Spans of lenses

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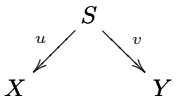
ABSTRACT

Corresponding to the variety of notions of asymmetric lens, various notions of symmetric lens have been proposed. A common theory of the various asymmetric and symmetric lenses should result from a study of spans of asymmetric lenses. In order to define a category whose arrows are spans of asymmetric lenses, the fact that a cospan of asymmetric lenses may not have a pullback must be dealt with. In this article, after resolving that problem we develop the functors which exhibit a category whose arrows are spans of well-behaved lenses as a retract of a category whose arrows are the corresponding symmetric lenses. We relate them to the symmetric lenses of Hofmann, Pierce and Wagner.

1. INTRODUCTION

A *span* is a pair of functions, or more generally of arbitrary morphisms of a given kind, with common domain:



Such a span is often described as a "span from X to Y", and denoted $u : X \longleftarrow S \longrightarrow Y : v$. The object S is sometimes called the *peak* of the span and the arrows u and v are called the *legs* of the span.

Spans have been used in a variety of fields as diverse as the abstract theory of relations and the design of circuits and systems. Naturally the mathematical theory of spans is well-developed. Of particular importance: The composition of a span from X to Y and a span from Y to Z is a span from X to Z calculated by constructing a pullback, and two spans from X to Y are *span-equivalent* when there is an isomorphism between their peaks which commutes with their legs.

Spans of transformations arise widely in areas related to Bidirectional Transformations (Bx) too. Examples include model driven engineering, triple graph grammars, and symmetric lenses. However, in Bx the classical theory of spans is harder to apply. Even in the case of basic (asymmetric) lenses, the classical theory doesn't apply because in the category whose morphisms are lenses, what one might expect to be the pullback of lenses need not satisfy the universal property of a pullback — the difficulty that arises is that the universally provided comparison morphism may not have a lens structure.

Even if we neglect for a moment the difficulty in properly defining composition of spans of lenses, there is a further complication: Span-equivalence does not seem to be the right notion of equivalence for spans of lenses. As Hofmann et al remarked in discussing spans of asymmetric lenses in [5] (full version), "in the span presentation there does not seem to be a natural and easy-to-use candidate for ... equivalence."

The main goal of this paper is to develop the mathematical foundations required to support the use of spans in Bx. We show that

1. While categories whose morphisms are lenses may not have pullbacks, there is a framework that frequently allows us to work as if they did by finding canonical lens structures on the pullbacks of the Get functions.

2. While classical span-equivalence is far too strong a condition for equivalence of spans of lenses, there is a natural generalisation (replacing isomorphisms with suitably non-empty lenses) which seems right.

Combining these we have the mathematical foundations required. To demonstrate this we apply them here to the study of the symmetric lenses of [5], and in future work to a unified theory of symmetric lenses of many kinds.

A remark on the technical content. The paper is necessarily quite mathematical – we are after all building a mathematical foundation, and the measure of its utility or even correctness is its ability to provide precise proofs and insightful clarifications of mathematical results about spans of lenses. Many proofs have been omitted because, once the correct formulation has been found, the proof is relatively routine for those with category theoretic skills, and is unlikely to be very enlightening for those with less category theoretic experience. Other proofs have been sketched especially where an unusual approach might be required. The main contributions of this work are not the proofs themselves, but rather finding the right formulations (eg Proposition 5 for dealing with the missing pullbacks, and the equivalence $\equiv_G$) which make the proofs feasible.

The structure of the paper is as follows. In Section 2 we review some details about asymmetric lenses and show how to canonically construct lenses on the pullbacks of the get functions of asymmetric lenses. The following section, Section 3, lays out the general theory that we use to systematically deal with these pullback like constructions (that aren't pullbacks) and introduces the equivalence relation which we will use on spans of asymmetric lenses. In Section 4 we introduce rl lenses — the approach to symmetric lenses that parallels that used in [5]. We show how to generalise that so that it can be applied in any category with products, making available immediately notions of symmetric lenses for graphs, categories, and ordered sets for example. In Section 5 we develop an equivalence for rl lenses motivated by, but different from, the equivalence for set based rl lenses in [5], and begin the development of two functors **A** and **S** which are used to compare span based and rl based symmetric lenses. Finally, in Section 6 we specialise to rl lenses with a pointed complement. These lenses correspond exactly to the lenses of [5] and we compare Hofmann et al's equivalence of such lenses with the equivalences proposed in this paper.

2. ASYMMETRIC LENSES

In this section we collect information about various notions of asymmetric lens along with some basic results that we will use later.

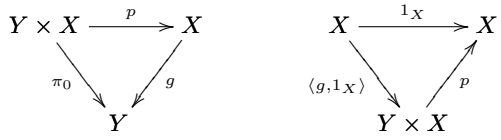
Let $\mathbf{C}$ be a category with finite products. Categories such as the category of sets and functions, ordered sets and monotone mappings, and categories and functors, are some of the examples we have in mind. We recall the definition of asymmetric lens in $\mathbf{C}$ [2, 5, 7].

DEFINITION 1. *For objects X, Y in $\mathbf{C}$, an* asymmetric lens *in $\mathbf{C}$ from X to Y, denoted $L : X \longrightarrow Y$ is $L = (X, Y, g, p)$ where $g : X \longrightarrow Y$ is called the* Get *morphism and $p : Y \times X \longrightarrow X$ is called the* Put *morphism. A lens is called* well-behaved *if it satisfies:*

(i) (PutGet) the Get of a Put is the projection: $gp = \pi_0$

(ii) (GetPut) the Put for a trivially updated state is trivial: $p\langle g, 1_X\rangle = 1_X$

Diagrammatically, two commutative triangles:

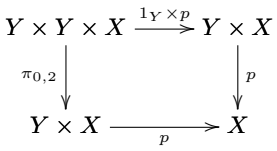


PutGet *GetPut*

A well-behaved lens is called very well-behaved *if it satisfies:*

(iii) (PutPut) composing Puts does not depend on the first update: $p(1_Y \times p) = p\pi_{0,2}$

Diagrammatically, a commutative square:



PutPut

We showed in [7] that, up to isomorphism, a very well-behaved asymmetric lens L in $\mathbf{C}$ has $X = Y \times C$ for an object C of "complements" and then g is the projection $g : Y \times C \longrightarrow Y$ while the put is defined by $p : Y \times (Y \times C) \longrightarrow Y \times C = \pi_{0,2}$, the projection onto the first and third factors. That generalises to categories with products (for example, those where the states of X and Y are, rather than mere sets, ordered sets or even more generally graphs or categories) the classical theory of constant complement updating [1]. Such lenses are algebras for a monad on $\mathbf{C}/Y$. Thus there is a well-defined notion of morphisms between lenses. However, our interest in this article is rather to treat various kinds of lenses as arrows of categories, so we will need a composition of lenses themselves.

Given lenses $L = (X, Y, g_1, p_1)$ and $M = (Y, Z, g_2, p_2)$, the *composite lens* is $ML = (X, Z, g, p)$ where $g = g_2 g_1$ and p is

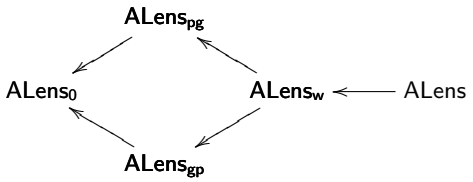
$$Z \times X \xrightarrow{1_Z \times \langle g_1, 1_X\rangle} Z \times Y \times X \xrightarrow{p_2 \times 1_X} Y \times X \xrightarrow{p_1} X$$

or as an elementary formula: $p(z, x) = p_1(p_2(z, g_1 x), x)$. For X in $\mathbf{C}$ there is an identity lens $\mathbf{1}_X = (X, X, 1_X, p_0)$ with p_0 the first projection, π_0. We will assume that products in $\mathbf{C}$ are associative. With that assumption, composition of lenses in $\mathbf{C}$ is associative and the identity lens acts as an identity.

The first additional properties of lenses are stable under composition:

LEMMA 2. *Suppose that L, M and ML are defined as above. If L and M both satisfy PutGet, respectively GetPut, PutPut, then ML satisfies PutGet, respectively GetPut and, provided L satisfies PutGet, respectively PutPut.*

Thus, there are categories we denote $\mathsf{ALens}_0(\mathbf{C})$, $\mathsf{ALens}_{\mathsf{pg}}(\mathbf{C})$, $\mathsf{ALens}_{\mathsf{gp}}(\mathbf{C})$, $\mathsf{ALens}_{\mathsf{w}}(\mathbf{C})$ and $\mathsf{ALens}(\mathbf{C})$ of asymmetric lenses, respectively lenses satisfying PutGet, GetPut, well-behaved lenses, and very well-behaved lenses in $\mathbf{C}$. In each case the objects are those of $\mathbf{C}$ and the arrows are asymmetric lenses with the corresponding property. When $\mathbf{C}$ is understood we will sometimes write $\mathsf{ALens}_{\mathsf{w}}$, for example, for $\mathsf{ALens}_{\mathsf{w}}(\mathbf{C})$. There are faithful, but evidently not full, inclusion functors



There are some further results about asymmetric lenses we need to record. First, finite product preserving functors preserve lens structures:

PROPOSITION 3. *Suppose that $\mathbf{C}$ and $\mathbf{D}$ are categories with finite products and $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a finite product preserving functor. If $L = (X, Y, g, p)$ is an asymmetric lens in $\mathbf{C}$, and respectively satisfying PutGet, GetPut, a well-behaved lens or a very well-behaved lens, then $FL = (FX, FY, Fg, Fp)$ is an (asymmetric) lens in $\mathbf{D}$, respectively satisfying PutGet, GetPut, a well-behaved lens, or a very well-behaved lens. If M is a lens in $\mathbf{C}$ composable with L, then $F(ML) = (FM)(FL)$, and we obtain a functor, also denoted F,*

$$F : \mathsf{ALens}_0(\mathbf{C}) \longrightarrow \mathsf{ALens}_0(\mathbf{D})$$

and respectively from $\mathsf{ALens}_{\mathsf{pg}}(\mathbf{C})$, $\mathsf{ALens}_{\mathsf{gp}}(\mathbf{C})$, $\mathsf{ALens}_{\mathsf{w}}(\mathbf{C})$ *and* $\mathsf{ALens}(\mathbf{C})$.

We denote the product-preserving inclusion of sets as discrete ordered sets by I, and of ordered sets as small categories by J in:

$$\mathbf{set} \xrightarrow{I} \mathbf{ord} \xrightarrow{J} \mathbf{cat}$$

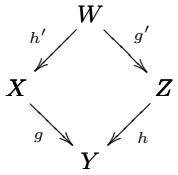
Thus there are faithful functors:

$$\mathsf{ALens}_0(\mathbf{set}) \xrightarrow{I} \mathsf{ALens}_0(\mathbf{ord}) \xrightarrow{J} \mathsf{ALens}_0(\mathbf{cat})$$

and similarly for the PutGet, GetPut, well-behaved and very well-behaved cases.

Next we consider pullbacks in $\mathbf{C}$ and the various categories above. Remember that the lenses are *morphisms*. For example, when very well-behaved lenses are viewed as algebras for a monad on $\mathbf{C}/Y$, pullbacks exist in the algebras whenever they do in the base. Our case here is different. Since we need products and pullbacks in the base, we assume all finite limits.

PROPOSITION 4. *Suppose that* $\mathbf{C}$ *is a category with finite limits and let* $L = (X, Y, g, p)$ *be a morphism of* $\mathsf{ALens}_{\mathsf{pg}}$, *resp.* $\mathsf{ALens}_{\mathsf{w}}$ *and* ALens, *and* $h : Z \longrightarrow Y$ *be a morphism in* $\mathbf{C}$. *Let*



be a pullback in $\mathbf{C}$. *Then* $L' = (W, Z, g', p')$ *is in* $\mathsf{ALens}_{\mathsf{pg}}$, *resp.* $\mathsf{ALens}_{\mathsf{w}}$ *and* ALens, *where* $p' : Z \times W \longrightarrow W$ *is the unique morphism into the pullback determined by* $p(h \times h') : Z \times W \longrightarrow X$ *and* $\pi_0 : Z \times W \longrightarrow Z$. *If also* $M = (Z, Y, h, q)$ *is in* $\mathsf{ALens}_{\mathsf{pg}}$, *resp.* $\mathsf{ALens}_{\mathsf{w}}$ *and* ALens, *then for* $M' = (W, X, h', q')$ *the corresponding lens we have* $LM' = ML'$.

In the proof, the PutGet law gives $gp(h \times h') = \pi_0(h \times h') = h\pi_0$ which is used to define p'. When PutGet is available, the remaining properties for L' follow routinely from those for L. When M' is defined, it is routine to check that the composite lenses are equal.

Note that we say nothing about ALens_0 nor $\mathsf{ALens}_{\mathsf{gp}}$. It appears that at least the PutGet condition is needed both to define a sensible p' and to show that the square of lenses commutes.

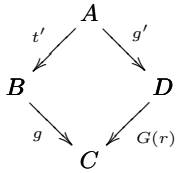
Finally, we make some comments regarding lenses involving initial objects. For any set Y there is a unique well-behaved asymmetric lens whose Get has empty domain in the category **set** of sets and functions. The Put is the unique $Y \times 0 \cong 0 \longrightarrow 0$. On the other hand, when the Get for a lens is a split epimorphism in $\mathbf{C}$ then it has a section s satisfying $gs = 1_Y$. We call an asymmetric lens *split* if its Get is a split epimorphism. Thus, an asymmetric lens which is split and has an inhabited codomain also has an inhabited domain. If the codomain is not inhabited (for example, is empty in the **set** case) then *its domain is also not inhabited*. In that case there is, of course, a unique lens structure taking the unique (identity) endo-function of the empty set for the Get.

3. CONSTRUCTION OF SP(G)

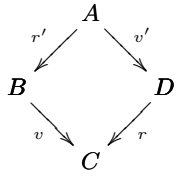
The various categories of asymmetric lenses usually do not have pullbacks. However, since the base category $\mathbf{C}$ is assumed to have finite limits, a cospan of asymmetric lens Gets has a pullback in the *base* category $\mathbf{C}$. Moreover, the pullback projections in $\mathbf{C}$ are themselves canonically the Gets of arrows (lenses) in the corresponding asymmetric lens category. We are interested in defining categories whose arrows are spans of the various sorts of asymmetric lenses. That motivates the following constructions.

Let $\mathbf{C}$ be a category with finite limits and $G : \mathbf{A} \longrightarrow \mathbf{C}$ an identity on objects functor. The reader should think of $\mathbf{A}$ as a category of asymmetric lenses and G as the forgetful functor which remembers only the Gets.

We suppose that there is an operation P on cospans of the form $B \xrightarrow{g} C \xleftarrow{G(r)} D$ in $\mathbf{C}$ which outputs an arrow $P(g, r)$ of $\mathbf{A}$. We are thinking of r as an asymmetric lens and g as a $\mathbf{C}$ arrow with the same codomain. We require three properties of P. Suppose first that G reflects isomorphisms, that is, if Gf is an isomorphism then f is an isomorphism. Next, we assume there is a pullback in $\mathbf{C}$:



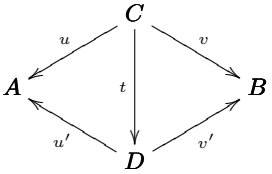
with $t' = G(r')$ where $r' = P(g, r)$ is in $\mathbf{A}$. Finally, if we also have that $g = G(v)$ then for $v' = P(G(r), v)$ the following square commutes in $\mathbf{A}$:



Note that the image under G of the square is still a pullback in $\mathbf{C}$.

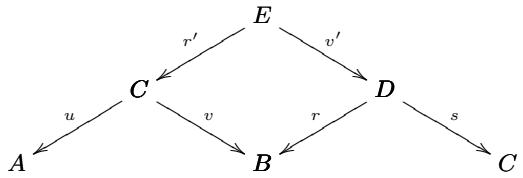
As noted above, the categories $\mathsf{ALens}_{\mathsf{pg}}(\mathbf{C})$, $\mathsf{ALens}_{\mathsf{w}}(\mathbf{C})$ and $\mathsf{ALens}(\mathbf{C})$ of various types of asymmetric lenses all satisfy the hypotheses for G and P.

Given G and P as above, we define a category $Sp(G)$. The objects of $Sp(G)$ are those of $\mathbf{A}$ (or $\mathbf{C}$). The arrows are equivalence classes for $\equiv_G$ of spans in $\mathbf{A}$. The equivalence relation $\equiv_G$ is generated by morphisms of spans in $\mathbf{A}$ such as



in which $u = u't$ and $v = v't$ (this is what it means to be a morphism of spans) and *for which* $G(t)$ *is a split epimorphism*. This condition on $G(t)$ is important since $G(t)$ will be a Get and we want to avoid cases where a lens whose Get has empty domain can make two spans equivalent. Two spans are equivalent, written $(u, v) \equiv_G (u', v')$ if there is a

"zig-zag" (a string of length zero or more of arrows, adjacent members of which meet head to head or tail to tail) of the span morphisms described above between them. Composition in $Sp(G)$ is defined by span composition in $\mathbf{C}$ of representatives. That is, the composite of the $\equiv_G$ equivalence classes of spans $(u, v), (r, s)$ (in $\mathbf{A}$) is the $\equiv_G$ equivalence class of the span (ur', sv') where $r' = P(G(v), r)$ and $v' = P(G(r), v)$, as in the diagram in $\mathbf{A}$:



PROPOSITION 5. *With the data just defined, $Sp(G)$ is a category.*

The proof is largely routine but we do note that it uses the fact that a split epimorphism "pulls back" to (that is, it has as an opposite pullback projection) a split epimorphism.

4. SYMMETRIC LENSES

As for asymmetric lenses, symmetric lenses were first defined in **set** and the concept can be generalized. The asymmetric lens concept is expressible in any category with finite products. The same is true for the symmetric lenses (in **set**) of Hofmann, Pierce and Wagner [5].

DEFINITION 6. *Let $\mathbf{C}$ be a category with finite products (including the empty product). For objects X, Y in $\mathbf{C}$, an* rl lens *from X to Y is a quintuple, $L = (X, Y, C, r, l)$ where C is an object of $\mathbf{C}$ of "complements" and r and l are morphisms*

$$r : X \times C \longrightarrow Y \times C \quad \textit{and} \quad l : Y \times C \longrightarrow X \times C$$

satisfying the four equations

$\pi_X lr = \pi_X : X \times C \longrightarrow X; \pi_C lr = \pi_C r : X \times C \longrightarrow C$ *(PUTRL)*

and

$\pi_Y rl = \pi_Y : Y \times C \longrightarrow Y; \pi_C rl = \pi_C l : Y \times C \longrightarrow C$ *(PUTLR)*

An rl lens with a specified point $m : 1 \longrightarrow C$ (m is for "missing") in its complements is called a pointed-complement- *or* pc-symmetric lens *and denoted $L = (X, Y, C, r, l, m)$. If L is an rl lens there is an opposite lens $L^{op} = (Y, X, C, l, r)$ with the same complements and l and r interchanged.*

REMARKS. A pc-symmetric lens in **set** is what was called a symmetric lens in [5]. We will have more to say about them below, but for now we just note that the point m is not involved in the equations so it just ensures that C is inhabited.

For an rl lens in **set**, the equations in the definition are equivalent to the implications used to define a symmetric lens in [5]. Those implications are

$$r(x, c) = (y, c') \Rightarrow l(y, c') = (x, c')$$

and

$$l(y, c) = (x, c') \Rightarrow r(x, c') = (y, c')$$

First, assume the equations in the definition and $r(x, c) = (y, c')$. We need to show that $l(y, c') = (x, c')$. Now since $l(y, c') = lr(x, c)$ we have $\pi_C l(y, c') = \pi_C lr(x, c) = \pi_C r(x, c) = \pi_C(y, c') = c'$ using the second equation of (PUTRL). Similarly $\pi_X l(y, c') = \pi_X lr(x, c) = \pi_X(x, c) = x$ using the first (PUTRL) equation, so $l(y, c') = (x, c')$. The other implication follows similarly. Now assume the implications. The first shows that $lr(x, c) = (x, c')$ where $r(x, c) = (y, c')$. Projections from this equation are exactly the (PUTRL) equations in the definition. The (PUTLR) equations follow from the second implication.

We have the following:

PROPOSITION 7. *For an rl lens $L = (X, Y, C, r, l)$ in $\mathbf{C}$, the equations $rlr = r$ and $lrl = l$ hold.*

PROOF. Since $\pi_X lr = \pi_X$, we have $\pi_X lrl = \pi_X l : Y \times C \longrightarrow X$ and $\pi_C lr = \pi_C r$ implies $\pi_C lrl = \pi_C rl = \pi_C l$ using the fourth equation. Since lrl and l have the same projections to X and C, they are equal. That $rlr = r$ is similar. □

It appears that the equations in the preceding Proposition do not imply the rl lens equations.

For any $L : X \longrightarrow Y$ in $\mathsf{ALens_w}(\mathbf{set})$, say $L = (X, Y, g, p)$, which is split by $s : Y \longrightarrow X$, there is a pc-symmetric lens $T(L) = (X, Y, C, r, l, m)$ where $C = \{f : Y \longrightarrow X \mid gf = 1_Y\}$ is the set of sections of g, $r(x, f) = (g(x), p(-, x))$, $l(y, f) = (f(y), p(-, f(y))$, and $m = s$.

PROPOSITION 8. *[5] For an asymmetric lens L in $\mathsf{ALens_w}(\mathbf{set})$ which is split, $T(L)$ is a pc-symmetric lens in* **set** *from X to Y.* □

We are more interested in relating rl lenses with spans of asymmetric lenses.

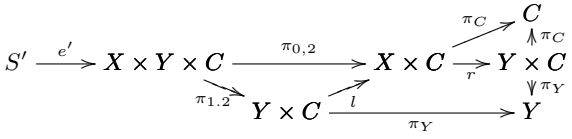
We begin by supposing that $L = (X, Y, C, r, l)$ is an rl lens in $\mathbf{C}$. If L is an rl lens in **set** it is easy to see that $\{(x, y, c) \mid r(x, c) = (y, c)\} = \{(x, y, c) \mid l(y, c) = (x, c)\}$. More generally,

PROPOSITION 9. *If $e : S \longrightarrow X \times Y \times C$ is an equalizer of*

$$r\pi_{0,2}, \pi_{1,2} : X \times Y \times C \longrightarrow Y \times C$$

then $e : S \longrightarrow X \times Y \times C$ is an equalizer of $l\pi_{1,2}, \pi_{0,2} : X \times Y \times C \longrightarrow X \times C$.

PROOF. Consider the diagram:



If $e' : S' \longrightarrow X \times Y \times C$ is an equalizer of $l\pi_{1,2}$ and $\pi_{0,2}$ then it equalizes $rl\pi_{1,2}$ and $r\pi_{0,2}$. Also, $\pi_Y rl = \pi_Y$ by assumption, so $r\pi_{0,2}e = \pi_{1,2}e$ and e' factors through the equalizer $e : S \longrightarrow X \times Y \times C$ of $r\pi_{0,2} = \pi_{1,2}$. Similarly, e factors through e', so e is also an equalizer of $l\pi_{1,2}$ and $\pi_{0,2}$. □

The equalizer S is called the *consistent triples*. In **set**, if (x, y, c) is a consistent triple for an rl lens then $r(x, c) = (y, c)$ and $l(y, c) = (x, c)$. The definition of S as an equalizer extends the concept of consistent triples to other categories with products.

Furthermore the construction of S allows us to define a span of well-behaved asymmetric lenses from an rl lens as follows.

PROPOSITION 10. *Suppose that $L = (X, Y, C, r, l)$ is an rl lens in $\mathbf{C}$. Let $e : S_L \rightarrowtail X \times Y \times C$ be an equalizer of $r\pi_{0,2}$ and $\pi_{1,2}$. Then there is a span*

$$L_l : X \leftarrow\!\!\shortmid S_L \longrightarrow Y : L_r$$

in $\mathsf{ALens_w}$ from X to Y with Gets defined by $g_l = \pi_X e$, $g_r = \pi_Y e$. The Put for L_l is defined by

$$p_l : X \times S_L \xrightarrow{1_X \times e} X \times X \times Y \times C \xrightarrow{\pi_{0,3}} X \times C$$

$$\xrightarrow{\Delta_X \times 1_C} X \times X \times C \xrightarrow{1_X \times r} S_L$$

The Put for L_r is similar.

We denote the span (L_l, L_r) by $A(L)$. In **set** the formula for the Put for L_l is $p_l(x', (x, y, c)) = (x', r(x', c))$.

Thus to every rl lens L we have associated a span of asymmetric well behaved lenses $A(L)$. Indeed the main purpose of this paper is to develop the machinery to allow us to, given a category of asymmetric lenses like $\mathsf{ALens_w}$, and its forgetful functor that remembers only the Gets, use the Sp construction to obtain the corresponding category of symmetric lenses. In future work we apply this construction to a range of types of lenses including delta lenses [3, 4], c-lenses [8], very well behaved lenses, and so on, to obtain categories of symmetric lenses of each kind.

Returning to our case in point, denote by U_w the forgetful functor from $\mathsf{ALens_w}$ to $\mathbf{C}$ which remembers only the Gets. The **category of symmetric well behaved lenses in C** is defined to be $Sp(U_w)$ and is denoted SLens_w.

Of course, we should compare in more detail the symmetric well behaved lenses in $\mathbf{C}$ with the rl lenses in $\mathbf{C}$. To do that properly we need to define composites of rl lenses, and an appropriate equivalence of rl lenses.

5. EQUIVALENCE OF RL LENSES

For rl lenses $L_1 = (X, Y, C_1, r_1, l_1)$ and $L_2 = (X, Y, C_2, r_2, l_2)$ we introduce a relation R and say that $L_1 R L_2$ if there is a well-behaved asymmetric lens $L = (C_1, C_2, t, p)$ from C_1 to C_2 with t a split epimorphism and such that L respects the operations of L_1 and L_2. That is,

$$r_2(X \times t) = (Y \times t)r_1 \text{ and } l_2(Y \times t) = (X \times t)l_1$$

and

$$r_1(X \times p) = (Y \times p)(r_2 \times C_1) \text{ and } l_1(Y \times p) = (X \times p)(l_2 \times C_1).$$

(In other words L commutes with the rl structures.) The relation R on rl lenses from X to Y generates an equivalence relation (its reflexive, symmetric, transitive closure) denoted $\equiv_{rl}$ on rl lenses from X to Y. We denote the $\equiv_{rl}$ equivalence class of $L = (X, Y, C, r, l)$ by $[L]_{rl}$.

DEFINITION 11. *For rl lenses $L = (X, Y, C, r, l)$ and $M = (Y, Z, C', r', l')$ the* rl-composite lens *is*

$$LM = (X, Z, C'', r'', l'', m'')$$

where $C'' = C \times C'$,

$$r'' = \langle \pi_{0,2}, \pi_1 \rangle (r' \times 1_C) \langle \pi_{0,2}, \pi_1 \rangle (r \times 1_{C'}),$$

and

$$l'' = (l \times 1_{C'}) \langle \pi_{0,2}, \pi_1 \rangle (l' \times 1_C) \langle \pi_{0,2}, \pi_1 \rangle.$$

PROPOSITION 12. *For rl lenses L_1, L_2 from X to Y and M from Y to Z in $\mathbf{C}$, the rl-composite lens ML_1 is an rl lens from X to Z. Moreover, if $L_1 \equiv_{rl} L_2$ then $ML_1 \equiv_{rl} ML_2$.*

PROOF. The first statement is similar to that in [5] using a twist isomorphism. For the second point, it is straightforward to prove the statement for the relation R and then it follows for $\equiv_{rl}$. □

COROLLARY 13. *For rl lenses L_1, L_2 from X to Y and M_1, M_2 from Y to Z in $\mathbf{C}$, if $L_1 \equiv_{rl} L_2$ and $M_1 \equiv_{rl} M_2$ then $M_1 L_1 \equiv_{rl} M_2 L_2$.*

The corollary allows us to define an associative composition on $\equiv_{rl}$ classes. The proof of associativity is again like that in [5]. We can now define the **category of rl lenses**, RLLens. The objects are those of $\mathbf{C}$. The arrows from X to Y are the $\equiv_{rl}$ classes of rl lenses from X to Y with the composition just described.

PROPOSITION 14. *There is an identity on objects functor we call $\mathbf{A} : \mathsf{RLLens} \longrightarrow \mathsf{SLens}_w$ defined by $\mathbf{A}([L]_{rl}) = [A(L)]_{U_w}$.*

PROOF. The proof proceeds by showing that $\mathbf{A}$ is well-defined, independently of the choice of representative of the rl equivalence class $[L]_{rl}$, and that the composite of rl lenses is sent, up to $\equiv_{U_w}$, to the composite of the corresponding spans in $Sp(U_w)$. □

Given a span $L : X \leftarrow\!\!\shortmid S \longrightarrow Y : M$ in $\mathsf{ALens_w}$ (that is a representative for an equivalence class which is a morphism in SLens_w) we determine an rl lens denoted $S(L, M)$ as follows.

PROPOSITION 15. *Suppose that $L = (S, X, g_l, p_l)$ and that $M = (S, Y, g_r, p_r)$ form a span of well-behaved asymmetric lenses in $\mathbf{C}$. Define*

$$r = \langle g_r, 1 \rangle p_l : X \times S \longrightarrow Y \times S$$

and

$$l = \langle g_l, 1 \rangle p_r : Y \times S \longrightarrow X \times S$$

then $S(L, M) = (X, Y, S, r, l)$ is an rl lens in $\mathbf{C}$.

PROOF. The proof is a routine verification that r and l so defined satisfy the four equations called PUTRL and PUTLR in Definition 6. □

PROPOSITION 16. *Suppose that $L : X \leftarrow\!\!\shortmid S \longrightarrow Y : M$ is a span of well-behaved asymmetric lenses in $\mathbf{C}$ and denote $AS(L, M)$ by $L_l : X \leftarrow\!\!\shortmid S_L \longrightarrow Y : L_r$. There is an isomorphism $g : S \longrightarrow S_L$ that is a morphism of spans, and consequently $AS(L, M) \equiv_{U_w} (L, M)$, and they are isomorphic as spans of asymmetric lenses.*

PROOF. The main point in the proof is to show that the consistent triples $S_{S(L,M)}$ have, in the case when $\mathbf{C}$ is the

category of sets, the form $(g_l(s), g_r(s), s)$ where g_l and g_r are the gets of the asymmetric lenses L and M respectively. The required isomorphism g is then apparent, being an isomorphism it has a canonical lens structure, and as a lens it commutes with the four asymmetric lenses L, M, L_l and L_r. □

PROPOSITION 17. *Suppose that* $L : X \leftarrow\!\!\!\prec S \longrightarrow\!\!\!\!\rightarrow Y : M$ *and* $L' : X \leftarrow\!\!\!\prec S' \longrightarrow\!\!\!\!\rightarrow Y : M'$ *are* $\equiv_{U_w}$ *equivalent spans of well behaved asymmetric lenses in* $\mathbf{C}$. *Then* $S(L, M) \equiv_{rl} S(L', M')$ *and so* $\mathbf{S}([(L, M)]_{\equiv_{U_w}}) = [S(L, M)]_{rl}$ *defines a functor* $\mathbf{S} : \mathsf{SLens}_w \longrightarrow\!\!\!\!\rightarrow \mathsf{RLLens}$.

PROOF. For the first point, $S(L, M) \equiv_{rl} S(L', M')$ provided that $AS(L, M) \equiv_{U_w} AS(L', M')$, but certainly we have $AS(L, M) \equiv_{U_w} (L, M) \equiv_{U_w} (L', M') \equiv_{U_w} AS(L', M')$.

To see that $\mathbf{S}$ is a functor, we note first that it is identity on objects. The first part of the proposition shows that $\mathbf{S}$ is well-defined on morphisms. Finally, to see that $\mathbf{S}$ is compatible with composition it suffices to trace how composition works on each side, noting that Proposition 15 defines each of the r and l operations in terms of the lenses L and M. □

THEOREM 18. SLens_w *is a retraction of* RLLens *via the functors* $\mathbf{A}$ *and* $\mathbf{S}$.

PROOF. Both $\mathbf{A}$ and $\mathbf{S}$ are identity on objects.

If $L : X \leftarrow\!\!\!\prec S \longrightarrow\!\!\!\!\rightarrow Y : M$ is a span of well-behaved asymmetric lenses in $\mathbf{C}$, then $\mathbf{AS}([(L, M)]_{\equiv_{U_w}}) = [(L, M)]_{\equiv_{U_w}}$. □

This result shows that every rl lens can be normalised into one of the form $\mathbf{S}([L, M])$ for a span of assymetric well behaved lenses $L : X \leftarrow\!\!\!\prec S \longrightarrow\!\!\!\!\rightarrow Y : M$, and that the category of rl lenses of that form is equivalent to the category SLens_w.

6. PC-SYMMETRIC LENSES

There is a category of pc-symmetric lenses defined in [5] which we now review. Our goal is to relate the equivalence of symmetric well behaved lenses defined as spans, $\equiv_{U_w}$, with the equivalence of pc-symmetric lenses used in [5].

The main difficulty we face is that the pointing turns out to be fundamentally important to the equivalence of pc-symmetric lenses, although it plays no role in the algebraic structure, and so has been excluded from the definition of rl lenses and symmetric well behaved lenses.

DEFINITION 19. *For pc-symmetric lenses* $L = (X, Y, C, r, l, m)$ *and* $M = (Y, Z, C', r', l', m')$ *the* pc-composite lens *is*

$$L \circ M = (X, Z, C'', r'', l'', m'')$$

where $C'' = C \times C'$,

$$r'' = \langle \pi_{0,2}, \pi_1 \rangle (r' \times 1_C) \langle \pi_{0,2}, \pi_1 \rangle (r \times 1_{C'}),$$

$$l'' = (l \times 1_{C'}) \langle \pi_{0,2}, \pi_1 \rangle (l' \times 1_C) \langle \pi_{0,2}, \pi_1 \rangle$$

and $m'' = (m, m')$. *We display* r'' (l'' *is similar):*

$$X \times C \times C' \xrightarrow{r \times 1_{C'}} Y \times C \times C' \xrightarrow{\langle \pi_{0,2}, \pi_1 \rangle} Y \times C' \times C$$

$$\xrightarrow{r' \times 1_C} Z \times C' \times C \xrightarrow{\langle \pi_{0,2}, \pi_1 \rangle} Z \times C \times C'$$

PROPOSITION 20. *For pc-symmetric lenses* L *and* M *in* $\mathbf{C}$, *the pc-composite lens* $L \circ M$ *is a pc-symmetric lens from* X *to* Z.

The proof for symmetric lenses in **set** is in [5], and is similar for symmetric lenses in $\mathbf{C}$.

In [5] an equivalence relation on pc-symmetric lenses from X to Y in **set** is defined as follows.

DEFINITION 21. *Suppose* $L_1 = (X, Y, C_1, r_1, l_1, m_1)$ *and* $L_2 = (X, Y, C_2, r_2, l_2, m_2)$ *are pc-symmetric lenses from* X *to* Y *and* $R \subseteq C_1 \times C_2$. *Say* L_1 *is* (R-)equivalent *to* L_2, *denoted* $L_1 \equiv_R L_2$ *iff* (m_1, m_2) *is in* R *and furthermore*

$$(c_1, c_2) \in R, x \in X, (y_1, c_3) = r_1(x, c_1), (y_2, c_4) = r_2(x, c_2) \text{ imply that } y_1 = y_2 \text{ and } (c_3, c_4) \in R \quad (1)$$

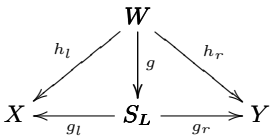
and corresponding symmetric statements hold for l_1, l_2. *We write* $L_1 \equiv_{pc} L_2$ *if there exists an* R *such that* $L_1 \equiv_R L_2$

Recall from [5] that $\equiv_{pc}$ is an equivalence relation which respects pc-symmetric lens composition. There is a category pcLens (called Lens in [5]) whose arrows are $\equiv_{pc}$ equivalence classes of pc-symmetric lenses with composition given by the pc-composite of representatives.

We want to relate the equivalence of spans of asymmetric lenses by $\equiv_{U_w}$ with equivalence between pc-symmetric lenses given by $\equiv_{pc}$. We start with a generator of $\equiv_{U_w}$. We will show that when two symmetric well behaved lenses are each pointed compatibly (choose any point in the peak of one, and use the equivalence to determine the corresponding point in the other) then their corresponding pc-symmetric lenses are $\equiv_{pc}$ equivalent.

PROPOSITION 22. *Suppose that* $L = (X, Y, C, r, l)$ *is an rl lens with* $A(L)$ *the span* $L_l : X \leftarrow\!\!\!\prec S_L \longrightarrow\!\!\!\!\rightarrow Y : L_r$ *of well-behaved asymmetric lenses. Suppose that* $M = (W, S_L, g, p)$ *is in* $\mathsf{ALens}_\mathsf{w}$ *and is split by* $s : S_L \longrightarrow\!\!\!\!\rightarrow W$. *Define* $h_l = g_l g, h_r = g_r g$, $q_l : X \times W \longrightarrow\!\!\!\!\rightarrow W$ *by* $q_l(x, w) = p(p_l(x, gw), w)$ *and* $q_r : Y \times W \longrightarrow\!\!\!\!\rightarrow W$ *by* $q_r(x, w) = p(p_l(x, gw), w)$, *defining asymmetric lenses* $M_l : X \leftarrow\!\!\!\prec W \longrightarrow\!\!\!\!\rightarrow Y : M_r$. *Define* $R = \{(w, \pi_C gw)\} \subseteq W \times C$, *and suppose* $m : 1 \longrightarrow\!\!\!\!\rightarrow W$ *is a point of* W *and hence* $(m, \pi_C gm)$ *is in* R. *Let* $L_{gm} = (X, Y, C, r, l, \pi_C g(m))$ *be the pc-symmetric lens pointed by* $\pi_C gm$. *Let* $S(M_l, M_r) = (X, Y, W, r_M, l_M)$ *be the rl lens determined above and* $L_m = (X, Y, W, r_M, l_M, m)$ *its corresponding pc-symmetric lens . Then* $L_m \equiv_R L_{gm}$.

PROOF. The situation is summed up in the following:



We first note that $r_M : X \times W \longrightarrow\!\!\!\!\rightarrow Y \times W$ is defined by $r_M(x, w) = (h_r q_l(x, w), q_l(x, w))$.

We are going to prove only implication (1), so suppose $(w, \pi_C gw) \in R, x \in X, (y_1, w_1) = r_M(x, w)$ and $(y_2, c_2) = r(x, \pi_C gw)$. We need to show that $y_1 = y_2$ and $(w_1, c_2) \in R$.

First

$$\begin{aligned} y_1 &= h_r q_l(x,w) \\ &= g_r g q_l(x,w) \\ &= g_r g p(p_l(x,gw),w) \quad \text{by def of } q_l \\ &= g_r p_l(x,gw) \quad \text{by PutGet} \\ &= g_r(x, r(x,\pi_C gw)) \quad \text{def of } p_l \\ &= \pi_Y(r(x,\pi_C gw) = y_2 \end{aligned}$$

Next $w_1 = q_l(x,w)$ by definition, so we want to show that $c_2 = \pi_C g(q_l(x,w))$ to get $(w_1, c_2) \in R$. Now $c_2 = \pi_C(r(x,\pi_C gw))$, so

$$\begin{aligned} c_2 &= \pi_C(r(x,\pi_C gw)) \\ &= \pi_C \pi_{Y,C} p_l(x,gw) \quad \text{def of } p_l \\ &= \pi_C p_l(x,gw) \quad \text{comp'n of projections} \\ &= \pi_C g p(p_l(x,gw),w) \quad \text{by PutGet} \\ &= \pi_C g q_l(x,w) \quad \text{def of } q_l \end{aligned}$$

That completes the proof. □

COROLLARY 23. *If* $L : X \leftarrow\!\!\!\!\cdot\; S \rightarrow\!\!\!\!\cdot\; Y : M$ *and* $L' : X \leftarrow\!\!\!\!\cdot\; S' \rightarrow\!\!\!\!\cdot\; Y : M'$ *are* $\equiv_{U_w}$ *equivalent spans in* SLens_w *via* $g : S \rightarrow\!\!\!\!\cdot\; S'$, $m : 1 \rightarrow\!\!\!\!\cdot\; S$ *and* $m' : 1 \rightarrow\!\!\!\!\cdot\; S'$ *where* m *and* $m' = g(m)$ *are related by* $\equiv_{U_w}$. *Then for the corresponding pc-symmetric lenses we have* $L'_{m'} \equiv_{pc} L_m$.

Since we have it on generators of the equivalence, the corollary shows that $\equiv_{U_w}$, appropriately pointed to permit a comparison, is a finer equivalence relation than $\equiv_{pc}$.

7. FUTURE WORK

This paper has been about developing the theory of spans of lenses and an appopriate equivalence relation for them which can then be used *parametrically* in the type of asymmetric lens of interest.

Having laid the mathematical foundations for a theory of symmetric lenses of type X as spans of asymmetric lenses of type X we are in a position to define new symmetric lenses, and to make detailed comparisons with the various extant notions of symmetric lenses.

We have currently carried through this project for the delta lenses of Diskin et al [3, 4, 6], and we are progressing well with c-lenses [8]. The overall goal is a unified treatment of all the various kinds of symmetric lenses that have been defined, along with guidance for the future development of new types of symmetric lenses.

In each case, we strive, as in the preceding section of this paper, to relate the work to extant structures, and to use the theory to shed more light on why they are the way that they are, or to reveal when there are choices that have been made that could have been made differently (thus for example, $\equiv_{pc}$ seems to be a coarser equivalence than is strictly needed, and we are studying this further).

In other future work we are exploring the lens-like aspects of our construction of $Sp(G)$. The operation $P(g,r)$ can itself be seen as a kind of put operation which satisfies its own PutGet, GetPut and PutPut conditions. It appears that if they are formulated correctly the resultant structure is a c-lens — further evidence of the utility of bidirectional transformations (even in the development of the theory of bidirectional transformations!).

8. CONCLUSIONS

We have presented foundations for a consistent and unified treatment of several different types of asymmetric and symmetric lenses. In that treatment, $Sp(U)$ is used to build categories of spans of asymmetric lenses. $Sp(U)$ provides a different treatment to the usual treatment for composing spans, thus overcoming the difficulty about categories of lenses frequently not having pullbacks. In addition, $Sp(U)$ uses a different equivalence from the one usually used in categories of spans. The new equivalence seems to be the natural generalisation for Bx as it replaces a requirement for an isomosphism between spans (a particularly restrictive kind of bidirectional transformation) with a lens.

The work has allowed us to compare the equivalence of the general theory with the equivalence proposed in [5]. The latter has some surprising features. It certainly achieves the outcomes its authors required, but it appears that the new equivalence can do the same without forcing so many different lenses to be equivalent.

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