Tannaka-Krein Reconstruction of Separable Frobenius Monoidal Functors

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For a monoidal functor $F : A \longrightarrow B$, we have a pair of maps, $Fx \otimes Fy \longrightarrow F(x \otimes y)$ and $e \longrightarrow Fe$, which we notate as follows:
Comonoidal Structure on a Functor $F$

Similarly, for a comonoidal $F$, we have maps $F(x \otimes y) \rightarrow Fx \otimes Fy$ and $Fe \rightarrow e$ which we notate in the obvious dual way, as follows:

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$F(x \otimes y)$  $Fx$  $Fe$  $e$

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$Fy$
Monoidal Functor Axioms

Graphically, the axioms for a monoidal functor are depicted as follows:

where, once again, the similar constraints for a comonoidal functor are exactly the above with composition read right-to-left instead of left-to-right.
A functor with these two properties has been called a Frobenius monoidal functor by Brian and Craig.
The object $E_F$

Let $F : A \longrightarrow B$ be a (not necessarily monoidal) functor between monoidal categories, where $B$ is assumed to be left closed. Define

$$E_F = \int_{a \in A} [Fa, Fa]$$

where I assume that $B$ is such that the indicated end exists. The notation $E_F$ is a shortening of $\text{End}^Y(F, F)$, by which the same object has been called before (by Ross, Brian, and possibly others). This is the Tannaka-Krein reconstruction object associated to $F$. 
There is a canonical action of $E_F$ on $F x$ for each object $x$ in $A$, using the $x$’th projection from the end followed by the evaluation $[F x, F x] \otimes F x \longrightarrow F x$ of the monoidal closed structure of $B$. The dinaturality of the end in $a$ gives rise to the naturality of the action on $F a$ in $a$, which we notate as:
Let us now assume that the closed structure of $B$ is given by left duals, that is, $[a, b] = La \otimes b$. If we also assume that $B$ is braided and that the tensor product coheres with the ends in $B$, then we obtain canonical actions of $E^n_F$ on $Fx_1 \otimes \ldots \otimes Fx_n$, defined inductively. For any map $f : X \longrightarrow E^n_F$, we then obtain a “discharged form” of $f$:

$$X \otimes Fx_1 \otimes \ldots \otimes x_n \longrightarrow Fx_1 \otimes Fx_1 \otimes \ldots \otimes Fx_n$$

Two maps are equal if and only if they have the same discharged forms.
Without assuming that $F$ bears a monoidal structure, one can define a monoid structure on $E_F$, as follows:

Note that this monoidal structure is associative and unital, without any assumption on $F$. 
Comonoidal Structure on $E_F$

Furthermore, if $F$ is known to be (lax) monoidal and comonoidal (without at the moment assuming any coherence between these structures) we can define a comonoid structure on $E_F$. 

\[ E_F \quad \xrightarrow{\varepsilon} \quad e \quad = \quad E_F \]

\[ E_F \quad \xrightarrow{x} \quad Fx \quad \xrightarrow{\otimes} \quad E_F \]

\[ Fx \quad \xrightarrow{y} \quad Fy \]

\[ E_F \quad \xrightarrow{\epsilon} \quad e \]

\[ E_F \quad \xrightarrow{e} \quad e \]
Finally, if $A$ is known to have left duals, we can define a canonical map $S : E_F \to E_F$ which we think of as a candidate for an antipode.

Notice in particular how the monoidal and comonoidal structures on $F$ permit one to consider the application of $F$ as not merely “boxes” but more like a flexible sheath.
Bialgebras

We can consider what extra conditions are required on $F$ to make the above into a bialgebra. A bialgebra in a braided monoidal category satisfies the following four axioms. First, the unit followed by the counit must be the identity:

\[
e \longrightarrow \eta \longrightarrow \epsilon \longrightarrow e = e \longrightarrow \epsilon \longrightarrow e
\]

Second and third, the unit and counit must respect the comultiplication and multiplication, respectively:

\[
\begin{array}{c}
\eta \\
E_F
\end{array}
\quad = \quad 
\begin{array}{c}
\eta \\
E_F
\end{array}
\quad \quad \quad 
\begin{array}{c}
\epsilon \\
E_F
\end{array}
\quad = \quad 
\begin{array}{c}
\epsilon \\
E_F
\end{array}
\]

Fourthly, the multiplication must cohere with the comultiplication, with the help of the braiding:

\[
\begin{array}{c}
E_F \\
E_F
\end{array}
\quad = \quad 
\begin{array}{c}
E_F \\
E_F
\end{array}
\quad = \quad 
\begin{array}{c}
E_F \\
E_F
\end{array}
\]
Bialgebra Axiom 1

For the first of these, we calculate:

\[ e \xrightarrow{\eta} e \xrightarrow{\epsilon} e \]

and we see that this composite is the identity on \( e \) precisely when \( e \xrightarrow{} Fe \xrightarrow{} e \) is the identity.
Bialgebra Axiom 2

For the second bialgebra axiom, we have the following two calculations:

\[ F(x \otimes y) \rightarrow F(x) \otimes F(y) \]

and so we see that these two are equal precisely when

\[ F(x \otimes y) \rightarrow F(x) \otimes F(y) \]

is the identity.
Bialgebra Axiom 3

For the third bialgebra axiom, we have the following two calculations:

\[ E_F \xrightarrow{\epsilon} e = E_F \xrightarrow{\epsilon} e \]

and we see that for these two to be equal, it suffices to have \( F e \xrightarrow{} e \xrightarrow{} F e \) be the identity, the use of which between the two actions in the first calculation gives the result.
Finally, for the final bialgebra axiom, the calculations shown below compute the discharged forms as

\[ F(x \otimes y) \rightarrow Fx \otimes Fy \rightarrow F(x \otimes y) \]

which shows that it suffices to request that

\[ F(x \otimes y) \rightarrow Fx \otimes Fy \rightarrow F(x \otimes y) \] should be the identity.
Bialgebra Left Hand Side, (1 of 3)
Bialgebra Left Hand Side, (2 of 3)
Bialgebra Right Hand Side, (1 of 2)
Demanding that $F$ be strong imposes the following four conditions on the monoidal/comonoidal structure:

\[
\begin{align*}
Fx & \otimes Fx = Fx \\
Fy & \otimes Fy = Fy
\end{align*}
\]

\[
\begin{align*}
Fx & \otimes Fx = Fx \\
Fy & \otimes Fy = Fy
\end{align*}
\]

\[
\begin{align*}
Fe & \otimes Fe = Fe \\
e & \otimes e = e \equiv e
\end{align*}
\]

Note that only the separability axiom preserves the number of connected components.
Weak Bialgebra Axioms

To move from a (strong) bialgebra to a weak one, the coherence of the multiplication with comultiplication is the only one which is retained. The coherence of the unit with the counit is discarded entirely, and the second and third axioms are replaced with the following four axioms:
Weak Unit Axioms

We first examine the unit axioms. In discharged form, the first unit expression is calculated as:

The calculations which follow show that the second and third unit expressions have the following discharged forms:

For these unit axioms, we see that it suffices to assume that \( F \) is Frobenius.
Calc. of 2nd Unit Expression (1 of 4)
Calc. of 2nd Unit Expression (2 of 4)
Calc. of 2nd Unit Expression (3 of 4)
Calc. of 2nd Unit Expression (4 of 4)
Calc. of 3rd Unit Expression (1 of 4)
Calc. of 3rd Unit Expression (2 of 4)
Calc. of 3rd Unit Expression (3 of 4)
Calc. of 3rd Unit Expression (4 of 4)
Weak Counit Axioms

As for the counit axioms, the discharged form of the first of these is easily calculated:

The discharged forms of the second and third counit expression are computed below as the same thing. Examining these calculations shows that the counit axioms follow merely from $F$ being both monoidal and comonoidal, without requiring Frobenius or separable.
Calc. of 2nd Counit Expression (1 of 6)
Calc. of 2nd Counit Expression (2 of 6)
Calc. of 2nd Counit Expression (3 of 6)
Calc. of 2nd Counit Expression (4 of 6)
Calc. of 2nd Counit Expression (5 of 6)
Calc. of 2nd Counit Expression (6 of 6)
Calc. of 3rd Counit Expression (1 of 6)
Calc. of 3rd Counit Expression (2 of 6)
Calc. of 3rd Counit Expression (4 of 6)
Calc. of 3rd Counit Expression (5 of 6)
Calc. of 3rd Counit Expression (6 of 6)
Moving from a bialgebra to a Hopf algebra, in both the strong and weak cases, involves adjoining an antipode, although with differing axioms. The strong antipode axioms request the following two equations:

\[ E_F \xrightarrow{S} E_F = E_F \xrightarrow{\epsilon} \eta \xrightarrow{E_F} = E_F \xrightarrow{S} E_F \]

This middle expression is discharged as:
Weak Hopf Algebras

On the other hand, the weak antipode axioms request the following three equations:

\[
\begin{align*}
E_F \xrightarrow{S} E_F &= \quad E_F \\
E_F \xrightarrow{S} E_F &= \quad E_F \\
E_F \xrightarrow{S} E_F &= \quad E_F
\end{align*}
\]

Since both the strong and weak axioms involve \( E_F \star S \) and \( S \star E_F \), we discharge those now.
Calculation of $E_F \star S$ (1 of 7)
Calculation of $E_F \star S$ (2 of 7)
Calculation of $E_F \star S$ (3 of 7)
Calculation of $E_F \star S$ (4 of 7)
Calculation of $E_F \star S$ (5 of 7)
Calculation of $E_F \star S$ (6 of 7)
Calculation of $E_F \star S$ (7 of 7)
Calculation of $S \star E_F$ (1 of 7)
Calculation of $S \star E_F$ (2 of 7)
Calculation of $S \star E_F$ (3 of 7)
Calculation of $S \star E_F$ (4 of 7)
Calculation of $\mathcal{S} \star E_F$ (5 of 7)
Calculation of $S \star E_F$ (6 of 7)
Calculation of $S \star E_F$ (7 of 7)
Strong Axioms

We see that it suffices to take $Fe \longrightarrow e \longrightarrow Fe$ to be the identity, which gives the strong axioms. As for the weak axioms, we compute the right-hand-sides of the first two weak antipode axioms.
Calc. of 1st Weak Antipode RHS (1 of 6)
Calc. of 1st Weak Antipode RHS (2 of 6)
Calc. of 1st Weak Antipode RHS (3 of 6)
Calc. of 1st Weak Antipode RHS (4 of 6)
Calc. of 1st Weak Antipode RHS (5 of 6)
Calc. of 1st Weak Antipode RHS (6 of 6)
Calc. of 2nd Weak Antipode RHS (1 of 6)

\[ E_F \xrightarrow{\eta} \xrightarrow{\epsilon} Fx \]

\[ = \]

\[ E_F \xrightarrow{\eta} \xrightarrow{e} Fx \]
Calc. of 2nd Weak Antipode RHS (2 of 6)
Calc. of 2nd Weak Antipode RHS (3 of 6)
Calc. of 2nd Weak Antipode RHS (4 of 6)
Calc. of 2nd Weak Antipode RHS (5 of 6)
Calc. of 2nd Weak Antipode RHS (6 of 6)
Weak Antipode Axioms

We see from the above calculations that the first two weak antipode axioms are satisfied for $F$ separable Frobenius. There is one additional antipode axiom which is imposed for a weak Hopf algebra, namely, that the convolution $S \ast E_F \ast S$ should equal $S$. Therefore, we discharge $S \ast E_F \ast S$.
Calculation of $S \star E_F \star S$ (1 of 12)
Calculation of $S \star E_F \star S$ (2 of 12)
Calculation of $S \star E_F \star S$ (3 of 12)
Calculation of $S \star E_F \star S$ (4 of 12)
Calculation of $S \star E_F \star S$ (5 of 12)
Calculation of $S \star E_F \star S$ (6 of 12)
Calculation of $S \star E_F \star S$ (7 of 12)
Calculation of $S \star E_F \star S$ (8 of 12)
Calculation of $S \star E_F \star S$ (9 of 12)
Calculation of $S \star E_F \star S$ (10 of 12)
Calculation of $S \rtimes E_F \rtimes S$ (11 of 12)
Calculation of $S \star E_F \star S$ (11 of 12)