "\Delta_f"] & ELf \end{tikzcd} \qquad \begin{tikzcd} Ef \arrow[r, "\Delta_f"] \arrow[d, "LRf"'] & ELf \arrow[d, "RLf"] \\ ERf \arrow[r, "\mu_f"] & Ef \end{tikzcd} \qquad \begin{tikzcd} ERf \arrow[r, "\mu_f"] \arrow[d, "RRf"'] & Ef \arrow[d, "Rf"] \\ B \arrow[r, "1"] & B \end{tikzcd} \; ; \tag{2.2}$$

while the conditions on δ in (iv) force its component at f to be given by the central square. Thus the only additional data beyond the underlying functorial factorisation (i) are maps $\Delta_f \colon Ef \to ELf$ and $\mu_f \colon ERf \to Ef$, satisfying suitable axioms. Note that the comonad L and monad R between them contain all of the data, so that we may unambiguously denote an AWFS simply by (L, R).

By examining (2.1) and (2.2), we see that the comonad L of an AWFS is a comonad *over* the domain functor $\mathrm{dom} \colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$. By this we mean that $\mathrm{dom} \cdot L = \mathrm{dom}$ and that $\mathrm{dom} \cdot \epsilon$ and $\mathrm{dom} \cdot \Delta$ are identity natural transformations. Dually, the monad R is a monad over the codomain functor.

2.3. **Coalgebras and algebras.** To an algebraic weak factorisation system (L, R), we may associate the categories L-**Coalg** and R-**Alg** of coalgebras and algebras for L and R; these are to be thought of as constituting the respective left and right classes of the AWFS. Because L is a comonad over the domain functor, a coalgebra structure $f \to Lf$ on $f \colon A \to B$ necessarily has its domain component an identity, and so is determined by a single map $s \colon B \to Ef$; we write such a coalgebra as $\boldsymbol{f} = (f, s) \colon A \to B$. Dually, an R-algebra structure on $g \colon C \to D$ is determined by a single map $p \colon Ef \to C$, and will be denoted $\boldsymbol{g} = (g, p) \colon C \to D$. If the base category $\mathcal{C}$ has an initial object 0, then we may speak of *algebraically cofibrant objects*, meaning L-coalgebras $0 \to X$; dually, if $\mathcal{C}$ has a terminal object 1, then an *algebraically fibrant object* is an R-algebra $X \to 1$.

2.4. **Lifting of coalgebras against algebras.** Given an L-coalgebra $\boldsymbol{f} = (f, s)$, an R-algebra $\boldsymbol{g} = (g, p)$ and a commuting square as in the centre of:

$$\begin{tikzcd} A' \arrow[r, "a"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B' \arrow[r, "b"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dotted] & D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "c"] \arrow[d, "g"'] & C' \arrow[d, "g'"] \\ D \arrow[r, "d"'] & D' \end{tikzcd} \tag{2.3}$$

there is a canonical diagonal filler $\Phi_{\boldsymbol{f},\boldsymbol{g}}(u,v)\colon B \to C$ given by the composite $p \cdot E(u,v) \cdot s\colon B \to Ef \to Eg \to C$. These fillers are natural with respect to morphisms of L-coalgebras and R-algebras; which is to say that if we have commuting squares as on the left and right above which underlie, respectively, an L-coalgebra morphism $\boldsymbol{f'} \to \boldsymbol{f}$ and an R-algebra morphism $\boldsymbol{g} \to \boldsymbol{g'}$, then composing (2.3) with these two squares preserves the canonical filler:

$$c \cdot \Phi_{\boldsymbol{f},\boldsymbol{g}}(u,v) \cdot b = \Phi_{\boldsymbol{f'},\boldsymbol{g'}}(cua, dvb)\ .$$

Writing $U\colon \mathsf{L}\text{-}\mathbf{Coalg} \to \mathcal{C}^{\mathbf{2}}$ and $V\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$ for the forgetful functors, this naturality may be expressed by saying that the canonical liftings constitute the components of a natural transformation

$$\Phi\colon \mathcal{C}^{\mathbf{2}}(U-, V?) \Rightarrow \mathcal{C}(\mathrm{cod}\ U-, \mathrm{dom}\ V?)\colon \mathsf{L}\text{-}\mathbf{Coalg}^{\mathrm{op}} \times \mathsf{R}\text{-}\mathbf{Alg} \to \mathbf{Set}\ . \tag{2.4}$$

2.5. **Factorisations with universal properties.** The two parts of the factorisation $f = Rf \cdot Lf$ of a map underlie the cofree L-coalgebra $\boldsymbol{L}f = (Lf, \Delta_f)\colon A \to Ef$ and the free R-algebra $\boldsymbol{R}f = (Rf, \mu_f)\colon Ef \to B$. The freeness of the latter says that, for any R-algebra $\boldsymbol{g} = (g,p)$ and morphism $(h,k)\colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$, there is a unique arrow ℓ such that the left-hand diagram in

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "Lf"] \arrow[dd, "f"'] & & C \arrow[dd, "g"] \\
& Ef \arrow[ur, dotted, "\exists!\ell"'] \arrow[d, "Rf"'] & \\
B \arrow[r, "1"] & B \arrow[r, "k"] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[dd, "f"'] & C \arrow[r, "1"] \arrow[d, "Lg"] & C \arrow[dd, "g"] \\
& Ef \arrow[dr, "Rg"] & \\
B \arrow[ur, dotted, "\exists!m"] \arrow[rr, "k"] & & D
\end{tikzcd} \tag{2.5}$$

commutes and such that (ℓ, k) is an algebra morphism $\boldsymbol{R}f \to \boldsymbol{g}$. The dual universal property, as on the right above, describes the co-freeness of $\boldsymbol{L}g$ with respect to maps out of an L-coalgebra $\boldsymbol{f} = (f,s)$.

Observe also the following canonical liftings involving (co)free (co)algebras:

$$\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "\boldsymbol{L}g"'] & A \arrow[d, "(g{,}p)"] \\
Eg \arrow[r, "Rg"'] \arrow[ur, dotted, "p"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "Lf"] \arrow[d, "(f{,}s)"'] & A \arrow[d, "\boldsymbol{R}f"] \\
B \arrow[r, "1"'] \arrow[ur, dotted, "s"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
Ef \arrow[r, "\Delta_f"] \arrow[d, "\boldsymbol{L}Rf"'] & ELf \arrow[d, "\boldsymbol{R}Lf"] \\
ERf \arrow[r, "\mu_f"'] \arrow[ur, dotted, "\Delta_f \cdot \mu_f"] & Ef\ .
\end{tikzcd}$$

(2.6) (2.7) (2.8)

Here, (2.6) implies that an R-algebra is uniquely determined by its liftings against L-coalgebras, and dually in (2.7); while (2.8) expresses precisely the non-trivial axiom of the distributive law $\delta\colon \mathsf{LR} \Rightarrow \mathsf{RL}$.

2.6. **Underlying weak factorisation system.** By the preceding two sections, the classes of maps in $\mathcal{C}$ that admit L-coalgebra or R-algebra structure satisfy all of the axioms needed to be the two classes of a weak factorisation system, except maybe for closure under retracts. On taking the retract-closures of these two classes, we thus obtain a weak factorisation system $(\mathcal{L}, \mathcal{R})$, the *underlying weak factorisation system* of (L, R).

2.7. **Algebras and coalgebras from liftings.** A *coalgebra lifting operation* on a map $g\colon C \to D$ is a function $\varphi_{\text{–}g}$ assigning to each L-coalgebra $\boldsymbol{f}$ and each commuting square $(u, v)\colon f \to g$ as in (2.3) a diagonal filler $\varphi_{\boldsymbol{f},g}(u, v)\colon B \to C$, naturally in maps of L-coalgebras. Maps equipped with coalgebra lifting operations are the objects of a category $\mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ over $\mathcal{C}^{\mathbf{2}}$—the nomenclature will be explained in Section 5.1 below—whose maps $(g, \varphi_{\text{–}g}) \to (g', \varphi_{\text{–}g'})$ are maps $(c, d)\colon g \to g'$ of $\mathcal{C}^{\mathbf{2}}$ for which $c \cdot \varphi_{\boldsymbol{f},g}(u, v) = \varphi_{\boldsymbol{f},g'}(cu, dv)$. Any R-algebra $\boldsymbol{g}$ induces the lifting operation $\Phi_{\text{–}\boldsymbol{g}}$ on its underlying map, and by (2.4) any algebra map respects these liftings; so we have a functor $\bar{\Phi}\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ over $\mathcal{C}^{\mathbf{2}}$.

Lemma 1. $\bar{\Phi}\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ *is injective on objects and fully faithful, and has in its image just those* $(g, \varphi_{\text{–}g})$ *such that* $\varphi_{\boldsymbol{L}f,g}(u, v)\cdot\mu_f = \varphi_{\boldsymbol{LR}f,g}(\varphi_{\boldsymbol{L}f,g}(u, v), v\cdot\mu_f)$ *for all maps* $(u, v)\colon Lf \to g$*:*

$$
\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "\boldsymbol{L}f"'] & A \arrow[r, "u"] \arrow[dd, "\boldsymbol{L}f"] & C \arrow[dd, "g"] \\
Ef \arrow[d, "\boldsymbol{LR}f"'] & & \\
ERf \arrow[r, "\mu_f"'] \arrow[urr, dotted, "\varphi(u,v)\cdot\mu_f"] & Ef \arrow[r, "v"'] \arrow[ur, dotted, "\varphi(u,v)"] & D
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\boldsymbol{L}f"'] & C \arrow[dd, "g"] \\
Ef \arrow[d, "\boldsymbol{LR}f"'] \arrow[ur, dotted, "\varphi(u,v)"] & \\
ERf \arrow[r, "v\cdot\mu_f"'] \arrow[uur, dotted, "\varphi(\varphi(u,v),v\cdot\mu_f)"'] & D
\end{tikzcd}
\tag{2.9}
$$

Proof. We observed in Section 2.5 that an R-algebra is determined by its liftings against coalgebras, whence $\bar{\Phi}$ is injective on objects. For full fidelity, let $\boldsymbol{g} = (g, p)$ and $\boldsymbol{h} = (h, q)$ be R-algebras and $(c, d)\colon (g, \Phi_{\text{–}\boldsymbol{g}}) \to (h, \Phi_{\text{–}\boldsymbol{h}})$ a map of underlying lifting operations; we must show (c, d) is an R-algebra map. Consider the diagram

$$
\begin{tikzcd}
C \arrow[r, "1"] \arrow[d, "\boldsymbol{L}g"'] & C \arrow[r, "c"] \arrow[d, "\boldsymbol{g}"] & D \arrow[d, "\boldsymbol{h}"] \\
Eg \arrow[r, "Rg"'] \arrow[ur, dotted, "p"] & C \arrow[r, "d"'] & C'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "\boldsymbol{L}g"'] & C' \arrow[r, "1"] \arrow[d, "\boldsymbol{L}h"] & C' \arrow[d, "\boldsymbol{h}"] \\
Eg \arrow[r, "E(c,d)"'] & Eh \arrow[r, "Rh"'] \arrow[ur, dotted, "q"] & D
\end{tikzcd}
\;.
$$

By (2.6) $\Phi_{\text{–}\boldsymbol{g}}$ assigns the filler p to the far left square; so as (c, d) respects liftings, $\Phi_{\text{–}\boldsymbol{h}}$ assigns the filler $c \cdot p$ to the composite left rectangle. Likewise $\Phi_{\text{–}\boldsymbol{h}}$ assigns the filler q to the far right square; so as the square to its left is a (cofree) map of L-coalgebras, $\Phi_{\text{–}\boldsymbol{h}}$ assigns the filler $q \cdot E(c, d)$ to the composite right rectangle. Thus $c \cdot p = q \cdot E(c, d)$ and (c, d) is an R-algebra map as required.

We next show that each $(g, \Phi_{\text{–}\boldsymbol{g}})$ in the image of $\bar{\Phi}$ satisfies (2.9). By universality (2.5) and naturality (2.4), it suffices to take $\boldsymbol{g} = \boldsymbol{R}Lf$ and $(u, v) = (LLf, 1)$, and now

$$
\begin{aligned}
\Phi_{\boldsymbol{L}f,\boldsymbol{R}Lf}(LLf, 1) \cdot \mu_f = \Delta_f \cdot \mu_f &= \Phi_{\boldsymbol{LR}f,\boldsymbol{R}Lf}(\Delta_f, \mu_f) \\
&= \Phi_{\boldsymbol{LR}f,\boldsymbol{R}Lf}(\Phi_{\boldsymbol{L}f,\boldsymbol{R}Lf}(LLf, 1), \mu_f)
\end{aligned}
$$

by (2.6) and (2.8), as required. Finally, we show that $\bar{\Phi}$ is surjective onto those pairs $(g, \varphi_{\text{–}g})$ satisfying (2.9). Given such a pair, we define $p = \varphi_{\boldsymbol{L}g,g}(1, Rg)$; now by universality (2.5) and naturality (2.4), the pair $(g, \varphi_{\text{–}g})$ will be the image under $\bar{\Phi}$ of $\boldsymbol{g} = (g, p)$ so long as (g, p) is in fact an R-algebra. The unit axiom is already inherent in p's being a lifting; as for the multiplication axiom, we have

$$\begin{aligned} p \cdot \mu_g &= \varphi_{\boldsymbol{L}g,g}(1, Rg) \cdot \mu_g = \varphi_{\boldsymbol{L}Rg,g}(\varphi_{\boldsymbol{L}g,g}(1, Rg), Rg \cdot \mu_g) \\ &= \varphi_{\boldsymbol{L}Rg,g}(p, RRg) = \varphi_{\boldsymbol{L}Rg,g}(p, Rg \cdot E(p, 1)) \\ &= \varphi_{\boldsymbol{L}g,g}(1, Rg) \cdot E(p, 1) = p \cdot E(p, 1) \end{aligned}$$

by definition of p, (2.9), and naturality of φ_{-g} with respect to the L-coalgebra morphism $(p, E(p, 1)) \colon \boldsymbol{L}Rg \to \boldsymbol{L}g$. $\square$

It is not hard to see that the category $\mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ is in fact isomorphic to the category (R, η)-**Alg** of algebras for the mere pointed endofunctor (R, η) underlying the monad R, with the functor $\bar{\Phi}$ corresponding under this identification to the natural inclusion of R-**Alg** into (R, η)-**Alg**. Of course, all these results have a dual form characterising coalgebras in terms of their liftings against algebras.

2.8. **Double categories of algebras and coalgebras.** In a weak factorisation system the left and right classes of maps are closed under composition and contain the identities. We now describe the analogue of this for AWFS. For binary composition, given R-algebras $\boldsymbol{g} \colon A \to B$ and $\boldsymbol{h} \colon B \to C$ we may define a coalgebra lifting operation $\Phi_{-,\boldsymbol{h}\cdot\boldsymbol{g}}$ on the composite underlying map $h \cdot g$, whose liftings are obtained by first lifting against $\boldsymbol{h}$ and then against $\boldsymbol{g}$, as on the left in:

$$(2.10) \qquad \Phi_{\boldsymbol{f},\boldsymbol{h}\cdot\boldsymbol{g}}(u, v) = \Phi_{\boldsymbol{f},\boldsymbol{g}}(u, \Phi_{\boldsymbol{f},\boldsymbol{h}}(gu, v)) \qquad\qquad \Phi_{\boldsymbol{f},\boldsymbol{1}_A}(u, v) = v \ .$$

It is easy to verify that this is a coalgebra lifting operation satisfying (2.9), so that by Lemma 1, it is the canonical lifting operation associated to a unique R-algebra $\boldsymbol{h} \cdot \boldsymbol{g} \colon A \to C$, the composite of $\boldsymbol{g}$ and $\boldsymbol{h}$. As for nullary composition, each identity map 1_A bears a *unique* coalgebra lifting operation as on the right above, which is easily seen to satisfy (2.9); so each identity map bears an R-algebra structure $\boldsymbol{1}_A$ which is in fact *unique*.

This composition law for algebras is associative and unital: to see associativity, we check that $\boldsymbol{k} \cdot (\boldsymbol{h} \cdot \boldsymbol{g})$ and $(\boldsymbol{k} \cdot \boldsymbol{h}) \cdot \boldsymbol{g}$ have the same lifting operations, and apply Lemma 1; unitality is similar. We may also use Lemma 1 to verify that each $(a, a) \colon \boldsymbol{1}_A \to \boldsymbol{1}_{A'}$ is a map of R-algebras, and that if $(a, b) \colon \boldsymbol{g} \to \boldsymbol{g}'$ and $(b, c) \colon \boldsymbol{h} \to \boldsymbol{h}'$ are maps of R-algebras, then so too is $(a, c) \colon \boldsymbol{h} \cdot \boldsymbol{g} \to \boldsymbol{h}' \cdot \boldsymbol{g}'$. Consequently, there is a double category $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ whose objects and horizontal arrows are those of $\mathcal{C}$, and whose vertical arrows and squares are the R-algebras and the maps thereof. There is a forgetful double functor $U^{\mathsf{R}} \colon \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ into the double category of commutative squares in $\mathcal{C}$, which, displayed as an internal functor between internal categories in **CAT**, is as on the left in:

$$\begin{tikzcd} \mathsf{R}\text{-}\mathbf{Alg} \times_{\mathcal{C}} \mathsf{R}\text{-}\mathbf{Alg} \arrow[r, "U^{\mathsf{R}} \times_{\mathcal{C}} U^{\mathsf{R}}"] \arrow[d, shift right=2, "p"'] \arrow[d, "m"] \arrow[d, shift left=2, "q"] & \mathcal{C}^{\mathbf{3}} \arrow[d, shift right=2, "p"'] \arrow[d, "m"] \arrow[d, shift left=2, "q"] \\ \mathsf{R}\text{-}\mathbf{Alg} \arrow[r, "U^{\mathsf{R}}"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & \mathcal{C}^{\mathbf{2}} \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\ \mathcal{C} \arrow[r, "1"'] \arrow[u, "i"] & \mathcal{C} \arrow[u, "i"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \arrow[r, "V_1 \times_{V_0} V_1"] \arrow[d, shift right=2, "p"'] \arrow[d, "m"] \arrow[d, shift left=2, "q"] & \mathcal{C}^{\mathbf{3}} \arrow[d, shift right=2, "p"'] \arrow[d, "m"] \arrow[d, shift left=2, "q"] \\ \mathcal{A}_1 \arrow[r, "V_1"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & \mathcal{C}^{\mathbf{2}} \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\ \mathcal{A}_0 \arrow[r, "V_0 = 1"'] \arrow[u, "i"] & \mathcal{C} \arrow[u, "i"] \end{tikzcd} \ .$$

Note that this double functor has object component *an identity* and arrow component a *faithful functor*. By a *concrete double category over* $\mathcal{C}$, we mean a double functor $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ (as on the right above) whose V has these two properties. For example, the L-coalgebras also constitute a concrete double category L-$\mathbb{C}\mathbf{oalg}$ over $\mathcal{C}$.

We note before continuing that the equality (2.9) may now be re-expressed as saying that each square as on the left below is one of L-$\mathbb{C}\mathbf{oalg}$, and each square on the right is one of R-$\mathbb{A}\mathbf{lg}$. These squares will be important in what follows.

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "\boldsymbol{L}f"'] & A \arrow[dd, "\boldsymbol{L}f"] \\ Ef \arrow[d, "\boldsymbol{L}Rf"'] & \\ ERf \arrow[r, "\mu_f"'] & Ef \end{tikzcd} \qquad\qquad \begin{tikzcd} Ef \arrow[r, "\Delta_f"] \arrow[dd, "\boldsymbol{R}f"'] & ELf \arrow[d, "\boldsymbol{R}Lf"] \\ & Ef \arrow[d, "\boldsymbol{R}f"] \\ B \arrow[r, "1"'] & B \end{tikzcd} \tag{2.11}$$

2.9. **Morphisms of AWFS.** Given AWFS (L,R) and $(\mathsf{L}',\mathsf{R}')$ on categories $\mathcal{C}$ and $\mathcal{D}$, a *lax morphism* of AWFS $(F,\alpha)\colon (\mathcal{C},\mathsf{L},\mathsf{R}) \to (\mathcal{D},\mathsf{L}',\mathsf{R}')$ comprises a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a natural family of maps α_f rendering commutative each square as on the left in:

$$\begin{tikzcd} & FA \arrow[dl, "L'Ff"'] \arrow[dr, "FLf"] & \\ E'Ff \arrow[rr, "\alpha_f"] \arrow[dr, "R'Ff"'] & & FEf \arrow[dl, "FRf"] \\ & FB & \end{tikzcd} \qquad\qquad \begin{tikzcd} E'Ff \arrow[r, "\alpha_f"] \arrow[d, "E'(\gamma_A,\gamma_B)"'] & FEf \arrow[d, "\gamma_{Ef}"] \\ E'Gf \arrow[r, "\beta_f"'] & GEf \end{tikzcd} \tag{2.12}$$

and such that the induced $(\alpha,1)\colon R'F^{\mathbf{2}} \Rightarrow F^{\mathbf{2}}R$ and $(1,\alpha)\colon L'F^{\mathbf{2}} \Rightarrow F^{\mathbf{2}}L$ are respectively a lax monad morphism $\mathsf{R} \to \mathsf{R}'$ and a lax comonad morphism $\mathsf{L} \to \mathsf{L}'$ over $F^{\mathbf{2}}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{D}^{\mathbf{2}}$ (i.e., a *monad functor* and a *comonad opfunctor* in the terminology of [48]). A transformation $(F,\alpha) \Rightarrow (G,\beta)$ between lax morphisms is a natural transformation $\gamma\colon F \Rightarrow G$ rendering commutative the square above right for each $f\colon A \to B$ in $\mathcal{C}$. Algebraic weak factorisation systems, lax morphisms and transformations constitute a 2-category $\mathbf{AWFS}_{\mathrm{lax}}$. Dually, an *oplax morphism* of AWFS involves maps α_f with the opposing orientation to (2.12), and with the induced $(\alpha,1)$ and $(1,\alpha)$ now being *oplax* monad and comonad morphisms over $F^{\mathbf{2}}$. With a similar adaptation on 2-cells, this yields a 2-category $\mathbf{AWFS}_{\mathrm{oplax}}$.

2.10. **Adjunctions of AWFS.** Given AWFS (L,R) and $(\mathsf{L}',\mathsf{R}')$ on categories $\mathcal{C}$ and $\mathcal{D}$, and an adjunction $F \dashv G\colon \mathcal{C} \to \mathcal{D}$, there is a bijection between 2-cells α exhibiting G as a lax AWFS morphism and 2-cells β exhibiting F as oplax; this is the *doctrinal adjunction* of [35]. From α we determine the components of β by

$$\beta_f = FE'f \xrightarrow{FE'(\eta_A,\eta_B)} FE'GFf \xrightarrow{F\alpha_{Ff}} FGEFf \xrightarrow{\epsilon_{EFf}} EFf \ ,$$

so that β is the *mate* [38] of α under the adjunctions $F \dashv G$ and $F^{\mathbf{2}} \dashv G^{\mathbf{2}}$. The functoriality of this correspondence is expressed through an identity-on-objects isomorphism of 2-categories $\mathbf{AWFS}_{\mathrm{radj}}{}^{\mathrm{coop}} \cong \mathbf{AWFS}_{\mathrm{ladj}}$, where $\mathbf{AWFS}_{\mathrm{radj}}$ is defined identically to $\mathbf{AWFS}_{\mathrm{lax}}$ except that its 1-cells come equipped with chosen left adjoints, and where $\mathbf{AWFS}_{\mathrm{ladj}}$ is defined from $\mathbf{AWFS}_{\mathrm{oplax}}$ dually. By an *adjunction of* AWFS, we mean a morphism of one of these isomorphic 2-categories; thus, a pair of a lax AWFS morphism (G, α) and an oplax AWFS morphism (F, β) whose underlying functors are adjoint, and whose 2-cell data determine each other by mateship. In this situation, an easy calculation shows that

$$\Phi_{\boldsymbol{f}, G\boldsymbol{g}}(u, v) = \Phi_{F\boldsymbol{f}, \boldsymbol{g}}(\bar{u}, \bar{v}) \tag{2.13}$$

for any L'-coalgebra $\boldsymbol{f}$, any R-algebra $\boldsymbol{g}$, and any $(u, v)\colon f \to Gg$ in $\mathcal{D}^{\mathbf{2}}$ with adjoint transpose $(\bar{u}, \bar{v})\colon Ff \to g$ in $\mathcal{C}^{\mathbf{2}}$.

2.11. **Semantics 2-functors.** A lax morphism of AWFS $(F, \alpha)\colon (\mathcal{C}, \mathsf{L}, \mathsf{R}) \to (\mathcal{C}', \mathsf{L}', \mathsf{R}')$ has an underlying lax monad morphism, yielding as in [30, Lemma 1] a lifted functor as to the left in

$$\begin{tikzcd} \mathsf{R}\text{-}\mathbf{Alg} \arrow[r, dotted, "\bar{F}"] \arrow[d, "U^{\mathsf{R}}"'] & \mathsf{R}'\text{-}\mathbf{Alg} \arrow[d, "U^{\mathsf{R}'}"] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "F^{\mathbf{2}}"'] & \mathcal{D}^{\mathbf{2}} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \arrow[r, dotted, "\bar{F}"] \arrow[d, "U^{\mathsf{R}}"'] & \mathsf{R}'\text{-}\mathbb{A}\mathbf{lg} \arrow[d, "U^{\mathsf{R}'}"] \\ \mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathbf{q}(F)"'] & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{tikzcd} \tag{2.14}$$

whose action on algebras we will abusively denote by $\boldsymbol{g} \mapsto F\boldsymbol{g}$. A short calculation from the fact that $(1, \alpha)$ is a lax comonad morphism shows that $F\Phi_{\boldsymbol{L}f, \boldsymbol{g}}(u, v) \cdot \alpha_f = \Phi_{\boldsymbol{L}'Ff, F\boldsymbol{g}}(Fu, Fv \cdot \alpha_f)$ for each R-algebra $\boldsymbol{g}$. From this and (2.10) it follows that $\Phi_{\boldsymbol{L}'Ff, F\boldsymbol{h}\cdot F\boldsymbol{g}}(Fu, Fv \cdot \alpha_{hg}) = \Phi_{\boldsymbol{L}'Ff, F(\boldsymbol{h}\cdot\boldsymbol{g})}(Fu, Fv \cdot \alpha_{hg})$ for all composable R-algebras $\boldsymbol{g}$ and $\boldsymbol{h}$; now taking $f = hg$ and $(u, v) = (1, Rhg)$ and applying (2.6), we conclude that $F\boldsymbol{h} \cdot F\boldsymbol{g} = F(\boldsymbol{h} \cdot \boldsymbol{g})$. Thus the lifted functor on the left of (2.14) preserves algebra composition; it must also preserve the (unique) algebra structures on identities, and so underlies a double functor as on the right. Moreover, each 2-cell $\gamma\colon (F, \alpha) \to (G, \beta)$ in $\mathbf{AWFS}_{\mathrm{lax}}$ has an underlying lax monad transformation, so that $\gamma\colon F \to G$ may be lifted to a transformation on categories of algebras, which—by concreteness—lifts further to a horizontal transformation on double categories.

The preceding constructions have evident duals involving $\mathbf{AWFS}_{\mathrm{oplax}}$ and coalgebras; and in this way, we obtain the left and right *semantics* 2*-functors*:

$$(\text{–})\text{-}\mathbb{C}\mathbf{oalg}\colon \mathbf{AWFS}_{\mathrm{oplax}} \to \mathbf{DBL}^{\mathbf{2}} \qquad (\text{–})\text{-}\mathbb{A}\mathbf{lg}\colon \mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{DBL}^{\mathbf{2}}\ , \tag{2.15}$$

where $\mathbf{DBL}$ denotes the 2-category of double categories, double functors and horizontal transformations. The following result is now [43, Lemma 6.9]:

Proposition 2. *The* 2*-functors* $(\text{–})\text{-}\mathbb{A}\mathbf{lg}$ *and* $(\text{–})\text{-}\mathbb{C}\mathbf{oalg}$ *are* 2*-fully faithful.*

Proof. By duality, we need only deal with the case of $(\text{–})\text{-}\mathbb{A}\mathbf{lg}$. Note that each double functor $\mathbb{S}\mathbf{q}(\mathcal{C}) \to \mathbb{S}\mathbf{q}(\mathcal{D})$ must be of the form $\mathbb{S}\mathbf{q}(F)$ for some $F\colon \mathcal{C} \to \mathcal{D}$; thus given $(\mathcal{C}, \mathsf{L}, \mathsf{R})$ and $(\mathcal{D}, \mathsf{L}', \mathsf{R}')$ in $\mathbf{AWFS}_{\mathrm{lax}}$, a morphism between their images in $\mathbf{DBL}^{\mathbf{2}}$ amounts to a square as on the right of (2.14). Each such has its underlying action on vertical arrows and squares as on the left in (2.14), and so must be induced by a unique lax monad morphism $\gamma \colon \mathsf{R} \to \mathsf{R}'$ over $F^{\mathbf{2}}$. As R and R' are monads over the codomain functor, we must have $\gamma = (\alpha, 1)$ for natural maps α_f as in (2.12). It remains to show that $(1, \alpha)$ is a lax comonad morphism $\mathsf{L} \to \mathsf{L}'$ over $F^{\mathbf{2}}$. The compatibilities in (2.12) already show that $(1, \alpha)$ commutes with counits; we need to establish the same for the comultiplications. Consider the diagrams:

$$
\begin{tikzcd}
E'Ff \arrow[r, "\Delta'_{Ff}"] \arrow[dd, "\boldsymbol{R}'Ff"] & E'L'Ff \arrow[r, "E'(1,\alpha_f)"] \arrow[d, "\boldsymbol{R}'L'Ff"] & E'FLf \arrow[r, "\alpha_{Lf}"] \arrow[d, "\boldsymbol{R}'FLf"] & FELf \arrow[d, "F\boldsymbol{R}Lf"'] \\
 & E'Ff \arrow[r, "\alpha_f"'] \arrow[d, "\boldsymbol{R}'Ff"] & FEf \arrow[r, "1"'] \arrow[d, "F\boldsymbol{R}f"] & FEf \arrow[d, "F\boldsymbol{R}f"'] \\
FB \arrow[r, "1"'] & FB \arrow[r, "1"'] & FB \arrow[r, "1"'] & FB
\end{tikzcd}
\qquad
\begin{tikzcd}
E'Ff \arrow[r, "\alpha_f"] \arrow[dd, "\boldsymbol{R}'Ff"] & FEf \arrow[r, "F\Delta_f"] \arrow[dd, "F\boldsymbol{R}f"] & FELf \arrow[d, "F\boldsymbol{R}Lf"'] \\
 & & FEf \arrow[d, "F\boldsymbol{R}f"'] \\
FB \arrow[r, "1"'] & FB \arrow[r, "1"'] & FB\,.
\end{tikzcd}
$$

Every region here is a square of R'-$\mathbb{A}\mathbf{lg}$: the leftmost as it is of the form (2.11), the rightmost as it is the image under the lifted double functor of another such square in R-$\mathbb{A}\mathbf{lg}$, and all the rest because $(\alpha, 1)$ is a lax monad morphism. So the exterior squares are also squares in R'-$\mathbb{A}\mathbf{lg}$; but as both have the same composite $FLLf$ with the unit morphism $(L'Ff, 1) \colon Ff \to R'Ff$, they must, by freeness of $\boldsymbol{R}'Ff$, coincide; now the equality of their domain-components expresses precisely the required compatibility with comultiplications.

This completes the proof of full fidelity on 1-cells; on 2-cells, it is easy to see that in the commuting diagram of 2-functors

$$
\begin{tikzcd}
\mathbf{AWFS}_{\mathrm{lax}} \arrow[r, "(\text{–})\text{-}\mathbb{A}\mathbf{lg}"] \arrow[d, "(\mathcal{C}, \mathsf{L}, \mathsf{R}) \mapsto (\mathcal{C}^{\mathbf{2}}, \mathsf{R})"'] & \mathbf{DBL}^{\mathbf{2}} \arrow[d, "(\text{–})_1"] \\
\mathbf{MND}_{\mathrm{lax}} \arrow[r, "(\text{–})\text{-}\mathbf{Alg}"] & \mathbf{CAT}^{\mathbf{2}}
\end{tikzcd}
$$

the left and bottom edges are locally fully faithful, and that the right edge is locally fully faithful on the concrete double categories in the image of (–)-$\mathbb{A}\mathbf{lg}$; whence the top edge is also locally fully faithful. □

2.12. **Orthogonal factorisation systems.** Recall that an *orthogonal factorisation system* [18] is a weak factorisation systems $(\mathcal{L}, \mathcal{R})$ wherein liftings (1.1) of $\mathcal{L}$-maps against $\mathcal{R}$-maps are unique. The following result characterises those AWFS arising from orthogonal factorisation systems: it improves on Theorem 3.2 of [25] by requiring idempotency of only *one* of L or R. This resolves the open question posed in Remark 3.3(a) of *ibid.*

Proposition 3. *Let* (L, R) *be an* AWFS *on* $\mathcal{C}$*. The following are equivalent:*

(i) L-$\mathbf{Coalg} \to \mathcal{C}^{\mathbf{2}}$ *is fully faithful;*
(ii) L *is an idempotent comonad;*
(iii) For each L*-coalgebra* $\boldsymbol{f} \colon A \to B$*, there is a coalgebra map* $(f, 1) \colon \boldsymbol{f} \to \mathbf{1}_B$*;*
(iv) R-$\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$ *is fully faithful;*

(v) R *is an idempotent monad;*
(vi) *For each* R*-algebra* $\boldsymbol{g}\colon C \to D$*, there is an algebra map* $(1, g)\colon \mathbf{1}_C \to \boldsymbol{g}$*;*
(vii) *Liftings of* L*-coalgebras against* R*-algebras are unique;*
(viii) *The underlying weak factorisation system* $(\mathcal{L}, \mathcal{R})$ *is orthogonal.*

Under these circumstances, moreover, the AWFS (L, R) *is determined up to isomorphism by the underlying* $(\mathcal{L}, \mathcal{R})$*.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (i) by standard properties of idempotent comonads, and (i) $\Rightarrow$ (iii) is trivial. We next prove (iii) $\Rightarrow$ (vii). So given $\boldsymbol{f} \in \mathsf{L}\text{-}\mathbf{Coalg}$ and $\boldsymbol{g} \in \mathsf{R}\text{-}\mathbf{Alg}$, we must show that any diagonal filler j for a square $(u, v)\colon f \to g$, as on the left below, is equal to the canonical filler $\Phi_{\boldsymbol{f},\boldsymbol{g}}(u, v)$.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dotted, "j"] & D \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[r, "j"] \arrow[d, "1"'] & C \arrow[d, "g"] \\ B \arrow[r, "1"'] & B \arrow[r, "v"'] & D \end{tikzcd} .$$

So factorise (u, v) as on the right above; by (iii), the left-hand square is a coalgebra map $\boldsymbol{f} \to \mathbf{1}_B$, whence by naturality (2.4) we have $j = \Phi_{\mathbf{1}_B,\boldsymbol{g}}(j, v) = \Phi_{\boldsymbol{f},\boldsymbol{g}}(u, v)$ as required. We next prove (vii) $\Rightarrow$ (iv); so for algebras $\boldsymbol{g}$ and $\boldsymbol{h}$, we must show that each $(c, d)\colon g \to h$ underlies an algebra map $\boldsymbol{g} \to \boldsymbol{h}$. By Lemma 1, it suffices to show that (c, d) commutes with the coalgebra lifting functions; in other words, that for each coalgebra $\boldsymbol{f}$ and each $(u, v)\colon f \to g$, we have $c \cdot \Phi_{\boldsymbol{f},\boldsymbol{g}}(u, v) = \Phi_{\boldsymbol{f},\boldsymbol{h}}(cu, dv)$. This is so by (vii) since both these maps fill the square $(cu, dv)\colon f \to h$.

Dual arguments now prove that (iv) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (i); it remains to show that (i)–(vii) are equivalent to (viii). Clearly (viii) $\Rightarrow$ (vii); on the other hand, if (i) and (iv) hold, then any retract of an L-coalgebra or R-algebra is again a coalgebra or algebra—being given by the splitting of an idempotent in $\mathsf{R}\text{-}\mathbf{Alg}$ or $\mathsf{L}\text{-}\mathbf{Coalg}$—so that any map in $\mathcal{L}$ or $\mathcal{R}$ is the underlying map of an L-coalgebra or R-algebra; whence by (vii) liftings of $\mathcal{L}$-maps against $\mathcal{R}$-maps are unique.

Finally, if (i)–(viii) hold, then we can reconstruct L and R up to isomorphism from $\mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$ and $\mathsf{L}\text{-}\mathbf{Coalg} \to \mathcal{C}^{\mathbf{2}}$, and can reconstruct these in turn from $\mathcal{L}$ and $\mathcal{R}$ as the full subcategories of $\mathcal{C}^{\mathbf{2}}$ on the $\mathcal{L}$-maps and the $\mathcal{R}$-maps. ☐

3. A Beck theorem for AWFS

In this section, we give our first main result—Theorem 6 below—which provides an elementary characterisation of the concrete double categories in the essential image of the semantics 2-functors (2.15). This will allow us to construct an AWFS simply by exhibiting a double category of an appropriate form to be one's double category of algebras or coalgebras. As explained in the introduction, the essential images of the corresponding semantics 2-functors for monads and comonads are characterised by Beck's (co)monadicity theorem, and so we term our result a "Beck theorem" for algebraic weak factorisation systems.

3.1. **Reconstruction.** There are two main aspects to the Beck theorem. The first is the following reconstruction result; in the special case where R is *idempotent*, this is [29, Proposition 5.9], while the general case is given as [6, Theorem 4.15]; for the sake of a self-contained presentation, we include a proof here, which improves on that of [6] only in trivial ways.

Proposition 4. *Let $\mathsf{R}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ be a monad over the codomain functor. The right semantics 2-functor (2.15) induces a bijection between extensions of R to an algebraic weak factorisation system (L, R) and extensions of the diagram*

$$\begin{tikzcd} \mathsf{R}\text{-}\mathbf{Alg} \arrow[r, "U^{\mathsf{R}}"] \arrow[d, shift right, "dU^{\mathsf{R}}"'] \arrow[d, shift left, "cU^{\mathsf{R}}"] & \mathcal{C}^{\mathbf{2}} \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ \mathcal{C} \arrow[r, "1"] & \mathcal{C} \end{tikzcd} \tag{3.1}$$

to a concrete double category over $\mathcal{C}$.

Proof. To give an extension of (3.1) to a concrete double category over $\mathcal{C}$ is to give a composition law $\boldsymbol{h}, \boldsymbol{g} \mapsto \boldsymbol{h} \star \boldsymbol{g}$ on R-algebras which is associative, unital, and compatible with R-algebra maps. For each such, we must exhibit a unique (L, R) whose induced R-algebra composition is $\star$. As in Section 2.1, the monad R determines the (L, ϵ) underlying L, and so we need only give the maps Δ_f satisfying appropriate axioms. Consider the diagram on the left in:

$$\begin{tikzcd} A \arrow[rr, "LLf"] \arrow[dd, "f"'] \arrow[dr, "Lf"] & & ELf \arrow[d, "RLf"] \\ & Ef \arrow[ur, dotted, "\Delta_f"] \arrow[d, "Rf"'] & Ef \arrow[d, "Rf"] \\ B \arrow[r, "1"] & B \arrow[r, "1"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] \arrow[dd, "f"'] \arrow[dr, "Lf"] & & B \arrow[dd, "1_B"] \\ & Ef \arrow[ur, dotted, "Rf"] \arrow[d, "Rf"'] & \\ B \arrow[r, "1"] & B \arrow[r, "1"] & B \end{tikzcd} .$$

The outer square commutes, and the right edge bears the R-algebra structure $\boldsymbol{R}f \star \boldsymbol{R}Lf$; now applying (2.5) yields the map Δ_f, with the induced square forming an R-algebra morphism $\boldsymbol{R}f \to \boldsymbol{R}f \star \boldsymbol{R}Lf$. If $\star$ did arise from an AWFS, this induced square would be exactly the right square of (2.11); whence these Δ_f's are the unique possible choice for L's comultiplication. By pasting together appropriate squares as in the proof of Proposition 2—involving algebra squares as on the left above, and also ones $(Rf, 1)\colon \boldsymbol{R}f \to \mathbf{1}_B$ induced by universality as on the right above—we may now check that these Δ_f's satisfy the comonad and distributivity axioms; we thus have the desired (L, R), and it remains only to show that the composition law it induces coincides with $\star$. So let $\boldsymbol{g}$ and $\boldsymbol{h}$ be composable R-algebras, write $f = hg$ for the composite of the underlying maps, and consider the left-hand diagram in:

$$\begin{tikzcd}
A \arrow[rr, "1"] \arrow[dr, "LLf"'] \arrow[dd, "Lf"'] & & A \arrow[d, "g"] \\
& ELf \arrow[ur, dotted, "\exists!\ell"'] \arrow[d, "RLf"'] & B \arrow[dd, "h"] \\
Ef \arrow[r, "1"'] \arrow[d, "Rf"'] & Ef \arrow[ur, dotted, "\exists!m"'] \arrow[d, "Rf"'] & \\
C \arrow[r, "1"] & C \arrow[r, "1"] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
Ef \arrow[r, "\Delta_f"] \arrow[dd, "Rf"'] & ELf \arrow[r, "\ell"] \arrow[d, "RLf"'] & A \arrow[d, "g"] \\
& Ef \arrow[r, "m"] \arrow[d, "Rf"'] & B \arrow[d, "h"] \\
C \arrow[r, "1"] & C \arrow[r, "1"] & C\,.
\end{tikzcd}$$

The dotted maps m and ℓ are obtained by successively applying (2.5) to the maps $(g,1)\colon f \to h$ and $(1,m)\colon Lf \to g$; thus the two squares on the far right above are maps of R-algebras $\boldsymbol{RLf} \to \boldsymbol{g}$ and $\boldsymbol{Rf} \to \boldsymbol{h}$. The rectangle to the left of these squares is a map of R-algebras $\boldsymbol{Rf} \to \boldsymbol{Rf} \star \boldsymbol{RLf}$ by definition, and also one $\boldsymbol{Rf} \to \boldsymbol{Rf} \cdot \boldsymbol{RLf}$, by (2.11). Thus the composite square can be seen both as a map $\boldsymbol{Rf} \to \boldsymbol{RLf} \cdot \boldsymbol{Rf} \to \boldsymbol{h} \cdot \boldsymbol{g}$ and as one $\boldsymbol{Rf} \to \boldsymbol{RLf} \star \boldsymbol{Rf} \to \boldsymbol{h} \star \boldsymbol{g}$. Since precomposing further with the unit $f \to Rf$ yields the identity square on $f = hg$, we conclude by universality of $\boldsymbol{Rf}$ that $(\ell\Delta_f, 1)$ is both the R-algebra structure of $\boldsymbol{h} \cdot \boldsymbol{g}$ and that of $\boldsymbol{h} \star \boldsymbol{g}$, which thus coincide. ☐

3.2. **Monads over the codomain functor.** What is missing from the last result is a characterisation of when a monad on an arrow category is over the codomain functor. Our next result provides this; it generalises the characterisation in [29, Proposition 5.1] of *idempotent* monads over the codomain functor.

Proposition 5. *A monad* R *on* $\mathcal{C}^{\mathbf{2}}$ *is isomorphic to one over the codomain functor if and only if:*

(a) Each identity map has an R*-algebra structure* $\mathbf{1}_A$*;*
(b) For each $f\colon A \to B$*, there is an algebra map* $(f,f)\colon \mathbf{1}_A \to \mathbf{1}_B$*;*
(c) For each R*-algebra* $\boldsymbol{g}\colon A \to B$*, there is an algebra map* $(g,1)\colon \boldsymbol{g} \to \mathbf{1}_B$*.*

Moreover, in this situation, the algebra structures on identity arrows are unique.

Proof. As the properties (a)–(c) are clearly invariant under monad isomorphism, we may assume in the "only if" direction that R is a monad over the codomain functor; now a short calculation shows that $\mathbf{1}_A = (1_A, R1_A)\colon A \to A$ is the unique R-algebra structure on 1_A, and that (b)–(c) are then satisfied. Conversely, let R satisfy (a)–(c), and consider the unit map $(r,s)\colon f \to Rf$, as on the left in:

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[d, "Rf"] \\
B \arrow[r, "s"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & C \arrow[d, "1"] \\
B \arrow[r, "1"] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "t"] \arrow[d, "Rf"] & B \arrow[d, "1"] \\
B \arrow[r, "s"] & D \arrow[r, "u"] & B\,.
\end{tikzcd}$$

We claim that s is invertible. Since Rf underlies the free R-algebra on f, and 1_C has by (a) the algebra structure $\mathbf{1}_C$, the commuting square in the middle above induces a unique algebra map $(t,u)\colon \boldsymbol{Rf} \to \mathbf{1}_C$ making the rightmost diagram commute. In particular $us = 1$, and it remains to show that $su = 1$. Using (b), we have the algebra map $(s,s) \cdot (t,u)\colon \boldsymbol{Rf} \to \mathbf{1}_B \to \mathbf{1}_D$, and using (c) we have $(Rf, 1)\colon \boldsymbol{R}f \to \mathbf{1}_D$. These maps coincide on precomposition with the unit $(r, s)\colon f \to Rf$, as on the left of:

$$
\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "t"] \arrow[d, "Rf"] & B \arrow[r, "s"] \arrow[d, "1"] & D \arrow[d, "1"] \\
B \arrow[r, "s"'] & D \arrow[r, "u"'] & B \arrow[r, "s"'] & D
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "Rf"] \arrow[d, "Rf"] & D \arrow[d, "\mathbf{1}"] \\
B \arrow[r, "s"'] & D \arrow[r, "1"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "1"] \arrow[d, "Rf"'] & C \arrow[d, "R'f"] \\
D \arrow[r, "u"'] & B
\end{tikzcd}
$$

So they must themselves agree; in particular, $su = 1$ and s and u are isomorphisms. So defining $R'f = uRf$, we have isomorphisms $Rf \cong R'f$ for each $f \in \mathcal{C}^{\mathbf{2}}$ as on the right above; now successively transporting the functor R and the monad structure thereon along these isomorphisms yields a monad $\mathsf{R}' \cong \mathsf{R}$ over the codomain functor, as required. $\square$

3.3. **The Beck theorem.** Combining the preceding two results, we obtain our first main theorem, characterising the concrete double categories in the essential image of the semantics 2-functor (–)-$\mathbb{A}\mathbf{lg}$. Of course, we have also the dual result, which we do not trouble to state, characterising the essential image of (–)-$\mathbb{C}\mathbf{oalg}$.

Theorem 6. *The 2-functor (–)-$\mathbb{A}\mathbf{lg}\colon \mathbf{AWFS}_{\text{lax}} \to \mathbf{DBL}^{\mathbf{2}}$ has in its essential image exactly those concrete double categories $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ such that:*

(i) The functor $V_1\colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$ on vertical arrows and squares is strictly monadic;
(ii) For each vertical arrow $\boldsymbol{f}\colon A \to B$ of $\mathbb{A}$, the following is a square of $\mathbb{A}$:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\boldsymbol{f}"'] & B \arrow[d, "\mathbf{1}"] \\
B \arrow[r, "1"'] & B
\end{tikzcd} . \tag{3.2}
$$

Proof. Any $U^{\mathsf{R}}\colon \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ clearly has property (i) while property (ii) follows from Proposition 5. Suppose conversely that the concrete $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ satisfies (i) and (ii). As V_1 is strictly monadic, $\mathcal{A}_1$ is isomorphic over $\mathcal{C}^{\mathbf{2}}$ to the category of algebras for the monad R induced by V_1 and its left adjoint. This R satisfies (a)–(c) of Proposition 5: (c) by virtue of (ii) above, and (a) and (b) using the vertical identities of the double category $\mathbb{A}$. Thus we have a monad $\mathsf{R}' \cong \mathsf{R}$ over the codomain functor; now transporting the double category structure of $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ along the isomorphisms $\mathcal{A}_1 \cong \mathsf{R}\text{-}\mathbf{Alg} \cong \mathsf{R}'\text{-}\mathbf{Alg}$ and $\mathcal{C}^{\mathbf{2}}$ yields $V'\colon \mathbb{A}' \to \mathbb{S}\mathbf{q}(\mathcal{C})$ which, by Proposition 4, is in the image of (–)-$\mathbb{A}\mathbf{lg}$. $\square$

We will call a concrete double category *right-connected* if it satisfies (ii) above, and *monadic right-connected* if it satisfies (i) and (ii). Now combining Theorem 6 with Proposition 4 and the remarks preceding it yields the following result; again, there is a dual form which we do not state characterising $\mathbf{AWFS}_{\text{oplax}}$.

Corollary 7. *The 2-category $\mathbf{AWFS}_{\text{lax}}$ is equivalent to the full sub-2-category of $\mathbf{DBL}^{\mathbf{2}}$ on the monadic right-connected concrete double categories.*

We conclude this section by describing two variations on our main result which will come in useful from time to time.

3.4. **Discrete pullback-fibrations.** Under mild conditions on the base category $\mathcal{C}$, we may rephrase Proposition 5 so as to obtain a more intuitive characterisation of the concrete double categories over $\mathcal{C}$ in the essential image of the semantics functor. Let us define a functor $p\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$ to be a *discrete pullback-fibration* if, for every $\boldsymbol{g} \in \mathcal{A}$ over $g \in \mathcal{C}^{\mathbf{2}}$ and every pullback square $(h,k)\colon f \to g$, there is a unique arrow $\varphi\colon \boldsymbol{f} \to \boldsymbol{g}$ in $\mathcal{A}$ over (h,k), and this arrow is cartesian:

$$
\exists! \boldsymbol{f} \xrightarrow{\exists! \varphi} \boldsymbol{g} \qquad \longmapsto \qquad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, "g"] \\
B \arrow[r, "k"'] & D
\end{tikzcd} . \tag{3.3}
$$

If $\mathcal{C}$ has all pullbacks, then the codomain functor $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ is a fibration, with the pullback squares in $\mathcal{C}^{\mathbf{2}}$ as its cartesian arrows; now the fact that the lifts (3.3) are cartesian implies that the composite functor $\mathrm{cod} \cdot p\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ is also a Grothendieck fibration, with cartesian liftings preserved and reflected by p.

Proposition 8. *If $\mathcal{C}$ has pullbacks, then a monad R on $\mathcal{C}^{\mathbf{2}}$ is isomorphic to one over the codomain functor if and only if $U^{\mathsf{R}}\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$ is a discrete pullback-fibration.*

Under the hypotheses of this proposition, the composite $\mathrm{cod} \cdot U^{\mathsf{R}}\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}$ is a Grothendieck fibration; which, loosely speaking, is the statement that "R-algebra structure is stable under pullback".

Proof. First let R be over the codomain functor. Given an R-algebra $\boldsymbol{g} = (g,p)\colon C \to D$ and pullback square as in (3.3), form the unique map $q\colon Ef \to A$ with $fq = Rf$ and $hq = p \cdot E(h,k)$. This easily yields an R-algebra $(f,q)\colon A \to B$ for which $(h,k)\colon (f,q) \to (g,p)$ is a cartesian algebra map; moreover, any q' for which (h,k) were a map $(f,q') \to (g,p)$ would have to satisfy the defining conditions of q; whence q is unique as required. Conversely, suppose that U^{R} is a discrete pullback-fibration. Arguing as in Proposition 5, it suffices to show that each unit-component $(r,s)\colon f \to Rf$ has s invertible. Form the pullback of Rf along s and induced map t as on the left in:

$$
\begin{tikzcd}
A \arrow[r, dotted, "t"] \arrow[rr, bend left, "r"] \arrow[d, "f"'] & E \arrow[r, "u"] \arrow[d, "g"] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, "Rf"] \\
B \arrow[r, "1"'] & B \arrow[r, "s"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "t"] \arrow[d, "f"'] & E \arrow[d, "g"] \\
B \arrow[r, "1"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "v"] \arrow[d, "Rf"'] & E \arrow[d, "g"] \\
B \arrow[r, "s"'] & D \arrow[r, "w"'] & B
\end{tikzcd} .
$$

As U^{R} is a discrete pullback-fibration, there is a unique R-algebra structure $\boldsymbol{g}$ on g making $(u,s)\colon \boldsymbol{g} \to \boldsymbol{R}f$ an algebra map; now applying freeness of $\boldsymbol{R}f$ to the central square above yields a unique algebra map $(v,w)\colon \boldsymbol{R}f \to \boldsymbol{g}$ making the right-hand diagram commute. In particular, $ws = 1$; on the other hand, the algebra map $(uv, sw)\colon \boldsymbol{R}f \to \boldsymbol{g} \to \boldsymbol{R}f$ precomposes with the unit $(r,s)\colon f \to Rf$ to yield (r,s); whence $(uv,sw) = (1,1)$ and s is invertible as required. $\square$

The evident adaptation of the proof of Theorem 6 now yields:

Theorem 9. *If $\mathcal{C}$ admits pullbacks, then a concrete double category $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ is in the essential image of $(\text{–})\text{-}\mathbb{A}\mathbf{lg}\colon \mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{DBL}^{\mathbf{2}}$ just when:*

(i) $V_1\colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic;
(ii) $V_1\colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$ is a discrete pullback-fibration.

Remark 10. Note that for the "if" direction of this result, $\mathcal{C}$ does not need to have *all* pullbacks; a closer examination of the proof of Proposition 8 shows that only pullbacks along (the underlying maps of) R-algebras are needed. This is useful in practice; categories with pullbacks of this restricted kind arise in the study of stacks [47], or in the categorical foundations of Martin–Löf type theory [20].

3.5. **Intrinsic concreteness.** Above, we have chosen to view the semantics 2-functors (2.15) as landing in the 2-category $\mathbf{DBL}^{\mathbf{2}}$. We have done this so as to stress the analogy with the situation for monads and comonads, and also because this is the most practically useful form of our results. Yet this presentation has some redundancy, as we now explain.

In the presence of the remaining hypotheses, condition (ii) of Theorem 6 is easily seen to be equivalent to the requirement that the codomain functor $c\colon \mathcal{A}_1 \to \mathcal{A}_0$ of the double category $\mathbb{A}$ be a left adjoint left inverse for the identities functor $i\colon \mathcal{A}_0 \to \mathcal{A}_1$ (such *lalis* will appear again in Section 4.2 below). This is a property of $\mathbb{A}$, rather than extra structure; namely the property that

$$\text{each } c_{\boldsymbol{f},iB}\colon \mathcal{A}_1(\boldsymbol{f}, iB) \to \mathcal{A}_0(c\boldsymbol{f}, B) \text{ is invertible.} \tag{3.4}$$

But given only this, we may reconstruct the double functor $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{A}_0)$. First we reconstruct the components (3.2) of the unit $\eta\colon 1 \Rightarrow ic$ using (3.4); now we reconstruct $V_1\colon \mathcal{A}_1 \to \mathcal{A}_0^{\mathbf{2}}$ as the functor classifying the natural transformation

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, bend left, "1"] \arrow[r, bend right, "ic"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{A}_1 \arrow[r, "d"] & \mathcal{A}_0 \end{tikzcd}\ ; \tag{3.5}$$

finally, by pasting together unit squares and using (3.4), we verify that $V = (\mathrm{id}, V_1)$ preserves vertical composition, and so is a double functor. If $W\colon \mathbb{B} \to \mathbb{S}\mathbf{q}(\mathcal{B}_0)$ is another concrete double category whose W is determined in this manner, then *any* double functor $F\colon \mathbb{A} \to \mathbb{B}$ will by (3.4) satisfy $F_1 \cdot \eta_{\mathbb{A}} = \eta_{\mathbb{B}} \cdot F_1$, and so $\mathbb{S}\mathbf{q}(F_0) \cdot V = W \cdot F$; that is, any double functor $\mathbb{A} \to \mathbb{B}$ is concrete.

Let us, therefore, define a double category $\mathbb{A}$ to be *right-connected* if it satisfies (3.4), and *monadic right-connected* if in addition the induced functor $\mathcal{A}_1 \to (\mathcal{A}_0)^{\mathbf{2}}$ is strictly monadic. The preceding discussion now shows that:

Proposition 11. *The assignation $(\mathcal{C}, \mathsf{L}, \mathsf{R}) \mapsto \mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ gives an equivalence of 2-categories between $\mathbf{AWFS}_{\mathrm{lax}}$ and the full sub-2-category of $\mathbf{DBL}$ on the monadic right-connected double categories.*

4. Double categories at work

As a first illustration of the usefulness of the double categorical approach and our Theorem 6, we use it to construct a variety of algebraic weak factorisation systems, some well-known, and some new.

4.1. **Split epimorphisms.** Given a category $\mathcal{C}$, we write $\mathbf{SplEpi}(\mathcal{C})$ for the category of split epimorphisms therein: objects are pairs of a map $g\colon A \to B$ of $\mathcal{C}$ together with a section p of g, while morphisms $(g,p) \to (h,q)$ are serially commuting diagrams as on the left in:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, shift right, "g"'] & C \arrow[d, shift right, "h"'] \\ B \arrow[r, "s"'] \arrow[u, shift right, "p"'] & D \arrow[u, shift right, "q"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} 1 \arrow[r, "m"] \arrow[dr, "1"'] & 0 \arrow[d, "e"] \arrow[dr, "me"] & \\ & 1 \arrow[r, "m"'] & 0\,. \end{tikzcd} \tag{4.1}$$

Split epimorphisms compose—by composing the sections—so that we have a double category $\mathbb{S}\mathbf{plEpi}(\mathcal{C})$ which is concrete over $\mathcal{C}$ and easily seen to be right-connected. It will thus be the double category of algebras of an AWFS on $\mathcal{C}$ whenever $U\colon \mathbf{SplEpi}(\mathcal{C}) \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic.

Now, we may identify U with $\mathcal{C}^j\colon \mathcal{C}^{\mathcal{S}} \to \mathcal{C}^{\mathbf{2}}$, where $\mathcal{S}$ is the *free split epimorphism*, drawn above right, and $j\colon \mathbf{2} \to \mathcal{S}$ is the evident inclusion. Thus U strictly creates colimits, and so will be strictly monadic whenever it has a left adjoint: which will be so whenever $\mathcal{C}$ is cocomplete enough to admit left Kan extensions along j. Using the Kan extension formula one finds that only binary coproducts are required; the free split epi Rf on $f\colon A \to B$ is $\langle f, 1\rangle\colon A+B \to B$ with section ι_B, while the unit $f \to Rf$ is given by:

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "f"'] & A+B \arrow[d, shift right, "\langle f,1\rangle"'] \\ B \arrow[r, "1"'] & B\,. \arrow[u, shift right, "\iota_B"'] \end{tikzcd} \tag{4.2}$$

The remaining structure of the AWFS for split epis can be calculated using Proposition 4. We find that each coproduct inclusion admits a L-coalgebra structure; while if $\mathcal{C}$ is lextensive, then this coalgebra structure is *unique*, and the category of L-coalgebras is precisely the category of coproduct injections and pullback squares. This was proved in [25, Proposition 4.2], but see also Example 12(iii) below.

Finally, let us observe an important property of the split epi AWFS. Although $\mathsf{R} = (R, \eta, \mu)$ is a monad, its algebras—the split epis—are simultaneously the algebras for its underlying pointed endofunctor (R, η); which is to say that the monad R is *algebraically-free* [36, §22] on its underlying pointed endofunctor. We will make use of this property in Proposition 16 below.

4.2. **Lalis.** In the terminology of Gray [26], a *lali* (left adjoint left inverse) in a 2-category $\mathcal{C}$ is a split epi $(g,p)\colon A \to B$ with the extra property that $g \dashv p$ with identity counit. This property may be expressed either by requiring that for each $x\colon A \to X$ and $y\colon B \to X$ in $\mathcal{C}$, the function

$$(\text{–}) \cdot p\colon \mathcal{C}(A,X)(x, yg) \to \mathcal{C}(B,X)(xp, y) \tag{4.3}$$

be invertible; or by requiring the provision of a—necessarily unique—unit 2-cell $\eta\colon 1 \Rightarrow pg$ satisfying $g\eta = 1$ and $\eta p = 1$. Lalis in $\mathcal{C}$ form a category $\mathbf{Lali}(\mathcal{C})$, wherein a morphism is a commuting diagram as on the left of (4.1); it is automatic by invertibility of (4.3) that such a morphism also commutes with the unit 2-cells.

Since split epis and adjoints compose, so too do lalis; thus—writing $\mathcal{C}_0$ for the underlying category of $\mathcal{C}$—lalis in $\mathcal{C}$ form a concrete sub-double category $\mathbb{L}\mathbf{ali}(\mathcal{C})$ of $\mathbb{S}\mathbf{plEpi}(\mathcal{C}_0)$ which, since it is full on cells, inherits right-connectedness. So $\mathbb{L}\mathbf{ali}(\mathcal{C})$ will be the double category of algebras for an AWFS on $\mathcal{C}_0$ whenever $U\colon \mathbf{Lali}(\mathcal{C}) \to (\mathcal{C}_0)^{\mathbf{2}}$ is strictly monadic. Now, we may identify U with restriction

$$\mathbf{2\text{-}CAT}(j, \mathcal{C})\colon \mathbf{2\text{-}CAT}(\mathcal{L}, \mathcal{C}) \to \mathbf{2\text{-}CAT}(\mathbf{2}, \mathcal{C})$$

along the inclusion $j\colon \mathbf{2} \to \mathcal{L}$ of $\mathbf{2}$ into the *free lali* $\mathcal{L}$—which has the same underlying category as the free split epimorphism $\mathcal{S}$ and a single non-trivial 2-cell $1 \Rightarrow me$; so as before, monadicity obtains whenever $\mathcal{C}$ is cocomplete enough (as a 2-category) to admit left Kan extensions along j. In this case, the colimits needed are *oplax colimits of arrows*: given $f\colon C \to B$, its oplax colimit is the universal diagram as on the left of:

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\ell"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[dl, "p"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{1}{\Rightarrow}"] & B \arrow[dl, "1"] \\ & B\,. & \end{tikzcd}$$

Applying the one-dimensional aspect of this universality to the cocone on the right yields a retraction $g\colon A \to B$ for p; now applying the two-dimensional aspect shows that the pair (g, p) verifies invertibility in (4.3), and so is a lali, the free lali on f, with as unit $f \to Rf$ the map $(\ell, 1)$.

When $\mathcal{C} = \mathbf{Cat}$, we may characterise the corresponding category of L-coalgebras as the subcategory of $\mathbf{Cat}^{\mathbf{2}}$ whose objects are the *categorical cofibrations*—those functors arising as pullbacks of the domain inclusion $\mathbf{1} \to \mathbf{2}$—and whose morphisms are the pullback squares; see example (iv) in Section 4.4 below.[1]

On reversing or inverting the non-trivial 2-cell of $\mathcal{L}$, it becomes the free *rali* (right adjoint left inverse) $\mathcal{L}^{\mathrm{co}}$ or the free *retract equivalence* $\mathcal{L}^{g}$; from which we obtain AWFS whose algebras are ralis or retract equivalences on any 2-category that admits lax colimits or pseudocolimits of arrows. When $\mathcal{C} = \mathbf{Cat}$, the respective L-coalgebra structures on a morphism f are unique, and exist just when f is a pullback of the codomain inclusion $\mathbf{1} \to \mathbf{2}$, respectively, when f is injective on objects.

4.3. **Via cocategories.** The preceding examples fit into a common framework. Let $\mathcal{V}$ be a suitable base for enriched category theory, and suppose that we are given a cocategory object $\mathbb{A}$ in $\mathcal{V}$-$\mathbf{Cat}$:

$$\mathbb{A}_0 = \mathbf{I} \;\overset{d}{\underset{c}{\rightrightarrows}}\!\!\!\overset{i}{\leftarrow}\; \mathbb{A}_1 \;\overset{p}{\underset{q}{\rightrightarrows}}\!\!\!\overset{m}{\leftarrow}\; \mathbb{A}_2$$

[1] This characterisation would extend to any suitably exact 2-category, except that there is no account of the appropriate two-dimensional exactness notion; it appears to be closely related to the one mentioned in the introduction to [49].

with object of co-objects the unit $\mathcal{V}$-category, with d and c jointly bijective on objects, and with $i \dashv c$ with identity unit. For any $\mathcal{V}$-category $\mathcal{C}$, the image of $\mathbb{A}$ under the limit-preserving $\mathcal{V}\text{-}\mathbf{CAT}(-,\mathcal{C})\colon \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$ is an internal category $\mathbb{A}(\mathcal{C})$ in $\mathbf{CAT}$—a double category—which is right-connected and concrete over $\mathcal{C}_0$, the underlying ordinary category of $\mathcal{C}$. The induced comparison functor $\mathbb{A}(\mathcal{C})_1 \to (\mathcal{C}_0)^{\mathbf{2}}$ is the functor

$$\mathcal{V}\text{-}\mathbf{CAT}(j,\mathcal{C})\colon \mathcal{V}\text{-}\mathbf{CAT}(\mathbb{A}_1,\mathcal{C}) \to \mathcal{V}\text{-}\mathbf{CAT}(\mathbf{2},\mathcal{C}) \tag{4.4}$$

where $j\colon \mathbf{2} \to \mathbb{A}_1$ is the functor from the free $\mathcal{V}$-category on an arrow which classifies the d-component of the unit $\eta\colon 1 \Rightarrow ci\colon \mathbb{A}_1 \to \mathbb{A}_1$. This j is bijective on objects, and so (4.4) will be monadic whenever $\mathcal{C}$ is sufficiently cocomplete that each $\mathcal{V}$-functor $\mathbf{2} \to \mathcal{C}$ admits a left Kan extension along j; whereupon Theorem 6 induces an AWFS on $\mathcal{C}$ with $\mathbb{A}(\mathcal{C})$ as its double category of algebras. Note that the assignation $\mathcal{C} \mapsto \mathbb{A}(\mathcal{C})$ is clearly 2-functorial in $\mathcal{C}$, so that by Proposition 11, it underlies a 2-functor $\mathcal{V}\text{-}\mathbf{CAT}' \to \mathbf{AWFS}_{\mathrm{lax}}$ defined on all $\mathcal{V}$-categories with sufficient colimits and all $\mathcal{V}$-functors (not necessarily cocontinuous) between them.

Of course, the preceding examples are instances of this framework on taking $\mathcal{V} = \mathbf{Set}$ with $\mathbb{A}_1 = \mathcal{S}$ (for split epis); or taking $\mathcal{V} = \mathbf{Cat}$ with $\mathbb{A}_1 = \mathcal{L}$, $\mathcal{L}^{\mathrm{co}}$ or $\mathcal{L}^g$ (for lalis, ralis or retract equivalences).

4.4. **Stable classes of monics.** Let $\mathcal{C}$ be a category with pullbacks, and consider a class $\mathcal{M}$ of monics which contains the isomorphisms, is closed under composition, and is stable under pullback along maps of $\mathcal{C}$; we call this a *stable class of monics*. The class $\mathcal{M}$ is said to be *classified* if the category $\mathcal{M}_{\mathrm{pb}}$ of pullback squares between $\mathcal{M}$-maps has a terminal object, which we call a *generic $\mathcal{M}$-map*. A standard argument [31, p. 24] shows that the domain of a generic $\mathcal{M}$-map must be terminal in $\mathcal{C}$.

Since $\mathcal{M}$-maps and pullback squares compose, the category $\mathcal{M}_{\mathrm{pb}}$ underlies a double category $\mathbb{M}_{\mathrm{pb}}$ concrete over $\mathcal{C}$. The monicity of the $\mathcal{M}$'s ensures that for each $m \in \mathcal{M}$, the square on the left below is a pullback and so in $\mathcal{M}_{\mathrm{pb}}$. Thus $\mathbb{M}_{\mathrm{pb}}$ is left-connected, and so will comprise the L-coalgebras of an AWFS on $\mathcal{C}$ as soon as the forgetful $U\colon \mathcal{M}_{\mathrm{pb}} \to \mathcal{C}^{\mathbf{2}}$ is comonadic.

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "1"'] & A \arrow[d, "m"] \\ A \arrow[r, "m"'] & B \end{tikzcd} \qquad\qquad \begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "a"'] & B_1 \arrow[r, shift left, "g_1"] \arrow[r, shift right, "h_1"'] \arrow[d, "b"'] & C_1 \arrow[d, "c"'] \\ A_2 \arrow[r, "f_2"'] & B_2 \arrow[r, shift left, "g_2"] \arrow[r, shift right, "h_2"'] & C_2 \end{tikzcd}$$

First we note that U always strictly creates equalisers. Indeed, if $b \rightrightarrows c$ is a parallel pair in $\mathcal{M}_{\mathrm{pb}}$ as on the far right above, then on forming the equaliser $a \to b$ in $\mathcal{C}^{\mathbf{2}}$, as to the left, the equalising square will be a pullback—since c is monic and both rows are equalisers—and so will lift uniquely to $\mathcal{M}_{\mathrm{pb}}$. Thus U will be strictly comonadic as soon as it has a right adjoint.

For this, we must assume that the generic $\mathcal{M}$-map $p\colon 1 \to \Sigma$ is exponentiable; it follows that every $\mathcal{M}$-map is exponentiable, since exponentiable maps are pullback-stable in any category with pullbacks (see [50, Corollary 2.6], for example). In particular, each map $B \times p\colon B \to B \times \Sigma$ is exponentiable, and so for any $f\colon A \to B$

of $\mathcal{C}$, we may form the object $\Pi_{B\times p}(f)$ of $\mathcal{C}/(B\times\Sigma)$; which has the universal property that, in the category $\mathcal{P}_f$ whose objects are pullback diagrams as on the left in

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\ell"'] & A \arrow[r, "f"] & B \arrow[d, "B\times p"] \\ Y \arrow[rr, "k"'] & & B\times\Sigma \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\varepsilon"] \arrow[d, "\eta"'] & A \arrow[r, "f"] & B \arrow[d, "B\times p"] \\ D \arrow[rr, "\Pi_{B\times p}(f)"'] & & B\times\Sigma \end{tikzcd}$$

and whose maps $(X,Y,h,k,\ell)\to(X',Y',h',k',\ell')$ are maps $X\to X'$ and $Y\to Y'$ satisfying the obvious equalities, there is a terminal object as on the right. Now, an object of $\mathcal{P}_f$ determines and is determined by a pair of squares

$$\begin{tikzcd} X \arrow[r, "h_1"] \arrow[d, "\ell"'] & A \arrow[r, "f"] & B \arrow[d, "1"] \\ Y \arrow[rr, "k_1"'] & & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "!"] \arrow[d, "\ell"'] & 1 \arrow[d, "p"] \\ Y \arrow[r, "k_2"'] & \Sigma \end{tikzcd}$$

with the right-hand one a pullback. This ensures that ℓ—as a pullback of p—is an $\mathcal{M}$-map, whereupon by genericity of p, the map k_2 is uniquely determined. Thus we may identify $\mathcal{P}_f$ with the comma category $U\downarrow f$, and the terminal object therein provides the value $\eta\colon C\to D$ of the desired right adjoint at f.

For the (L,R) induced in this way, the restriction of the monad R to the slice over B is the *partial $\mathcal{M}$-map classifier monad* on $\mathcal{C}/\mathcal{B}$; see [46] and the references therein. Note also that, in the terminology of [17], the $\Pi_{B\times p}(f)$ constructed above is a *pullback complement* of f and $B\times p$.

Examples 12. (i) Let $\mathcal{E}$ be an elementary topos, and $\mathcal{M}$ the stable class of all monics. $\mathcal{M}$ is classified by $\top\colon 1\to\Omega$, which—like any map in a topos—is exponentiable; and so we have an AWFS on $\mathcal{E}$ for which L-**Coalg** is the category of monomorphisms and pullback squares. The corresponding R-algebras are discussed in [39]. More generally, we can let $\mathcal{E}$ be any quasitopos, and $\mathcal{M}$ the class of strong monomorphisms.

(ii) Let $\mathcal{E}$ be an elementary topos, j a Lawvere-Tierney topology on $\mathcal{E}$, and $\mathcal{M}$ the class of j-dense monomorphisms. This is a stable class of monics classified by $\top\colon 1\to J$, where J is the equaliser of $j,\top\colon\Omega\to\Omega$. So j-dense monomorphisms and their pullbacks form the coalgebras of an AWFS on $\mathcal{E}$. An algebraically fibrant object is a "weak sheaf": an object equipped with coherent, but not unique, choices of patchings for j-covers. An object X is a sheaf if and only if both $X\to 1$ and $X\to X\times X$ admit R-algebra structure.

(iii) Let $\mathcal{C}$ be a lextensive category and $\mathcal{M}$ the stable class of coproduct injections. This is classified by $\iota_1\colon 1\to 1+1$; which by extensivity is always exponentiable, with $\Pi_{\iota_1}=(-)+1\colon\mathcal{C}\to\mathcal{C}/(1+1)$. More generally, the right adjoint to pullback along $B\times\iota_1$ is $(-)+B\colon\mathcal{C}/B\to\mathcal{C}/(B+B)$; whence the AWFS generated on $\mathcal{C}$ is that for split epis. This re-proves the fact that, in any lextensive category, the coalgebras of the AWFS for split epis are the coproduct injections and pullback squares.

(iv) Consider, as in Section 4.2, the stable class of categorical cofibrations in **Cat**. This is classified by the domain functor $d\colon \mathbf{1} \to \mathbf{2}$, which is exponentiable in **Cat**, so that we have an AWFS wherein L-coalgebras are the categorical cofibrations and pullback squares. For any category B, the functor $\Pi_{B\times d}\colon \mathbf{Cat}/B \to \mathbf{Cat}/(B\times \mathbf{2})$ sends $f \in \mathbf{Cat}/B$ to the induced map between the oplax colimits of f and of 1_B—the latter being simply $B \times \mathbf{2}$—and it follows that this AWFS is the one for lalis. This proves the claim about its coalgebras made in Section 4.2 above.

(v) By a *left cofibration* of simplicial sets, we mean a pullback of the face inclusion $\delta_1\colon \Delta[0] \to \Delta[1]$. The left cofibrations constitute a stable class of monics, classified by δ_1; and since **SSet** is a topos, this δ_1 is exponentiable. We thus have an AWFS on simplicial sets whose L-coalgebras are left cofibrations and pullbacks. The corresponding R-coalgebras we call the *simplicial lalis*; we will have more to say about them in Section 8.2 below.

4.5. **Projective and injective liftings.** Our next example is a construction which, under suitable circumstances, allows us to lift an AWFS $(\mathcal{D}, \mathsf{L}, \mathsf{R})$ along a functor $F\colon \mathcal{C} \to \mathcal{D}$ to yield an AWFS on $\mathcal{C}$. There are two ways of doing this, well known from the model category literature: we either lift the algebras (as in [16]) or the coalgebras (as in [8]). In the former case, we form the pullback on the left in:

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, "V"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \arrow[d, "U^{\mathsf{R}}"] \\ \mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathbf{q}(F)"] & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_1 \arrow[r] \arrow[d, "V_1"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{R}\text{-}\mathbf{Alg} \arrow[d, "U^{\mathsf{R}}"] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "F^{\mathbf{2}}"] & \mathcal{D}^{\mathbf{2}} \end{tikzcd} . \tag{4.5}$$

The double category $\mathbb{A}$ therein, as a pullback of right-connected double categories, is itself right-connected; while the component functor V_1 of V as on the right above is a pullback of the strictly monadic U^{R}, whence easily seen to strictly create coequalisers for V_1-absolute coequaliser pairs. So whenever V_1 admits a left adjoint, $\mathbb{A}$ will by Theorem 6 constitute the algebra double category of an AWFS $(\mathsf{L}', \mathsf{R}')$ on $\mathcal{C}$, the *projective lifting* of (L, R) along F. The left square of (4.5) then amounts to a lax morphism of AWFS $(\mathcal{C}, \mathsf{L}', \mathsf{R}') \to (\mathcal{D}, \mathsf{L}, \mathsf{R})$, which is easily seen to be a cartesian lifting of F along the forgetful functor $\mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{CAT}$.

Dually, we may form the pullback of double categories on the left in

$$\begin{tikzcd} \mathbb{B} \arrow[r] \arrow[d, "W"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg} \arrow[d, "U^{\mathsf{L}}"] \\ \mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathbf{q}(F)"] & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}_1 \arrow[r] \arrow[d, "W_1"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{L}\text{-}\mathbf{Coalg} \arrow[d, "U^{\mathsf{L}}"] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "F^{\mathbf{2}}"] & \mathcal{D}^{\mathbf{2}} \end{tikzcd} ; \tag{4.6}$$

if on doing so, the component W_1 of W as to the right has a right adjoint, then $\mathbb{B}$ will be the coalgebra double category of an AWFS on $\mathcal{C}$, the *injective lifting* of (L, R) along F. The left square of (4.6) now corresponds to an oplax map of AWFS providing a cartesian lifting of F with respect to the forgetful $\mathbf{AWFS}_{\mathrm{oplax}} \to \mathbf{CAT}$.

Sometimes, it may be that the required adjoints to V_1 or W_1 simply exist. For example, the projective and injective liftings of an AWFS $(\mathcal{C}, \mathsf{L}, \mathsf{R})$ along a forgetful functor $\mathcal{C}/X \to \mathcal{C}$ always exist, and coincide; the factorisations of the resultant *slice* AWFS are given by

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "a"'] & & B \arrow[dl, "b"] \\ & X & \end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd} A \arrow[r, "Lf"] \arrow[dr, "a"'] & Ef \arrow[r, "Rf"] \arrow[d, "b \cdot Rf"] & B \arrow[dl, "b"] \\ & X & \end{tikzcd} .$$

More typically, we will appeal to a general result like the following, which ensures that the desired adjoint exists without necessarily giving a closed formula for it. As in the introduction, we call an AWFS on a locally presentable category *accessible* if its comonad L and monad R are so, in the sense of preserving κ-filtered colimits for some regular cardinal κ; in fact, it is easy to see that accessibility of the comonad implies that of the monad, and vice versa.

Proposition 13. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally presentable categories, and (L, R) an accessible* AWFS *on $\mathcal{D}$.*

(a) The projective lifting of (L, R) along any right adjoint $F \colon \mathcal{C} \to \mathcal{D}$ exists and is accessible;

(b) The injective lifting of (L, R) along any left adjoint $F \colon \mathcal{C} \to \mathcal{D}$ exists and is accessible.

Proof. For (a), let $F \colon \mathcal{C} \to \mathcal{D}$ be a right adjoint. To show the projective lifting exists, we must prove that the functor V_1 to the right of (4.5) has a left adjoint. Now, $\mathcal{C}^{\mathbf{2}}$ and $\mathcal{D}^{\mathbf{2}}$ are locally presentable since $\mathcal{C}$ and $\mathcal{D}$ are, by [19, §7.2(h)], while R-**Alg** is locally presentable since R is accessible, by [19, Satz 10.3]; moreover, both $F^{\mathbf{2}}$ and U^{R} are right adjoints. Since U^{R} has the isomorphism-lifting property, its pullback against $F^{\mathbf{2}}$ is also a bipullback [34]; but by [10, Theorem 2.18], the 2-category of locally presentable categories and right adjoint functors is closed under bilimits in **CAT**, whence V_1, like $F^{\mathbf{2}}$ and U^{R}, lies in this 2-category; thus it has a left adjoint K_1 and is accessible by [19, Satz 14.6]. So the projective lifting of (L, R) along F exists, and is accessible as its monad $V_1 K_1$ is so.

For (b), let $F \colon \mathcal{C} \to \mathcal{D}$ be a left adjoint; we must show that W_1 to the right of (4.6) has a right adjoint. We now argue using [10, Theorem 3.15], which states that the 2-category of locally presentable categories and left adjoint functors is closed under bilimits in **CAT**. As before, $F^{\mathbf{2}} \colon \mathcal{C}^{\mathbf{2}} \to \mathcal{D}^{\mathbf{2}}$ lies in this 2-category, and $U^{\mathsf{L}} \colon \mathsf{L}\text{-}\mathbf{Coalg} \to \mathcal{D}^{\mathbf{2}}$ will so long as L-**Coalg** is in fact locally presentable. But L is an accessible comonad on the accessible category $\mathcal{D}^{\mathbf{2}}$; by [41, Theorem 5.1.6], the 2-category of accessible categories and accessible functors is closed in **CAT** under all bilimits, in particular under Eilenberg-Moore objects of comonads, and so L-**Coalg** is accessible; it is also cocomplete (since U^{L} creates colimits) and so is locally presentable, as required. We thus conclude that the functor W_1, like U^{L} and $F^{\mathbf{2}}$, is a left adjoint between locally presentable categories; while by [19, Satz 14.6], its right adjoint G_1 is accessible. So the projective lifting of (L, R) along F exists, and is accessible as its comonad $W_1 G_1$ is so. $\square$

In fact, we may drop the requirement of local presentability from the first part of the preceding proposition if we strengthen the hypotheses on U.

Proposition 14. *Let $\mathcal{D}$ be a cocomplete category, let T be an accessible monad on $\mathcal{D}$ and let (L, R) be an accessible* AWFS *on $\mathcal{D}$. Then the projective lifting of (L, R) along the forgetful functor $U^{\mathsf{T}} \colon \mathsf{T}\text{-}\mathbf{Alg} \to \mathcal{D}$ exists and is accessible.*

Proof. As in (4.5), we form the pullback of double categories as to the left in

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, "V"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \arrow[d, "U^{\mathsf{R}}"] \\ \mathbb{S}\mathbf{q}(\mathsf{T}\text{-}\mathbf{Alg}) \arrow[r, "\mathbb{S}\mathbf{q}(U^{\mathsf{T}})"] & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_1 \arrow[r] \arrow[d, "V_1"'] & \mathsf{R}\text{-}\mathbf{Alg} \arrow[d] \\ (\mathsf{T}\text{-}\mathbf{Alg})^{\mathbf{2}} \arrow[r, "(U^{\mathsf{T}})^{\mathbf{2}}"] & \mathcal{D}^{\mathbf{2}} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{S} & \mathsf{R} \arrow[l, "\sigma"'] \\ \mathsf{T}^{\mathbf{2}} \arrow[u, "\nu"] & \mathrm{id} \arrow[l, "!"] \arrow[u, "!"'] \end{tikzcd}$$

and must show that the underlying 1-component V_1, as in the centre, has a left adjoint. Now, the monad R on $\mathcal{D}^{\mathbf{2}}$ is accessible by assumption, while $\mathsf{T}^{\mathbf{2}}$ is accessible since T is so; whence by [36, Theorem 27.1] the coproduct $\mathsf{S} = \mathsf{T}^{\mathbf{2}} + \mathsf{R}$ exists in $\mathbf{MND}(\mathcal{D}^{\mathbf{2}})$ and is accessible. This coproduct is equally the pushout on the right above, and [36] guarantees that the functor $(–)$-$\mathbf{Alg}$ sends this square to a pullback of categories, so that we may identify V_1 with $\nu^* \colon \mathsf{S}\text{-}\mathbf{Alg} \to \mathsf{T}^{\mathbf{2}}\text{-}\mathbf{Alg} = (\mathsf{T}\text{-}\mathbf{Alg})^{\mathbf{2}}$. Now as $\mathcal{D}^{\mathbf{2}}$ is cocomplete and S is accessible, this ν^* has a left adjoint by [36, Theorem 25.4]. □

A standard application of the preceding results is to the construction of AWFS on diagram categories. If the cocomplete $\mathcal{C}$ bears an accessible AWFS (L, R), and $\mathcal{I}$ is a small category, then the functor $U \colon \mathcal{C}^{\mathcal{I}} \to \mathcal{C}^{\mathrm{ob}\,\mathcal{I}}$ given by precomposition with $\mathrm{ob}\,\mathcal{I} \to \mathcal{I}$ has a left adjoint (given by left Kan extension) and is strictly monadic; so that by Proposition 14 we may projectively lift the pointwise AWFS on $\mathcal{C}^{\mathrm{ob}\,\mathcal{I}}$ along U to obtain the *projective* AWFS on $\mathcal{C}^{\mathcal{I}}$. If $\mathcal{C}$ is moreover complete, then U also has a right adjoint given by right Kan extension; if it is moreover locally presentable then we may apply Proposition 13(b) to injectively lift the pointwise AWFS along U, so obtaining the *injective* AWFS on $\mathcal{C}^{\mathcal{I}}$.

5. Cofibrant generation by a category

For our next application of Theorem 6, we use it to give a simplified treatment of *cofibrantly generated* algebraic weak factorisation systems. An AWFS is cofibrantly generated when its R-algebras are precisely morphisms equipped with a choice of liftings against some small category $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ of “generating cofibrations”. This extends and tightens the familiar notion of cofibrant generation of a weak factorisation system by a set of generating cofibrations; and the main result of [22] extends and tightens Quillen’s small object argument to show that, under commonly-satisfied conditions on $\mathcal{C}$, the AWFS cofibrantly generated by any small $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ exists. The proof given there is quite involved; we will give an simpler one exploiting Theorem 6. First we recall the necessary background.

5.1. **Lifting operations.** Given categories $U \colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ and $V \colon \mathcal{K} \to \mathcal{C}^{\mathbf{2}}$ over $\mathcal{C}^{\mathbf{2}}$, a $(\mathcal{J}, \mathcal{K})$-*lifting operation* is a natural family of functions $\varphi_{j,k}$ assigning to each

$j \in \mathcal{J}$ and $k \in \mathcal{K}$ and each commuting square

$$\begin{tikzcd} \mathrm{dom}\, Uj \arrow[r, "u"] \arrow[d, "Uj"'] & \mathrm{dom}\, Vk \arrow[d, "Vk"] \\ \mathrm{cod}\, Uj \arrow[r, "v"] \arrow[ur, dotted, "\varphi_{j,k}(u,v)"] & \mathrm{cod}\, Vk \end{tikzcd}$$

a diagonal filler as indicated making both triangles commute. Naturality here expresses that the $\varphi_{j,k}$'s are components of a natural transformation $\varphi\colon \mathcal{C}^{\mathbf{2}}(U-, V?) \Rightarrow \mathcal{C}(\mathrm{cod}\, U-, \mathrm{dom}\, V?)\colon \mathcal{J}^{\mathrm{op}} \times \mathcal{K} \to \mathbf{Set}$. Thus Section 2.4 describes the canonical (L-**Coalg**, R-**Alg**)-lifting operation associated to any AWFS.

The assignation taking $(\mathcal{J}, \mathcal{K})$ to the collection of $(\mathcal{J}, \mathcal{K})$-lifting operations is easily made the object part of a functor $\mathbf{Lift}\colon (\mathbf{CAT}/\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \times (\mathbf{CAT}/\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \to \mathbf{SET}$; we now have, as in [22, Proposition 3.8]:

Proposition 15. *Each functor* $\mathbf{Lift}(\mathcal{J}, -)$ *and* $\mathbf{Lift}(-, \mathcal{K})$ *is representable, so that we induce an adjunction*

$$(\mathbf{CAT}/\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \underset{(-)^{\pitchfork}}{\overset{{}^{\pitchfork}(-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{CAT}/\mathcal{C}^{\mathbf{2}} \ . \tag{5.1}$$

Proof. The category $\mathcal{J}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ representing $\mathbf{Lift}(\mathcal{J}, -)$, has as objects, pairs of $g \in \mathcal{C}^{\mathbf{2}}$ and φ_{-g} a $(\mathcal{J}, g)$-lifting operation; and as maps $(g, \varphi_{-g}) \to (h, \varphi_{-h})$, those $g \to h$ of $\mathcal{C}^{\mathbf{2}}$ which commute with the lifting functions. Dually, the representing category ${}^{\pitchfork}\mathcal{K} \to \mathcal{C}^{\mathbf{2}}$ for $\mathbf{Lift}(-, \mathcal{K})$ comprises objects $f \in \mathcal{C}^{\mathbf{2}}$ equipped with $(f, \mathcal{K})$-lifting operations, together with the maps of $\mathcal{C}^{\mathbf{2}}$ between them that commute with the lifting functions. □

We have seen a particular instance of this result in Section 2.7 above; the category L-$\mathbf{Coalg}^{\pitchfork}$ described there is the representing object for $\mathbf{Lift}(\mathsf{L}\text{-}\mathbf{Coalg}, -)$, and the functor $\bar{\Phi}\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ is the one induced by this representation from the canonical lifting operation of Section 2.4.

5.2. **Cofibrant generation.** If the maps $g\colon C \to D$ and $h\colon D \to E$ of $\mathcal{C}$ are equipped with lifting operations φ_{-g} and φ_{-h} against a category $U\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$, then their composite hg also bears a lifting operation φ_{-hg}, defined as in (2.10) by $\varphi_{j,hg}(u, v) = \varphi_{j,g}(u, \varphi_{j,h}(gu, v))$. This composition, together with the equipment of an identity map with its *unique* lifting operation, provides the necessary vertical structure to make $\mathcal{J}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ into a concrete double category $\mathcal{J}^{\pitchfork\!\!\!\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$; dually, we can make ${}^{\pitchfork}\mathcal{K} \to \mathcal{C}^{\mathbf{2}}$ into a concrete double category ${}^{\pitchfork\!\!\!\pitchfork}\mathcal{K} \to \mathbb{S}\mathbf{q}(\mathcal{C})$.

We now define an AWFS (L, R) on $\mathcal{C}$ to be *cofibrantly generated* by a small $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ if $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \cong \mathcal{J}^{\pitchfork\!\!\!\pitchfork}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$. If this isomorphism is verified for a $\mathcal{J}$ which is large, we say instead that $\mathcal{C}$ is *class-cofibrantly generated* by $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$. There are dual notions of fibrant or class-fibrant generation, involving an isomorphism ${}^{\pitchfork\!\!\!\pitchfork}\mathcal{K} \cong \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$; however, these are markedly less prevalent than their duals in categories of mathematical interest.

5.3. **Existence of cofibrantly generated awfs.** In [22, Definition 3.9] is given the notion of an AWFS being "algebraically-free" on $U\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$: to which the notion of cofibrant generation given above, though apparently different in form, is in fact equivalent.[2] Theorem 4.4 of *ibid.* thus guarantees, among other things, that in a locally presentable category $\mathcal{C}$, the AWFS cofibrantly generated by *any* small $U\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ exists. We now use Theorem 6 to give a shorter proof of this.

Proposition 16. *If $\mathcal{C}$ is locally presentable then the* AWFS (L, R) *cofibrantly generated by any small $U\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ exists; its underlying monad R is algebraically-free on a pointed endofunctor and accessible.*

This result as stated is less general than [22, Theorem 4.4], which also deals with certain kinds of non-locally presentable $\mathcal{C}$. Though we have no need for this extra generality here, let us note that reincorporating it would simply be a matter of adapting the final paragraph of the following proof.

Proof. It is easy to see that $V\colon \mathcal{J}^{\pitchfork\!\!\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ is right-connected; so by Theorem 6, it will comprise the double category of algebras for the desired AWFS as soon as $V_1\colon \mathcal{J}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic. Now an object of $\mathcal{J}^{\pitchfork}$ is equally a map g of $\mathcal{C}$ together with a section φ_{-g} of the natural transformation

$$\psi_{-g}\colon \mathcal{C}(\mathrm{cod}\ U{-}, \mathrm{dom}\ g) \Rightarrow \mathcal{C}^{\mathbf{2}}(U{-}, g)\colon \mathcal{J}^{\mathrm{op}} \to \mathbf{Set} \tag{5.2}$$

whose j-component sends $m\colon \mathrm{cod}\ Uj \to C$ to $(m \cdot Uj, gm)\colon Uj \to g$; while a map $(g, \varphi_{-g}) \to (h, \varphi_{-h})$ of $\mathcal{J}^{\pitchfork}$ is a map $g \to h$ of $\mathcal{C}^{\mathbf{2}}$ for which the induced $\psi_{-g} \to \psi_{-h}$ in $[\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]^{\mathbf{2}}$ commutes with the sections. We thus have a pullback as on the left in

$$\begin{tikzcd}
\mathcal{J}^{\pitchfork} \arrow[r] \arrow[d, "V_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{SplEpi}([\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]) \arrow[d] \\
\mathcal{C}^{\mathbf{2}} \arrow[r, "\psi"] & {[\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]^{\mathbf{2}}}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
K\psi \arrow[r, "K\eta\psi"] \arrow[d, "\varepsilon"'] & KT\psi \arrow[d] \\
1 \arrow[r, "\rho"] & P \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{5.3}$$

We first show that ψ has a left adjoint. The composite $\mathrm{cod} \cdot \psi\colon \mathcal{C}^{\mathbf{2}} \to [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]$ is the singular functor $\mathcal{C}^{\mathbf{2}}(U, 1)$ which has left adjoint F_1 given by the left Kan extension of $U\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ along the Yoneda embedding; while $\mathrm{dom} \cdot \psi$ is the functor $\mathcal{C}(\mathrm{cod} \cdot U, \mathrm{dom}) \cong \mathcal{C}^{\mathbf{2}}(\mathrm{id} \cdot \mathrm{cod} \cdot U, 1)$ (as $\mathrm{id} \dashv \mathrm{dom}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$) and so also has a left adjoint F_2. The natural transformation $\mathrm{cod} \cdot \psi \to \mathrm{dom} \cdot \psi$ induces one $\alpha\colon F_2 \to F_1$, using which ψ admits a left adjoint K sending $f\colon X \to Y$ to the pushout of $F_2 f\colon F_2 X \to F_2 Y$ along $\alpha_X\colon F_2 X \to F_1 X$.

Now as we noted in Section 4.1 above, $\mathbf{SplEpi}([\mathcal{J}^{\mathrm{op}}, \mathbf{Set}])$ may be identified with the category of algebras for a pointed endofunctor (T, η) on $[\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]^{\mathbf{2}}$ with unit $\eta_f\colon f \to Tf$ given by (4.2). It follows that $\mathcal{J}^{\pitchfork}$ is isomorphic over $\mathcal{C}^{\mathbf{2}}$ to the category of algebras for the pointed endofunctor (P, ρ) in the pushout square above right (c.f. [53, Theorem 2.1]). An easy consequence of this identification is that V_1 strictly creates coequalisers of V_1-absolute coequaliser pairs; so it will be strictly monadic whenever it has a left adjoint—that is, whenever free (P, ρ)-algebras exist. We show this existence using results of [36].

[2] This follows from Proposition 22 below.

Since $\mathcal{C}$ is locally presentable, there is a regular cardinal κ such that the domain and codomain of each Uj is κ-presentable; thus the domain and codomain of each $\psi_{j-}\colon \mathcal{C}(\mathrm{cod}\ Uj, \mathrm{dom}\ -) \Rightarrow \mathcal{C}^{\mathbf{2}}(Uj, -)$ preserves κ-filtered colimits, so that ψ does so too. Now K, being a left adjoint, is cocontinuous, while T is so by inspection of (4.2); whence the P of (5.3) is a pushout of functors preserving κ-filtered colimits, and so also preserves them. Thus by [36, Theorem 22.3], free (P, ρ)-algebras exist, which is to say that V_1 has a left adjoint F_1 as required. Since P preserves κ-filtered colimits, V_1 creates them, and so the induced monad $\mathsf{R} = V_1 F_1$ preserves them; it is thus accessible. Finally, it follows from [36, Proposition 22.2] that R is the algebraically-free monad on the pointed endofunctor (P, ρ). □

This result provides a rich source of algebraic weak factorisation systems; in particular, we may make any cofibrantly generated weak factorisation system on a locally presentable category into an AWFS by applying the result to its set J of generating cofibrations, seen as a discrete category over $\mathcal{C}^{\mathbf{2}}$. For applications that make serious use of the algebraicity so obtained, see [21, 23]; for applications utilising the extra generality of cofibrant generation by a small *category*, rather than a small set of morphisms, see [5, 7].

5.4. **Inadequacy of cofibrant generation by a category.** Proposition 16 tells us that the monad of an AWFS cofibrantly generated by a small category has two specific properties: it is accessible, and it is algebraically-free on a pointed endofunctor. Our experience of monads suggests that the former property should be more common than the latter, and in fact this is the case: many AWFS of practical interest verify the accessibility, but not the freeness. For example:

Proposition 17. *The monad of the* AWFS *for lalis on* **Cat** *is accessible, but not algebraically-free on a pointed endofunctor; in particular, this* AWFS *is not cofibrantly generated by a small category.*

Proof. The monad R at issue is given by left Kan extension and restriction along the 2-functor $j\colon \mathbf{2} \to \mathcal{L}$ of Section 4.2; it is thus cocontinuous and in particular, accessible. On the other hand, if it were algebraically-free on a pointed endofunctor, then its category of algebras—$\mathbf{Lali}(\mathbf{Cat})$—would be isomorphic to the category of algebras for a pointed endofunctor, and as such would be retract-closed: meaning that, for any lali $f \dashv u$ and any retract g of f in $\mathbf{Cat}^{\mathbf{2}}$, there would exist lali structure on g. Now, to equip the unique functor $\mathcal{A} \to 1$ with the structure of a lali is to specify a terminal object of $\mathcal{A}$; so it is enough to describe a category $\mathcal{A}$ with a terminal object, and a retract $\mathcal{B}$ of $\mathcal{A}$ without one. To this end, take $\mathcal{B}$ to be the free idempotent $t^2 = t$ as on the left in:

$$
\begin{tikzcd} 0 \arrow[loop left, "t"] \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} 0 \arrow[loop left, "t"] \arrow[r, "!"] & 1 \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} 0 \arrow[loop left, "t"] \end{tikzcd}
$$

and take $\mathcal{A}$ to be $\mathcal{B}$ with a terminal object 1 freely adjoined. The inclusion of $\mathcal{B}$ into $\mathcal{A}$ admits a retraction, which fixes t and sends $!\colon 0 \to 1$ to the identity on 0. Yet $\mathcal{B}$ does not admit a terminal object. □

In particular, this result tells us that, since lalis are not closed under retracts, they cannot be characterised as the right class of maps for a mere weak factorisation system; thus the algebraicity is, in this case, essential.

6. Cofibrant generation by a double category

In light of Proposition 17, it is natural to ask whether there is a more refined notion of cofibrant generation which encompasses such examples of AWFS as the one for lalis on **Cat**. In this section, we describe such a notion; it involves lifting properties against a small *double* category, rather than a small category, of generating cofibrations.

6.1. **Double-categorical lifting operations.** Let $U\colon \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ and $V\colon \mathbb{K} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ be double categories over $\mathbb{S}\mathbf{q}(\mathcal{C})$. We define a $(\mathbb{J}, \mathbb{K})$-*lifting operation* to be a $(\mathcal{J}_1, \mathcal{K}_1)$-lifting operation in the sense of Section 5.1 which is also compatible with vertical composition in $\mathbb{J}$ and $\mathbb{K}$, in the sense that

$$\begin{aligned} \varphi_{j,\ell\cdot k}(u,v) &= \varphi_{j,k}(u, \varphi_{j,\ell}(Vk\cdot u, v)) \\ \text{and} \quad \varphi_{j\cdot i,k}(u,v) &= \varphi_{j,k}(\varphi_{i,k}(u, v\cdot Uj), v) \end{aligned} \tag{6.1}$$

for all vertically composable maps $j\cdot i\colon A \to B \to C$ and $\ell \cdot k\colon D \to E \to F$ in $\mathbb{J}$ and in $\mathbb{K}$. For example, by virtue of (2.10) and its dual, the lifting operation (2.4) associated to an AWFS is an $(\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}, \mathsf{R}\text{-}\mathbb{A}\mathbf{lg})$-lifting operation. As before, the assignation sending $\mathbb{J}$ and $\mathbb{K}$ to the collection of $(\mathbb{J}, \mathbb{K})$-lifting operations underlies a functor $\mathbb{L}\mathbf{ift}\colon (\mathbf{DBL}/\mathbb{S}\mathbf{q}(\mathcal{C}))^{\mathrm{op}} \times (\mathbf{DBL}/\mathbb{S}\mathbf{q}(\mathcal{C}))^{\mathrm{op}} \to \mathbf{SET}$; and also as before, we have:

Proposition 18. *Each functor* $\mathbb{L}\mathbf{ift}(\mathbb{J}, \text{–})$ *and* $\mathbb{L}\mathbf{ift}(\text{–}, \mathbb{K})$ *is representable, so that the adjunction* (5.1) *extends to one*

$$(\mathbf{DBL}/\mathbb{S}\mathbf{q}(\mathcal{C}))^{\mathrm{op}} \underset{(\text{–})^{\pitchfork}}{\overset{{}^{\pitchfork}(\text{–})}{\underset{\perp}{\leftrightarrows}}} \mathbf{DBL}/\mathbb{S}\mathbf{q}(\mathcal{C}) \ . \tag{6.2}$$

Proof. Given $U\colon \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$, we have as in Section 5.2 the concrete double category ${\mathcal{J}_1}^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$. The representing object $\mathbb{J}^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ for $\mathbb{L}\mathbf{ift}(\mathbb{J}, \text{–})$ is the sub-double category of ${\mathcal{J}_1}^{\pitchfork}$ with the same objects and horizontal arrows, and just those vertical arrows (g, φ_{-g}) whose lifting operations respect vertical composition in $\mathbb{J}$, together with all cells between them. The representing object ${}^{\pitchfork}\mathbb{K} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ for $\mathbb{L}\mathbf{ift}(\text{–}, \mathbb{K})$ is defined dually. $\square$

Note that the 2-functor $(\text{–})_1\colon \mathbf{DBL} \to \mathbf{CAT}$ sending a double category to its vertical category is represented by the free vertical arrow $\mathbf{2}_v$, and consequently has a left adjoint, sending $\mathcal{C}$ to the product of $\mathbf{2}_v$ with the free *horizontal* double category on $\mathcal{C}$. This lifts to a left adjoint $\mathbb{F}$ for the induced functor $(\text{–})_1\colon \mathbf{DBL}/\mathbb{S}\mathbf{q}(\mathcal{C}) \to \mathbf{CAT}/\mathcal{C}^{\mathbf{2}}$ on slice categories; and as there are no non-trivial vertical composites in $\mathbb{F}\mathcal{J}$, we have that in fact $\mathcal{J}^{\pitchfork} = (\mathbb{F}\mathcal{J})^{\pitchfork}$; thus the double category structure on $\mathcal{J}^{\pitchfork}$, for which we offered no abstract justification in Section 5.2 above, is now explained in terms of the adjunction (6.2).

6.2. **Cofibrant generation by a double category.** An algebraic weak factorisation system (L, R) is said to be *cofibrantly generated* by a small double category $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ if $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \cong \mathbb{J}^{\pitchfork}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$; if we have this isomorphism for a large $\mathbb{J}$, then we call (L, R) *class-cofibrantly* generated by $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$. Once again, we have the dual notions of fibrant and class-fibrant generation by a double category.

If, as in the preceding section, we identify a category over $\mathcal{C}^{\mathbf{2}}$ with the free double category thereon, then this definition is a conservative extension of that given in Section 5.2 above. However, by contrast with Proposition 17, we have:

Proposition 19. *The* AWFS *for lalis on* **Cat** *is cofibrantly generated by a small double category.*

Proof. We first claim that to equip a functor $f\colon A \to B$ with lali structure is equally to give:

- A section $u\colon \mathrm{ob}\, B \to \mathrm{ob}\, A$ of the action of f on objects; and
- For each $a \in A$ and $b \in B$, a section $\gamma_a\colon B(fa, b) \to A(a, ub)$ of the action of f on morphisms; such that
- $\gamma_a(\beta \cdot f\alpha) = \gamma_{a'}(\beta) \cdot \alpha$ for all $\alpha\colon a \to a'$ in A and $\beta\colon fa' \to b$ in B; and
- $\gamma_{ub}(1_b) = 1_{ub}$ for all $b \in B$.

Indeed, if f is part of a lali $f \dashv p$ with unit η, then we obtain these data by taking u to be the action of p on objects, and taking $\gamma_a(\beta) = p(\beta) \cdot \eta_a$. Conversely, given these data, we define a section p of f on objects by $p(b) = u(b)$ and on morphisms by $p(\beta\colon b \to b') = \gamma_{ub}(\beta)\colon ub \to ub'$, and define a unit $\eta\colon 1 \Rightarrow pf$ with components $\eta_a = \gamma_a(1_{fa})$. This proves the claim.

We now define a small $V\colon \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathbf{Cat})$ for which $\mathbb{J}^{\pitchfork} \cong \mathbb{L}\mathbf{ali}(\mathbf{Cat})$. The objects of $\mathbb{J}$ are the ordinals $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$, and its horizontal arrows are order-preserving maps; on these, V acts as the identity. The vertical arrows of $\mathbb{J}$ are freely generated by morphisms $k\colon \mathbf{0} \to \mathbf{1}$, $\ell\colon \mathbf{1} \to \mathbf{2}$ and $m\colon \mathbf{2} \to \mathbf{3}$, which are sent by V to the appropriate initial segment inclusions between ordinals. Its squares are freely generated by the following four:

$$
\begin{tikzcd}
\mathbf{0} \arrow[r, "!"] \arrow[d, "k"'] \arrow[dr, phantom, "\text{(a)}"] & \mathbf{1} \arrow[d, "\ell"] \\
\mathbf{1} \arrow[r, "\delta_0"'] & \mathbf{2}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \arrow[r, "\delta_0"] \arrow[d, "\ell"'] \arrow[dr, phantom, "\text{(b)}"] & \mathbf{2} \arrow[d, "m"] \\
\mathbf{2} \arrow[r, "\delta_0"'] & \mathbf{3}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \arrow[r, "\delta_1"] \arrow[d, "\ell"'] \arrow[dr, phantom, "\text{(c)}"] & \mathbf{2} \arrow[d, "m"] \\
\mathbf{2} \arrow[r, "\delta_1"'] & \mathbf{3}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{0} \arrow[r, "\mathrm{id}"] \arrow[d, "k"'] & \mathbf{0} \arrow[dd, "k"] \\
\mathbf{1} \arrow[d, "\ell"'] \arrow[r, phantom, "\text{(d)}"] & \\
\mathbf{2} \arrow[r, "!"'] & \mathbf{1}
\end{tikzcd}
,
$$

where (following standard simplicial notation) δ_0 and δ_1 denote the order-preserving injections omitting 0 and 1 respectively. We claim that $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathbf{Cat})$ cofibrantly generates the AWFS for lalis; in other words, that $\mathbb{J}^{\pitchfork} \cong \mathbb{L}\mathbf{ali}(\mathbf{Cat})$ over $\mathbb{S}\mathbf{q}(\mathbf{Cat})$. Indeed, given a functor $f\colon A \to B$, we see that:

- To equip f with a Vk-lifting operation is to give a section u of its action on objects;
- To equip f with a $V\ell$-lifting operation is to give, for every $a \in A$ and $\beta\colon fa \to b$ in B, a map $\gamma_a(\beta)\colon a \to \bar{b}$ in A over β; to ask for the compatibility (a) with the Vk-lifting operation is to ask that, in fact, $\bar{b} = ub$.
- To equip f with a Vm-lifting operation is to give, for every $\alpha\colon a \to a'$ in A and $\beta\colon fa' \to b$, a map $\psi_\alpha(\beta)\colon a' \to \bar{b}$ in A over β. The compatibility (b) now forces $\psi_\alpha(\beta) = \gamma_{a'}(\beta)$, while that in (c) forces $\psi_\alpha(\beta) \cdot \alpha = \gamma_a(\beta \cdot f\alpha)$; so ψ is determined by γ, and we have $\gamma_a(\beta \cdot f\alpha) = \gamma_{a'}(\beta) \cdot \alpha$.

- Finally, the compatibility (d) forces $\gamma_{ub}(1_b) = 1_{ub}$.

This verifies that vertical arrows of $\mathbb{J}^{\pitchfork\!\!\pitchfork}$ are lalis in **Cat**; similar arguments show that squares of $\mathbb{J}^{\pitchfork\!\!\pitchfork}$ are ones preserving the lali structure, and that composition is composition of lalis, as required. □

6.3. **Canonical class-cofibrant generation.** As further evidence for the adequacy of the notion of double-categorical cofibrant generation, we have the following result, which tells us that any AWFS is class-cofibrantly generated by its (typically large) double category of coalgebras.

Proposition 20. *Any* AWFS *is class-cofibrantly generated by* $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ *and class-fibrantly generated by* $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$.

Proof. By duality, we need prove only the first statement. As above, (2.4) is an $(\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}, \mathsf{R}\text{-}\mathbb{A}\mathbf{lg})$-lifting operation, and so by Proposition 18 corresponds to a concrete double functor $\Lambda \colon \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork\!\!\pitchfork}$ over $\mathcal{C}$. Of course, Λ is the identity on objects and horizontal arrows; while its component $\Lambda_1 \colon \mathsf{R}\text{-}\mathbf{Alg} \to (\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork\!\!\pitchfork})_1$ on vertical arrows and squares postcomposes with the inclusion $(\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork\!\!\pitchfork})_1 \hookrightarrow \mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ to yield the $\bar{\Phi}$ of Lemma 1. As $\bar{\Phi}$ is injective on objects and fully faithful, so too is Λ_1; while since the left square of (2.11) is one of $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}$, the condition (6.1) defining the objects in $(\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork\!\!\pitchfork})_1$ implies that in (2.9), so that Λ_1 is also surjective on objects, and thus an isomorphism. □

The fact of an isomorphism $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg} \cong {}^{\pitchfork\!\!\pitchfork}\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ is often a useful tool for calculation. Many AWFS that arise in practice are cofibrantly generated by double categories in a natural way; as such, we have a concrete understanding of the R-algebras. The above isomorphism offers one technique for obtaining the corresponding L-coalgebras without explicit calculation of the values of L or R.

6.4. **Freeness of cofibrantly generated awfs.** Above, we have defined cofibrant generation of an AWFS in terms of a universal property of its double category of algebras. We conclude this section by showing that this implies another universal property: that its double category of coalgebras is *freely generated* by the given $\mathbb{J}$ with respect to left adjoint functors between concrete double categories. In the case of cofibrant generation by a mere category, this was shown in [43, Theorem 6.22]; our result generalises this, and simplifies the proof.

The key step will be to extend the adjunction (6.2) to account for change of base. To this end, we consider the 2-category $\mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})$, whose objects are double functors $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$, whose 1-cells are squares as on the left of

$$(6.3) \qquad \begin{tikzcd} \mathbb{J} \arrow[r, "\bar{F}"] \arrow[d, "U"'] & \mathbb{J}' \arrow[d, "U'"] \\ \mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathbf{q}(F)"'] & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{J} \arrow[r, bend left, "\bar{F}"] \arrow[r, bend right, "\bar{F}'"'] \arrow[r, phantom, "\Downarrow\bar{\alpha}"] \arrow[d, "U"'] & \mathbb{J}' \arrow[d, "U'"] \\ \mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, bend left, "\mathbb{S}\mathbf{q}(F)"] \arrow[r, bend right, "\mathbb{S}\mathbf{q}(F')"'] \arrow[r, phantom, "\Downarrow\mathbb{S}\mathbf{q}(\alpha)"] & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{tikzcd}$$

together with a chosen right adjoint G for F, and whose 2-cells are diagrams as on the right above. We define the 2-category $\mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}})$ dually.

Proposition 21. *The adjunction* (6.2) *extends to a* 2*-adjunction*

$$\begin{tikzcd} (\mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}}))^{\mathrm{coop}} \arrow[r, shift right, "(-)^{\pitchfork}"'] \arrow[r, phantom, "\perp"] & \mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}}) \arrow[l, shift right, "{}^{\pitchfork}(-)"'] \end{tikzcd} \ , \tag{6.4}$$

with respect to which the canonical isomorphisms $\Lambda\colon \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork}$ *and* $\bar{\Lambda}\colon \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg} \to {}^{\pitchfork}\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ *of Proposition 20 are* 2*-natural.*

Proof. The 2-functor $(-)^{\pitchfork}$ is defined as before on objects, and as before on 1-cells (6.3) whose bottom edge is an identity. To complete the definition on 1-cells, it thus suffices to consider squares whose *top* edge is an identity, as on the left in

$$\begin{tikzcd} \mathbb{J} \arrow[r, "1"] \arrow[d, "U"'] & \mathbb{J} \arrow[d, "\mathbb{S}\mathbf{q}(F)\cdot U"] \\ \mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathbf{q}(F)"] & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{tikzcd} \qquad \begin{tikzcd} (F\mathbb{J})^{\pitchfork} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{J}^{\pitchfork} \arrow[d] \\ \mathbb{S}\mathbf{q}(\mathcal{D}) \arrow[r, "\mathbb{S}\mathbf{q}(G)"] & \mathbb{S}\mathbf{q}(\mathcal{C}) \end{tikzcd} \ . \tag{6.5}$$

Let $F\mathbb{J}$ denote the double category $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{D})$ over $\mathbb{S}\mathbf{q}(\mathcal{D})$ down the right edge of this square, and let G be the chosen right adjoint for F. We claim that there is in fact a *pullback* square as displayed right above: which we will take to be the value of $(-)^{\pitchfork}$ at the left-hand square. For this, we must show that for each $g\colon C \to D$ of $\mathcal{D}$, the $(F\mathbb{J}, g)$-lifting operations $\varphi_{-,g}$ correspond functorially with $(\mathbb{J}, Gg)$-lifting operations $\bar{\varphi}_{-,Gg}$. But such lifting operations are given by natural transformations

$$\varphi_{-,g}\colon \mathcal{C}(\mathrm{cod}\ FU-, \mathrm{dom}\ g) \Rightarrow \mathcal{C}^{\mathbf{2}}(FU-, g)\colon \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$$
$$\bar{\varphi}_{-,Gg}\colon \mathcal{C}(\mathrm{cod}\ U-, \mathrm{dom}\ Gg) \Rightarrow \mathcal{C}^{\mathbf{2}}(U-, Gg)\colon \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$$

satisfying axioms; so as $F \dashv G$, these are in a clear bijective correspondence, under which the maps of lifting operations also correspond as required.

Finally, we must define $(-)^{\pitchfork}$ on 2-cells. Given a diagram as on the right of (6.3), the 2-cell $\alpha\colon F \Rightarrow F'$ therein transposes under adjunction to one $\beta\colon G' \Rightarrow G$. As the forgetful double functors $(\mathbb{J}')^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{D})$ and $\mathbb{J}^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ are concrete, there is a unique possible 2-cell $\bar{\beta}\colon (\bar{F}')^{\pitchfork} \Rightarrow \bar{F}^{\pitchfork}$ lifting β; a short calculation shows that this lifting indeed exists, so that we may take $(\alpha, \bar{\alpha})^{\pitchfork} = (\beta, \bar{\beta})$. This completes the definition of $(-)^{\pitchfork}$; that of ${}^{\pitchfork}(-)$ is dual.

To verify that the extended $(-)^{\pitchfork}$ and ${}^{\pitchfork}(-)$ are 2-adjoint, we observe that for $\mathbb{J}$ and $\mathbb{K}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$ and $\mathbb{S}\mathbf{q}(\mathcal{D})$, a map $\mathbb{J} \to {}^{\pitchfork}\mathbb{K}$ in $\mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})$ is an adjunction $F \dashv G\colon \mathcal{D} \to \mathcal{C}$ together with a $(F\mathbb{J}, \mathbb{K})$-lifting operation, while a map $\mathbb{K} \to \mathbb{J}^{\pitchfork}$ in $\mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}})$ is an adjunction $F \dashv G$ together with a $(\mathbb{J}, G\mathbb{K})$-lifting operation; by the above argument, these data are in bijective correspondence. We argue similarly for the bijection on 2-cells. Finally, the naturality of the isomorphisms Λ and $\bar{\Lambda}$ is exactly (2.13); their 2-naturality now follows by concreteness. $\square$

Using the result, we now give our promised generalisation of [43, Theorem 6.22].

Proposition 22. *If* (L, R) *is (class-)cofibrantly generated by* $U\colon \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$*, then it provides a left* 2*-adjoint at* $\mathbb{J}$ *for* $(-)\text{-}\mathbb{C}\mathbf{oalg}\colon \mathbf{AWFS}_{\mathrm{ladj}} \to \mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})$.

Proof. We have natural isomorphisms

$$\begin{aligned}
\mathbf{AWFS}_{\mathrm{ladj}}((\mathsf{L},\mathsf{R}),\ (\mathsf{L}',\mathsf{R}')) &\cong \mathbf{AWFS}_{\mathrm{radj}}((\mathsf{L}',\mathsf{R}'),\ (\mathsf{L},\mathsf{R}))^{\mathrm{op}}\\
&\cong \mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}})(\mathsf{R}'\text{-}\mathbb{A}\mathbf{lg}, \mathsf{R}\text{-}\mathbb{A}\mathbf{lg})^{\mathrm{op}}\\
&\cong \mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}})(\mathsf{R}'\text{-}\mathbb{A}\mathbf{lg}, \mathbb{J}^{⫛})^{\mathrm{op}}\\
&\cong \mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})(\mathbb{J}, {}^{⫛}\mathsf{R}'\text{-}\mathbb{A}\mathbf{lg})\\
&\cong \mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})(\mathbb{J}, \mathsf{L}'\text{-}\mathbb{C}\mathbf{oalg})\ ,
\end{aligned}$$

by, respectively, the isomorphism $\mathbf{AWFS}_{\mathrm{ladj}} \cong \mathbf{AWFS}_{\mathrm{radj}}{}^{\mathrm{coop}}$; full fidelity of $(-)$-$\mathbb{A}\mathbf{lg}$; the assumed isomorphism $\mathbb{J}^{⫛} \cong \mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$; adjointness (6.4); and the natural isomorphisms of Proposition 20. ☐

7. A characterisation of accessible awfs

We are now ready to give our second main result, Theorem 25, which gives a characterisation of the accessible AWFS on a locally presentable category. We will show that, for a locally presentable $\mathcal{C}$, any small double category over $\mathbb{S}\mathbf{q}(\mathcal{C})$ cofibrantly generates an accessible AWFS, and, moreover, that every accessible AWFS on $\mathcal{C}$ arises in this way.

7.1. **Existence of cofibrantly generated AWFS.** We begin by extending Proposition 16 from cofibrant generation by categories to that by double categories.

Proposition 23. *If $\mathcal{C}$ is locally presentable, then the* AWFS *cofibrantly generated by any small double category $U\colon \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ exists and is accessible.*

Proof. Write $\mathcal{J}_2 = \mathcal{J}_1 \times_{\mathcal{J}_0} \mathcal{J}_1$ for the category of vertically composable pairs in $\mathbb{J}$, and $m\colon \mathcal{J}_2 \to \mathcal{J}_1$ for the vertical composition. Now the triangle on the left in

$$\begin{tikzcd}
\mathcal{J}_2 \arrow[rr, "m"] \arrow[dr, "U_1 m"'] & & \mathcal{J}_1 \arrow[dl, "U_1"] \\
& \mathcal{C}^{\mathbf{2}} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{J}_1{}^{⫛} \arrow[rr, "m^{⫛}"] \arrow[dr] & & \mathcal{J}_2{}^{⫛} \arrow[dl] \\
& \mathbb{S}\mathbf{q}(\mathcal{C}) &
\end{tikzcd}$$

induces a morphism $m^{⫛}$ of concrete double categories over $\mathcal{C}$ as on the right, which on vertical arrows sends $(g, \varphi_{-g})\colon D \to E$ to $(g, \psi_{-g})\colon D \to E$, with $(\mathcal{J}_2, g)$-lifting operation ψ_{-g} defined as on the left in

$$\psi_{(j,i),g}(u,v) = \varphi_{j\cdot i,g}(u,v) \qquad\qquad \theta_{(j,i),g}(u,v) = \varphi_{j,g}(\varphi_{i,g}(u, v\cdot Uj), v)\ .$$

We also have the $(\mathcal{J}_2, g)$ lifting operation θ_{-g} as on the right above, and the assignation $(g, \varphi_{-g}) \mapsto (g, \theta_{-g})$ in fact gives the action on vertical arrows of a further double functor $\delta_{\mathbb{J}}\colon \mathcal{J}_1{}^{⫛} \to \mathcal{J}_2{}^{⫛}$ concrete over $\mathcal{C}$. It is easy to see that we now have an equaliser of concrete double categories over $\mathcal{C}$ as on the left in

$$\begin{tikzcd}
\mathbb{J}^{⫛} \arrow[r] \arrow[dr] & \mathcal{J}_1{}^{⫛} \arrow[r, shift left, "m^{⫛}"] \arrow[r, shift right, "\delta_{\mathbb{J}}"'] \arrow[d] & \mathcal{J}_2{}^{⫛} \arrow[dl] \\
& \mathbb{S}\mathbf{q}(\mathcal{C}) &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{J}_1{}^{\pitchfork} \arrow[r, shift left, "m^{\pitchfork}"] \arrow[r, shift right, "(\delta_{\mathbb{J}})_1"'] \arrow[d, "\cong"'] & \mathcal{J}_2{}^{\pitchfork} \arrow[d, "\cong"] \\
\mathsf{R}_1\text{-}\mathbf{Alg} \arrow[r, shift left, "s^*"] \arrow[r, shift right, "t^*"'] & \mathsf{R}_2\text{-}\mathbf{Alg}\ .
\end{tikzcd}
\tag{7.1}$$

Since ${\mathcal{J}_1}^{\pitchfork\pitchfork}$ and ${\mathcal{J}_2}^{\pitchfork\pitchfork}$ are right-connected, so too will $\mathbb{J}^{\pitchfork\pitchfork}$ be; whence by Theorem 6, the AWFS cofibrantly generated by $\mathbb{J}$ will exist if and only if $(\mathbb{J}^{\pitchfork\pitchfork})_1 \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic. As $\mathcal{J}_1$ and $\mathcal{J}_2$ are small, Proposition 16 ensures that there are AWFS $(\mathsf{L}_1, \mathsf{R}_1)$ and $(\mathsf{L}_2, \mathsf{R}_2)$ on $\mathcal{C}$ with ${\mathcal{J}_1}^{\pitchfork} \cong \mathsf{R}_1\text{-}\mathbf{Alg}$ and ${\mathcal{J}_2}^{\pitchfork} \cong \mathsf{R}_2\text{-}\mathbf{Alg}$ over $\mathcal{C}^{\mathbf{2}}$; now by full fidelity of the assignment $\mathbf{Mnd}(\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \to \mathbf{CAT}/\mathcal{C}^{\mathbf{2}}$, there are unique monad maps $s, t \colon \mathsf{R}_2 \rightrightarrows \mathsf{R}_1$ rendering commutative the diagram above right. It follows that $(\mathbb{J}^{\pitchfork\pitchfork})_1$ is isomorphic over $\mathcal{C}^{\mathbf{2}}$ to the equaliser of the lower row. Since R_1 and R_2 are accessible, the parallel pair of monad morphisms s, t admits by [36, Theorem 27.1] a coequaliser $q \colon \mathsf{R}_1 \to \mathsf{R}$ which is again accessible, and for which q^* is the equaliser of s^* and t^*. Thus $(\mathbb{J}^{\pitchfork\pitchfork})_1 \cong \mathsf{R}\text{-}\mathbf{Alg}$ over $\mathcal{C}^{\mathbf{2}}$ so that $(\mathbb{J}^{\pitchfork\pitchfork})_1 \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic for an accessible monad, as required. $\square$

We now wish to show that every accessible AWFS on a locally presentable category arises in the manner of the preceding proposition. The key idea is to show that any accessible AWFS has a small dense subcategory of L-coalgebras, and then deduce the result using Proposition 20 and the following lemma:

Lemma 24. *Given a morphism of categories over $\mathcal{C}^{\mathbf{2}}$*

$$\begin{tikzcd} \mathcal{J} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{K} \arrow[dl, "V"] \\ & \mathcal{C}^{\mathbf{2}} & \end{tikzcd} ,$$

the induced $F^{\pitchfork} \colon \mathcal{K}^{\pitchfork} \to \mathcal{J}^{\pitchfork}$ is an isomorphism of categories whenever the identity 2-cell exhibits V as the pointwise Kan extension $\mathrm{Lan}_F U$. In particular, this is the case whenever F is dense and V preserves the colimits exhibiting this density.

Proof. As in the proof of Proposition 16, an element of $\mathcal{K}^{\pitchfork}$ is a pair (g, φ_{-g}) where $g \colon C \to D$ in $\mathcal{C}$ and φ_{-g} is a section of the $\psi_{-g} \colon \mathcal{C}(\mathrm{cod}\, V{-}, C) \to \mathcal{C}^{\mathbf{2}}(V{-}, g)$ of (5.2); or equally (as $\mathrm{cod} \dashv \mathrm{id} \colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$) a section of $(1, g) \cdot ({-}) \colon \mathcal{C}^{\mathbf{2}}(V{-}, 1_C) \to \mathcal{C}^{\mathbf{2}}(V{-}, g)$. In these terms, a map $(g, \varphi_{-g}) \to (h, \varphi_{-h})$ of $\mathcal{K}^{\pitchfork}$ is a map $g \to h$ of $\mathcal{C}^{\mathbf{2}}$ for which the induced $\mathcal{C}^{\mathbf{2}}(V{-}, g) \to \mathcal{C}^{\mathbf{2}}(V{-}, h)$ commutes with the given sections. The map $\mathcal{K}^{\pitchfork} \to \mathcal{J}^{\pitchfork}$ is induced by the restriction functor $[F^{\mathrm{op}}, 1] \colon [\mathcal{K}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]$, and sends (g, φ_{-g}) to the pair (g, φ') with second component

$$\varphi' = \varphi_{-g} F \colon \mathcal{C}^{\mathbf{2}}(U{-}, g) \to \mathcal{C}^{\mathbf{2}}(U{-}, 1_C) \ .$$

So to prove $F^{\pitchfork} \colon \mathcal{K}^{\pitchfork} \to \mathcal{J}^{\pitchfork}$ is an isomorphism, it will suffice to show that $[F^{\mathrm{op}}, 1]$ becomes fully faithful when restricted to the full subcategory of $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ on the presheaves of the form $\mathcal{C}^{\mathbf{2}}(V{-}, g)$. Now, to say that the identity 2-cell exhibits V as $\mathrm{Lan}_F U$ is equally to say that it exhibits V^{op} as $\mathrm{Ran}_{F^{\mathrm{op}}} U^{\mathrm{op}}$; the continuous $\mathcal{C}^{\mathbf{2}}({-}, g) \colon (\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \to \mathbf{Set}$ preserves this pointwise Kan extension, and so the identity 2-cell exhibits $\mathcal{C}^{\mathbf{2}}(V{-}, g)$ as the right Kan extension of $\mathcal{C}^{\mathbf{2}}(U{-}, g)$ along F^{op}. The universal property of this right Kan extension now immediately implies that the action of $[F^{\mathrm{op}}, 1]$ on morphisms

$$[\mathcal{K}^{op}, \mathbf{Set}](\mathcal{C}^{\mathbf{2}}(V{-}, f), \mathcal{C}^{\mathbf{2}}(V{-}, g)) \to [\mathcal{J}^{op}, \mathbf{Set}](\mathcal{C}^{\mathbf{2}}(U{-}, f), \mathcal{C}^{\mathbf{2}}(U{-}, g))$$

is invertible for any f and g in $\mathcal{C}^{\mathbf{2}}$, as required. Finally, note that if F is dense, then the identity 2-cell exhibits $1_{\mathcal{K}}$ as $\mathrm{Lan}_F F$, and that if V preserves the colimits exhibiting this density, then it preserves this left Kan extension, so that the identity 2-cell exhibits V as $\mathrm{Lan}_F U$. $\square$

Theorem 25. *An* AWFS *on a locally presentable category is accessible if and only if it is cofibrantly generated by some small double category.*

Proof. One direction is Proposition 23; for the other, let (L, R) be an accessible AWFS on the locally presentable $\mathcal{C}$. By Proposition 20 we have an isomorphism $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork} \cong \mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$; it will thus suffice to find a small sub-double category $\mathsf{L}_\lambda\text{-}\mathbb{C}\mathbf{oalg} \hookrightarrow \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}$ with the induced $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork} \to \mathsf{L}_\lambda\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork}$ invertible. To this end, we consider the following diagram, whose left column is yet to be defined:

$$\begin{tikzcd}
\mathsf{L}_\lambda\text{-}\mathbf{Coalg}_2 \arrow[r, "j_2"] \arrow[d, shift right=2] \arrow[d] \arrow[d, shift left=2] & \mathsf{L}\text{-}\mathbf{Coalg}_2 \arrow[r, "V_2"] \arrow[d, shift right=2, "p_{\mathsf{L}}"'] \arrow[d, "m_{\mathsf{L}}" description] \arrow[d, shift left=2, "q_{\mathsf{L}}"] & \mathcal{C}^{\mathbf{3}} \arrow[d, shift right=2, "p"'] \arrow[d, "m" description] \arrow[d, shift left=2, "q"] \\
\mathsf{L}_\lambda\text{-}\mathbf{Coalg} \arrow[r, "j_1"] \arrow[d, shift right=2] \arrow[d, shift left=2] & \mathsf{L}\text{-}\mathbf{Coalg} \arrow[r, "V_1"] \arrow[d, shift right=2, "d_{\mathsf{L}}"'] \arrow[d, shift left=2, "c_{\mathsf{L}}"] & \mathcal{C}^{\mathbf{2}} \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\
\mathcal{C}_\lambda \arrow[r, "j_0"] \arrow[u] & \mathcal{C} \arrow[r, "1"] \arrow[u, "i_{\mathsf{L}}" description] & \mathcal{C} \arrow[u, "i" description]
\end{tikzcd} .$$

As colimits are pointwise in functor categories the right column is composed of cocontinuous functors between cocomplete categories; since V_1 creates colimits, the same is true for $\mathsf{L}\text{-}\mathbf{Coalg}$ and each of d_{L}, i_{L} and c_{L}. Moreover d_{L} has the isomorphism-lifting property since V_1 and d do, whence by [34] the pullback $\mathsf{L}\text{-}\mathbf{Coalg}_2$ of d_{L} along c_{L} is also a bipullback. Consequently, $\mathsf{L}\text{-}\mathbf{Coalg}_2$ is cocomplete and the projections q_{L} and p_{L} cocontinuous; so too is m_{L}, since d_{L} and c_{L} jointly create colimits; and similarly for V_2. Thus the right two columns are composed of cocomplete categories and cocontinuous functors.

Now $\mathcal{C}$ is locally presentable, whence also $\mathcal{C}^{\mathbf{2}}$ and $\mathcal{C}^{3}$; and since L is accessible, $\mathsf{L}\text{-}\mathbf{Coalg}$ is locally presentable as in the proof of Proposition 13. Finally, $\mathsf{L}\text{-}\mathbf{Coalg}_2$ is a bipullback of cocontinuous functors between locally presentable categories, and so itself locally presentable by [10, Theorem 3.15]. So the right two columns are composed of locally presentable categories and cocontinuous functors.

Any cocontinuous functor between locally presentable categories has an accessible right adjoint [19, Satz 14.6]; thus there is a regular λ so that the right two columns are composed of locally λ-presentable categories and functors with λ-accessible right adjoints; it follows that each left adjoint preserves λ-presentable objects. We now define $\mathsf{L}_\lambda\text{-}\mathbb{C}\mathbf{oalg}$[3] as the full sub-double category of $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}$ with as objects, isomorphism-class representatives of the λ-presentable objects of $\mathcal{C}$. This category is small, and the inclusion $j_0 \colon \mathcal{C}_\lambda \to \mathcal{C}$ is dense; we claim that j_1 and j_2 are too. For j_1, we consider the full dense subcategory $(\mathsf{L}\text{-}\mathbf{Coalg})_\lambda$ of λ-presentable objects in $\mathsf{L}\text{-}\mathbf{Coalg}$. Because d_{L} and c_{L} preserve λ-presentability it follows that each λ-presentable of $\mathsf{L}\text{-}\mathbf{Coalg}$ has λ-presentable domain and codomain; so the fully faithful dense functor $(\mathsf{L}\text{-}\mathbf{Coalg})_\lambda \to \mathsf{L}\text{-}\mathbf{Coalg}$ factors through the full inclusion $j_1 \colon \mathsf{L}_\lambda\text{-}\mathbf{Coalg} \to \mathsf{L}\text{-}\mathbf{Coalg}$, whence, by [37, Theorem 5.13] j_1 is dense. The

[3]Our notation $\mathsf{L}_\lambda\text{-}\mathbb{C}\mathbf{oalg}$ acknowledges the fact that this is equally the double category of coalgebras associated to the restricted AWFS on $\mathcal{C}_\lambda$.

density of j_2 follows in the same manner, by considering the full dense subcategory $(\mathsf{L}\text{-}\mathbf{Coalg}_2)_\lambda$ of $\mathsf{L}\text{-}\mathbf{Coalg}_2$ and arguing that it is contained in $(\mathsf{L}_\lambda\text{-}\mathbf{Coalg})_2$.

Since both j_1 and j_2 are dense, and since all of the functors to their right are cocontinuous it follows from Lemma 24 that both ${j_1}^{\pitchfork}$ and ${j_2}^{\pitchfork}$ are isomorphisms of categories; whence ${j_1}^{\text{⫛}}$ and ${j_2}^{\text{⫛}}$ as displayed in the right two columns of

$$\begin{tikzcd}
\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\text{⫛}} \arrow[r] \arrow[d, "j^{\text{⫛}}"'] & \mathsf{L}\text{-}\mathbf{Coalg}^{\text{⫛}} \arrow[r, shift left, "m^{\text{⫛}}"] \arrow[r, shift right, "\delta"'] \arrow[d, "{j_1}^{\text{⫛}}"] & (\mathsf{L}\text{-}\mathbf{Coalg}_2)^{\text{⫛}} \arrow[d, "{j_2}^{\text{⫛}}"'] \\
\mathsf{L}_\lambda\text{-}\mathbb{C}\mathbf{oalg}^{\text{⫛}} \arrow[r] & \mathsf{L}_\lambda\text{-}\mathbf{Coalg}^{\text{⫛}} \arrow[r, shift left, "m^{\text{⫛}}"] \arrow[r, shift right, "\delta"'] & (\mathsf{L}_\lambda\text{-}\mathbf{Coalg}_2)^{\text{⫛}}
\end{tikzcd}$$

are isomorphisms of double categories. But both rows are, as in (7.1), equalisers, and it follows that the induced double functor $j^{\text{⫛}}$ on the left is invertible as claimed. □

8. Fibre squares and enriched cofibrant generation

We conclude this paper with a further batch of examples based on a surprisingly powerful modification of the small object argument of Proposition 16. Given a well-behaved monoidal category $\mathcal{V}$ equipped with an AWFS (L, R), a well-behaved $\mathcal{V}$-category $\mathcal{C}$, and a small category $U\colon \mathcal{J} \to (\mathcal{C}_0)^{\mathbf{2}}$ over $(\mathcal{C}_0)^{\mathbf{2}}$, it constructs the "free (L, R)-enriched AWFS cofibrantly generated by $\mathcal{J}$". This is an AWFS on $\mathcal{C}_0$ whose algebras $\boldsymbol{g}\colon C \to D$ are maps equipped with, for every $Uj\colon A \to B$ in the image of U, a choice of R-map structure on the canonical comparison map $\mathcal{C}(B, C) \to \mathcal{C}(B, D) \times_{\mathcal{C}(A,D)} \mathcal{C}(A, C)$ in $\mathcal{V}$, naturally in j.

The "enriched small object argument" which builds AWFS of this kind is most properly understood in the context of *monoidal algebraic weak factorisation systems* [44]; and as indicated in the introduction, a comprehensive treatment along those lines must await a further paper. Here, we content ourselves with giving the construction and a range of applications.

8.1. **Fibre squares.** The cleanest approach to enriched cofibrant generation makes use of the following *fibre square* construction. It associates to any AWFS (L, R) on a category $\mathcal{C}$ with pullbacks an AWFS on the arrow category $\mathcal{C}^{\mathbf{2}}$ whose algebras are the R-*fibre squares*: maps $(u, v)\colon f \to g$ of $\mathcal{C}^{\mathbf{2}}$ as to the left of

$$\begin{tikzcd}
A \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "f"'] \arrow[dr, dotted, "k"] & & \\
& B \times_D C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
& B \arrow[r, "v"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Fibre}_{\mathsf{R}}(\mathcal{C}) \arrow[r] \arrow[d, "V"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{R}\text{-}\mathbf{Alg} \arrow[d, "U^{\mathsf{R}}"] \\
(\mathcal{C}^{\mathbf{2}})^{\mathbf{2}} \arrow[r, "(u,v)\mapsto k"'] & \mathcal{C}^{\mathbf{2}}
\end{tikzcd}$$

equipped with an R-algebra structure on the comparison map $k\colon A \to B \times_D C$. The R-fibre squares are the objects of a category $\mathbf{Fibre}_{\mathsf{R}}(\mathcal{C})$, whose morphisms are determined by way of the pullback square on the right above.

We may compose R-fibre squares: given $(u, v)\colon f \to g$ with R-algebra structure $\boldsymbol{k}\colon A \to B \times_D C$ and $(s, t)\colon g \to h$ with structure $\boldsymbol{\ell}\colon C \to D \times_F E$, we can pull back $\boldsymbol{\ell}$ along $v \times_F E$ and use Proposition 8 to obtain an algebra structure $v^*\boldsymbol{\ell}\colon B \times_D C \to B \times_F E$; the composite algebra $v^*\boldsymbol{\ell} \cdot \boldsymbol{k}\colon A \to B \times_D C \to B \times_F E$ now equips $(s,t)\cdot(u,v)\colon f \to h$ with R-fibre square structure. In this way we obtain a double category $\mathbb{F}\mathbf{ibre}_{\mathsf{R}}(\mathcal{C}) \to \mathbb{S}\mathbf{q}(\mathcal{C}^{\mathbf{2}})$ which is concrete over $\mathcal{C}^{\mathbf{2}}$ and easily seen to be right-connected; so it will be the double category of algebras for an AWFS on $\mathcal{C}^{\mathbf{2}}$ as soon as $V\colon \mathbf{Fibre}_{\mathsf{R}} \to (\mathcal{C}^{\mathbf{2}})^{\mathbf{2}}$ is strictly monadic. As a pullback of the strictly monadic $U^{\mathsf{R}}\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$, this V will always create V-absolute coequaliser pairs; so it suffices to show that it has a left adjoint. Given a square in $\mathcal{C}$ as on the left below, we factor the induced $k\colon A \to B \times_D C$ as $Rk \cdot Lk\colon A \to Ek \to B \times_D C$, yielding the factorisation

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "Lk"] \arrow[d, "f"'] & Ek \arrow[r, "\pi_2 \cdot Rk"] \arrow[d, "\pi_1 \cdot Rk"'] & C \arrow[d, "g"] \\ B \arrow[r, "1"'] & B \arrow[r, "v"'] & D \end{tikzcd} \tag{8.1}$$

on the right. The rightmost square, when equipped with the algebra structure $\boldsymbol{Rk}\colon Ek \to B \times_D C$ on the comparison map, now comprises the free R-fibre square on (u,v). This shows that V has a left adjoint, whence by Theorem 6, $\mathbb{F}\mathbf{ibre}_{\mathsf{R}}(\mathcal{C}) \to \mathbb{S}\mathbf{q}(\mathcal{C}^{\mathbf{2}})$ is the double category of algebras for an AWFS on $\mathcal{C}^{\mathbf{2}}$; it is not hard to calculate that the corresponding coalgebras are maps $(u,v)\colon f \to g$ for which u is equipped with L-coalgebra structure and v is invertible. Note that if (L,R) is an accessible AWFS on a locally presentable category, then the fibre square AWFS on $\mathcal{C}^{\mathbf{2}}$ is also accessible, since its factorisations (8.1) are constructed using the accessible L and R and finite limits in $\mathcal{C}$.

8.2. **Enriched cofibrant generation.** Suppose now that $\mathcal{V}$ is a locally presentable symmetric monoidal closed category: thus a suitable base for enriched category theory. Let (L,R) be an accessible AWFS on $\mathcal{V}$, and let $\mathcal{C}$ be a $\mathcal{V}$-category whose underlying category $\mathcal{C}_0$ is locally presentable; finally, let $U\colon \mathcal{J} \to (\mathcal{C}_0)^{\mathbf{2}}$ be a small category over $(\mathcal{C}_0)^{\mathbf{2}}$. By applying its monad and comonad pointwise, we may lift the AWFS (L,R) on $\mathcal{V}$ to an accessible AWFS on $[\mathcal{J}^{\mathrm{op}},\mathcal{V}]$; now applying the fibre square construction of the preceding section to this yields an accessible AWFS on $[\mathcal{J}^{\mathrm{op}},\mathcal{V}]^{\mathbf{2}}$. There is a right adjoint functor $\tilde{U}\colon \mathcal{C}_0 \to [\mathcal{J}^{\mathrm{op}},\mathcal{V}]^{\mathbf{2}}$ sending $x \in \mathcal{C}_0$ to the arrow $\mathcal{C}(\mathrm{cod}\,U-,x) \to \mathcal{C}(\mathrm{dom}\,U-,x)$ of $[\mathcal{J}^{\mathrm{op}},\mathcal{V}]$; and we now define the *enriched* AWFS *cofibrantly generated by* $\mathcal{J}$ to be the projective lifting of the fibre square AWFS on $[\mathcal{J}^{\mathrm{op}},\mathcal{V}]^{\mathbf{2}}$ along $\tilde{U}$; note that the existence of this lifting is guaranteed by Proposition 13(a). The double category of algebras of the AWFS $(\mathsf{L}_{\mathcal{J}},\mathsf{R}_{\mathcal{J}})$ so obtained is thus defined by a pullback square

$$\begin{tikzcd} \mathsf{R}_{\mathcal{J}}\text{-}\mathbb{A}\mathbf{lg} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{F}\mathbf{ibre}_{[\mathcal{J}^{\mathrm{op}},\mathcal{V}]}([\mathcal{J}^{\mathrm{op}},\mathcal{V}]) \arrow[d] \\ \mathbb{S}\mathbf{q}(\mathcal{C}_0) \arrow[r, "\mathbb{S}\mathbf{q}(\tilde{U})"'] & \mathbb{S}\mathbf{q}([\mathcal{J}^{\mathrm{op}},\mathcal{V}]^{\mathbf{2}}) \end{tikzcd} ;$$

so that in particular, an algebra structure on a map $g\colon C \to D$ is given by the choice, for each $j \in \mathcal{J}$ with image $Uj\colon A \to B$ in $\mathcal{C}^{\mathbf{2}}$, of an R-algebra structure on the comparison map $\mathcal{C}(B,C) \to \mathcal{C}(A,C) \times_{\mathcal{C}(A,D)} \mathcal{C}(B,D)$, naturally in j; we may say that g is equipped with an *enriched lifting operation* against $\mathcal{J}$.

8.3. **Examples.** The remainder of the paper will be given over to a range of examples of enriched cofibrant generation. All these examples will in fact be generated by a mere *set* of maps J, seen as a discrete subcategory $J \hookrightarrow (\mathcal{C}_0)^{\mathbf{2}}$ of a locally presentable $\mathcal{C}_0$.

Examples 26. (i) Let $\mathcal{V} = \mathbf{Set}$ and (L, R) be the AWFS for split epis thereon; then the notion of enriched cofibrant generation reduces to the unenriched one of Section 5.2. A full treatment of enriched cofibrant generation would in fact generalise each aspect of the theory developed in Sections 5 and 6; but as we have said above, this must await another paper.

(ii) More generally, let $\mathcal{V}$ be any locally presentable symmetric monoidal closed category, and let (L, R) be the split epi AWFS thereon. The enriched AWFS cofibrantly generated by $J \hookrightarrow \mathcal{C}^{\mathbf{2}}$ is precisely that obtained by the "enriched small object argument" of [45, Proposition 13.4.2].

(iii) Let $\mathcal{V} = \mathbf{Set}$ and let (L, R) be the initial AWFS thereon; its algebra category is the full subcategory of $\mathbf{Set}^{\mathbf{2}}$ on the isomorphisms. In this case, the algebra category of the enriched AWFS $(\mathsf{L}_J, \mathsf{R}_J)$ generated by $J \hookrightarrow \mathcal{C}^{\mathbf{2}}$ may be identified with the full subcategory of $\mathcal{C}^{\mathbf{2}}$ on those maps with the *unique* right lifting property against each map in J. In particular, since R_J-$\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$ is fully faithful, Proposition 3 applies, so that $(\mathsf{L}_J, \mathsf{R}_J)$ in fact describes the *orthogonal* factorisation system whose left class is generated by J [18, §2.2].

Examples 27. For our next examples of enriched cofibrant generation, we take $\mathcal{V} = \mathbf{Cat}$ equipped with the AWFS (L, R) for retract equivalences of Section 4.2.

(i) Take $\mathcal{C} = \mathbf{Cat}$ and let J comprise the single functor $!\colon \mathbf{0} \to \mathbf{1}$. It's easy to see that the enriched AWFS generated by J is again that for retract equivalences.

(ii) Take $\mathcal{C} = \mathbf{Cat}$ and let J comprise the single functor $\top\colon \mathbf{1} \to \mathbf{2}$ picking out the terminal object of $\mathbf{2}$. An algebra for the enriched AWFS generated by J is a functor $G\colon C \to D$ such that the induced $C^{\mathbf{2}} \to D \downarrow G$ bears retract equivalence structure, or equivalently, is fully faithful and equipped with a section on objects. Full fidelity corresponds to the requirement that every arrow of C be cartesian over D, whereupon a section on objects amounts to a choice of cartesian liftings: so an algebra is a cloven fibration whose fibres are groupoids. We find further that maps of algebras are squares strictly preserving the cleavage, and that composition of algebras is the usual composition of fibrations.

(iii) Let $(\mathcal{C}, j)$ be a small site, let $\mathcal{E} = \mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$ be the 2-category of pseudofunctors, strong transformations and modifications $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, and let $y\colon \mathcal{C} \to \mathcal{E}$ be the Yoneda embedding. If $(U_i \to U : i \in I)$ is a covering family in $\mathcal{C}$, then the associated *covering 2-sieve* $f : \varphi \to yU$ in $\mathcal{E}$ is the second half of the pointwise (bijective on objects, fully faithful) factorisation

$\sum_i yU_i \to \varphi \to yU$; recall that an object $X \in \mathcal{E}$ is called a *stack* if the restriction functor $\mathcal{E}(f, X) \colon \mathcal{E}(yU, X) \to \mathcal{E}(\varphi, X)$ is an equivalence of categories for each covering 2-sieve.

Consider now the 2-category $\mathcal{E}_s = \mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})_s$ of pseudofunctors, 2-natural transformations and modifications. By the general theory of [11], there is an accessible 2-monad T on $\mathbf{Cat}^{\mathrm{ob}\,\mathcal{C}}$ whose 2-category $\mathsf{T}\text{-}\mathbf{Alg}_s$ of strict algebras and strict algebra morphisms is $\mathcal{E}_s$—so that in particular, $\mathcal{E}_s$ is locally presentable—and whose 2-category $\mathsf{T}\text{-}\mathbf{Alg}$ of strict algebras and pseudomorphisms is $\mathcal{E}$. It follows by [11, Theorem 3.13] that the inclusion 2-functor $\mathcal{E}_s \to \mathcal{E}$ has a left adjoint $Q \colon \mathcal{E}_s \to \mathcal{E}$. Thus we may identify stacks with pseudofunctors X such that the restriction functor $\mathcal{E}_s(Qf, X) \colon \mathcal{E}_s(QyU, X) \to \mathcal{E}_s(Q\varphi, X)$ is an equivalence for each covering 2-sieve f.

As $\mathcal{E}_s$ is locally presentable, we may consider the enriched AWFS thereon generated by the set $\{Qf : f \text{ a covering 2-sieve in } \mathcal{E}\}$, and by the above, we see that its algebraically fibrant objects are a slight strengthening of the notion of stack; namely, they are pseudofunctors $X \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ such that each $\mathcal{E}(f, X) \colon \mathcal{E}(yU, X) \to \mathcal{E}(\varphi, X)$ is given the structure of a *retract* equivalence.

(iv) We may make arbitrary stacks into the algebraically fibrant objects of an AWFS by way of the following observations. Given $F \colon C \to D$ in $\mathbf{Cat}$, we may form its pseudolimit $D \downarrow_{\cong} F$—the category whose objects are triples $(d \in D, c \in C, \theta \colon d \cong Fc)$—and choices of equivalence section for $D \downarrow_{\cong} F \to D$ now correspond to choices of equivalence pseudoinverse for F. It follows that if $f \colon X \to Y$ is a map in a finitely cocomplete 2-category $\mathcal{E}$, and $\bar{f} \colon X \to \bar{Y}$ is the injection of X into the pseudocolimit of f, then for each $F \in \mathcal{E}$, retract equivalence structures on $\mathcal{E}(\bar{f}, F)$ correspond with equivalence structures on $\mathcal{E}(f, F)$.

In particular, if $(\mathcal{C}, j)$ is a small site and $\mathcal{E}$ and $\mathcal{E}_s$ are as above, then we may consider the enriched AWFS on $\mathcal{E}_s$ cofibrantly generated by the set $\{\overline{Qf} : f \text{ a covering 2-sieve in } \mathcal{E}\}$. Its algebraically fibrant objects are now stacks in the usual sense; to be precise, they are pseudofunctors X such that each $\mathcal{E}(yU, X) \to \mathcal{E}(\varphi, X)$ is provided with a chosen equivalence pseudoinverse.

Examples 28. Our next collection of examples again take $\mathcal{V} = \mathbf{Cat}$, but now with (L, R) the AWFS for lalis of Section 4.2.

(i) Let $\mathcal{C} = \mathbf{Cat}$ and let J comprise the single functor $! \colon \mathbf{0} \to \mathbf{1}$; then the enriched AWFS generated by J is simply that for lalis.

(ii) Let $\mathcal{C} = \mathbf{Cat}$ and let J comprise the single functor $\top \colon \mathbf{1} \to \mathbf{2}$ picking out the terminal object of $\mathbf{2}$. To equip $G \colon C \to D$ with algebra structure for the enriched AWFS this generates is to equip the induced $C^{\mathbf{2}} \to D \downarrow G$ with a right adjoint right inverse; which by [26], is equally to equip G with the structure of a cloven fibration. As before, maps of algebras are squares strictly preserving the cleavage; and composition of algebras is the usual composition of fibrations.

(iii) Generalising the previous two examples, let Φ be some set of small categories, and let J comprise the class of functors $\{A \to A_\perp : A \in \Phi\}$ obtained by freely adjoining an initial object to each A. To equip $G\colon C \to D$ with algebra structure for the induced AWFS is to give, for each $A \in \Phi$, a right adjoint section for the comparison functor $[A_\perp, C] \to [A_\perp, D] \times_{[A,D]} [A, C]$. Since a functor out of $A_\perp$ is the same thing as a functor out of A together with a cone over it, a short calculation shows that this amounts to giving, for each $A \in \Phi$, each diagram $F\colon A \to C$, and each cone $p\colon \Delta V \to GF$ in D, a cone $q\colon \Delta W \to F$ in C with $Gq = p$, such that for any cone $q'\colon \Delta W' \to F$, each factorisation of Gq' through p via a map $k\colon GW' \to V$ lifts to a unique factorisation of q' through q.[4] In particular, an algebraically fibrant object for this AWFS is a category equipped with a choice of limits for all diagrams indexed by categories in Φ.

(iv) Generalising the preceding example, let $\mathcal{C}$ be any locally presentable 2-category. For any arrow $f\colon A \to B$ of $\mathcal{C}$, let $\bar f\colon A \to \bar B$ denote the injection into the lax colimit of the arrow f, and for any set J of arrows in $\mathcal{C}$, let $\bar J = \{\bar f : f \in J\}$. By using the universal property of the colimit, we may calculate that an algebra structure on a morphism $g\colon C \to D$ for the AWFS induced by $\bar J$ is given by the choice, for every $f\colon A \to B$ in J and every 2-cell θ as on the left below, of a map k and 2-cell γ as in the centre with $gk = v$ and $g\gamma = \theta$, and such that γ is *initial* over g; thus, for every diagram as on the right with $g\alpha = \theta \cdot \beta f$, there exists a unique 2-cell $\delta\colon \ell \Rightarrow k$ with $g\delta = \beta$ and $\alpha = \gamma \cdot \delta f$.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\Uparrow\theta"] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, "k"'] \arrow[u, phantom, shift right=4, "\Uparrow\gamma"] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, "\ell"] \arrow[u, phantom, shift right=4, "\Uparrow\alpha"] & D \arrow[u, phantom, shift left=4, "\Downarrow\beta"] \end{tikzcd} .$$

In particular, the algebraically fibrant objects for this AWFS are those $C \in \mathcal{C}$ such that, for each $f\colon A \to B$ in J, the functor $\mathcal{C}(f, C)\colon \mathcal{C}(B, C) \to \mathcal{C}(A, C)$ is equipped with a right adjoint; that is, such that each morphism $A \to C$ in $\mathcal{C}$ is equipped with a chosen right Kan extension along f.

Of course, we may construct examples dual to the preceding ones—dealing with ralis, opfibrations, final liftings of cocones, categories with colimits, and left Kan extensions—by replacing the AWFS for lalis in the above by that for ralis.

Example 29. Let $\mathcal{V}$ be the category $\mathbf{SSet} = [\Delta^{\mathrm{op}}, \mathbf{Set}]$ of simplicial sets, equipped with the AWFS for *trivial Kan fibrations*, cofibrantly generated by the set of boundary inclusions $\{\partial\Delta[n] \to \Delta[n] : n \in \mathbb{N}\}$. Given a set J of maps in the locally presentable $\mathbf{SSet}$-category $\mathcal{C}$, the algebraically fibrant objects of the enriched AWFS cofibrantly generated by J are those $X \in \mathcal{C}$ such that, for each $f\colon A \to B$ in J, the induced map $\mathcal{C}(B, X) \to \mathcal{C}(A, X)$ bears algebraic trivial fibration structure. Most typically, one would consider this in the situation where

[4] Note that when A is a discrete category, this is precisely the notion of G-*initial lifting* [1, Definition 10.57] of a discrete cone.

$\mathcal{C}$ is a simplicial model category and the J's are a class of cofibrations; then the algebraically fibrant objects above which are also fibrant for the underlying model structure are precisely the *J-local objects* [28, Definition 3.1.4]. The enriched small object argument in this particular case was described in [14, §7].

Our final examples bear on the theory of *quasicategories* [32]. Quasicategories are simplicial sets with fillers for all inner horns; they are a particular model for $(\infty, 1)$-categories—weak higher categories whose cells of dimension > 1 are weakly invertible—and have a comprehensive theory [33, 40] paralleling the classical theory of categories. In particular, there are good notions of *Grothendieck fibration*, *limit*, and *Kan extension* for quasicategories; and by analogy with Examples 27, we may hope to obtain these notions as fibrations or fibrant objects for AWFS constructed by enriched cofibrant generation, so long as we can find a good quasicategorical analogue of the AWFS for lalis.

For this we propose the AWFS of Examples 12(v) above, whose R-algebras we called the *simplicial lalis*. By using the explicit formulae of Section 4.4, we may calculate the monad R at issue, and so obtain the following concrete description of its algebras. First some notation: given a simplicial set X, we write $x\colon a \rightsquigarrow b$ to denote a simplex $x \in X_{n+1}$ whose $(n+1)$st face is a and whose final vertex is b. Now by a *simplicial lali*, we mean a simplicial map $f\colon A \to B$ together with:

- A section $u\colon B_0 \to A_0$ of the action of f on 0-simplices;
- For each $a \in A_n$ and $x\colon fa \rightsquigarrow b$ in B, a simplex $\gamma_a(x)\colon a \rightsquigarrow ub$ over x;

subject to the coherence conditions that:

- $\gamma_a(x) \cdot \delta_i = \gamma_{a\cdot\delta_i}(x \cdot \delta_i)$ and $\gamma_a(x) \cdot \sigma_i = \gamma_{a\cdot\sigma_i}(x \cdot \sigma_i)$;
- $\gamma_{ub}(b \cdot \sigma_0) = ub \cdot \sigma_0$; and
- $\gamma_{\gamma_a(x)}(x \cdot \sigma_{n+1}) = \gamma_a(x) \cdot \sigma_{n+1}$.

By comparing with Proposition 19, we see that if $A = N\mathcal{C}$ and $B = N\mathcal{D}$ are the nerves of small categories, then a simplicial lali $A \to B$ is the same thing as a lali $\mathcal{C} \to \mathcal{D}$. This, of course, does not in itself justify the notion of simplicial lali; in order to do so, we will relate it to a notion introduced by Riehl and Verity in [52, Example 4.4.7], which for the present purposes we shall refer to as a *quasicategorical lali*. A simplicial map $f\colon A \to B$ is a quasicategorical lali if equipped with a strict section $u\colon B \to A$, a simplicial homotopy $\eta\colon 1 \Rightarrow uf$ satisfying $f\eta = 1_f$ (on the nose), and a 2-homotopy

$$\begin{tikzcd} & u \arrow[dr, "1_u"] \arrow[d, phantom, "\theta"] & \\ u \arrow[ur, "\eta u"] \arrow[rr, "1_u"'] & {} & u \end{tikzcd}$$

satisfying $f\theta = 1_{1_B}$. We have established the following result; its proof is beyond the scope of this paper, but note that it is a quasicategorical analogue of the correspondence between the two views of ordinary lalis described in Proposition 19.

Proposition 30. *Let $f\colon A \to B$ be a map of simplicial sets. If f is a simplicial lali, then it is a quasicategorical lali; if it is a quasicategorical lali and an inner Kan fibration, then it is a simplicial lali.*

Since Riehl and Verity are able to use quasicategorical lalis to describe various aspects of the theory of quasicategories, the above proposition allows us to describe these same aspects using simplicial lalis, and, hence, using the theory of enriched AWFS.

Examples 31. We consider enriched cofibrant generation in the case where $\mathcal{V} =$ **SSet** and (L, R) is the AWFS for simplicial lalis.

(i) *Limits.* Let A be a quasicategory, and let $A_\perp$ denote the quasicategory $\Delta[0] \oplus A$ obtained by adjoining an initial vertex to A. Riehl and Verity show in [52, Corollary 5.2.19] that a quasicategory X admits all limits of shape A—in the sense of [32, Definition 4.5]—just when the simplicial map $X^{A_\perp} \to X^A$ bears quasicategorical lali structure. It follows that, among the quasicategories, those equipped with a choice of all limits of shape A may be realised as the algebraically fibrant objects of an AWFS on **SSet**; namely, the enriched AWFS cofibrantly generated by the single map $A \to A_\perp$. By enriched cofibrant generation with respect to the set of maps $J = \{A \to A_\perp : A \text{ a finite quasicategory}\}$ we may capture quasicategories with all finite limits as algebraically fibrant objects.

(ii) *Grothendieck fibrations.* Riehl and Verity are in the process of analysing the quasicategorical Grothendieck fibration—the "Cartesian fibrations" of [40, §2.4]—and have shown [51] that an isofibration $g\colon C \to D$ of quasicategories bears such a structure just when the comparison functor $C^{\mathbf{2}} \to D \downarrow g$ bears quasicategorical lali structure. Here, $C^{\mathbf{2}}$ denotes the exponential $C^{\Delta[1]}$, while $D \downarrow g$ is the pullback of $g\colon C \to D$ along $D^{\delta_0}\colon D^{\Delta[1]} \to D$. Thus, among the isofibrations of quasicategories, the Grothendieck fibrations can be realised as the algebras of an AWFS on **SSet**—namely, that obtained by enriched cofibrant generation with respect to the single map $\delta_0\colon \Delta[0] \to \Delta[1]$.

(iii) *Right Kan extensions.* By [52, Example 5.0.4 and Proposition 5.1.19], a morphism of quasicategories $f\colon C \to D$ admits a right adjoint if and only if the projection $f \downarrow D \to D$ admits simplicial lali structure. So suppose that J is a set of morphisms between quasicategories; for each $f\colon A \to B$ in J, define $\bar{B}$ to be the the pushout of f along $\delta_1 \times A\colon A \to \Delta[1] \times A$, and define $\bar{f}\colon A \to \bar{B}$ to be the composite of $\delta_0 \times A\colon A \to \Delta[1] \times A$ with the pushout injection $\Delta[1] \times A \to \bar{B}$. Then a quasicategory X is an algebraically fibrant object for the enriched AWFS cofibrantly generated by $\{\bar{f} : f \in J\}$ just when each functor $X^f\colon X^B \to X^A$ has a right adjoint; that is, just when each functor $A \to X$ admits a right Kan extension along f.

Of course, by considering the AWFS for simplicial *ralis* in place of simplicial lalis, we may capture notions such as colimits, opfibrations and left Kan extensions.

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