Strings and Stripes
Graphical Calculus
for Monoidal Functors and Monads

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Purpose

- Spread propaganda about graphical notation for functors between monoidal categories.
- The category of algebras C^T for a comonoidal monad T on C is itself a monoidal category.
- If A is a monoidal category with duals, and $\gamma : f \longrightarrow g$ is a monoidal natural transformation between strong monoidal functors $f, g : A \longrightarrow B$, then γ is invertible.

Graphical Calculus of Monoidal Functors

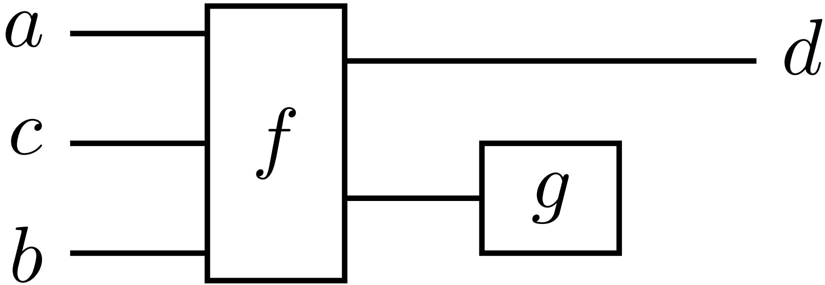
Let:

$$F : A \longrightarrow B$$

be a functor between monoidal categories.

Graphical Calculus for Monoidal Categories

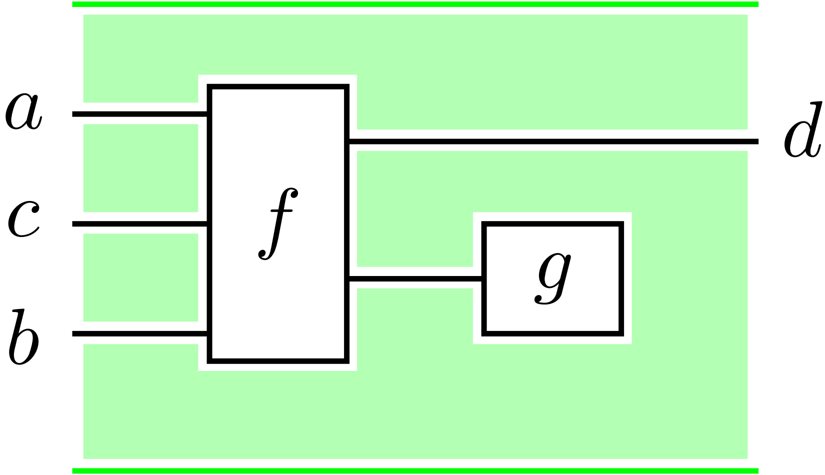
Consider the following composite in A:



$$a \otimes c \otimes b \xrightarrow{f} d \otimes x \xrightarrow{d \otimes g} d \otimes e \simeq d$$

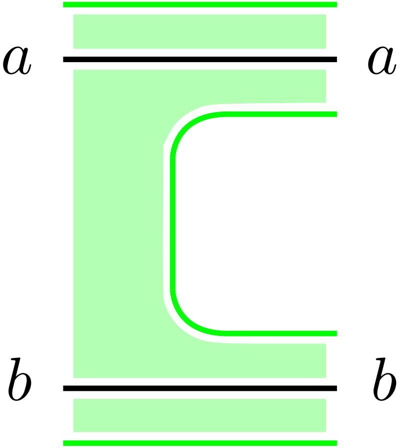
Graphical Calculus for Monoidal Categories

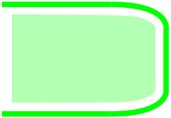
We apply F to obtain its image in B:



$$F(a \otimes c \otimes b) \xrightarrow{Ff} F(d \otimes x) \xrightarrow{F(d \otimes g)} F(d \otimes e) \simeq Fd$$

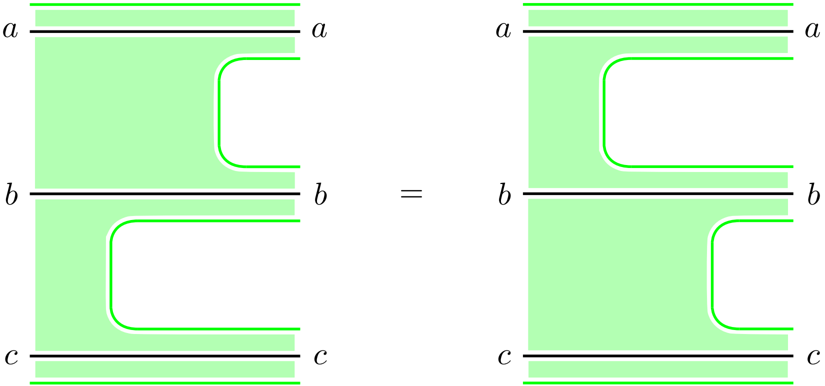
Comonoidal Structure





$F(a \otimes b) \longrightarrow Fa \otimes Fb \qquad = \qquad Fe \longrightarrow e$

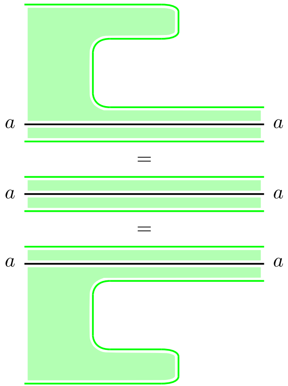
Comonoidal Coherence



$$F(a \otimes b \otimes c) \xrightarrow{\phi} F(a \otimes b) \otimes Fc \xrightarrow{\phi \otimes Fc} Fa \otimes Fb \otimes Fc \quad = \quad F(a \otimes b \otimes c) \xrightarrow{\phi} Fa \otimes F(b \otimes Fc) \xrightarrow{Fa \otimes \phi} Fa \otimes Fb \otimes Fc$$

Comonoidal Coherence

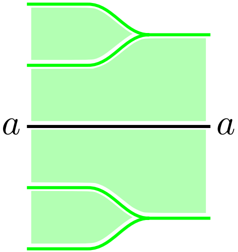
$$Fa \simeq F(e \otimes a) \xrightarrow{\phi} Fe \otimes Fa \xrightarrow{\otimes Fa} e \otimes Fa \simeq Fa$$

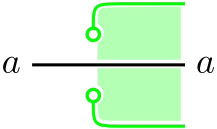


$$Fa \simeq F(e \otimes e) \xrightarrow{\phi} Fa \otimes Fe \xrightarrow{Fa \otimes} Fa \otimes e \simeq Fa$$

Monads

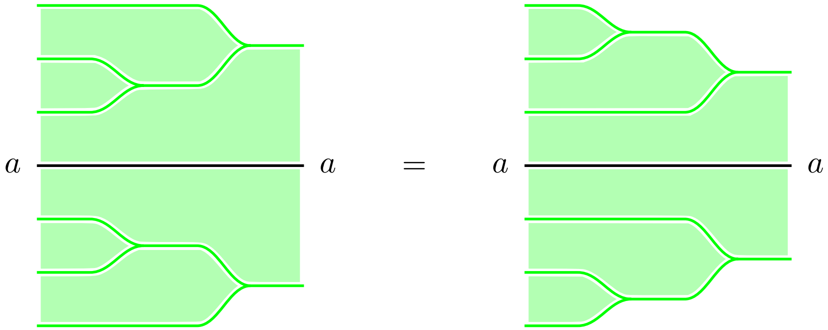
Suppose that $T : C \longrightarrow C$ is a monad:





$$TTa \xrightarrow{\mu a} Ta \qquad = \qquad a \xrightarrow{\eta a} Ta$$

Monad Coherence

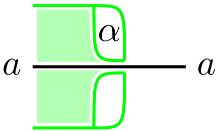


$$TTTa \xrightarrow{T\mu a} TTa \xrightarrow{\mu a} Ta \quad = \quad TTTa \xrightarrow{\mu Ta} TTa \xrightarrow{\mu a} Ta$$

Monad Coherence

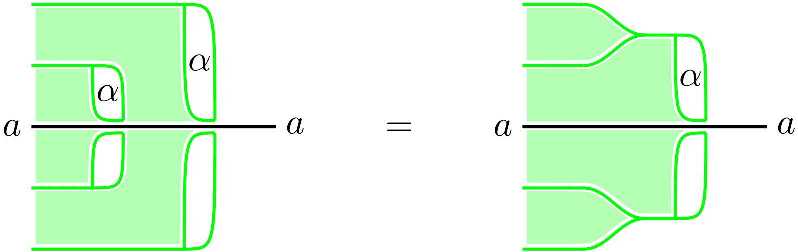
$$Ta \xrightarrow{T\eta a} TTa \xrightarrow{\mu a} Ta = \quad Ta \xrightarrow{Ta} Ta \quad = Ta \xrightarrow{\eta Ta} TTa \xrightarrow{\mu a} Ta$$

Algebras for Monads



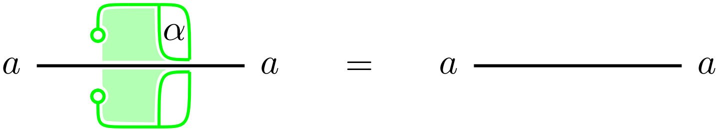
$$Ta \xrightarrow{\alpha} a$$

Algebra Coherence



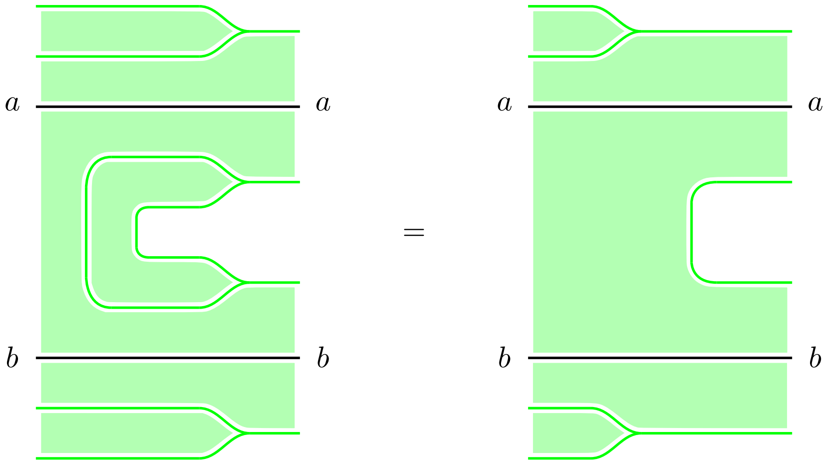
$$TTa \xrightarrow{T\alpha} Ta \xrightarrow{\alpha} a \quad = \quad TTa \xrightarrow{\mu a} Ta \xrightarrow{\alpha} a$$

Algebra Coherence



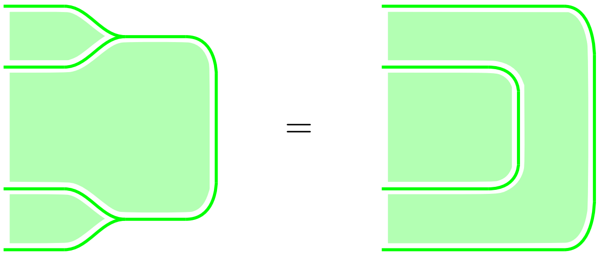
$$a \xrightarrow{\eta a} Ta \xrightarrow{\alpha} a \quad = \quad a \xrightarrow{a} a$$

Comonoidal Monads



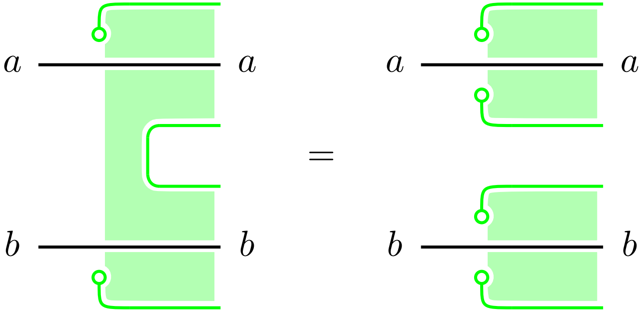
$$\left(TT(a \otimes b) \xrightarrow{T\phi} T(Ta \otimes Tb) \xrightarrow{\phi} TTa \otimes TTb \xrightarrow{\mu a \otimes \mu b} Ta \otimes Tb \right) = \left(TT(a \otimes b) \xrightarrow{\mu(a \otimes b)} T(a \otimes b) \xrightarrow{\phi} Ta \otimes Tb \right)$$

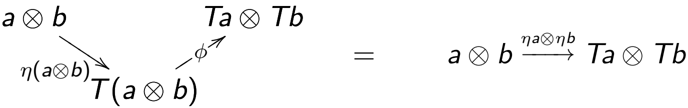
Comonoidal Monads



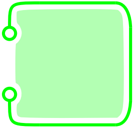
$$TTe \xrightarrow{\mu e} Te \xrightarrow{\phi_0} e \quad = \quad TTe \xrightarrow{T\phi_0} Te \xrightarrow{\phi_0} e$$

Comonoidal Monads





Comonoidal Monads

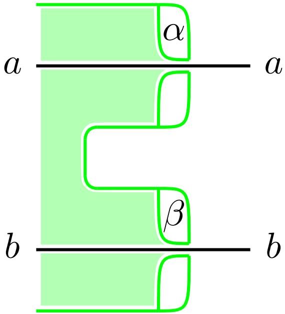
 $=$

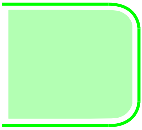
$$e \xrightarrow{\eta e} Te \xrightarrow{\phi_0} e \quad = \quad e \xrightarrow{e} e$$

Monoidal Structure on C^T

If $T : C \longrightarrow C$ is a comonoidal monad, then C^T inherits a natural monoidal structure.

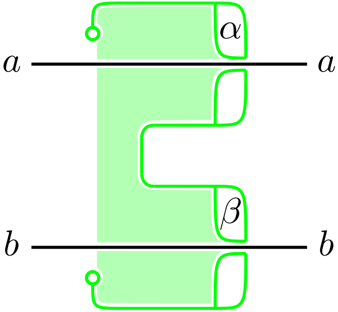
Monoidal Structure on C^T





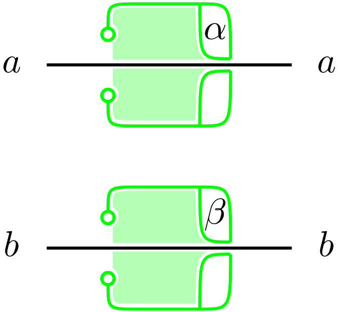
$$T(a \otimes b) \xrightarrow{\phi} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b \qquad Te \xrightarrow{\phi_0} e$$

Unitality of Algebra Structure on $a \otimes b$



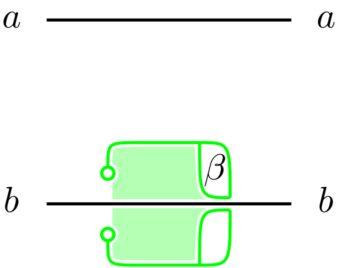
$$a \otimes b \xrightarrow{\eta(a \otimes b)} T(a \otimes b) \xrightarrow{\phi_0} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

Unitality of Algebra Structure on $a \otimes b$



$$a \otimes b \xrightarrow{\eta a \otimes \eta b} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

Unitality of Algebra Structure on $a \otimes b$

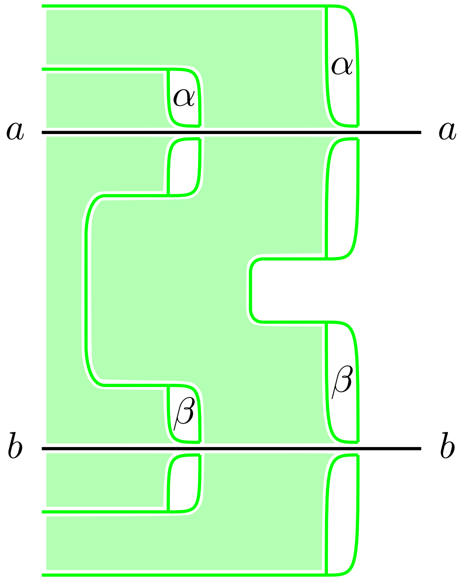


$$a \otimes b \xrightarrow{a \otimes \eta b} a \otimes Tb \xrightarrow{a \otimes \beta} a \otimes b$$

Unitality of Algebra Structure on $a \otimes b$

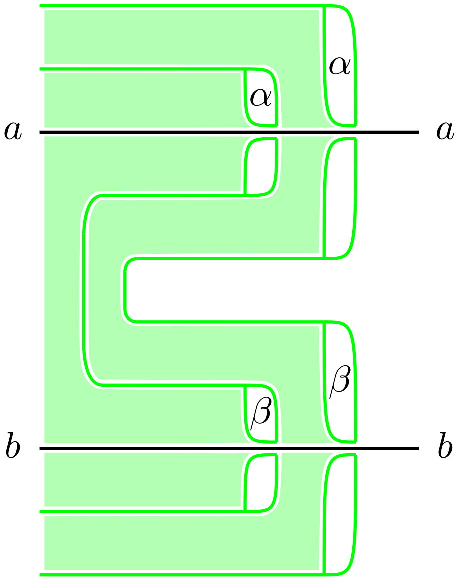
$$a \otimes b \xrightarrow{a \otimes b} a \otimes b$$

Multiplicativity of Algebra Structure on $a \otimes b$



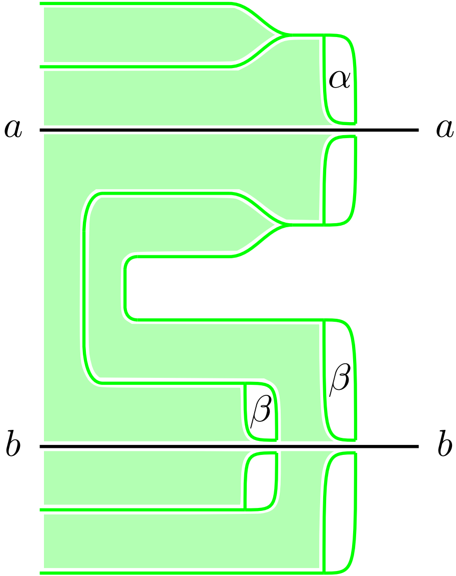
$$TT(a \otimes b) \xrightarrow{T\phi_0} T(Ta \otimes Tb) \xrightarrow{T(\alpha \otimes \beta)} T(a \otimes b) \xrightarrow{\phi} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

Multiplicativity of Algebra Structure on $a \otimes b$



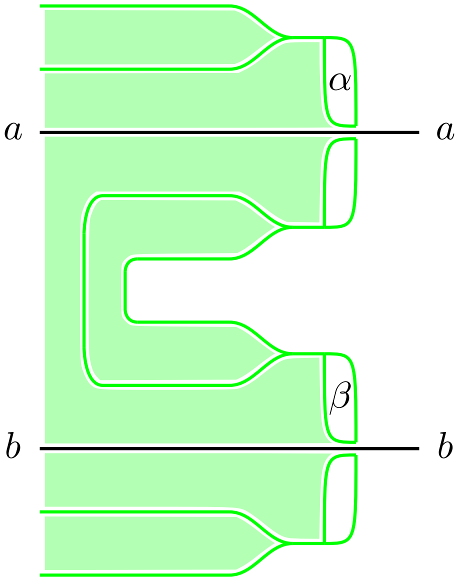
$$TT(a \otimes b) \xrightarrow{T\phi} T(Ta \otimes Tb) \xrightarrow{\phi} TTa \otimes TTb \xrightarrow{T\alpha \otimes T\beta} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

Multiplicativity of Algebra Structure on $a \otimes b$



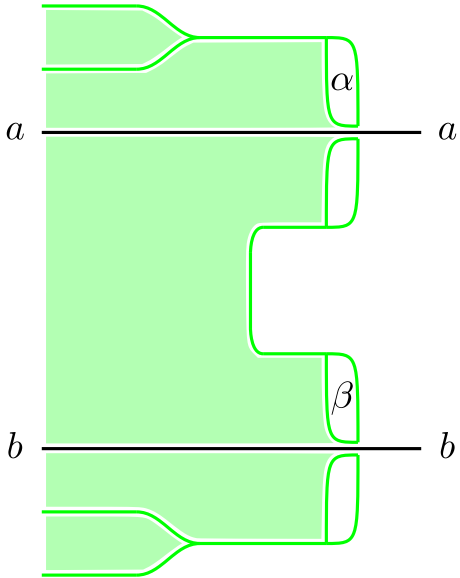
$$TT(a \otimes b) \xrightarrow{T\phi} T(Ta \otimes Tb) \xrightarrow{\phi} TTa \otimes TTb \xrightarrow{\mu a \otimes T\beta} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

Multiplicativity of Algebra Structure on $a \otimes b$



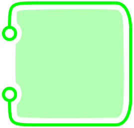
$$TT(a\otimes b) \xrightarrow{T\phi} T(Ta\otimes Tb) \xrightarrow{\phi} TTa\otimes TTb \xrightarrow{\mu a\otimes \mu b} Ta\otimes Tb \xrightarrow{\alpha\otimes\beta} a\otimes b$$

Multiplicativity of Algebra Structure on $a \otimes b$



$$TT(a \otimes b) \xrightarrow{\mu(a \otimes b)} T(a \otimes b) \xrightarrow{\phi} Ta \otimes Tb \xrightarrow{\alpha \otimes \beta} a \otimes b$$

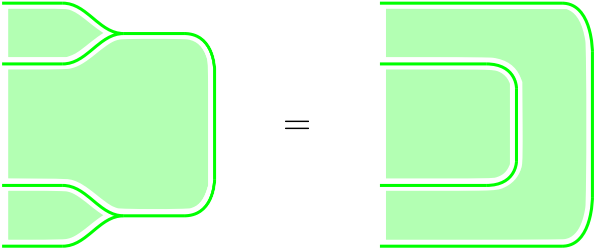
Unitality of Algebra Structure on e



$=$

$$e \xrightarrow{\eta e} Te \xrightarrow{\phi_0} e \quad = \quad e \xrightarrow{e} e$$

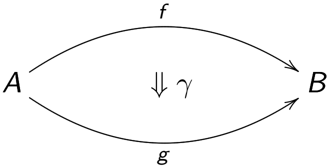
Multiplicativity of Algebra Structure on e



$$TTe \xrightarrow{\mu e} Te \xrightarrow{\phi_0} e \quad = \quad TTe \xrightarrow{T\phi_0} Te \xrightarrow{\phi_0} e$$

Duals Invert

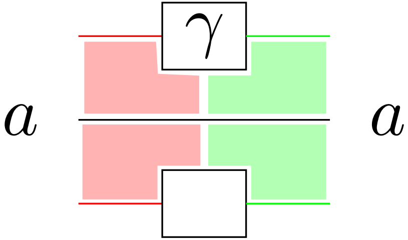
Suppose that A and B are monoidal categories, that f and g are strong monoidal functors, and that γ is a monoidal natural transformation.



If A has duals, then γ is invertible.

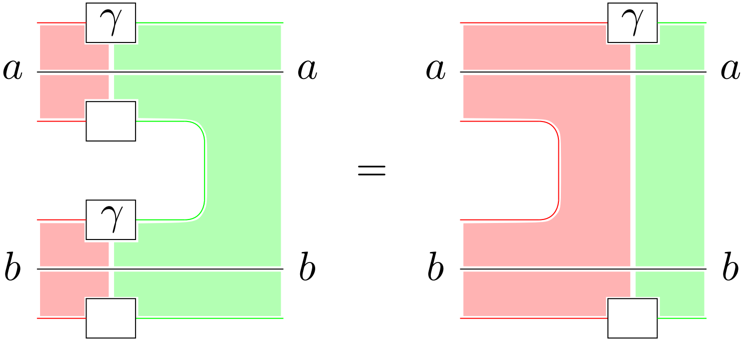
Natural Transformations

If $\gamma : f \longrightarrow g$ is a natural transformation, it will have components which we denote as:



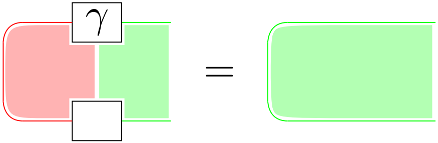
$$fa \xrightarrow{\gamma a} ga$$

Monoidal Natural Transformations



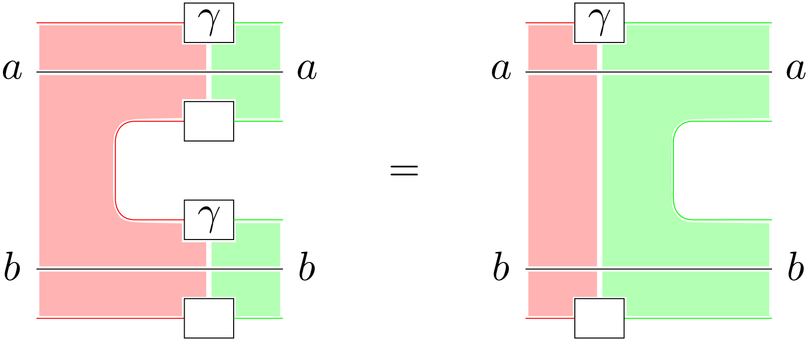
$$\left(fa \otimes fb \xrightarrow{\gamma a \otimes \gamma b} ga \otimes gb \xrightarrow{\psi} g(a \otimes b)\right) = \left(fa \otimes fb \xrightarrow{\psi} f(a \otimes b) \xrightarrow{\gamma(a \otimes b)} g(a \otimes b)\right)$$

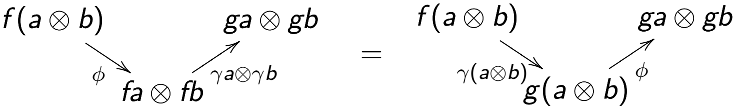
Monoidal Natural Transformations



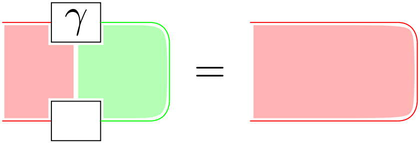
$$e \xrightarrow{\psi_0} fe \xrightarrow{\gamma e} ge \qquad = \qquad e \xrightarrow{\psi_0} ge$$

Comonoidal Natural Transformations





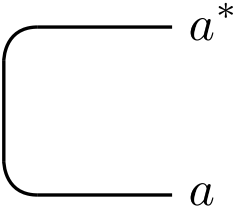
Comonoidal Natural Transformations



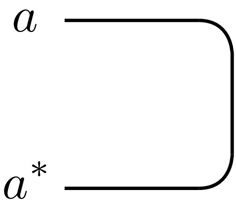
$$fe \xrightarrow{\gamma e} ge \xrightarrow{\phi_0} e \qquad = \qquad fe \xrightarrow{\phi_0} e$$

Duals for A

Suppose now that A has (right) duals; that is, for every object $a \in A$ there is an object a^* furnished with maps:

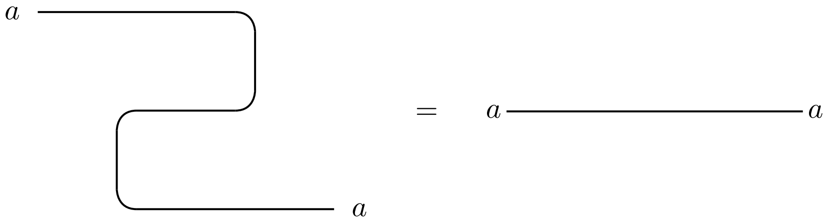


$e \xrightarrow{\eta a} a^* \otimes a$

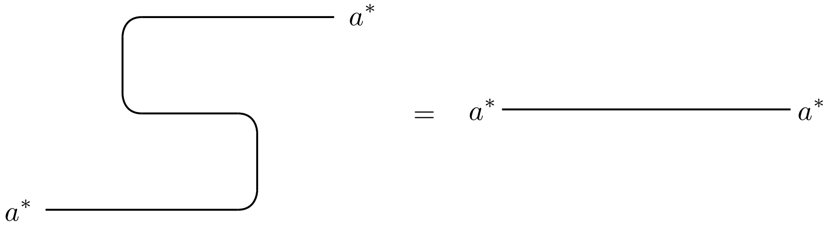


$a \otimes a^* \xrightarrow{\epsilon a} e$

Duals for A

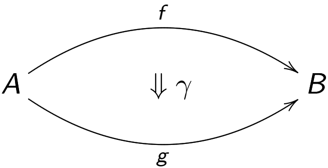


Duals for A



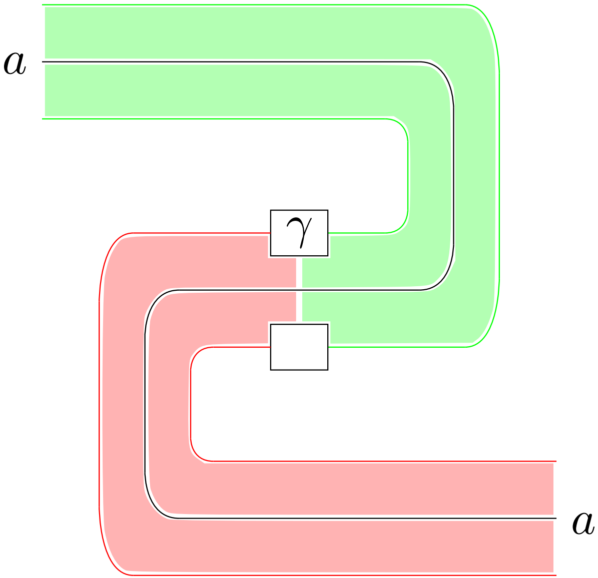
Duals Invert

Suppose that A and B are monoidal categories, that f and g are strong monoidal functors, and that γ is a monoidal natural transformation.

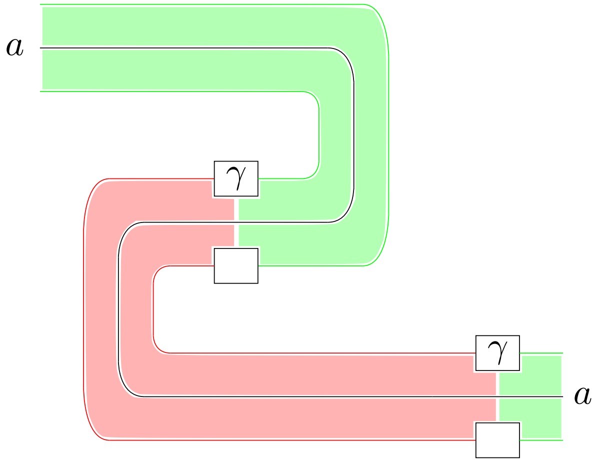


If A has duals, then γ is invertible.

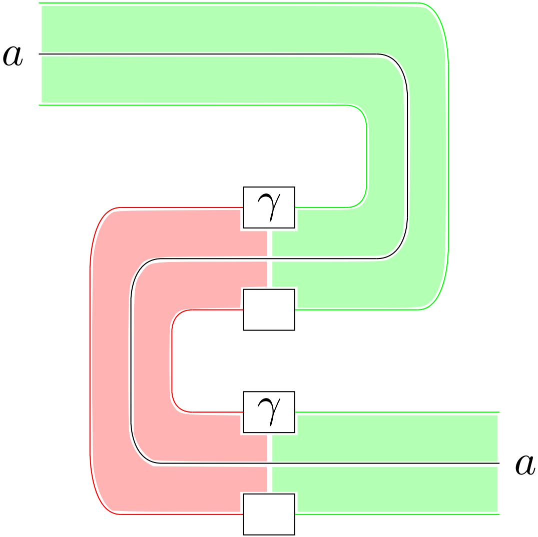
Definition of $\gamma^{-1}a$



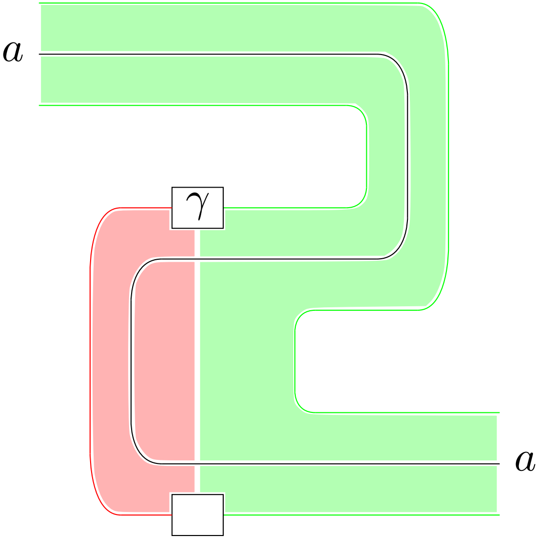
Duals Invert



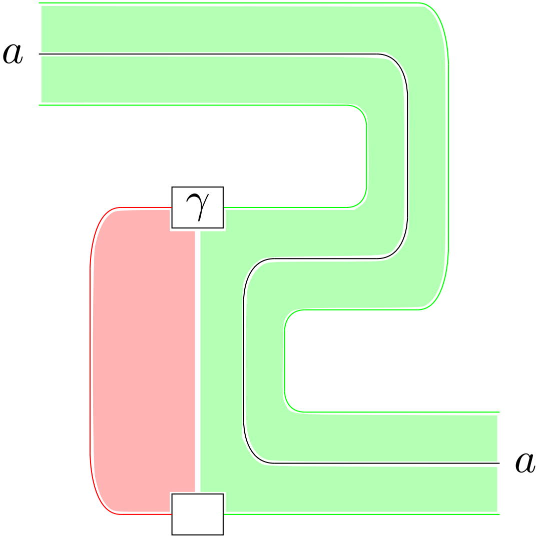
Duals Invert



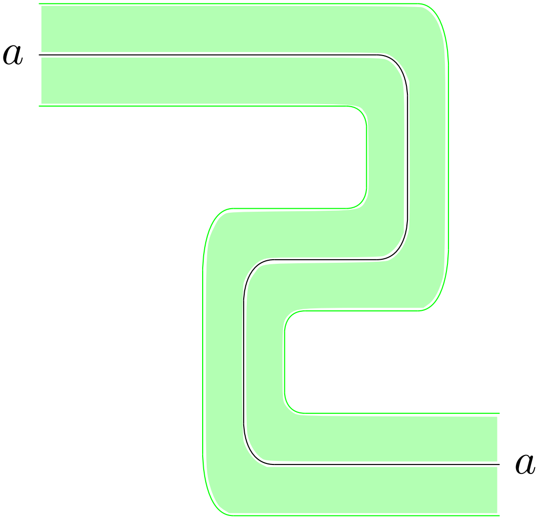
Duals Invert



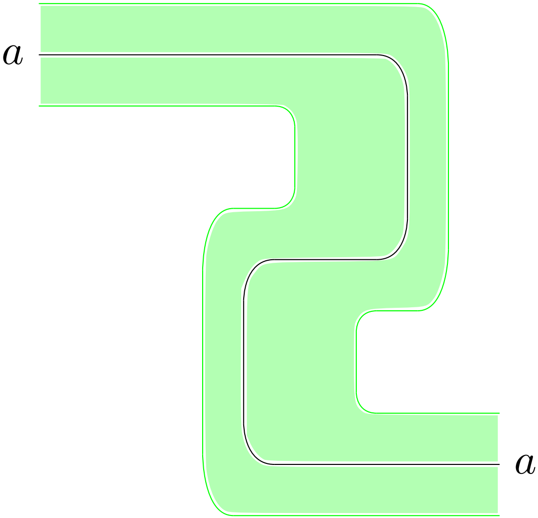
Duals Invert



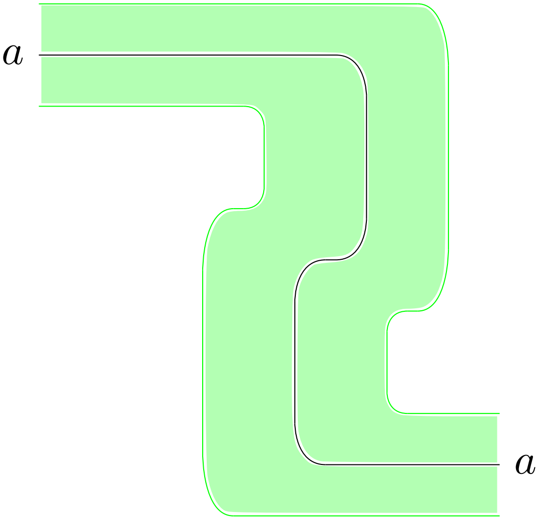
Duals Invert



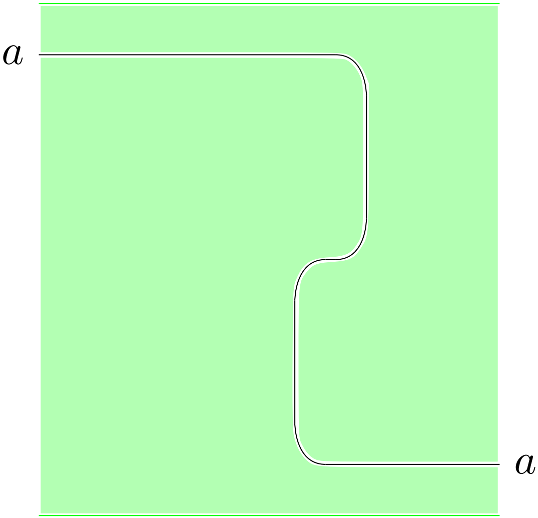
Duals Invert



Duals Invert



Duals Invert



Duals Invert