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Oriented Combinatorial Topology and Concurrency ⋆

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Abstract

Higher dimensional automata (HDA) provide valuable models of concurrent processes. Much current research related to HDA aims to further develop algebraic topological notions required to analyse HDA in order to determine computer scientific properties including deadlock, safety, unreachable states, etc. It is well-known that classical algebraic topology will not suffice since the sequences of actions represented by (1-dimensional) paths need to be monotone with respect to a multi-dimensional coordinate system (the coordinate system might be thought of as time, or its coordinates can be thought of separately as progress with respect to particular processes). The extent to which higher dimensional paths inherit an orientation as a result of either the coordinate system, or the definition of homotopy, varies according to the precise notions of directed algebraic topology that are utilised. This paper considers an extreme position in which all higher dimensional paths, like 1-dimensional paths, are oriented and can only be composed when orientations are compatible. This point of view has arisen both from software engineering considerations and from considerations of the history of classical combinatorial topology.

1 Introduction

When Vaughan Pratt first introduced higher dimensional automata (drafts of [19]) the higher dimensional cells were all oriented and although cells were cubical the underlying graphical structures were general pasting schemes [17].

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More recently, following the seminal work of Goubault [10], HDA have been studied extensively. Significant progress has been attained in further work by Goubault and coauthors, for example [13] and [12], in a series of papers by Gaucher [5], [6], [7], [8], [9] and in work by Fajstrup [4], Grandis [14], [15], [16] and Raussen [21], [22] amongst others. The HDA have variously been based upon n-categories [19], cubical complexes [20], and pasting schemes [6]. Nevertheless, the bulk of the work explores homotopies between oriented paths in an ambient unoriented space.

In this paper we discuss some advantages of using combinatorial topology with oriented n-cells of arbitrary shape. The underlying graphical structure that we use was introduced by Buckland-Johnson-Verity [3], and was designed to support explicit choice higher dimensional automata (ECHDA) [1] which were first proposed in [1] and were further developed in [2].

In the full paper we discuss

(i) The introduction of oriented higher dimensional cells in the foundation of algebraic topology and the effect it had on clarifying the subject (leading to the study of simplicial objects in general, and to the widespread use of the singular homology for example)

(ii) How the analogous introduction of oriented higher dimensional cells in a geometric representation of concurrent processes can be interpreted

(iii) The software engineering advantages that might be expected from serious development of the oriented higher dimensional approach to HDA

Most importantly, we trace through several detailed examples showing the interaction of concurrency and choice in ECHDA. These illustrate the expected software engineering examples, as well as clarifying the oriented higher dimensional approach.

2 Structure

The full paper has, after the introduction, five sections which cover

(i) Historical remarks on oriented combinatorial topology

(ii) A proposal for the analogous treatment of HDA with remarks about the software engineering advantages of taking this approach

(iii) A definition of ECHDA. ECHDA are one possible approach to oriented combinatorial HDA

(iv) Examples

(v) A discussion of free constructions.

[1] ECHDA is a plural abbreviation. When we need to refer to an explicit choice higher dimensional automat*on* we will call it an ECHDon.

Here we will make no further remarks about any except the last of these, but free constructions do warrant some further explanation.

3 Free constructions

It might be argued that HDA are already “oriented”. Does not providing an order on 1-dimensional paths correspond to an order on the vertices of a simplex?

This is certainly true, but it does not suffice for the analogy being explored in this paper. The singular complex of a space is a canonical representation of its structure because it is the free simplicial object on the space (calculated in the sense of [18]). Similarly, the collection of paths (of arbitrary dimension) in an ECHDon ought to carry the structure of the free ω-category on the ECHDon.

We explain this in further detail in the final section of the paper, and discuss its software engineering implications: A path (of arbitrary dimension) in an ECHDon represents a process obtained by partial evaluation within the process represented by the ECHDon. The partial evaluation may be with respect to values of variables or choices (as in common applications of partial evaluation in computer science), or it may be with respect to concurrency or scheduling.

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