

Strings and Stripes Graphical Calculus for Monoidal Functors and Monads

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Purpose

- ▶ Spread propaganda about graphical notation for functors between monoidal categories.
- ▶ The category of algebras C^T for a comonoidal monad T on C is itself a monoidal category.
- ▶ If A is a monoidal category with duals, and $\gamma : f \rightarrow g$ is a monoidal natural transformation between strong monoidal functors $f, g : A \rightarrow B$, then γ is invertible.

Graphical Calculus of Monoidal Functors

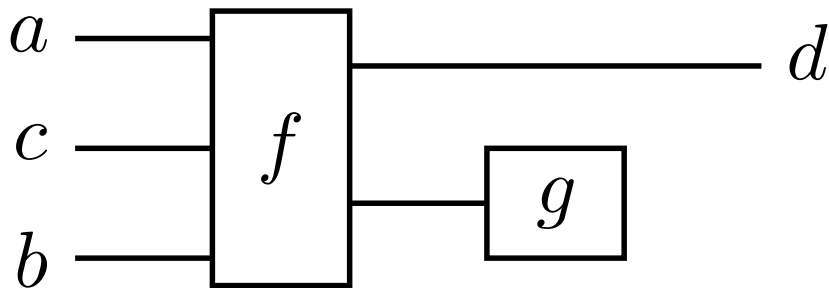
Let:

$$F : A \longrightarrow B$$

be a functor between monoidal categories.

Graphical Calculus for Monoidal Categories

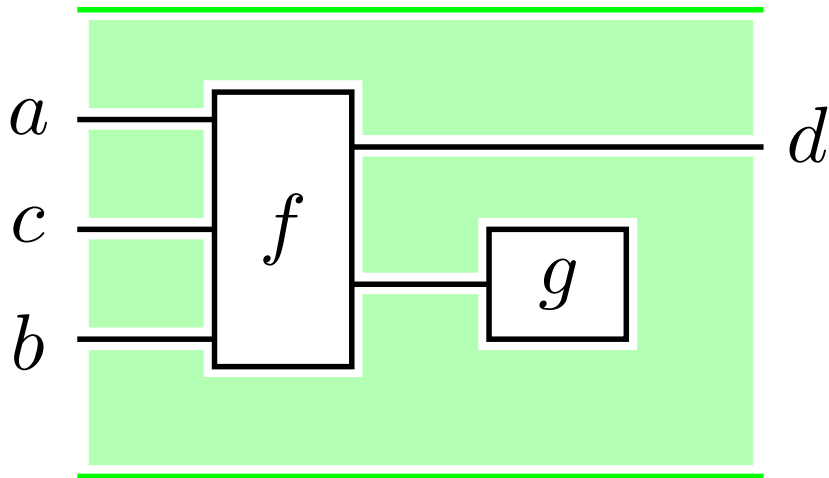
Consider the following composite in A :



$$a \otimes c \otimes b \xrightarrow{f} d \otimes x \xrightarrow{d \otimes g} d \otimes e \simeq d$$

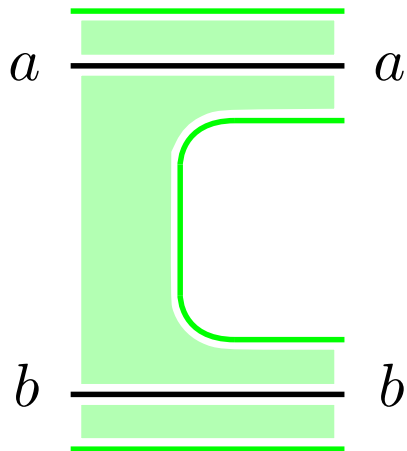
Graphical Calculus for Monoidal Categories

We apply F to obtain its image in B :



$$F(a \otimes c \otimes b) \xrightarrow{Ff} F(d \otimes x) \xrightarrow{F(d \otimes g)} F(d \otimes e) \simeq Fd$$

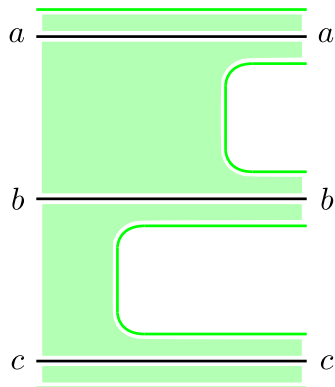
Comonoidal Structure



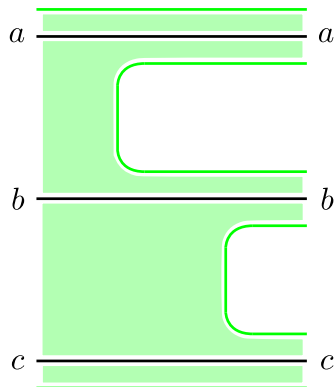
$$F(a \otimes b) \longrightarrow Fa \otimes Fb \quad =$$

$$Fe \longrightarrow e$$

Comonoidal Coherence



=



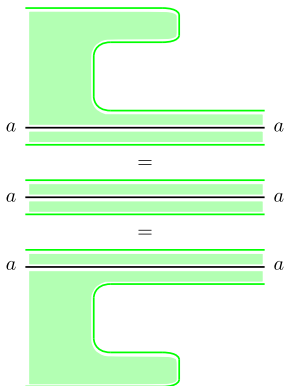
$$\begin{array}{ccc}
 F(a \otimes b \otimes c) & & Fa \otimes Fb \otimes Fc \\
 \searrow \phi & & \nearrow \phi \otimes Fc \\
 F(a \otimes b) \otimes Fc & &
 \end{array}$$

=

$$\begin{array}{ccc}
 F(a \otimes b \otimes c) & & Fa \otimes Fb \otimes Fc \\
 \searrow \phi & & \nearrow Fa \otimes \phi \\
 Fa \otimes F(b \otimes Fc) & &
 \end{array}$$

Comonoidal Coherence

$$Fa \simeq F(e \otimes a) \xrightarrow{\phi} Fe \otimes Fa \xrightarrow{\otimes Fa} e \otimes Fa \simeq Fa$$



$$Fa \simeq F(e \otimes e) \xrightarrow{\phi} Fa \otimes Fe \xrightarrow{Fa \otimes} Fa \otimes e \simeq Fa$$

Monads

Suppose that $T : C \rightarrow C$ is a monad:

$$TTa \xrightarrow{\mu^a} Ta \quad = \quad a \xrightarrow{\eta^a} Ta$$

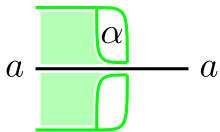
Monad Coherence

$TTTa \xrightarrow{T\mu a} TTa \xrightarrow{\mu a} Ta = TTTa \xrightarrow{\mu Ta} TTa \xrightarrow{\mu a} Ta$

Monad Coherence

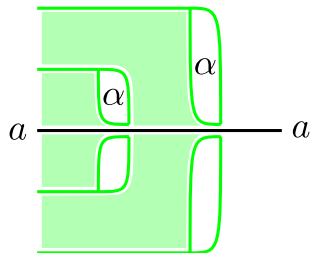
$$Ta \xrightarrow{T\eta a} TTa \xrightarrow{\mu a} Ta = Ta \xrightarrow{T a} Ta = Ta \xrightarrow{\eta T a} TTa \xrightarrow{\mu a} Ta$$

Algebras for Monads



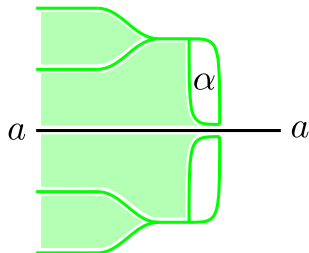
$Ta \xrightarrow{\alpha} a$

Algebra Coherence



$$TTa \xrightarrow{T\alpha} Ta \xrightarrow{\alpha} a$$

=

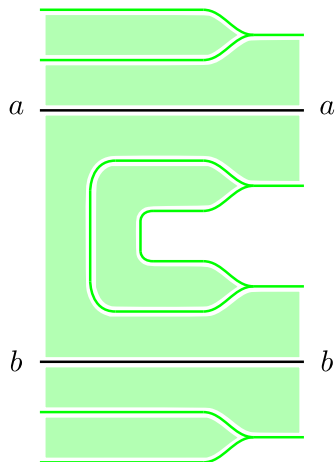


$$TTa \xrightarrow{\mu a} Ta \xrightarrow{\alpha} a$$

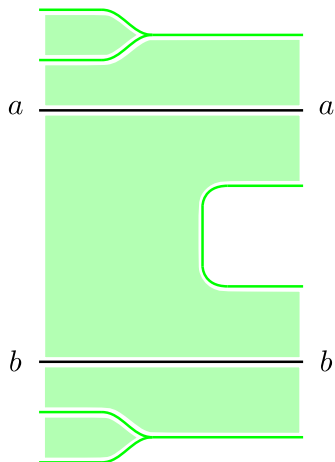
Algebra Coherence

$$\begin{array}{ccc} \begin{array}{c} a \text{ --- } a \\ \begin{array}{c} \text{Green box with } \alpha \text{ and two circles} \end{array} \end{array} & = & a \text{ --- } a \\ a \xrightarrow{\eta^a} Ta \xrightarrow{\alpha} a & = & a \xrightarrow{a} a \end{array}$$

Comonoidal Monads



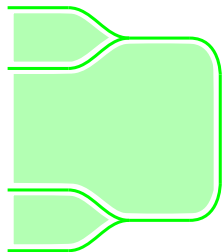
=



$$\begin{array}{ccc}
 TT(a \otimes b) & & Ta \otimes Tb \\
 T\phi \downarrow & & \uparrow \mu_{a \otimes b} \\
 T(Ta \otimes Tb) & \xrightarrow{\phi} & TTa \otimes TTb
 \end{array} =$$

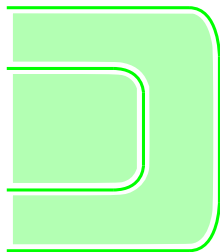
$$\begin{array}{ccc}
 TT(a \otimes b) & & Ta \otimes Tb \\
 \mu_{(a \otimes b)} \searrow & & \nearrow \phi \\
 & T(a \otimes b) &
 \end{array}$$

Comonoidal Monads



$$TTe \xrightarrow{\mu e} Te \xrightarrow{\phi_0} e$$

=



=

$$TTe \xrightarrow{T\phi_0} Te \xrightarrow{\phi_0} e$$

Comonoidal Monads

The diagram illustrates the naturality of the comultiplication map ϕ in a comonoidal monad. It consists of two main parts, separated by an equals sign, and a commutative diagram below.

Left side: A vertical green rectangle with two horizontal black lines. The top line is labeled a on both ends. The bottom line is labeled b on both ends. A green wire enters from the top left, loops around the top edge, and exits at the top right. A second green wire enters from the bottom left, loops around the bottom edge, and exits at the bottom right. A third green wire enters from the middle left, loops around the right edge, and exits at the middle right.

Right side: A vertical green rectangle with two horizontal black lines. The top line is labeled a on both ends. The bottom line is labeled b on both ends. A green wire enters from the top left, loops around the top edge, and exits at the top right. A second green wire enters from the middle left, loops around the left edge, and exits at the middle right. A third green wire enters from the bottom left, loops around the bottom edge, and exits at the bottom right.

Commutative diagram below:

$$\begin{array}{ccc} a \otimes b & & Ta \otimes Tb \\ & \searrow \eta(a \otimes b) & \nearrow \phi \\ & T(a \otimes b) & \end{array}$$

Equation below:

$$a \otimes b \xrightarrow{\eta_{a \otimes b}} Ta \otimes Tb$$

The overall equation is: $\text{Left Diagram} = \text{Right Diagram} = \text{Equation below}$

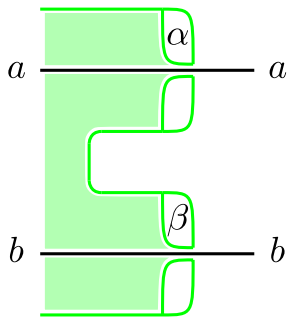
Comonoidal Monads

$$e \xrightarrow{\eta^e} T e \xrightarrow{\phi_0} e = e \xrightarrow{e} e$$

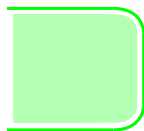
Monoidal Structure on C^T

If $T : C \rightarrow C$ is a comonoidal monad, then C^T inherits a natural monoidal structure.

Monoidal Structure on C^T

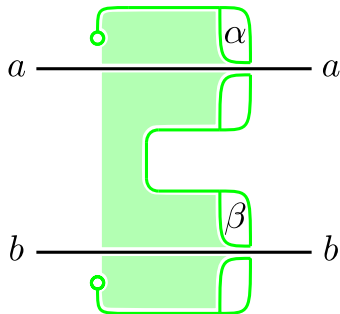


$$T(a \otimes b) \xrightarrow{\phi} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$



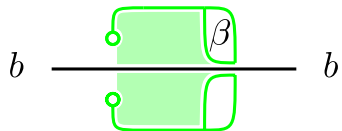
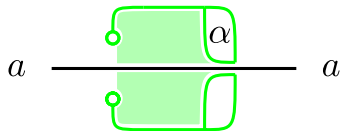
$$Te \xrightarrow{\phi_0} e$$

Unitality of Algebra Structure on $a \otimes b$



$$a \otimes b \xrightarrow{\eta(a \otimes b)} T(a \otimes b) \xrightarrow{\phi_0} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

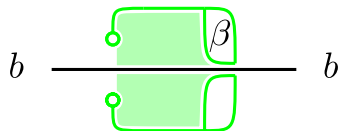
Unitality of Algebra Structure on $a \otimes b$



$$a \otimes b \xrightarrow{\eta^a \otimes \eta^b} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

Unitality of Algebra Structure on $a \otimes b$

$$a \text{ ————— } a$$



$$a \otimes b \xrightarrow{a \otimes \eta b} a \otimes Tb \xrightarrow{a \otimes \beta} a \otimes b$$

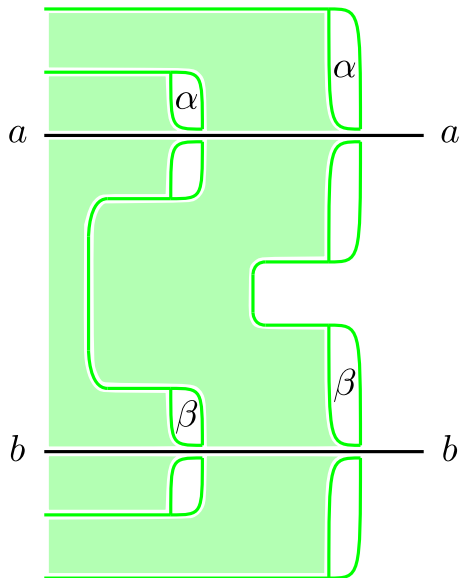
Unitality of Algebra Structure on $a \otimes b$

$$a \text{ ————— } a$$

$$b \text{ ————— } b$$

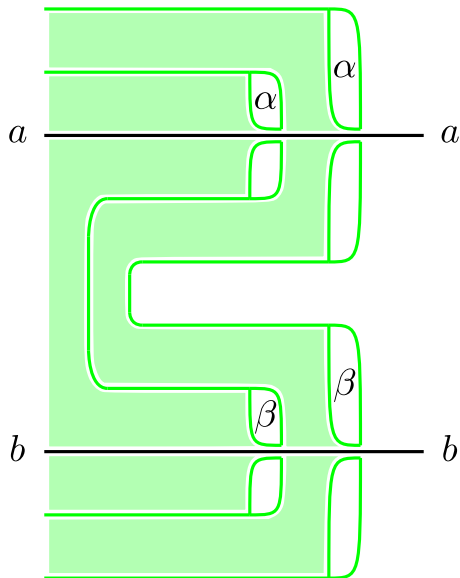
$$a \otimes b \xrightarrow{a \otimes b} a \otimes b$$

Multiplicativity of Algebra Structure on $a \otimes b$



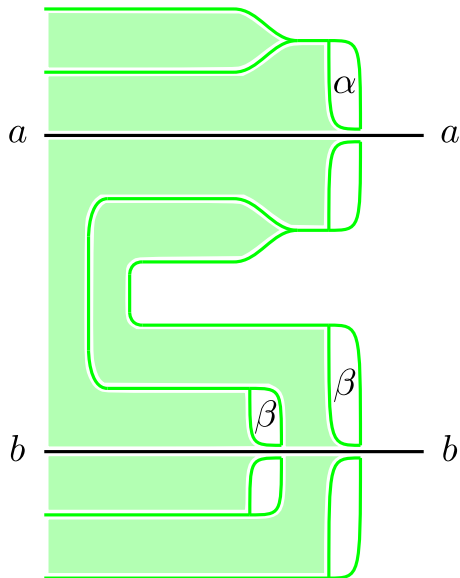
$$TT(a \otimes b) \xrightarrow{T\phi_0} T(Ta \otimes Tb) \xrightarrow{T(\alpha \otimes \beta)} T(a \otimes b) \xrightarrow{\phi} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

Multiplicativity of Algebra Structure on $a \otimes b$



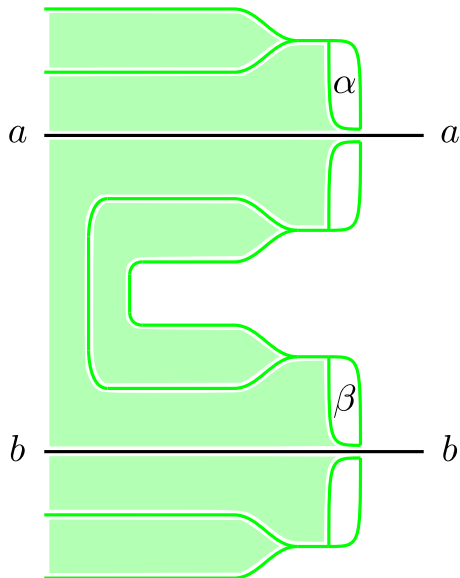
$$TT(a \otimes b) \xrightarrow{T\phi} T(Ta \otimes Tb) \xrightarrow{\phi} TTa \otimes TTb \xrightarrow{T\alpha \otimes T\beta} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

Multiplicativity of Algebra Structure on $a \otimes b$



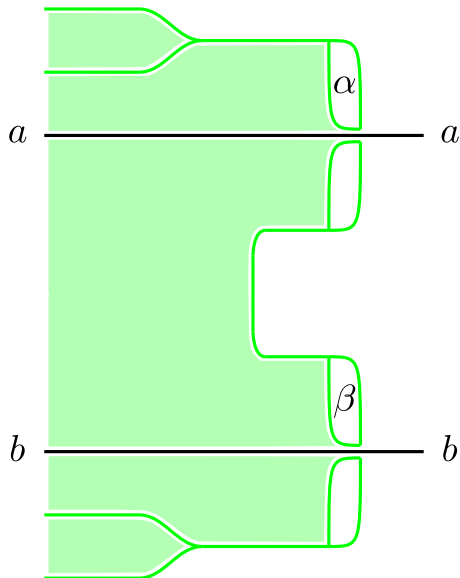
$$TT(a \otimes b) \xrightarrow{T\phi} T(Ta \otimes Tb) \xrightarrow{\phi} TTa \otimes TTb \xrightarrow{\mu_{a \otimes b} \otimes T\beta} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

Multiplicativity of Algebra Structure on $a \otimes b$



$$TT(a \otimes b) \xrightarrow{T\phi} T(Ta \otimes Tb) \xrightarrow{\phi} TTa \otimes TTb \xrightarrow{\mu a \otimes \mu b} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

Multiplicativity of Algebra Structure on $a \otimes b$

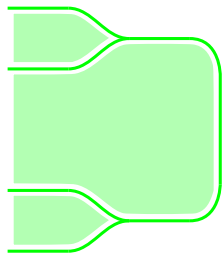


$$TT(a \otimes b) \xrightarrow{\mu^{(a \otimes b)}} T(a \otimes b) \xrightarrow{\phi} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

Unitality of Algebra Structure on e

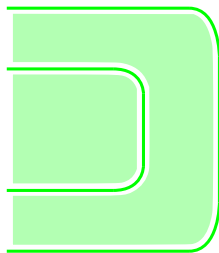
$$e \xrightarrow{\eta^e} Te \xrightarrow{\phi_0} e = e \xrightarrow{e} e$$

Multiplicativity of Algebra Structure on e



$$TTe \xrightarrow{\mu_e} Te \xrightarrow{\phi_0} e$$

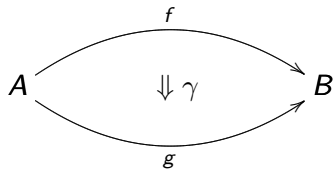
=



$$TTe \xrightarrow{T\phi_0} Te \xrightarrow{\phi_0} e$$

Duals Invert

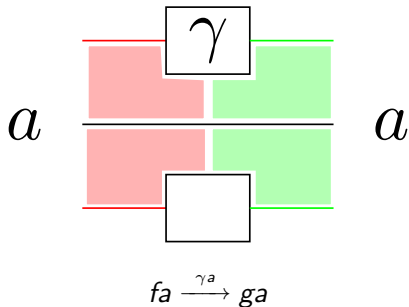
Suppose that A and B are monoidal categories, that f and g are strong monoidal functors, and that γ is a monoidal natural transformation.



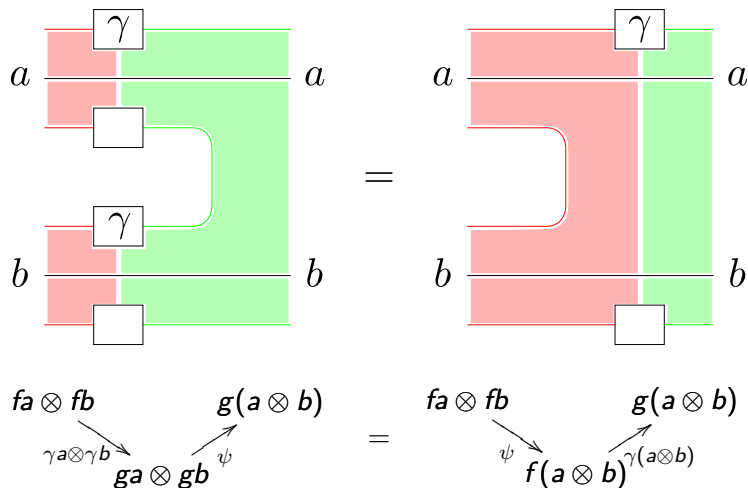
If A has duals, then γ is invertible.

Natural Transformations

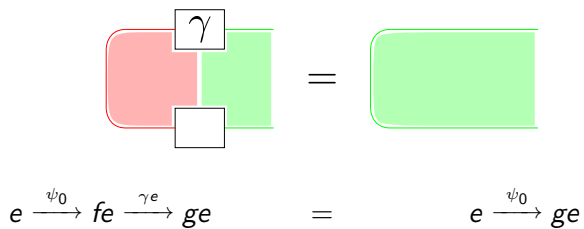
If $\gamma : f \rightarrow g$ is a natural transformation, it will have components which we denote as:



Monoidal Natural Transformations



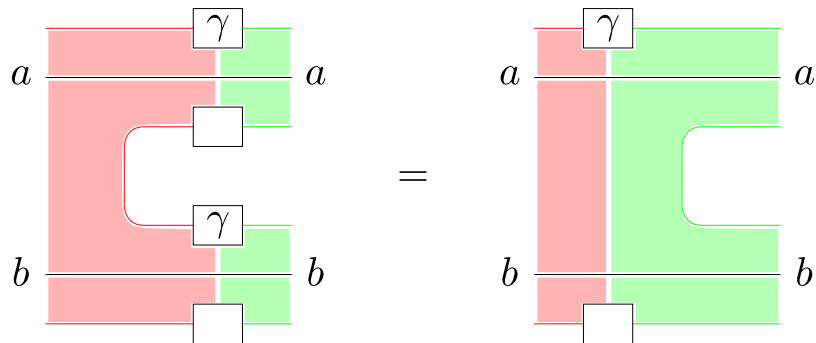
Monoidal Natural Transformations



The diagram illustrates the naturality condition for a monoidal natural transformation γ . It consists of two parts separated by an equals sign. The left part shows a sequence of two boxes: a red box on the left and a green box on the right. A white box labeled γ is positioned above the red box, and another white box is positioned below it. The red box has a rounded left side and a straight right side, while the green box has a straight left side and a rounded right side. Below this sequence is the equation $e \xrightarrow{\psi_0} fe \xrightarrow{\gamma e} ge$. The right part of the diagram shows a single green box with rounded corners on both the left and right sides. Below this box is the equation $e \xrightarrow{\psi_0} ge$. The entire diagram is set against a white background.

$$e \xrightarrow{\psi_0} fe \xrightarrow{\gamma e} ge = e \xrightarrow{\psi_0} ge$$

Comonoidal Natural Transformations

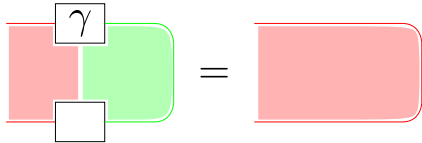


$$\begin{array}{ccc}
 f(a \otimes b) & & ga \otimes gb \\
 \searrow \phi & & \nearrow \gamma_a \otimes \gamma_b \\
 & fa \otimes fb &
 \end{array}$$

=

$$\begin{array}{ccc}
 f(a \otimes b) & & ga \otimes gb \\
 \searrow \gamma(a \otimes b) & & \nearrow \phi \\
 & g(a \otimes b) &
 \end{array}$$

Comonoidal Natural Transformations

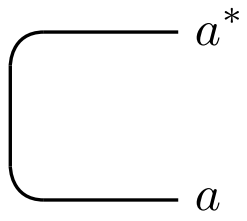


The diagram shows an equality between two comonoid structures. On the left, a red rectangle is followed by a green rounded rectangle. A box containing the Greek letter γ is positioned above the junction, and an empty box is positioned below it. On the right, a single red rounded rectangle is shown. An equals sign is placed between the two structures.

$$fe \xrightarrow{\gamma_e} ge \xrightarrow{\phi_0} e = fe \xrightarrow{\phi_0} e$$

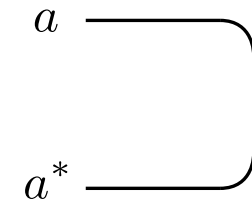
Duals for A

Suppose now that A has (right) duals; that is, for every object $a \in A$ there is an object a^* furnished with maps:



A commutative diagram with two objects, a^* at the top and a at the bottom. A horizontal line connects them on the left, which then curves down and right to form a U-shaped path. Below the diagram is the equation $e \xrightarrow{\eta^a} a^* \otimes a$.

$$e \xrightarrow{\eta^a} a^* \otimes a$$



A commutative diagram with two objects, a at the top and a^* at the bottom. A horizontal line connects them on the right, which then curves down and left to form a U-shaped path. Below the diagram is the equation $a \otimes a^* \xrightarrow{\epsilon_a} e$.

$$a \otimes a^* \xrightarrow{\epsilon_a} e$$

Duals for A

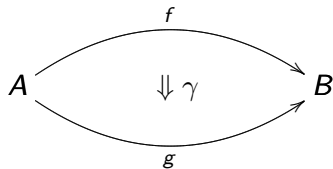
$$a \text{ --- } \left[\begin{array}{c} \text{---} \\ \downarrow \\ \text{---} \\ \downarrow \\ \text{---} \end{array} \right] a = a \text{ --- } a$$

Duals for A

The diagram illustrates a simplification of a path. On the left, a complex path starts at a point labeled a^* on the left, moves horizontally to the right, then vertically up, then horizontally to the right, then vertically down, and finally horizontally to the right to a point labeled a^* on the right. This path is shown to be equivalent to a single straight horizontal line connecting the two a^* points, as indicated by the equals sign (=) between the two diagrams.

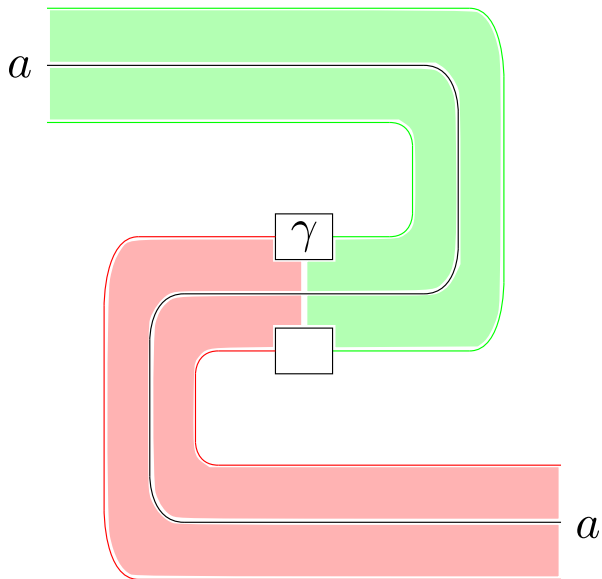
Duals Invert

Suppose that A and B are monoidal categories, that f and g are strong monoidal functors, and that γ is a monoidal natural transformation.

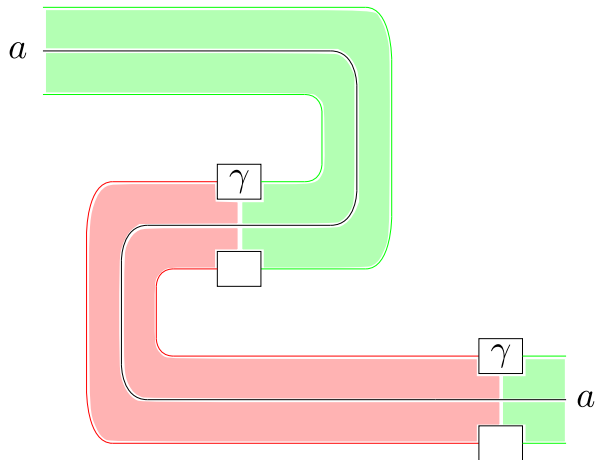


If A has duals, then γ is invertible.

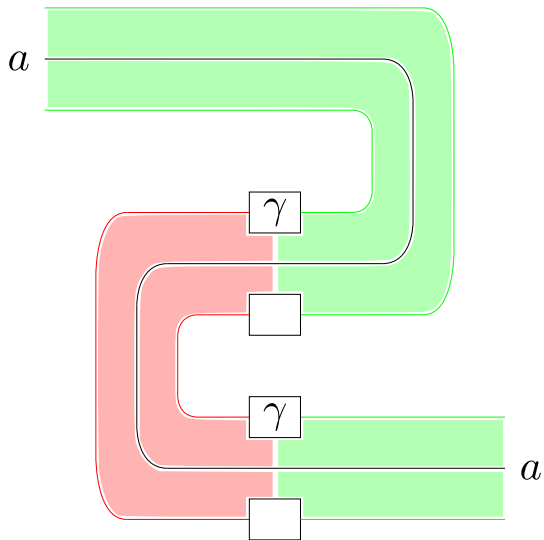
Definition of $\gamma^{-1}a$



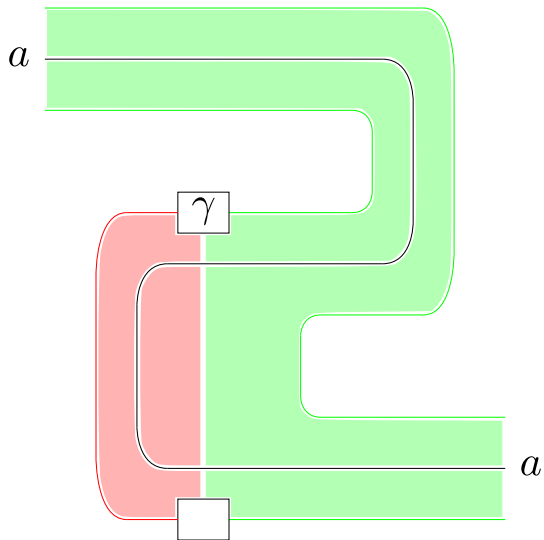
Duals Invert



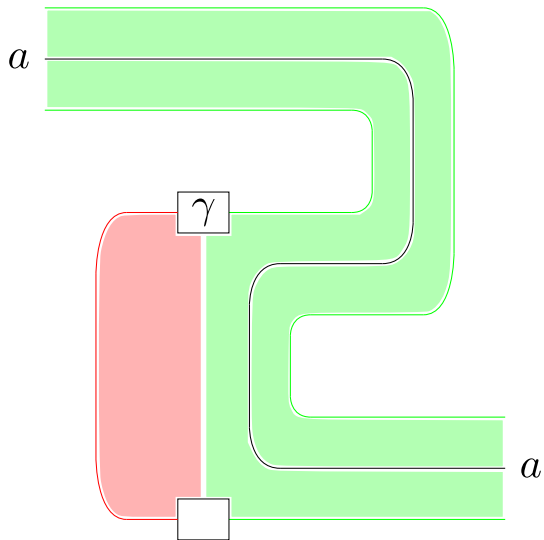
Duals Invert



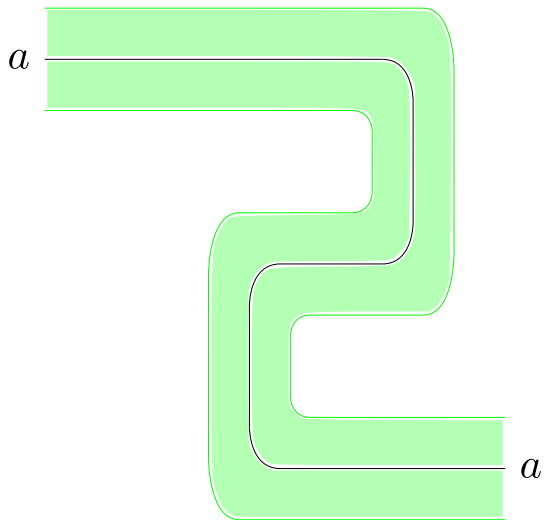
Duals Invert



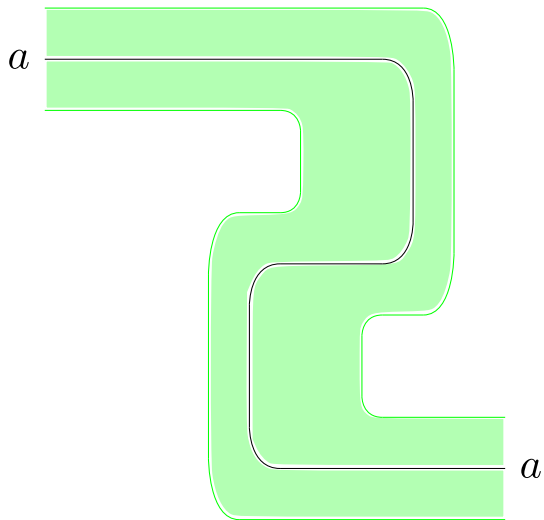
Duals Invert



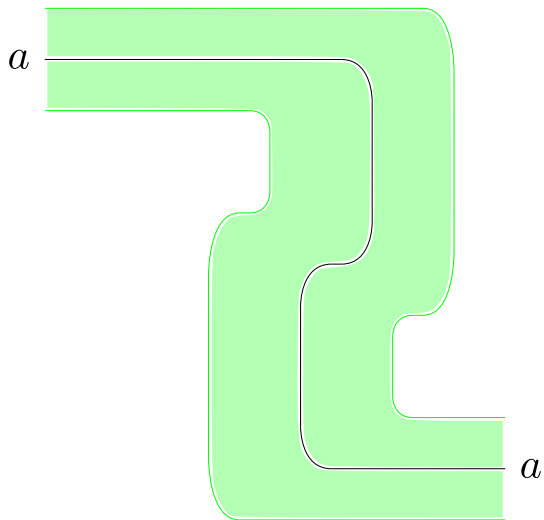
Duals Invert



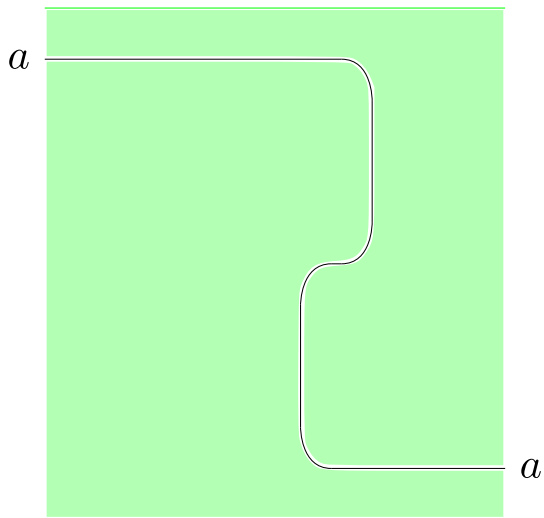
Duals Invert



Duals Invert



Duals Invert



Duals Invert

