Pasting Diagrams in n-Categories with Applications to Coherence Theorems and Categories of Paths

Michael Johnson
Department of Pure Mathematics
University of Sydney
Australia 2006

June, 1987

Contents

Acknowledgements

This thesis reports research carried out between November 1984 and December 1986 (except for Theorem 3.7 which was discovered in April 1987). During that time I have been encouraged and influenced by a number of mathematicians.

First and foremost I am pleased to record my gratitude to my supervisor, Bob Walters, who has shared with me the joys and crises, and has given unreservedly his time as well as his advice and encouragement.

Other people who have had a significant impact on this work include Ross Street, Max Kelly (and indeed the whole Sydney Category Seminar), John Gray, Sammy Eilenberg, Bob Paré and Steve Schanuel. In addition I have benefitted from conversations with Jean Bénabou, Aurelio Carboni, André Joyal, Stefano Kasangian, Bill Lawvere and Saunders Mac Lane.

This work has been supported financially by the Department of Pure Mathematics of The University of Sydney, by a James King of Irrawang Scholarship, and by two University of Sydney travel grants. It has also been assisted indirectly by an Australian Research Grants Scheme Programme Grant. This document was typeset using LaTeX on hardware provided by the Department of Pure Mathematics.

Chapter 1 below records known results and no originality is claimed for it. Otherwise, except as explicitly stated in the text, this thesis is entirely original.

Introduction

With the rise in interest in n-categories, at least for $n = 2$, the operation of *pasting* has been recognized as a valuable tool in understanding categorical diagrams involving several different compositions. Nevertheless, the operation and the pasting diagrams themselves, have never been given a precise formal description. Of course, one can work with pasting diagrams without worrying about their foundation (much as many 19th Century group theorists worked productively in groups of substitutions without an axiomatization of groups), but broader applications and more general results need a firm footing.

This thesis attempts to provide an axiomatic foundation for *pasting diagrams*, and to take advantage of the clearer formulation in the following applications.

Pasting diagrams in 2-categories were described in a 1971 lecture of Walters [23], where the importance of the *pasting theorem* was recognized. The theorem asserts that any "composable" diagram in a 2-category determines a unique cell, independently of the choice of the order of the compositions involved. Unfortunately the theorem has never been proved, chiefly because of the lack of a sufficient formalization of the notion of pasting diagram. Chapter 2 below sets down a definition of pasting diagrams, isolates the "composable" diagrams (called *well-formed pasting diagrams*), and presents a proof of the general n-category pasting theorem.

Street, in [20], introduced *orientals*: The nth oriental is "the free n-category on the n-simplex". Chapter 3 presents a new description of Street's orientals. The cells of the free n-category on the n-simplex should be made up of simplexes freely composed, and the recognition of a natural pasting diagram structure on the (combinatorial) ω-simplex makes such cells easy to define. They turn out to be simply the well formed (= "composable") subpasting diagrams of the simplex, and the fact that they form a free ω-category follows from generalities about pasting diagrams.

The first *coherence theorems* were proved by Mac Lane [15] in 1963. Mac Lane has described coherence theorems as assertions that all of a certain class of diagrams commute, but in most of the classical examples the diagrams involved are diagrams of *natural transformations*, which are most naturally treated as pasting diagrams in the 2- (or even 3-) category **Cat**, rather than ordinary diagrams in an ordinary category. Chapter 4 takes this point of view and describes a fairly general setting for the analysis of coherence questions. Many classical results follow easily from the n-categorical geometry of the diagrams and new results—the higher coherence conditions—are equally simple.

Algebraic topology frequently constructs algebraic objects from topological objects by taking "paths" in the topological object, with composition given by some form of concatenation of paths, usually modulo a homotopy relation. One version of this construction which doesn't use homotopy is called the Moore category of a space and works by having many different domains for paths. Moore's construction only applies to 1-dimensional paths and it has not been clear how to extend it to higher dimensions because the necessary collection of domains for paths seems quite complicated and because Moore's construction has not been well understood in general settings (homotopy groups are groups because S^n is a *cogroup* in the homotopy category, but Moore's domains of paths do *not* form a cocategory).

The nature of the Moore construction is analysed in Chapter 5 which introduces parametrized theories. The algebras for a parametrized theory of (say) categories are categories with appropriate extra structure—examples include Lawvere's *categories with duration*, Bénabou's *fibrations*, and Paré-Schumacher's *indexed categories*. Moore's domains of paths are a co- parametrized-category and so, for classical reasons, his paths form a parametrized-category. The extra structure is a 'length' for each cell; forgetting the length gives the usual Moore category. Furthermore, the collection of freely composed simplexes (*well-formed simplicial sets*) described in Chapter 3 forms a co-parametrized-ω-category, and so provides the desired generalization of the Moore construction to higher dimensional paths.

History

The problems described above have been "in the air" for a number of years. Researchers have suspected that they are interrelated but precise relations have not been found. This section briefly reviews the recent history of each of the problems.

The categorical operation of pasting was first used by Bénabou in his treatment of bicategories [3]. Independently, Walters in his thesis [22] used pasting, and he introduced it to Sydney in his 1971 lecture [23]. The latter is also important because it recognized the importance of the pasting theorem, although Walters stated it as part of an alternative axiomatics for 2-categories. Later pasting played an important role in the joint work of Street and Walters [21].

An expository description of pasting can be found in the review of Kelly and Street [13] which also includes a description of the pasting theorem. Despite a suggested technique, and a number of attempts, no proof of the theorem has appeared.

The investigation of the "free n-category on the n-simplex" was begun by Roberts [18]. Street and Duskin worked on the problem for a number of years and recently Street [20] has obtained a solution. Walters has argued that there should be a more geometric description of such a geometric construction and Aitchison [1] has analysed the geometry of the "free n-category on the n-cube", but his work did not achieve Walters' hoped for simplification of Street's results.

The relationship between orientals and higher coherence conditions has been particularly tantalizing. Mac Lane proved the first coherence theorems in 1963 [15] and his theorems have been important in the development of enriched category theory by Kelly [11] and others. The development of the theory of categories enriched over a bicategory led Street and Walters to realize that the associative law is obtainable from the free 3-category on the 3-simplex, that coherence of associativity is obtainable from the free 4-category on the 4-simplex, and to conjecture that the higher coherence conditions arise analogously. However, there has been no indication as to why the known laws for an associativity should correspond to simplexes, nor any indication of why this should continue to be the case for higher conditions.

The final problem is more recent than the others. In 1985 Lawvere and Schanuel, following a lecture given by Walters in Buffalo, were interested in obtaining an n-dimensional version of the Moore construction to permit "homotopy theory without quotienting". In July the 2-dimensional case was still presenting difficulties and Schanuel brought the problem to the author's attention, suspecting that the theory of n-dimensional diagrams in ω-categories might also apply to n-dimensional paths in topological spaces. This corresponds well with Walters' observation that orientals should be geometrical (and the results that are reported below bear them both out).

Chapter 1

Preliminaries

This chapter presents known results (without motivation) and establishes notation. Sophisticated readers may like to skip directly to chapter 2 and refer to this chapter if and when notational difficulties arise.

1.1 Graded Sets

Sets may be identified with discrete categories and this identification allows us to speak of functors whose domain is a set. A *graded set* is a functor A from the set of natural numbers, $\omega = \{0, 1, \ldots\}$, into the category of sets. The image of the functor A at the natural number n is written A_n. Elements of the set A_n are called *n-dimensional*.

A graded set may be specified by giving a sequence of sets $(A_i)_{i\in\omega}$. We adopt the convention that if a particular member of the sequence, say A_n, is not specified then it is assumed to be empty. A graded set A is called *n-dimensional* if $A_k = \emptyset$ for all $k > n$, and $A_n \neq \emptyset$. The *n-dimensional skeleton* of a graded set A, denoted $|A|_n$, is the graded set obtained from A by setting $(|A|_n)_k = \emptyset$ for all $k > n$, and $(|A|_n)_k = A_k$ for $k \leq n$.

We say that A is a subgraded set of B, written $A \subset B$, when, for each i, $A_i \subset B_i$. Subgraded sets inherit the usual operations on subsets by defining them pointwise. Thus, for $A = (A_i)$, and $B = (B_i)$, $A \cup B$ is defined to be $(A_i \cup B_i)_{i\in\omega}$. Similarly for intersection $(A \cap B)$ and set theoretic difference $(A - B)$.

If A is a graded set say that x is an *element* of A, and write $x \in A$, when $x \in A_n$ for some n. An element $x \in A_n$ of a graded set A, will often be identified with the subgraded set X of A given by

$$\begin{aligned} X_k &= \{x\}, \quad \text{when } k = n \\ X_k &= \emptyset, \quad \text{when } k \neq n. \end{aligned}$$

Then, a definition couched in terms of subgraded sets of A, applies also to elements of A.

Write $[n] = \{0, 1, \ldots, n\}$ and let Δ be the category of finite ordinals and non-decreasing functions. A *simplicial set* is a functor $\Delta^{\mathrm{op}} \to \mathbf{Set}$. Composition with the functor $\omega \to \Delta^{\mathrm{op}}$, given by $n \mapsto [n]$, gives the underlying graded set of a simplicial set. The extra structure possessed by a simplicial set, but not by a graded set, is the information about the images of the morphisms of Δ.

For each n and each $i \in [n]$, denote by ∂_i^n the morphism $[n] \to [n+1]$ of Δ which is injective and whose image in $[n+1]$ does not include i, and by σ_i^n the morphism $[n+1] \to [n]$ which is surjective and has $\sigma_i^n(i) = \sigma_i^n(i+1) = i$. When there is no danger of confusion we suppress the superscript n's. The ∂_i and their images in a simplicial set are called *face operators*, and the σ_i and their images are called *degeneracy operators*. The morphisms of Δ are generated by its face and degeneracy operators (for details see e.g., [17] or [5]).

An element of a simplicial set is called *degenerate* if it occurs in the image of some degeneracy operator.

A *simplicial complex* is a finite or countably infinite set K, together with a collection $\mathcal{K}$ of subsets of K such that $A \in \mathcal{K}$ and $B \subset A$ implies $B \in \mathcal{K}$. Such a complex determines a graded set, also called K by

$$K_n = \{A \in \mathcal{K} : \text{the cardinality of } A \text{ is } n+1\}.$$

A simplicial complex is said to have an orientation when its elements are linearly ordered. A simplicial complex with an orientation may be equipped with face operators $\partial_i : K_n \to K_{n-1}$, $i \in [n]$, defined on $A = \{a_0, a_1, \ldots, a_n : a_i < a_{i+1}, i \in [n-1]\}$ by

$$\partial_i(A) = \{a_0, a_1, \ldots, \widehat{a_i}, \ldots, a_n\}$$

(where the circumflex indicates the absence of the character which it accents).

A simplicial complex which has an orientation generates a simplicial set whose non-degenerate elements are the same as the elements of the complex, and whose face operators are as just described, by freely adding degenerate elements subject to the usual simplicial identities. Specifically, if $(K, \mathcal{K})$ is such a complex then define a simplicial set A by

$$A_n = \{(k_0, k_1, \ldots, k_n) : k_i \in K, \text{ and } k_i \leq k_{i+1}\}$$

with face and degeneracy operators given by deleting, respectively repeating, the element in position i.

The *standard* n*-simplex* is the simplicial set which is generated by the simplicial complex $([n], \mathcal{P}[n])$, where $\mathcal{P}$ denotes the power set and $[n]$ bears the usual order. The *standard* ω*-simplex* is the simplicial set generated by the simplicial complex $S = (\omega, \mathcal{P}_f(\omega))$, where $\mathcal{P}_f(\omega)$ is the set of finite subsets of ω and ω bears the usual order.

The *vertices* of a simplicial set, simplicial complex, or graded set, are the elements of dimension zero.

1.2 Higher Dimensional Categories

This section is taken almost entirely from Street [20] pages 1, 2, 4 and 41. We begin with a one-sorted, arrows-only description of categories.

A *category* $(A, s, t, *)$ consists of a set A, functions $s, t : A \to A$ satisfying the equations

$$ss = ts = s, \quad tt = st = t,$$

and, a function $* : \{(a, b) \in A \times A : s(a) = t(b)\} \to A$, whose value $a * b$ at (a, b) satisfies the equations

$$s(a * b) = s(b), \quad t(a * b) = t(a),$$

such that the following axioms hold:

(right identity)	$s(a) = t(v) = v$	implies	$a * v = a$;
(left identity)	$u = s(u) = t(a)$	implies	$u * a = a$;
(associativity)	$s(a) = t(b)$, $s(b) = t(c)$	imply	$a * (b * c) = (a * b) * c$.

The functions $s, t, *$ are respectively called *source*, *target* and *composition*; by abuse of notation the category $(A, s, t, *)$ is often denoted by A. Elements of A are called *arrows* and the notation $a : u \to v$ is used to denote that $s(a) = u$ and $t(a) = v$. An arrow u is in the image of s if and only if it is in the image of t; such arrows are called *identities* (or *objects*) and satisfy $s(u) = t(u) = u$. A pair (a, b) of arrows is called *composable* when $s(a) = t(b)$.

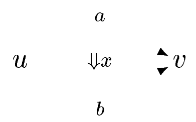
A 2-*category* $(A, s_0, t_0, *_0, s_1, t_1, *_1)$ is given by giving two category structures on A $(A, s_0, t_0, *_0)$ and $(A, s_1, t_1, *_1)$ satisfying the following conditions:

(i) $s_1 s_0 = s_0 = s_0 s_1 = s_0 t_1, \quad t_1 t_0 = t_0 = t_0 t_1 = t_0 s_1$;

(ii) $s_0(a) = t_0(a')$ implies $s_1(a *_0 a') = s_1(a) *_0 s_1(a')$ and $t_1(a *_0 a') = t_1(a) *_0 t_1(a')$;

(iii) $s_1(a) = t_1(b)$, $s_1(a') = t_1(b')$, $s_0(a) = t_0(a')$ imply
$(a *_1 b) *_0 (a' *_1 b') = (a *_0 a') *_1 (b *_0 b')$.

The identities for $*_0$ are called 0-*cells* and the identities for $*_1$ are called 1-*cells*. The notation



is used to denote that $x \in A$, $s_1(x) = a$, $t_1(x) = b$, $s_0(x) = u$ and $t_0(x) = v$.

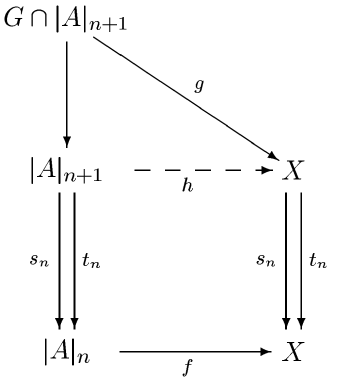
An ω-*category* $(A, (s_n, t_n, *_n)_{n \in \omega})$ consists of category structures $(A, s_n, t_n, *_n)$ on A for each $n \in \omega$ such that $(A, s_m, t_m, *_m, s_n, t_n, *_n)$ is a 2-category for all $m < n$.

The identities for $*_n$ are called n-cells. We write A_n for the set of n-cells of the ω-category A.

For $r \in \omega$, an r-*category* is an ω-category for which all elements are r-cells. We write $|A|_r$ for the r-category consisting of the r-cells of A.

An ω-functor $f : (A, (s_n, t_n, *_n)_{n \in \omega}) \to (A', (s'_n, t'_n, *'_n)_{n \in \omega})$ is a function $f : A \to A'$ which respects all sources, targets and compositions.

An ω-category A is *freely generated* by a subset G of A when, for all ω-categories X, for all $n \in \omega$, for all ω-functors $f : |A|_n \to X$, and, for all functions $g : G \cap |A|_{n+1} \to X$ such that $s_n g = f s_n$, $t_n g = f t_n$, there exists a unique ω-functor $h : |A|_{n+1} \to X$ whose restriction to $|A|_n$ is f and whose restriction to $G \cap |A|_{n+1}$ is g (see diagram below).



1.3 Theories, Algebras and Families

One of the great advantages of category theory is the possibility of instantiating the axioms of equational mathematical theories in particular categories, usually themselves called theories. Algebras for such a theory are then functors from the particular category into the category of sets which preserve appropriate structure of the theory. Usually the correct morphisms for such algebras are simply natural transformations.

More generally the category of sets may be replaced by any category $\mathcal{C}$ and we consider algebras in $\mathcal{C}$. For example a topological group is a group (an algebra for the theory of a group) in the category **Top** of topological spaces.

In Chapter 5 we will have reason to investigate algebras for finite limit theories (hereafter called theories since we will treat no others) in the category Fam $\mathcal{C}$ of families of objects of a category $\mathcal{C}$. This section reviews the definitions and some elementary properties.

A *theory* T is a small finitely complete category. If $\mathcal{C}$ is any category a T-algebra in $\mathcal{C}$ is a functor $F : \mathsf{T} \to \mathcal{C}$ which preserves finite limits (a *left exact functor*). A T-algebra in **Set** is often just referred to as a T-algebra.

Most of the applications below involve *single sorted theories* in which T has a generic object G—all other objects of T are limits of diagrams in which G is the only object. When T is a single sorted theory with generic object G, F a left exact functor $\mathsf{T} \to \mathcal{C}$, and $D = FG$ the image of G, we will often abuse notation and refer to D as a T-algebra in $\mathcal{C}$.

Recall that covariant representable functors $\mathcal{C}(X,-)$ preserve finite limits. Thus if F is a T-algebra in $\mathcal{C}$ then the composite

$$\mathsf{T} \xrightarrow{F} \mathcal{C} \xrightarrow{\mathcal{C}(X,-)} \mathbf{Set}$$

is a T-algebra (in $\mathbf{Set}$). For example if $\mathcal{C} = \mathbf{Top}$ and T is the theory of a group then we rediscover that "homming from any topological space X into a topological group yields a natural group structure" usually described as the *pointwise* group structure on the hom set.

If T is a theory a T-*coalgebra* in a category $\mathcal{C}$ is a T-algebra in $\mathcal{C}^{\mathrm{op}}$. As before, if H is a coalgebra in $\mathcal{C}$ then

$$\mathsf{T} \xrightarrow{H} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathcal{C}^{\mathrm{op}}(X,-)} \mathbf{Set}$$

is a T-algebra, but $\mathcal{C}^{\mathrm{op}}(X,-) = \mathcal{C}(-,X)$ so "homming *out* of a *co*group yields a natural group structure". There are not many cogroups in nature but one well known example occurs in $\mathbf{Hot}$, the category of topological spaces and homotopy classes of continuous maps: For $n \geq 1$, S^n is a cogroup in $\mathbf{Hot}$. Thus $\mathbf{Hot}(S^n, X)$ is a group, usually called the nth homotopy group of X.

The original, non-categorical, formulation of this duality is due to Eckmann-Hilton. See for example Hilton [6].

Suppose $\mathcal{C}$ is a category. The category of families of objects of $\mathcal{C}$, $\mathrm{Fam}\,\mathcal{C}$, is defined as follows. An object of $\mathrm{Fam}\,\mathcal{C}$ is a small set I and an I-indexed family $(A_i)_{i\in I}$ of objects of $\mathcal{C}$. An arrow of $\mathrm{Fam}\,\mathcal{C}$ from $(A_i)_{i\in I}$ to $(B_j)_{j\in J}$ consists of a function $\phi : I \to J$ and a family of arrows of $\mathcal{C}$, $f_i : A_i \to B_{\phi(i)}$.

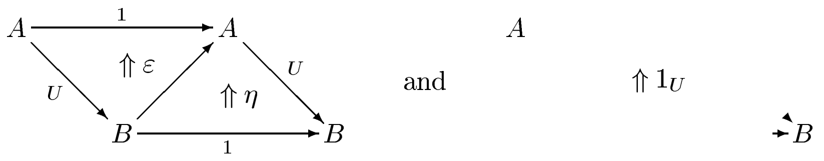
Notice that $\mathcal{C}$ is a full subcategory of $\mathrm{Fam}\,\mathcal{C}$, the inclusion being given by sending an object to the singleton family containing it, and that $\mathrm{Fam}\,\mathcal{C}$ is fibred over $\mathbf{Set}$, with $p : \mathrm{Fam}\,\mathcal{C} \to \mathbf{Set}$ given by $(A_i)_{i\in I} \mapsto I$ and if $(\phi, f_i) : (A_i)_{i\in I} \to (B_j)_{j\in J}$ then $p(\phi, f_i) = \phi : I \to J$.

$\mathrm{Fam}\,\mathcal{C}$ may be characterized universally as the free coproduct completion of $\mathcal{C}$. Yet another description of $\mathrm{Fam}\,\mathcal{C}$ is: If I is the inclusion $\mathbf{Set} \hookrightarrow \mathbf{Cat}$ then $\mathrm{Fam}\,\mathcal{C}$ is the lax comma category $I/\mathcal{C}$.

Chapter 2

Pasting Diagrams

Category theorists make extensive use of diagrams. A diagram in a category has been defined to be a graph morphism from some graph into the underlying graph of the category [2]. In 2-category theory, pasting diagrams like

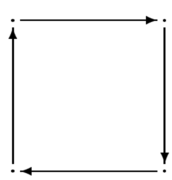


(2.1)

(the equality of which expresses one of the triangular equations of an adjunction) play an important role. If a diagram in a 2-category were to be a 2-graph morphism [4] from some 2-graph into the underlying 2-graph of the 2-category then the left hand side of (2.1) would not be a diagram (although the right hand side would be a diagram).

Street [19], recognizing this difficulty, introduced *computads*. A computad $\mathcal{G}$ is a graph $|\mathcal{G}|$ together with a second graph structure whose edges are called 2-cells and whose vertices are elements of the free category on $|\mathcal{G}|$. Furthermore, in the second graph structure, two vertices can be connected by an edge only if they share the same domain and codomain as elements of the free category on $|\mathcal{G}|$. Street defined the underlying computad of a 2-category in which a 2-cell from the 2-category appears between every possible factorization of its domain and codomain. A diagram in a 2-category may be taken to be a computad morphism into the underlying computad of the 2-category.

We take the view that the above use of graphs and computads defines a diagram by a *parametrization*—compare with "A path in a topological space is a continuous map from the unit interval ...". However, a parametrizing object is usually in some sense 'loop free' or 'non-singular' and there is no such requirement above. So, for example, the definitions allow



to occur as a parametrizing object and hence a similar square of morphisms forms a diagram in a category. It is not at all clear how to interpret such a diagram.

In this chapter we introduce *pasting schemes*. A *loop-free pasting scheme* is an appropriate parametrizing object for a diagram in an n-category. A *realization* of a loop-free pasting scheme in a particular n-category is the map which defines the parametrization. Because the free n-category on an n-dimensional loop-free pasting scheme has a particularly simple structure our realizations will be functors rather than pasting scheme morphisms into the underlying pasting scheme of the n-category. A *well-formed pasting scheme* is the parametrizing object for a composable diagram—sometimes referred to as a 'leg' in a diagram in an ordinary category. The *n-category pasting theorem* states that a well-formed pasting scheme with a given realization has a unique composite.

2.1 Pasting Schemes

In this section we set down the technical details needed in order to be precise about diagrams like (2.1) above. Such a diagram will be determined by a realization of a pasting scheme in a category. A pasting scheme will be a graded set $(A_i)_{i\in\omega}$, where for each i, A_i represents the set of i-cells in the diagram. The actual arrangement of the cells relative to one another will be determined by two collections of relations $\mathsf{E}^i_j, \mathsf{B}^i_j : A_i \to A_j$ which may be thought of as describing which j-cells are at the 'end', respectively 'beginning', of each of the i-cells.

Let $A = (A_i)_{i\in\omega}$ be a graded set, E^i_j, $i, j \in \omega$, $j \le i$, a collection of relations with E^i_j a relation between the sets A_i and A_j. Let X be a subgraded set of A of dimension n. Write $\mathsf{E}^k(X)$ for the graded set $\mathsf{E}^k(X)_i = \{y \in A_i : \text{there exists } x \in X_k,\ x\,\mathsf{E}^k_i\,y\}$ and $\mathsf{E}(X)$ for $\mathsf{E}^n(X)$. If E^i_j, B^i_j are two such collections of relations let R^i_j be the relation between A_i and A_j given by $x\,\mathsf{R}^i_j\,y$ when there exists a sequence $x = x_1, x_2, \ldots x_j = y$ of elements of A satisfying $x_k\,\mathsf{D}^p_q\,x_{k+1}$ for $k = 1, 2, \ldots j-1$ and $\mathsf{D}^p_q = \mathsf{E}^p_q$ or B^p_q.

We will often position the grading subscript on the relation writing $\mathsf{E}_i(X)$ rather than $\mathsf{E}(X)_i$. The relation E^i_j is called *finitary* when, for any $x \in A_i$, $\mathsf{E}^i_j(x)$ is finite.

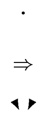
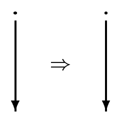
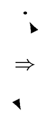
In what follows, the E^i_j will be 'end' relations and the B^i_j 'beginning' relations and we have a *duality*: If P is a proposition then $dual_k\,P$ stands for the proposition obtained from P by replacing all occurrences of E^k by B^k and vice versa.

A *pasting scheme* $(A, \mathsf{E}, \mathsf{B})$ is a graded set (A_i) together with finitary relations E^i_j, B^i_j, $j \le i$, such that

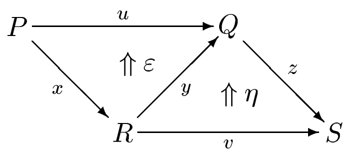
1. E^i_j is a relation between A_i and A_j
2. E^i_i is the identity relation on A_i
3. For $k > 0$ and any $x \in A_k$ there exists $y \in A_{k-1}$ with $x\,\mathsf{E}^k_{k-1}\,y$
4. For $k < n$, $w\,\mathsf{E}^n_k\,x$ if and only if there exists u, v such that $w\,\mathsf{E}^n_{n-1}\,u\,\mathsf{E}^{n-1}_k\,x$ and $w\,\mathsf{E}^n_{n-1}\,v\,\mathsf{B}^{n-1}_k\,x$
5. If $w\,\mathsf{E}^n_{n-1}\,z\,\mathsf{E}^{n-1}_k\,x$ then either $w\,\mathsf{E}^n_k\,x$ or there is a v with $w\,\mathsf{B}^n_{n-1}\,v\,\mathsf{E}^{n-1}_k\,x$

and dually (notice that there are four dual forms of condition 5). We will allow A to ambiguously denote either the pasting scheme or its graded set.

Informally, condition 3 says that every k-cell ends at at least one $k-1$ cell, and dually begins at at least one $k-1$ cell. Condition 4 ensures that low dimensional ends occur between higher dimensional ends—see for instance $Q \in \mathsf{E}(\eta)$ in Example 2.1 below. Finally, condition 5 ensures that a cells beginnings and ends 'close up' and that their orientations agree:

This , rather than this  or this .

Example 2.1 The diagram



is a representation of the pasting scheme $(A, \mathsf{E}, \mathsf{B})$ given by

$$\begin{aligned} A_0 &= \{P, Q, R, S\} \\ A_1 &= \{u, v, x, y, z\} \\ A_2 &= \{\varepsilon, \eta\} \\ A_k &= \emptyset, \qquad k > 2 \end{aligned}$$

$$\begin{array}{lcllcl} \mathsf{E}_2^2 &=& \{(\varepsilon,\varepsilon),(\eta,\eta)\} & \mathsf{B}_2^2 &=& \{(\varepsilon,\varepsilon),(\eta,\eta)\} \\ \mathsf{E}_1^2 &=& \{(\varepsilon,u),(\eta,y),(\eta,z)\} & \mathsf{B}_1^2 &=& \{(\varepsilon,x),(\varepsilon,y),(\eta,v)\} \\ \mathsf{E}_0^2 &=& \{(\eta,Q)\} & \mathsf{B}_0^2 &=& \{(\varepsilon,R)\} \\ \mathsf{E}_1^1 &=& \{(u,u),(v,v),(x,x),(y,y),(z,z)\} & \mathsf{B}_1^1 &=& \{(u,u),(v,v),(x,x),(y,y),(z,z)\} \\ \mathsf{E}_0^1 &=& \{(u,Q),(v,S),(x,R),(y,Q),(z,S)\} & \mathsf{B}_0^1 &=& \{(u,P),(v,R),(x,P),(y,R),(z,Q)\} \\ \mathsf{E}_0^0 &=& \{(P,P),(Q,Q),(R,R),(S,S)\} & \mathsf{B}_0^0 &=& \{(P,P),(Q,Q),(R,R),(S,S)\} \end{array}$$

In a pasting scheme A define a relation $\triangleleft_A$ (written as $\triangleleft$ when there is no danger of confusion) as follows: for any k, and for any $a, b \in A_k$, say $a \triangleleft b$ if there is a sequence

$$a = a_0, a_1, \ldots, a_j = b,\ j > 0,$$

of elements of A_k with, for all $i < j$, $\mathsf{E}_{k-1}(a_i) \cap \mathsf{B}_{k-1}(a_{i+1}) \neq \emptyset$. As usual, if X is a subgraded set of A, we write $\triangleleft^k(X)$ for $\{b \in A : \text{there exists } x \in X_k,\ b \triangleleft x\ \}$, and if X is n-dimensional, $\triangleleft(X)$ for $\triangleleft^n(X)$.

A pasting scheme A is said to have *no direct loops* when, for any k and for any $a, b \in A_k$, $\mathsf{B}(a) \cap \mathsf{E}(a) = \{a\}$ and $a \triangleleft b$ implies $\mathsf{B}(a) \cap \mathsf{E}(b) = \emptyset$.

If A is a pasting scheme and X a finite subgraded set of A, define the *domain* of X, $\operatorname{dom} X$ by $X - \mathsf{E}(X)$ and the *codomain* of X, $\operatorname{cod} X$ by $X - \mathsf{B}(X)$.

Lemma 2.2 *If A is a finite, k-dimensional pasting scheme with no direct loops, then* $\operatorname{dom} A$ *is a $k-1$ dimensional graded set.*

Proof. The domain of A is at most $k-1$ dimensional since $(\operatorname{dom} A)_k = A_k - \mathsf{E}_k(A_k) = \emptyset$. To see that $\operatorname{dom} A$ is $k-1$ dimensional choose some $a_0 \in A_k$ and some $y_0 \in \mathsf{B}_{k-1}(a_0)$. If $y_0 \notin \operatorname{dom} A$ it can only be because $y_0 \in \mathsf{E}(a_1)$ for some $a_1 \in A_k$. Now choose any $y_1 \in \mathsf{B}_{k-1}(a_1)$, and repeat. Since A has no direct loops, $a_i \neq a_j$ for $i \neq j$ and so, since A_k is finite, we must eventually locate a $y_m \in (\operatorname{dom} A)_{k-1}$.□

Theorem 2.3 *If A is a finite pasting scheme with no direct loops then*

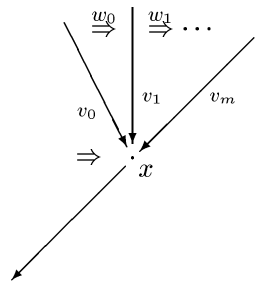
$$\operatorname{dom}\operatorname{dom} A = \operatorname{dom}\operatorname{cod} A.$$

Proof. Notice

$$\begin{aligned} \operatorname{dom}\operatorname{dom} A &= (A - \mathsf{E}(A)) - \mathsf{E}(A - \mathsf{E}(A)) &&= A - (\mathsf{E}(A) \cup \mathsf{E}(A - \mathsf{E}(A))) \\ \operatorname{dom}\operatorname{cod} A &= (A - \mathsf{B}(A)) - \mathsf{E}(A - \mathsf{B}(A)) &&= A - (\mathsf{B}(A) \cup \mathsf{E}(A - \mathsf{B}(A))) \end{aligned}$$

so it suffices to show $\mathsf{E}(A) \cup \mathsf{E}(A - \mathsf{E}(A)) = \mathsf{B}(A) \cup \mathsf{E}(A - \mathsf{B}(A))$ which is clear in dimensions greater than or equal to $\dim A = n$ say.

$\subset$: Suppose $x \in \mathsf{E}(A)$, x of dimension $k < n$. If $x \in \mathsf{B}(A)$ then $x \in$ RHS so suppose $x \notin \mathsf{B}(A)$. By pasting scheme condition 4 there exists v_0 with $v_0 \, \mathsf{E}_k^{n-1} \, x$. If $v_0 \in A - \mathsf{B}(A)$ then $x \in$ RHS. If $v_0 \notin A - \mathsf{B}(A)$ then there must be a w_0 with $w_0 \, \mathsf{B}_{n-1}^n \, v_0 \, \mathsf{E}_k^{n-1} \, x$ whence by the *dual*$_n$ of condition 5 there exists $v_1 \in \mathsf{E}(w_0)$ with $v_1 \, \mathsf{E}_k^{n-1} \, x$. Repeating we eventually obtain $v_m \in A - \mathsf{B}(A)$ and $x \in \mathsf{E}(v_m)$.



Suppose $x \in \mathsf{E}(A - \mathsf{E}(A))$, say $v_0 \, \mathsf{E}_k^{n-1} \, x$ with $v_0 \in A - \mathsf{E}(A)$ and suppose $x \notin \mathsf{B}(A)$. As before, if $v_0 \in A - \mathsf{B}(A)$ then $x \in$ RHS; otherwise we apply condition 5 until we obtain some $v_m \in A - \mathsf{B}(A)$ with $x \in \mathsf{E}(v_m)$.
$\supset$: The converse inclusion is precisely the *dual*$_n$ of the above.$\Box$

2.2 Well-Formed Pasting Schemes

We have shown that finite pasting schemes with no direct loops have sensible notions of domain and codomain which satisfy the basic equation dom dom $=$ dom cod . If a finite pasting scheme parametrizes a composable diagram then its highest dimensional elements must agree in orientation. In this section we describe *well-formed* pasting schemes—those in which the arrangements of the highest dimensional cells in the scheme, and in all of its domains and codomains, are *compatible.*

If A is a k-dimensional pasting scheme with no direct loops write

$$\begin{array}{rcll} s_n(A) & = & t_n(A) = A & \text{if } n \geq k \\ s_n(A) & = & \text{dom}^{k-n} A & \text{if } n < k \\ t_n(A) & = & \text{cod}^{k-n} A & \text{if } n < k. \end{array}$$

Notice that if $n < k$ then $s_n(A)$ and $t_n(A)$ are n-dimensional by Lemma 2.2. We call $s_n(A)$ the *n-source* of A, and $t_n(A)$ the *n-target* of A.

A pasting scheme A of dimension $k > 0$ is called *compatible* when for any $x, y \in A_k$, if $x \neq y$ then $\mathsf{B}_{k-1}(x) \cap \mathsf{B}_{k-1}(y) = \emptyset$ and $\mathsf{E}_{k-1}(x) \cap \mathsf{E}_{k-1}(y) = \emptyset$. A zero dimensional pasting scheme is called compatible if it is a singleton.

A subgraded set X of a pasting scheme A is called a *subpasting scheme* of A if $y \in \mathsf{R}(X)$ implies $y \in X$.

A finite pasting scheme A with no direct loops is called *well formed* if

1. A is compatible
2. For all $n \geq 0$ both $s_n(A)$ and $t_n(A)$ are compatible subpasting schemes of A.

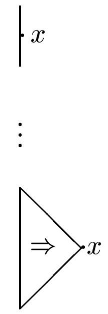
Example 2.4 1. The pasting scheme of Example 1 is well formed.

2. Any finite chain of abutting arrows (head to tail and without loops) represents a well-formed pasting scheme.
3. All the diagrams involving 2-cells in Lecture Notes in Math. 420 may be expressed as well-formed pasting schemes or assert the equality of two subdiagrams which may be expressed as well-formed pasting schemes.

Examples of well-formed pasting schemes of dimension greater than two appear in Chapter 3.

2.3 Loop-Free Pasting Schemes

Pasting schemes with no direct loops, (and even well-formed pasting schemes with no direct loops), may still exhibit subtle looping behaviour like



where the lines should be thought of as k-dimensional, the double arrow as $k+1$ dimensional, x as $k-1$ dimensional, and the ellipsis $(\vdots)$ as j-dimensional with $j < k$. In this section we set down the conditions (again rather technical) which eliminate such behaviour. Schemes satisfying these conditions are called *loop free* and in the remainder of this chapter we show that loop-free schemes and well-formed subschemes of them, behave as we expect pasting schemes should.

A pasting scheme B is called *loop free* if

1. B has no direct loops
2. For any $x \in B$, $\mathsf{R}(x)$ is well formed
3. For any $k-1$ dimensional well-formed subpasting scheme A of B and any $x \in B_k$ with $\operatorname{dom} \mathsf{R}(x) \subset A$
   (a) $A \cap \mathsf{E}(x) = \emptyset$
   (b) if $y \in A$ and $\mathsf{B}(x) \cap \mathsf{R}(y) \neq \emptyset$ then $y \in \mathsf{B}(x)$
4. For any well-formed j-dimensional subpasting scheme A of B and any $x \in B$ with $s_j(\mathsf{R}(x)) \subset A$, if $u, u' \in s_j(\mathsf{R}(x))$ and, for some $v \in A_j$, $u \triangleleft_A v \triangleleft_A u'$, then $v \in s_j(\mathsf{R}(x))$

and dually.

Remark 2.5 In Chapter 3 we will prove that condition 3 is a consequence of the other three conditions. For now we include all four conditions because the presence of condition 3 simplifies the development of the theory which will be needed in the inductive proof that conditions 1, 2, and 4 imply condition 3.

Example 2.6 All the well-formed pasting schemes of Example 2.4 are loop free, as are the non-well-formed commutative schemes referred to there.

From now on we will consider only loop-free pasting schemes. In this and the next section we establish some of their properties.

Proposition 2.7 *Suppose B is a loop-free pasting scheme, $x \in B_k$, then*

$$\operatorname{dom} \mathsf{R}(x) = \mathsf{R}(\mathsf{B}_{k-1}(x)).$$

Proof. Firstly, $\operatorname{dom} \mathsf{R}(x) = \mathsf{R}(x) - \mathsf{E}(x) \subset \mathsf{R}(\mathsf{B}_{k-1}(x))$ because, using pasting scheme condition 4, $\mathsf{R}(x) = \mathsf{R}(\mathsf{B}_{k-1}(x)) \cup \mathsf{R}(\mathsf{E}_{k-1}(x)) \cup \{x\}$ and, using pasting scheme condition 5, $\mathsf{R}(\mathsf{E}_{k-1}(x)) \subset \mathsf{R}(\mathsf{B}_{k-1}(x)) \cup \mathsf{E}(x)$. But, since B has no direct loops,

$$\mathsf{B}_{k-1}(x) \subset \mathsf{R}(x) - \mathsf{E}(x)$$

and so, using loop free condition 2, $\mathsf{R}(\mathsf{B}_{k-1}(x)) \subset \mathsf{R}(\operatorname{dom} \mathsf{R}(x)) = \operatorname{dom} \mathsf{R}(x)$.□

Proposition 2.8 (Pasting On) *Suppose B is a loop-free pasting scheme. If A is a well-formed $k-1$ dimensional subpasting scheme of B and $x \in B_k$ satisfies* $\operatorname{dom} \mathsf{R}(x) \subset A$ *then* $A \cup \mathsf{R}(x)$ *is a well-formed subpasting scheme of B.*

Proof. The scheme $A \cup \mathsf{R}(x)$ has only a single k-dimensional element and so is compatible. Furthermore

$$\begin{aligned} s_{k-1}(A \cup \mathsf{R}(x)) &= A \cup \mathsf{R}(x) - \mathsf{E}(x) \\ &= (A - \mathsf{E}(x)) \cup (\mathsf{R}(x) - \mathsf{E}(x)) \\ &= A \cup \operatorname{dom} \mathsf{R}(x) \\ &= A \end{aligned}$$

which is well formed. Hence for all $n \neq k$, $s_n(A \cup \mathsf{R}(x))$ is well formed. Furthermore, since for $j < k-1$

$$\begin{aligned} t_j(A \cup \mathsf{R}(x)) &= t_j(t_{k-1}(A \cup \mathsf{R}(x)) \\ &= t_j(s_{k-1}(A \cup \mathsf{R}(x)), \end{aligned}$$

$t_n(A \cup \mathsf{R}(x))$ is well formed for all $n \neq k, k-1$.

It remains only to consider

$$\begin{aligned} t_{k-1}(A \cup \mathsf{R}(x)) &= A \cup \mathsf{R}(x) - \mathsf{B}(x) \\ &= (A - \mathsf{B}(x)) \cup (\mathsf{R}(x) - \mathsf{B}(x)) \\ &= (A - \mathsf{B}(x)) \cup \operatorname{cod} \mathsf{R}(x) \end{aligned}$$

which is a subpasting scheme since $\operatorname{cod} \mathsf{R}(x)$ and (using loop free condition 3) $A - \mathsf{B}(x)$ are subpasting schemes. Finally, $t_{k-1}(A \cup \mathsf{R}(x))$ is compatible since suppose not then there exists $z, w \in (t_{k-1}(A \cup \mathsf{R}(x)))_{k-1}$, $z \neq w$, such that there exists $a \in \mathsf{D}_{k-2}(z) \cap \mathsf{D}_{k-2}(w)$, $\mathsf{D} = \mathsf{E}$ or $\mathsf{D} = \mathsf{B}$. Now z, w are not both in $A - \mathsf{B}(x)$, since if it is $k-1$ dimensional then it must be compatible being a subpasting scheme of a compatible $k-1$ dimensional pasting scheme, nor in $\operatorname{cod} \mathsf{R}(x)$ since it is compatible. Hence, without loss of generality, suppose

$$w \in A - \mathsf{B}(x), \; z \in \operatorname{cod} \mathsf{R}(x).$$

Now $a \notin \mathsf{E}(x)$ since $a \in \mathsf{D}(w) \subset A$ and $A \cap \mathsf{E}(x) = \emptyset$ so, by pasting scheme condition 5, there exists $v \in \mathsf{B}_{k-1}(x)$ with $a \in \mathsf{D}_{k-2}(v)$ contradicting the compatibility of A.□

Theorem 2.9 *Suppose that Q is a loop-free pasting scheme and that A, B are well-formed subpasting schemes of Q with $s_n(B) = t_n(A)$, then $A \cap B = s_n(B)$.*

Proof. By induction over the dimension of $A \cup B$.

If $A \cup B$ is of dimension less than or equal to n then $s_n(B) = t_n(A)$ implies that $A = t_n(A) = s_n(B) = B = A \cap B$ as required.

Suppose $A \cup B$ is of dimension $n+1$ and $x \in A \cap B$ but $x \notin s_n(B)$ then A, B are both $n+1$ dimensional since otherwise $t_n(A) = A$ or $s_n(B) = B$, whence $x \notin s_n(A) = t_n(B)$ implies $x \notin A \cap B$. Thus $s_n(B) = \operatorname{dom} B$ and $x \notin s_n(B)$ implies $x \in \mathsf{E}(w)$ some $w \in B_{n+1}$.

In B let $Y = \lhd_B(w) = \{y \in B : y \lhd w\}$. We show that there is an enumeration $y_0, y_1, \ldots, y_r$ of the elements of Y such that

$$\mathsf{B}_n(y_i) \subset \operatorname{dom} B \cup \mathsf{E}(\{y_j : j < i\}) - \mathsf{B}(\{y_j : j < i\}).$$

Firstly, there exists s suitable y_0 since, choose any $y \in Y$, if $\mathsf{B}_n(y) \not\subset \operatorname{dom} B$ it can only be because there is some $y' \lhd y$ with $\mathsf{E}_n(y') \cap \mathsf{B}_n(y) \neq \emptyset$. Repeating we obtain $y'' \lhd y' \lhd y$ etc. Since B is finite and has no direct loops this process must terminate yielding some suitable $y^{(q)} = y_0$ say. Similarly, there exists $y_1 \in Y - \{y_0\}$ such that

$$\mathsf{B}_n(y_1) \subset \operatorname{dom} B \cup \mathsf{E}(y_0)$$

etc. Furthermore, because of the compatibility of Y (inherited from B), if

$$\begin{array}{rcl} \mathsf{B}_n(y_i) & \subset & \operatorname{dom} B \cup \mathsf{E}(\{y_j : j < i\}) \text{ then} \\ \mathsf{B}_n(y_i) & \subset & \operatorname{dom} B \cup \mathsf{E}(\{y_j : j < i\}) - \mathsf{B}(\{y_j : j < i\}). \end{array}$$

Now since $\operatorname{dom} \mathsf{R}(y_0) = \mathsf{R}(\mathsf{B}_n(y_0))$ (Proposition 2.7) we can apply Proposition 2.8 to conclude that

$$\operatorname{cod}(\operatorname{dom} B \cup \mathsf{R}(y_0)) = \operatorname{dom} B \cup \mathsf{R}(y_0) - \mathsf{B}(y_0) = \operatorname{dom} B \cup \mathsf{E}(y_0) - \mathsf{B}(y_0)$$

is well formed. Proceedingly inductively,

$$B' = \operatorname{dom} B \cup \mathsf{E}(Y) - \mathsf{B}(Y)$$

is well formed.

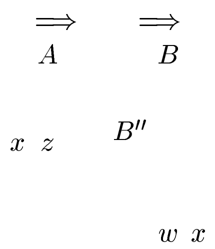
Similarly, $x \notin t_n(A)$ implies $x \in \mathsf{B}(z)$ some $z \in A_{n+1}$ and if

$$z \triangleleft_A z' \text{ and } v \in \mathsf{E}_n(z') \cap \operatorname{cod} A = \mathsf{E}_n(z') \cap \operatorname{dom} B$$

then $v \in B'$ since $v \in \mathsf{B}_n(y)$ some $y \in Y$ would give a direct loop. Thus

$$B'' = B' \cup \mathsf{B}(\triangleright_A(z) \cup \{z\}) - \mathsf{E}(\triangleright_A(z) \cup \{z\})$$

is well formed as above. But $\operatorname{dom} \mathsf{R}(w) \subset B''$, and $x \in \mathsf{E}(w) \cap B''$ which contradicts Q loop free.



Now suppose $h > n+1$ and that for all well-formed A, B with $A \cup B$ of dimension less than h and $s_n(B) = t_n(A)$ we have $A \cap B = s_n(B)$. Let A, B be well-formed subpasting schemes of Q with $A \cup B$ of dimension h. Once again suppose $x \in A \cap B$ but $x \notin s_n(B)$. We may suppose x is of dimension less than h since if not choose any $v \in \mathsf{E}_{h-1}(x)$ then $v \in A \cap B$, v is of dimension $h-1$ and $v \notin s_n(B)$, so v will do. Let $P = \{a \in A_h : x \in \mathsf{E}(a)\}$ and $Q = \{b \in B_h : x \in \mathsf{E}(b)\}$. Put

$$\begin{array}{rcl} A' & = & s_{h-1}(A) \cup \mathsf{E}(\triangleleft_A P \cup P) - \mathsf{B}(\triangleleft_A P \cup P) \\ B' & = & s_{h-1}(B) \cup \mathsf{E}(\triangleleft_B Q \cup Q) - \mathsf{B}(\triangleleft_B Q \cup Q). \end{array}$$

Then A', B' are well formed and $s_n(B') = s_n(B) = t_n(A) = t_n(A')$ but $x \in A' \cap B'$, $x \notin s_n(B') = t_n(A')$ and A', B' are of dimension less than h, contradicting the inductive hypothesis.□

2.4 Paring Down Well-Formed Schemes

A pasting scheme A of dimension $k > 0$ is called *strongly compatible* when, for any $x, y \in A_k$, $x \neq y$ implies $\mathsf{B}(x) \cap \mathsf{B}(y) = \emptyset$ and $\mathsf{E}(x) \cap \mathsf{E}(y) = \emptyset$. A zero dimensional pasting scheme is strongly compatible if it is a singleton.

Proposition 2.10 *Every loop-free well-formed pasting scheme is strongly compatible.*

Proof. Suppose that Q is a k-dimensional loop-free well-formed, pasting scheme. By way of contradiction, suppose $w, z \in A_k$, $w \neq z$, and $a \in \mathsf{E}(w) \cap \mathsf{E}(z)$. Let $Y = \triangleleft\{w, z\}$.

As in the proof of Theorem 2.9, $\operatorname{dom} Q \cup \mathsf{E}(Y) - \mathsf{B}(Y) = A$ say, is well formed. Furthermore, either $\mathsf{B}_{k-1}(w) \subset A$ and $\mathsf{B}_{k-1}(z) \subset A$ and hence $\operatorname{dom} \mathsf{R}(w) \subset A$ so using Proposition 2.8 $A' = A \cup \mathsf{E}(w) - \mathsf{B}(w)$ is well formed, or $w \in Y$ or $z \in Y$ (but not both); suppose without loss of generality $w \in Y$ and then let $A' = A$. Now in either case $a \in A'$ (since $a \in \mathsf{E}(w)$ and, because Q has no direct loops, for any $y \in Y \cup \{w\}$, $a \notin \mathsf{B}(y)$) and $\mathsf{B}_{k-1}(z) \subset A'$ hence $\operatorname{dom} \mathsf{R}(z) \subset A'$, but $a \in \mathsf{E}(z)$ contradicting Q loop free.□

Proposition 2.11 (Paring Down) *Suppose that Q is k-dimensional, loop free and well formed, and $y \in Q_k$ satisfies* $\operatorname{dom} \mathsf{R}(y) \subset \operatorname{dom} Q$ *then* $Q - \mathsf{B}(y)$ *is well formed.*

Proof. If $Q - \mathsf{B}(y)$ is $k-1$ dimensional then $Q - \mathsf{B}(y) = \operatorname{cod} Q$ which is well formed, so suppose $Q - \mathsf{B}(y)$ is k-dimensional. Then $Q - \mathsf{B}(y)$ is compatible since Q is, and it is a subpasting scheme since if not then there exists $a \in \mathsf{B}(y) \cap \mathsf{R}(z)$ some $z \in Q_k - \{y\}$. Furthermore, $a \notin \mathsf{E}(z)$ since $a \in \mathsf{B}(y) \subset \operatorname{dom} Q$, therefore $a \in \mathsf{R}(w)$ some $w \in \mathsf{B}_{k-1}(z)$ whence $w \in \operatorname{dom} Q$ or $\mathsf{E}(z_2)$ etc. to obtain $w \in \operatorname{dom} Q$ with $a \in \mathsf{R}(w)$, $w \in \mathsf{B}_{k-1}(z_r)$ but then Q loop free implies $w \in \mathsf{B}_{k-1}(y)$ and $z_r \neq y$ (because $a \notin \mathsf{R}(\mathsf{E}_{k-1}(y))$ but $a \in \mathsf{R}(\mathsf{E}_{k-1}(z_r))$) which contradicts the compatibility of Q.

Furthermore,

$$\begin{aligned}
\operatorname{cod}(Q - \mathsf{B}(y)) &= Q - \mathsf{B}(y) - \mathsf{B}(Q_k - \{y\}) \\
&= Q - (\mathsf{B}(y) \cup \mathsf{B}(Q_k - \{y\})) \\
&= Q - \mathsf{B}(Q_k) = \operatorname{cod} Q
\end{aligned}$$

is well formed and so $s_n(Q - \mathsf{B}(y))$, $t_n(Q - \mathsf{B}(y))$ are well formed for all $n \neq k, k-1$.

It remains only to show that $s_{k-1}(Q - \mathsf{B}(y)) = \operatorname{dom}(Q - \mathsf{B}(y))$ is a compatible pasting scheme. Now,

$$\begin{aligned}
\operatorname{dom}(Q - \mathsf{B}(y)) &= Q - \mathsf{B}(y) - \mathsf{E}(Q_k - \{y\}) \\
&= Q - \mathsf{E}(Q_k - \{y\}) - \mathsf{B}(y) \\
&= Q - \mathsf{E}(Q_k) \cup \mathsf{E}(y) - \mathsf{B}(y), \text{ Proposition 2.10} \\
&= \operatorname{dom} Q \cup \mathsf{E}(y) - \mathsf{B}(y) \\
&= \operatorname{cod}(\operatorname{dom} Q \cup \mathsf{R}(y))
\end{aligned}$$

which is a compatible pasting scheme by Proposition 2.8.□

2.5 Categories of Pasting Schemes

Well-formed pasting schemes parametrize 'composable' diagrams. If, in a loop-free pasting scheme, we have two well-formed subpasting schemes whose n-source and n-target match up, we should be able to paste them together to obtain another well-formed scheme. This is made precise in the following theorem.

Theorem 2.12 *Suppose that S is a loop-free pasting scheme and $\mathcal{P}$ the collection of well-formed subpasting schemes of S then $(\mathcal{P}, (s_i, t_i, \cup)_{i \in \omega})$ is an ω-category.*

Proof. The elementary properties of s_i and t_i follow from their definition in terms of dom and cod and Theorem 2.3; identity, associativity and middle four interchange laws follow from analogous properties of union (for identity $A \subset B$ implies $A \cup B = B$).

For the other composition axioms suppose $s_i(B) = t_i(A)$ for some $A, B \in \mathcal{P}$. We prove by induction over the dimension of $A \cup B$ that

(a) $s_i(A \cup B) = s_i(A)$,

(b) $s_j(A \cup B) = s_j(A) \cup s_j(B)$ for $j > i$, and

(c) $A \cup B$ is well formed.

If $A \cup B$ is of dimension less than or equal to i then $s_i(B) = t_i(A)$ implies $A = B$ so (a), (b) and (c) follow.

Suppose A, B are well formed, $A \cup B$ is of dimension $h > i$ and that for all well-formed A, B with $A \cup B$ of dimension less than h, $s_i(B) = t_i(A)$ implies (a), (b) and (c).

$$\begin{aligned}
\text{(a) If } h > i+1 \text{ then } s_i(A \cup B) &= s_i(s_{h-1}(A \cup B)) = s_i(\mathrm{dom}\,(A \cup B)) \\
&= s_i((A \cup B) - \mathsf{E}(A \cup B)) \\
&= s_i((A - \mathsf{E}(A_h) - \mathsf{E}(B_h)) \cup (B - \mathsf{E}(B_h) - \mathsf{E}(A_h))) \\
&= s_i((s_{h-1}(A) - \mathsf{E}(B_h)) \cup (s_{h-1}(B) - \mathsf{E}(A_h))) \\
&= s_i(s_{h-1}(A) \cup s_{h-1}(B)), \text{ since } A \cap B = s_i(B) \\
&= s_i(s_{h-1}(A)), \text{ by inductive hypothesis (a)} \\
&= s_i(A) \\
\text{If } h = i+1 \text{ then } s_i(A \cup B) &= s_i((s_{h-1}(A) - \mathsf{E}(B_h)) \cup (s_{h-1}(B) - \mathsf{E}(A_h))) \\
&= s_i(A), \text{ since } s_i(B) - \mathsf{E}(A_h) \subset s_i(A)
\end{aligned}$$

$$\begin{aligned}
\text{(b) If } j \geq h \text{ then } s_j(A \cup B) &= A \cup B = s_j(A) \cup s_j(B) \\
\text{If } j < h \text{ then } s_j(A \cup B) &= s_j(s_{h-1}(A \cup B)) = s_j(\mathrm{dom}\,(A \cup B)) \\
&= s_j(s_{h-1}(A) \cup s_{h-1}(B)), \text{ as above} \\
&= s_j(s_{h-1}(A)) \cup s_j(s_{h-1}(B)), \text{ inductive hypothesis (b)} \\
&= s_j(A) \cup s_j(B)
\end{aligned}$$

(c) $A \cup B$ is a compatible pasting scheme since A and B are and $\mathsf{E}^h(A) \cap \mathsf{E}^h(B) = \emptyset$ and $\mathsf{B}^h(A) \cap \mathsf{B}^h(B) = \emptyset$ (Theorem 2.9). Furthermore, if $h > i+1$ then $\mathrm{dom}\,(A \cup B) = s_{h-1}(A) \cup s_{h-1}(B)$ (by (b)), while if $h = i+1$ then $\mathrm{dom}\,(A \cup B) = s_i(A)$ (by (a)). In either case $\mathrm{dom}\,(A \cup B)$ is well formed. Similarly, using dual forms of (a) and (b), $\mathrm{cod}\,(A \cup B)$ is well formed.□

If S is a loop-free pasting scheme then the ω-category of Theorem 2.12 is called the *ω-category of components of S*. The pasting theorem, which asserts that all strategies for composing cells in a 'composable diagram' in an ω-category yield the same result, follows from the freeness of ω-categories of components.

Theorem 2.13 *Suppose S is a loop-free pasting scheme then the ω-category of components of S is the free ω-category generated by the $\mathsf{R}(x)$, $x \in S$.*

Proof. The fact that the ω-category of components of S is generated by the $\mathsf{R}(x)$, $x \in S$, follows by induction.

Suppose that A is a well-formed subpasting scheme of S of dimension k, $x \in A_k$ and $A \neq \mathsf{R}(x)$. We show that for some j there exists $y \in A_j$, $y \notin \mathsf{R}(x)$, with either $A - \mathsf{B}(y)$ and $s_{j-1}(A) \cup \mathsf{R}(y)$ well formed and $j-1$ composable with composite A, or $A - \mathsf{E}(y)$ and $t_{j-1}(A) \cup \mathsf{R}(y)$ well formed and $j-1$ composable with composite A.

Since $A \neq \mathsf{R}(x)$ there exists y_0 of maximum dimension say j, such that $y_0 \notin \mathsf{R}(x)$. Furthermore, by part 4 of the definition of loop-free,

$$\text{either } \triangleleft_A\{y_0\} \cap \mathsf{R}(x) = \emptyset, \text{ or } \triangleright_A\{y_0\} \cap \mathsf{R}(x) = \emptyset.$$

Suppose $\triangleleft_A\{y_0\} \cap \mathsf{R}(x) = \emptyset$ (the other case follows dually) then any $\triangleleft_A$-minimal element of $\triangleleft_A\{y_0\}$ will do for y since $s_{j-1}(A) \cup \mathsf{R}(y)$ is well formed (Proposition 2.8) and $A - \mathsf{B}(y)$ is well formed (Proposition 2.11, Proposition 2.8 and Theorem 2.12).

Freeness follows exactly as in Street [20, Theorem 18].□

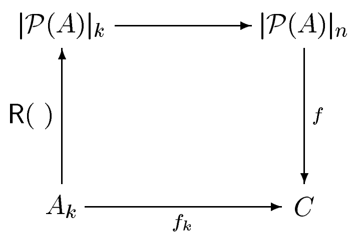
Remark 2.14 It is noteworthy that, despite our different context and greater generality, Street's proof [20, Theorem 18] generalizes with only notational modifications to our Theorem 2.13. In Chapter 3 we describe the particular pasting scheme which corresponds to Street's work.

2.6 The Pasting Theorem

We have described loop-free pasting schemes which are the appropriate parametrizing objects for diagrams in ω-categories. Among these are the well-formed loop-free pasting schemes which are the appropriate parametrizing objects for composable diagrams. It remains to describe the parametrizations themselves and to establish that a well-formed loop-free pasting scheme which is the domain of a given parametrization in some ω-category, determines a unique cell called the *composite* of the diagram in the ω-category. These two tasks are interwoven. We proceed inductively.

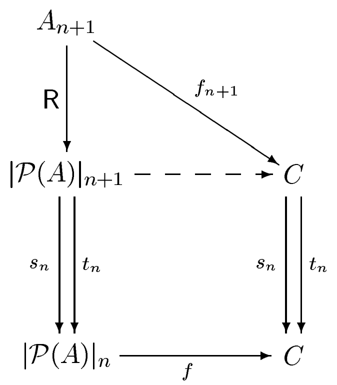
A *realization* (A, f_i) of a pasting scheme A in an ω-category C is a collection of functions $f_i : A_i \to C_i$, $i = 0, 1, \ldots$, which we will sometimes view as functions $f_i : A_i \to C$, into the underlying set of the ω-category C.

We write $\mathcal{P}(A)$ for the ω-category of components of A—its elements are well-formed subpasting schemes of A (Theorem 2.12). The j-category $|\mathcal{P}(A)|_j$ is the sub-ω-category of $\mathcal{P}(A)$ whose elements are well-formed subpasting schemes of A of dimension less than or equal to j. For each k we have a function $\mathsf{R}(\ ) : A_k \to |\mathcal{P}(A)|_k$ and functors (which we will not name) including $|\mathcal{P}(A)|_j$ in $|\mathcal{P}(A)|_k$, $j < k$. A realization (A, f_i) is said to be *n-extendable* when there exists a unique functor $f : |\mathcal{P}(A)|_n \to C$ such that the diagrams (of functions)



commute for all $k \leq n$.

Inductive Definition Any realization (A, f_i) will be called *zero-appropriate*, and is zero-extendable (f_0 is already a functor $|\mathcal{P}(A)|_0 \to C$). Suppose that every n-appropriate realization is n-extendable and suppose given an n-appropriate realization (A, f_i). Then we have



We say that (A, f_i) is $n+1$ *appropriate* if $s_n f_{n+1} = f s_n \mathsf{R}$ and $t_n f_{n+1} = f t_n \mathsf{R}$, whence, by the freeness of $\mathcal{P}(A)$, (A, f_i) is $n+1$ extendable. A realization is called *appropriate* if it is n-appropriate for all n.

Thus a realization is nothing more than an assignment, to each n-dimensional element of a pasting scheme, of an n-cell in an ω-category. A realization is appropriate if it respects s_k and t_k—if it's categorically sensible. A *diagram* (A, f_i) in an ω-category C is a loop-free pasting scheme A together with an appropriate realization $f_i : A_i \to C_i$. A *composable diagram* in an ω-category C is a well-formed loop-free pasting scheme A together with an appropriate realization $f_i : A_i \to C_i$.

Observation 2.15 (The pasting theorem) If A is an n-dimensional, well-formed loop-free pasting scheme then $A \in |\mathcal{P}(A)|_n$. Furthermore, appropriate realizations are extendable. Thus a composable diagram (A, f_i) in an ω-category C determines uniquely a cell of C called the *composite* of (A, f_i) by $f(A)$.

In the sequel, all realizations will be appropriate and all well-formed schemes will be loop-free pasting schemes. We will suppress the unnecessary adjectives.

Chapter 3

Well-Formed Simplicial Sets

This chapter is devoted to describing a particularly important class of pasting schemes—the well-formed simplicial sets. The remainder of this thesis arises, in one way or another, from an investigation of some of the applications of the well-formed simplicial sets.

We begin by describing relations E^i_j and B^i_j which make the standard ω-simplex into a pasting scheme. We then show that the n-simplexes are well-formed subschemes and that the ω-simplex is loop free. The well-formed subschemes of the ω-simplex are called *well-formed simplicial sets.*

The following consideration of well-formed simplicial sets was originally inspired by Street's *orientals* [20]. There is a correspondence between the well-formed simplicial sets and the elements of Street's O_ω, and as an application of Theorems 2.12 and 2.13 of Chapter 2 we obtain Street's main results [20, Theorems 14 and 18] in the well-formed simplicial set form. Street has applied his orientals to non-abelian cohomology while this work uses well-formed simplicial sets to investigate coherence conditions in Chapter 4 and to define an ω-category of paths in Chapter 5.

3.1 The ω-simplex as a pasting scheme

Let S be the simplicial complex $(\omega, \mathcal{P}_f(\omega))$, that is, the collection of non-degenerate elements of the ω-simplex with the induced face operators. In this chapter we will work in S because an element of S is a *non-degenerate* element of the ω-simplex. We will define a pasting scheme structure on S and isolate the well-formed subpasting schemes of S, which, being subsimplicial complexes, may be viewed as simplicial sets in the usual way.

Let x be a k-dimensional element of S and let A be any subset of $[k] = \{0, 1, 2, \ldots, k\}$, say $A = \{a_1, a_2, \ldots, a_i\}$ with $a_j < a_{j+1}$. Write $\mathsf{R}_A(x) = \partial_{a_1}\partial_{a_2}\ldots\partial_{a_i}(x)$ and if the a_i are all are all even (respectively all odd) then write $\mathsf{E}_A(x)$ (respectively $\mathsf{B}_A(x)$) for $\mathsf{R}_A(x)$. Thus $\mathsf{R}_A(x)$ is the $k-i$ subset obtained from x by *simultaneously* deleting the $a_1 st, a_2 nd, \ldots, a_i th$ elements of x.

Define relations E^i_j between S_i and S_j by $x\,\mathsf{E}^i_j\,y$ if there exists a set A with $y = \mathsf{E}_A(x)$, and similarly B^i_j. Then $(S, \mathsf{E}, \mathsf{B})$ satisfies conditions 1 to 3 for a pasting scheme and it satisfies conditions 4 and 5 by the following lemmas.

Lemma 3.1 *Suppose that S, E, and B are given as above and that $w, x \in S$ with $w\,\mathsf{E}^n_k\,x$ then there exists $u \in S$ with $w\,\mathsf{E}^n_{n-1}\,u\,\mathsf{E}^{n-1}_k\,x$, and dually.*

Proof. Suppose $x = \mathsf{E}_{\{a_1,a_2,\ldots,a_i\}}(w)$ then let $u = \mathsf{E}_{\{a_i\}}(w)$ and notice that we then have $x = \mathsf{E}_{\{a_1,a_2,\ldots,a_{i-1}\}}(u)$. For the $dual_{n-1}$ let $u = \mathsf{E}_{\{a_1\}}(w)$ and then $x = \mathsf{B}_{\{a_2-1,a_3-1,\ldots,a_i-1\}}(u)$.□

Lemma 3.2 $(S, \mathsf{E}, \mathsf{B})$ *satisfies pasting scheme condition 5.*

Proof. Suppose $w\,\mathsf{E}^n_{n-1}\,z\,\mathsf{E}^{n-1}_k\,x$, say $z = \mathsf{E}_{\{j\}}(w)$ and $x = \mathsf{E}_{\{a_1,a_2,\ldots,a_m\}}(z)$ then if $j > a_m$, $x = \mathsf{E}_{\{a_1,a_2,\ldots,a_m,j\}}(w)$. Otherwise, for some h, $j \le a_h$ but $j > a_{h-1}$, where for simplicity we set $a_0 = -1$. Then

$$x = \mathsf{E}_{\{a_1,a_2,\ldots,a_{h-1},j,a_{h+1},\ldots,a_m\}}(\mathsf{B}_{a_h+1}(w)).$$

To illustrate the duality in the ω-simplex we prove also the $dual_n$ of condition 5. Suppose $w \, \mathsf{B}^n_{n-1} \, z \, \mathsf{E}^{n-1}_k \, x$, say $z = \mathsf{B}_{\{j\}}(w)$ and $x = \mathsf{E}_{\{a_1,a_2,\ldots,a_m\}}(z)$ then if $j < a_1$,

$$x = \mathsf{B}_{\{j,a_1+1,a_2+1,\ldots,a_m+1\}}(w).$$

Otherwise, suppose $j \geq a_h$ but $j < a_{h+1}$, where for simplicity we set $a_{m+1} = n+1$. Then

$$x = \mathsf{E}_{\{a_1,a_2,\ldots,a_{h-1},j-1,a_{h+1},\ldots,a_m\}}(\mathsf{E}_{a_h}(w)).\Box$$

Thus $(S, \mathsf{E}, \mathsf{B})$ is a pasting scheme. In fact it is a loop-free pasting scheme, but for now we show only that it is free of direct loops.

Lemma 3.3 $(S, \mathsf{E}, \mathsf{B})$ *has no direct loops.*

Proof. Certainly, for any $a \in S$, $\mathsf{B}(a) \cap \mathsf{E}(a) = \emptyset$ since elements of $\mathsf{B}(a)$ are obtained by deleting odd positioned elements from a, while elements of $\mathsf{E}(a)$ are obtained by deleting even positioned elements from a.

It remains to show that if $a \triangleleft b$ then $\mathsf{B}(a) \cap \mathsf{E}(b) = \emptyset$. The proof is by induction over the common dimension of a and b, say k.

For $k = 1$, $a \triangleleft b$ implies that there exists $a_0 = a = \{a_0^0, a_0^1\}$, $a_1 = \{a_1^0, a_1^1\}$, $a_2 = \{a_2^0, a_2^1\}$, $\ldots$ $a_n = b = \{a_n^0, a_n^1\}$, with the a_i^j natural numbers, $a_i^0 < a_i^1$, and, in order to obtain $\mathsf{E}(a_i) \cap \mathsf{B}(a_{i+1}) \neq \emptyset$, it must be that $a_i^1 = a_{i+1}^0$. Hence a_n^1, the only end of b, is strictly greater than a_0^0, the only beginning of a.

Now suppose that a, b are k-dimensional and $a \triangleleft b$ then we have $a_0 = a = \{a_0^0, a_0^1, \ldots, a_0^k\}$, $a_1 = \{a_1^0, a_1^1, \ldots, a_1^k\}$, $a_2 = \{a_2^0, a_2^1, \ldots, a_2^k\}$, $\ldots$ $a_n = b = \{a_n^0, a_n^1, \ldots, a_n^k\}$. Furthermore, since $\mathsf{E}(a_i) \cap \mathsf{B}(a_{i+1}) \neq \emptyset$, a_{i+1} is obtainable from a_i by deleting an even positioned element and inserting an odd positioned one.

Now if $a_0^k = a_1^k = \cdots = a_n^k$ then writing $a_i - \{a_i^k\} = a_i'$ we obtain the $k-1$ dimensional sequence $a_0', a_1', \ldots, a_n'$ showing that $a_0' \triangleleft a_n'$ and use induction (since $\mathsf{B}(a_0) \cap \mathsf{E}(a_n) \neq \emptyset$ implies $\mathsf{B}(a_0') \cap \mathsf{E}(a_n') \neq \emptyset$).

If, on the other hand, $a_0^k \neq a_j^k$ for some j then, if k is odd, then $a_0^k < a_j^k \leq a_n^k$ and so $a_n^k \notin a_0$ but for any $x \in \mathsf{E}(a_n)$, $a_n^k \in x$ therefore $x \notin \mathsf{B}(a_0)$ hence $\mathsf{B}(a) \cap \mathsf{E}(b) = \emptyset$. Similarly, if k is even then $a_0^k > a_j^k \geq a_n^k$ and so $a_0^k \notin a_n$ but for any $x \in \mathsf{B}(a_0)$, $a_0^k \in x$ therefore $x \notin \mathsf{E}(a_n)$ hence $\mathsf{B}(a) \cap \mathsf{E}(b) = \emptyset.\Box$

3.2 Simplexes are Well Formed

In any pasting scheme $(A, \mathsf{E}, \mathsf{B})$, an element $z \in A$ generates the subpasting scheme $\mathsf{R}(z) \subset A$. When the pasting scheme is $(S, \mathsf{E}, \mathsf{B})$, as defined above, and $z \in S_n$, $\mathsf{R}(z)$ is an n-simplex. In this section we show that for any $z \in S$, $\mathsf{R}(z)$ is a well formed subpasting scheme of S.

Suppose $z \in S_n$. If $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of $[n]$ which is written in increasing order and satisfies a_i even whenever i is even and a_i odd whenever i is odd, write

$$\mathsf{A}^1_A(z) \text{ for } \mathsf{R}_A(z).$$

If the elements of A are alternatively even and odd, beginning with an even write

$$\mathsf{A}^0_A(z) \text{ for } \mathsf{R}_A(z)$$

(mnemonic: alternating removal, beginning with an element of parity zero). Just as for $\mathsf{E}_j(z)$, write $\mathsf{A}^1_j(z)$ for the set of all j-dimensional elements obtainable from z by removing elements of alternating parity, beginning with an odd, and

$$\mathsf{A}^1_j(Z) = \bigcup_{z \in Z} \mathsf{A}^1_j(z)$$

etc.

Our aim is to express $\mathrm{dom}^j(\mathsf{R}(z))$ in terms of $\mathsf{A}^1_{n-j}(z)$. We begin by characterizing the ends of $\mathsf{A}^1_{n-j}(z)$.

Lemma 3.4 *Suppose given A alternatively odd and even, beginning with an odd. For ease of exposition let $a_0 = -1$, and let $A = \{a_1, a_2, \ldots, a_j : a_i < a_{i+1},\ i \in [j-1]\}$. Then $x \in \mathsf{E}(\mathsf{A}^1_A(z))$ if and only if there exists a set B, disjoint from A, such that*

(a) if $a_{i-1} < b < a_i$ then $b \not\equiv a_i \pmod 2$;

(b) if $a_i \leq a_j < b$ for all i then $b \equiv a_j \pmod 2$,

and such that $x = \mathsf{R}_{A\cup B}(z)$.

Proof. Properties (a) and (b) may be interpreted as saying that x has the form

$$\mathsf{R}_{\{\underbrace{b,b,\ldots,b}_{\text{even}},a_1,\underbrace{b,b,\ldots,b}_{\text{odd}},a_2,\underbrace{b,b,\ldots,b}_{\text{even}},a_3,\underbrace{\ldots\ldots}_{\text{odd}}\ldots a_j,\underbrace{b,b,\ldots,b}\}}(z)$$

where the elements of B are all represented by b.

($\Leftarrow$) Such an x is an end of $\mathsf{A}^1_A(z)$ since $a_{i-1} < b < a_i$, $b \not\equiv a_i$ mod 2 implies that the element of z in position b will be in an even position in $\mathsf{A}^1_A(z)$. Similarly for $b > a_j$.

($\Rightarrow$) If $\mathsf{R}_{A\cup B}(z) = x$ and B does not satisfy the property, then there exists some $b \in B$ with $a_{i-1} < b < a_i$ but $a_i \equiv b$ mod 2, or with $b > a_j$ but $a_j \not\equiv b$ mod 2. Now the element of z in position b must be removed from $\mathsf{A}^1_A(z)$ to get x but it will be in an odd position so $x \notin \mathsf{E}(\mathsf{A}^1_A(z))$.□

Lemma 3.5 *Suppose z is an n-dimensional element of S then*

$$\mathrm{dom}^j(\mathsf{R}(z)) = \mathsf{R}(\mathsf{A}^1_{n-j}(z)).$$

Proof. By induction on j. True for $j = 0, 1$.

Suppose true for j then

$$\mathrm{dom}^{j+1}(\mathsf{R}(z)) = \mathrm{dom}\,(\mathsf{R}(\mathsf{A}^1_{n-j}(z)))$$

so it suffices to show

$$\mathrm{dom}\,(\mathsf{R}(\mathsf{A}^1_{n-j}(z))) = \mathsf{R}(\mathsf{A}^1_{n-(j+1)}(z)).$$

$\supset$: Suppose $x \in \mathsf{R}(\mathsf{A}^1_{n-j}(z))$ but $x \notin \mathrm{dom}\,\mathsf{R}(\mathsf{A}^1_{n-j}(z))$ then $x \in \mathsf{E}(\mathsf{R}(\mathsf{A}^1_{n-j}(z)))$ so $x \in \mathsf{E}(\mathsf{A}^1_A(z))$ some A of cardinality j, so there exists a B as in Lemma 3.4 with $x = \mathsf{R}_{A\cup B}(z)$ but then $x \notin \mathsf{R}(\mathsf{A}^1_{n-(j+1)}(z))$ since there can be no $j+1$ element alternating (beginning odd) set of vertices to be deleted.

$\subset$: Suppose $x \in \mathrm{dom}\,\mathsf{R}(\mathsf{A}^1_{n-j}(z))$ then $x \in \mathsf{R}(\mathsf{A}^1_A(z))$ for some $A = \{a_1, a_2, \ldots, a_j\}$ say, but $x \notin \mathsf{E}(\mathsf{A}^1_{n-j}(z))$. Write $x = \mathsf{R}_{A\cup B}(z)$. Now since $x \notin \mathsf{E}(\mathsf{A}^1_A(z))$, B does not satisfy the properties given in Lemma 3.4 so either there exists $b \in B$, $b > a_j$ with $b \not\equiv a_j$ mod 2 whence

$$x \in \mathsf{R}(\mathsf{A}^1_{\{a_1,a_2,\ldots,a_j,b\}}(z)) \subset \mathsf{R}(\mathsf{A}^1_{n-(j+1)}(z)),$$

or there exists some $b \in B$ such that for some $i \in [j]$, $a_{i-1} < b < a_i$ (allowing again $a_0 = -1$) and $b \equiv a_i$ mod 2. Furthermore, we can choose such a b so that there exists b' with $b < b' < a_i$ and $b' \not\equiv a_i$ mod 2 (if not then writing c_i for the least element of $A \cup B$ such that $a_{i-1} < c_i \leq a_i$ and $c_i \equiv a_i$ mod 2 we see that $x \in \mathsf{E}(\mathsf{A}^1_{\{c_1,c_2,\ldots,c_j\}}(z))$ contrary to assumption) and then

$$x \in \mathsf{R}(\mathsf{A}^1_{\{a_1,\ldots,a_{i-1},b,b',a_i,\ldots,a_{j-1}\}}(z)) \subset \mathsf{R}(\mathsf{A}^1_{n-(j+1)}(z)).\square$$

Theorem 3.6 *Suppose $z \in S$ then $\mathsf{R}(z)$ is well formed.*

Proof. Suppose that z is n-dimensional. Trivially $\mathsf{R}(z)$ is a compatible pasting scheme.

Furthermore, for all j, $\mathrm{dom}^j(\mathsf{R}(z))$ is a pasting scheme (by Lemma 3.5, $\mathrm{dom}^j(\mathsf{R}(z)) = \mathsf{R}(\mathsf{A}^1_{n-j}(z))$) and is compatible since if $x, y \in \mathsf{A}^1_{n-j}(z)$ with $x \neq y$ but $\partial_i x = \partial_h y$ then it must be that

$$x = \mathsf{A}^1_{\{a_1,a_2,\ldots,a_k,a_{k+1},a_{k+2},\ldots,a_j\}}(z)$$

and

$$y = \mathsf{A}^1_{\{a_1,a_2,\ldots,a_k,b_{k+1},a_{k+2},\ldots,a_j\}}(z)$$

whence $a_{k+1} \equiv k+1 \equiv b_{k+1} \bmod 2$ and so $i \not\equiv h \bmod 2$ (since if, without loss of generality, $a_{k+1} > b_{k+1}$ then $i \equiv k+1-k \equiv 1 \bmod 2$ and $h \equiv k+1-(k+1) \equiv 0 \bmod 2$). Thus $\mathsf{D}_{n-j-1}(x) \cap \mathsf{D}_{n-j-1}(y) \neq \emptyset$ implies one D is a B and the other is an E.

Similarly, for all j, $\operatorname{cod}^j(\mathsf{R}(z)) = \mathsf{R}(\mathsf{A}^0_{n-j}(z))$ is a compatible pasting scheme.□

3.3 An Improved Criterion for Loop Free

This section is devoted to showing that condition 3 in the definition of loop-free pasting schemes is superfluous—conditions 1, 2 and 4 imply condition 3. This result is very important for applications of the theory of pasting schemes because loop free condition 3 is hard to verify in practical situations. The result is used in Section 3.4 to show that $(S, \mathsf{E}, \mathsf{B})$ is loop free as well as in other applications not reported here (cf. Prospectus).

Theorem 3.7 *Suppose $(B, \mathsf{E}, \mathsf{B})$ is a pasting scheme such that*

1. *B has no direct loops*
2. *For any $x \in B$, $\mathsf{R}(x)$ is well formed*
4. *For any well-formed j-dimensional subpasting scheme A of B and any $x \in B$ with $s_j(\mathsf{R}(x)) \subset A$, if $u, u' \in s_j(\mathsf{R}(x))$ and, for some $v \in A_j$, $u \lhd_A v \lhd_A u'$, then $v \in s_j(\mathsf{R}(x))$, and dually,*

then

3. *For any $k-1$ dimensional well-formed subpasting scheme A of B and any $x \in B_k$ with $\operatorname{dom} \mathsf{R}(x) \subset A$*
   - *(a) $A \cap \mathsf{E}(x) = \emptyset$*
   - *(b) if $y \in A$ and $\mathsf{B}(x) \cap \mathsf{R}(y) \neq \emptyset$ then $y \in \mathsf{B}(x)$*

Proof. By induction over k.

If $k = 1$ then any well-formed $k-1$ dimensional subpasting scheme A of B has a single element, say a. Suppose $x \in B$, then, since $\mathsf{R}(x)$ is well formed and B has no direct loops, $\mathsf{R}_0(x)$ consists of exactly two elements $\mathsf{B}_0(x)$ and $\mathsf{E}_0(x)$, and $\operatorname{dom} \mathsf{R}(x) = \mathsf{B}_0(x)$, $\operatorname{cod} \mathsf{R}(x) = \mathsf{E}_0(x)$. Thus to say $\operatorname{dom} \mathsf{R}(x) \subset A$ is to say $a = \mathsf{B}_0(x)$, hence $\mathsf{E}(x) \cap A = \emptyset$. Furthermore, $\mathsf{B}(x) \cap \mathsf{R}(y) \neq \emptyset$ for some $y \in A$ implies $y = a \in \mathsf{B}(x)$.

Suppose true for all dimensions less than or equal to $k-1$ i.e., for $j \leq k-1$ and for all well-formed $j-1$ dimensional subpasting schemes of B and any $x \in B_j$ such that $\operatorname{dom} \mathsf{R}(x) \subset A$, (a) and (b) hold.

(b) Suppose that (b) is false for some $k-1$ dimensional well-formed A, and x a k-dimensional element with $\operatorname{dom} \mathsf{R}(x) \subset A$, then there exists $y \in A$, $y \notin \mathsf{B}(x)$, and $a \in \mathsf{B}(x) \cap \mathsf{R}(y)$. Now there are $w, z \in \mathsf{B}_{k-1}(x)$ with $a \in \mathsf{B}(w)$, and $a \in \mathsf{E}(z)$. Furthermore, $y \notin \mathsf{R}(x)$ since $a \in \mathsf{B}(x)$, $a \in \mathsf{R}(y)$, $y \in \mathsf{R}(x)$ contradicts $\operatorname{cod} \mathsf{R}(x)$ well formed. In fact, we may choose y to be $k-1$ dimensional since $a \notin \operatorname{dom} A$ (because $a \in \mathsf{E}(z)$, $z \in \mathsf{B}(x) \subset A$) and $\operatorname{dom} A$ is a well-formed pasting scheme so $y \notin \operatorname{dom} A$, therefore there is some $k-1$ dimensional $y' \in A$, with $y \in \mathsf{E}(y')$ and therefore with $a \in \mathsf{R}(y')$ and $y' \notin \mathsf{R}(x)$ (since $y \notin \mathsf{R}(x)$) hence $y' \notin \mathsf{B}(x)$.

So suppose y, z, and w are all $k-1$ dimensional. Now $y \lhd_A w$ since if not then $y \in A - \mathsf{B}(\lhd_A w \cup w) = A'$ say, which is well formed, but $a \in \mathsf{R}(y)$, $a \notin A'$, contradiction. Similarly $z \lhd_A y$. However, $z \lhd_A y \lhd_A w$, with $z, w \in \mathsf{B}(x)$ and $y \notin \mathsf{B}(x)$ contradicts 4.

(a) Suppose B satisfies 1, 2 and 4, A is a $k-1$ dimensional well formed subpasting scheme of B and $x \in B_k$ with $\operatorname{dom} \mathsf{R}(x) \subset A$. Required to show $\mathsf{E}(x) \cap A = \emptyset$ so suppose $a \in \mathsf{E}(x) \cap A$. The proof will follow from three lemmas.

Lemma 3.8 *Suppose that A, x and a are as above and that the theorem has been established for all dimensions less than k then $a \notin \operatorname{dom} A$.*

Proof. Suppose $a \in \operatorname{dom} A$. Since $a \in \mathsf{E}(x)$ there exists a $w \in \mathsf{E}_{k-1}(x)$ with $a \in \mathsf{E}(w)$. Put

$$W = \triangleleft_{\operatorname{cod} \mathsf{R}(x)}(w) \cup \{w\}.$$

Notice $(\operatorname{dom} \mathsf{R}(W))_{k-2} \subset \operatorname{dom} \operatorname{cod} \mathsf{R}(x) = \operatorname{dom} \operatorname{dom} \mathsf{R}(x) \subset A$. Let

$$Y = \{y \in A_{k-1} : \mathsf{E}_{k-2}(y) \cap (\operatorname{dom} \mathsf{R}(W))_{k-2} \neq \emptyset\}.$$

Since B satisfies 1, 2 and 4, so must $|B|_{k-1}$ the $k-1$ dimensional skeleton of B. Furthermore, since $|B|_{k-1}$ is $k-1$ dimensional it satisfies 3 also (since the theorem has been established for dimensions less than k) and so we may apply Propositions 2.8 and 2.11. Repeated application of Proposition 2.11 gives $A' = A - \mathsf{B}(\triangleleft_A(Y) \cup Y)$ well formed and hence $\operatorname{dom} A'$ well formed. Repeated application of Proposition 2.8 gives $A'' = \operatorname{dom} A' \cup \mathsf{E}(\triangleleft_{\operatorname{cod} \mathsf{R}(x)}(w)) - \mathsf{B}(\triangleleft_{\operatorname{cod} \mathsf{R}(x)}(w))$ well formed.

Now $a \in A''$ since $a \in \operatorname{dom} A$ and $a \notin \mathsf{B}(\triangleleft_A(Y) \cup Y)$, $a \notin \mathsf{B}(\triangleleft_{\operatorname{cod} \mathsf{R}(x)}(w))$ since B has no direct loops. Also, $\mathsf{B}_{k-2}(w) \subset A''$. But A'' is well formed $k-2$ dimensional, $\operatorname{dom} \mathsf{R}(w) \subset A''$ and $a \in A'' \cap \mathsf{E}(w)$ contradicting the theorem for dimension $k-1$.□

Lemma 3.9 *Suppose that A, x and a are as above and that the theorem has been established for dimensions less than k then a is not $k-2$ dimensional.*

Proof. Suppose a is $k-2$ dimensional and note that by Lemma 3.8 $a \notin \operatorname{dom} A$, and by $dual_{k-1}$ of Lemma 3.8 $a \notin \operatorname{cod} A$.

Now A is $k-1$ dimensional. If $j < k-1$ and $a, b \in A_j$, write $a <_A b$ when there exists $\alpha, \beta \in A_{k-1}$, $\alpha \triangleleft_A \beta$ with $a \in \mathsf{B}(\alpha)$, $b \in \mathsf{E}(\beta)$.

Let $Y = \triangleleft_A\{y \in A_{k-1} : a \in \mathsf{E}(y)\} \cup \{y \in A_{k-1} : a \in \mathsf{E}(y)\}$, suppose $w \in \mathsf{E}_{k-1}(x)$ with $a \in \mathsf{E}(w)$ and let $W = \triangleleft_{\operatorname{cod} \mathsf{R}(x)}(w) \cup \{w\}$. Notice that $A' = A - \mathsf{B}(Y)$ is, by $k-1$ dimensional paring down, well formed, $k-1$ dimensional and $a \in \operatorname{dom} A'$. Hence if $(\operatorname{dom} \mathsf{R}(W))_{k-2} \subset A'$ we may obtain a contradiction exactly as in the proof of Lemma 3.8. So suppose there exists $b \in (\operatorname{dom} \mathsf{R}(W))_{k-2}$ with $b <_A a$ (i.e., $b \in \mathsf{B}(Y)$). Dually, for $z \in \mathsf{E}_{k-1}(x)$ with $x \in \mathsf{B}(z)$, set $Z = \triangleright_{\operatorname{cod} \mathsf{R}(x)}(z) \cup \{z\}$ and there must exist $c \in (\operatorname{dom} \mathsf{R}(z))_{k-2}$ with $a <_A c$. But $b \triangleleft_A a$ means that there exists $\beta, \alpha \in A_{k-1}$, $\beta \triangleleft_A \alpha$ with $b \in \mathsf{B}_{k-2}(\beta)$, $a \in \mathsf{E}_{k-2}(\alpha)$, and $a \triangleleft_A c$ means that there exists $\alpha', \gamma \in A_{k-1}$, with $\alpha' \triangleleft \gamma$ and $a \in \mathsf{B}_{k-2}(\alpha')$, $c \in \mathsf{E}_{k-2}(\gamma)$. Notice $\beta \triangleleft_A \alpha \triangleleft_A \alpha' \triangleleft_A \gamma$ and, using the compatibility of A, $\beta, \gamma \in \mathsf{B}_{k-1}(x)$ hence by Lemma 3.11 $\alpha, \alpha' \in \mathsf{B}_{k-1}(x)$ too. But then $a \notin \mathsf{E}_{k-2}(x)$, contrary to assumption.□

Lemma 3.10 *Suppose that A and x are as above, and that the theorem has been established for dimensions less than k, then $A' = (A \cup \mathsf{R}(x)) - \mathsf{B}(x)$ is well formed $k-1$ dimensional.*

Proof. $A' = (A - \mathsf{B}(x)) \cup (\mathsf{R}(x) - \mathsf{B}(x))$ is a union of subpasting schemes of S (by Theorem 3.6 and k-dimensional (b) which has already been proved) and hence is a subpasting scheme.

A' is compatible since suppose $u, v \in A'_{k-1}$ with $\partial_i u = \partial_j v$, $i \equiv j \bmod 2$ then u, v are not both in $A - \mathsf{B}(x)$ nor $\mathsf{R}(x) - \mathsf{B}(x)$ by the compatibility of those schemes, so without loss of generality, suppose $u \in A - \mathsf{B}(x)$, $v \in \operatorname{cod}(\mathsf{R}(x))$. If $\partial_j v \in \operatorname{dom} \mathsf{R}(x)$ then proceed as in the proof of Theorem 2.8. If $\partial_j v \notin \operatorname{dom} \mathsf{R}(x)$ then $\partial_j v \in \mathsf{E}_{k-2}(x)$ but $\partial_j v = \partial_i u \in A - \mathsf{B}(x) \subset A$, contradicting Lemma 3.9.

Furthermore, $\operatorname{dom} A = \operatorname{dom} A'$ since $\operatorname{dom} A \subset \operatorname{dom} A'$ because $A' \supset A - \mathsf{B}(x) \supset \operatorname{dom} A$ (elements of $\mathsf{B}(x)$ are $k-1$ dimensional or ends of $k-1$ dimensional elements), $\mathsf{E}(A') \subset \mathsf{E}(A) \cup \mathsf{E}(x)$ (using pasting scheme condition 5), and $\mathsf{E}(A) \cap \operatorname{dom} A = \emptyset = \mathsf{E}(x) \cap \operatorname{dom} A$ (using Lemma 3.8); while $\operatorname{dom} A \supset \operatorname{dom} A'$ because $A \supset \operatorname{dom} A'$ (elements of $\mathsf{E}(x)$ are $k-1$ dimensional or ends of $k-1$ dimensional elements), $\mathsf{E}(A) \subset \mathsf{E}(A') \cup \mathsf{B}(x)$ (using a dual form of pasting scheme condition 5), and $\mathsf{E}(A') \cap \operatorname{dom} A' = \emptyset = \mathsf{B}(x) \cap \operatorname{dom} A'$. Dually (the $dual_{k-1}$), $\operatorname{cod} A' = \operatorname{cod} A$ and so all lower dimensional domains and codomains are compatible pasting schemes.□

Proof of Theorem 3.7 continued. Now finally let $a \in \mathsf{E}(x) \cap A$, $w \in \mathsf{E}_{k-1}(x)$, and $a \in \mathsf{E}(w)$. Then $a \in A \cup \mathsf{R}(x) - \mathsf{B}(x) = A'$ and $a \notin \operatorname{dom} A$ (Lemma 3.8) but $a \in A$ so $a \in \mathsf{E}(w')$ for some $w' \in A$. Notice $w' \notin \mathsf{B}(x)$ since $\operatorname{dom} \mathsf{R}(x)$ is well formed. Now $w \neq w'$ since $\mathsf{E}_{k-1}(x) \cap A = \emptyset$ (because if not either $\mathsf{E}_{k-2}(x) \cap A \neq \emptyset$ contradicting Lemma 3.9 or A is not compatible). But $w, w' \in A'$ contradicting Proposition 2.10.□

3.4 Well-Formed Simplicial Sets

We return to the pasting scheme $(S, \mathsf{E}, \mathsf{B})$.

Lemma 3.11 *Suppose that A is a well-formed j-dimensional subpasting scheme of S, $x \in S_n$ with $s_j(\mathsf{R}(x)) \subset A$, $u, u' \in s_j(\mathsf{R}(x))$, and $u \lhd_A v \lhd_A u'$ for some $v \in A_j$ then $v \in s_j(\mathsf{R}(x))$.*

Proof. Suppose $u = v_0 \lhd v_1 \lhd \cdots \lhd v_p = v$ and $v = v_p \lhd v_{p+1} \lhd \cdots \lhd v_q = u'$ with $v_i \in A$ and with $\mathsf{E}_{j-1}(v_i) \cap \mathsf{B}_{j-1}(v_{i+1}) \neq \emptyset$. We show only that $v_1 \in s_j(\mathsf{R}(x))$ and the lemma follows inductively.

By Theorem 3.5, $u, u' \in \mathsf{A}^1_j(x)$, say $u = \mathsf{A}^1_{\{a_1,a_2,\ldots,a_{n-j}\}}(x)$ and $u' = \mathsf{A}^1_{\{b_1,b_2,\ldots,b_{n-j}\}}(x)$.

We obtain v_{i+1} from v_i by deleting an even positioned element of v_i and inserting a new element in an odd position. Hence $b_1 > a_1$ because, if not, the element in position b_1 of x would be present in u and would need to be deleted before reaching u', but being in an odd position it could only be deleted if something were deleted or inserted before it, whence the *something* would need to be inserted/deleted which could only be done if something *more* were deleted/inserted before it etc. Similarly the deletions/insertions made in moving from v_i to v_{i+1} must occur in positions greater than a_1.

Now u and v_1 have a face in common which is an even face of u, say $\mathsf{E}_{\{b\}}(u)$, and an odd face of v_1, and $b > a_1$. Suppose that the element of x which is deleted from u by the application of $\mathsf{E}_{\{b\}}$ is in a position greater than or equal to a_h but less than a_{h+1} where, for simplicity we set $a_{n-j+1} = n + 1$. Then

$$\begin{aligned}\mathsf{E}_{\{b\}}(u) = \mathsf{E}_{\{b\}}\mathsf{A}^1_{\{a_1,a_2,\ldots,a_{n-j}\}}(x) &= \mathsf{B}_{\{a_h-(h-1)\}}\mathsf{A}^1_{\{a_1,\ldots,a_{h-1},b+h,a_{h+1},\ldots,a_{n-j}\}}(x) \\ &= \mathsf{B}_{\{a_h-(h-1)\}}(w) \text{ say,}\end{aligned}$$

which shows that $\mathsf{E}_{\{b\}}(u)$ is an odd face of some $w \in \mathsf{A}^1_j(x) \subset A$ which contradicts the compatibility of A unless $v_1 = w \in \mathsf{A}^1_j(x) \subset s_j(\mathsf{R}(x))$ as required.□

In view of Theorem 3.7, Lemma 3.11 completes the demonstration that $(S, \mathsf{E}, \mathsf{B})$ is a loop-free pasting scheme. The well-formed subpasting schemes of $(S, \mathsf{E}, \mathsf{B})$ are subcomplexes of S because if X is a subpasting scheme of S and $x \in X$ then $y \subset x$ implies $y \in \mathsf{R}(x)$, and so $y \in X$. Thus the subpasting schemes of S may be viewed as simplicial sets and the well-formed subpasting schemes are often called *well-formed simplicial sets.*

By Theorem 2.12 this collection of well-formed simplicial sets forms an ω-category with union (= pasting) as composition. In fact, by Theorem 2.13 it is the free ω-category generated by the $\mathsf{R}(x)$, i.e., by the simplexes.

The original construction of “the free ω-category on the ω-simplex” is due to Street [20]. In this chapter the pasting scheme $(S, \mathsf{E}, \mathsf{B})$ was defined so that the “orientation” of the elements of S would agree with Streets' choice of orientation—odd images are beginnings and even images are ends. However, in many applications, especially those presented in Chapter 4, the opposite orientation is desirable.

Let $(S', \mathsf{E}', \mathsf{B}')$ be the pasting scheme obtained from $(S, \mathsf{E}, \mathsf{B})$ by dualizing at all dimensions greater than one. Thus

$$(S', \mathsf{E}', \mathsf{B}') = dual_2 dual_3 \ldots \quad (S, \mathsf{E}, \mathsf{B})$$

Notice that the orientation of the 1-dimensional elements is the same in $(S, \mathsf{E}, \mathsf{B})$ and $(S', \mathsf{E}', \mathsf{B}')$: in both cases 1-dimensional elements begin at smaller numbers and end at larger numbers (this is just a matter of convenience).

Thus defined $(S', \mathsf{E}', \mathsf{B}')$ is a loop-free pasting scheme and its subpasting schemes are subcomplexes which may be viewed as simplicial sets. In the remainder of this work, “well-formed simplicial set” will mean “well-formed subpasting scheme of $(S', \mathsf{E}', \mathsf{B}')$”.

Of course the well-formed subpasting schemes of $(S', \mathsf{E}', \mathsf{B}')$ form a free ω-category as above. This ω-category will play an important role in Chapter 5 where it will arise as *the ω-category of sorts of a co-parametrized-ω-category* in the category of simplicial sets.

Chapter 4

Coherence

It is commonplace for mathematicians to identify isomorphic structures. For example, if R is a commutative ring, and A,B and C are R-modules then the two R-modules $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$ are distinct but isomorphic and rarely distinguished. However, if there is more than one isomorphism between two structures then, to be precise about our identification, we must choose one. The chosen isomorphism is often a 'canonical' isomorphism (in the example there are isomorphisms sending $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$ and $a \otimes (b \otimes c) \mapsto -(a \otimes b) \otimes c$ and the former is the usual canonical isomorphism). Often we will make many identifications at once using many isomorphisms. Roughly speaking, we call a collection of isomorphisms *coherent* when any two composites of elements of the collection, both of which are between the same two structures, are equal.

Thus coherence theorems are often portrayed as assertions that all of a certain class of diagrams commute (e.g. [16, page 161] but see also [10]). This description however, must be qualified in two important respects.

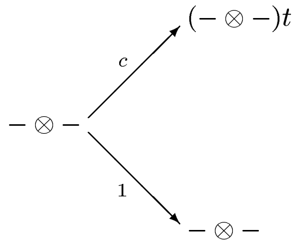
Firstly, returning to the example above, the tensor product of two modules is actually a functor $\otimes : R\text{-}\mathbf{Mod} \times R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ and so, if we wish to identify the two associations of the tensor products of three modules in general, the canonical maps

$$a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$

should be the components of a natural transformation $a : - \otimes (- \otimes -) \to (- \otimes -) \otimes -$. Then the diagrams whose commutativity is guaranteed by a coherence theorem are diagrams of *natural transformations* rather than diagrams of *components* of natural transformations. The difference is important: Commutative diagrams of natural transformations give rise to commutative diagrams of components but not all diagrams of components so arise—if the canonical isomorphisms include the usual commutativity $c_{AB} : A \otimes B \to B \otimes A$ and identity $1_{AB} : A \otimes B \to A \otimes B$ isomorphisms then the diagram

$$A \otimes A \underset{1_{AA}}{\overset{c_{AA}}{\rightrightarrows}} A \otimes A \qquad (4.1)$$

does not commute in general even though all diagrams manufactured from the natural transformations c, a and 1 do commute. This can happen because the diagram of natural transformations whose (A, A) component is (4.1) is not even closed being



(where t is the twist functor $R\text{-}\mathbf{Mod} \times R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod} \times R\text{-}\mathbf{Mod}$ defined by $(A, B) \mapsto (B, A)$).

Secondly, the diagrams of which we speak are *formal* diagrams independent of any particular realization. This matters because in a particular realization it might happen that functors which are the vertices of two logically distinct diagrams of natural transformations might fortuitously coincide resulting in a single amalgamated diagram. One would not expect this 'accidental' diagram to commute.

In this chapter we describe a fairly general setting for the analysis of coherence questions. Our approach is to set up machinery to prove assertions that all of a certain class of diagrams commute, taking account of the above two qualifications in the following way. Firstly, the diagrams which we investigate will be diagrams in n-categories. This allows us to deal explicitly with diagrams of natural transformations in their full context as 2- or even 3-cells rather than their 'shadows' of components. Secondly, pasting schemes (Chapter 2) are a notion of formal diagram for n-categories. Working with pasting schemes it is easy to determine which diagrams must exist in all realizations, and so to avoid the diagrams which arise fortuitously in a particular realization.

Our chief goal is to establish the "higher coherence conditions" for associativities and identities.

4.1 Coherent Situations

A relation R between sets A_i and A_j is usually defined to be a subset of the product $A_i \times A_j$ with $a \in A_i$ related to $b \in A_j$ if and only if $(a, b) \in R$. Then, if $A'_i \subset A_i$ and $A'_j \subset A_j$ the *restriction* of R to A'_i, A'_j, denoted $R|_{A'_i,A'_j}$, is the subset of $A'_i \times A'_j$ given by $(A'_i \times A'_j) \cap R$. If R is a relation between sets A_i and A_j and R' is a relation between sets A'_i and A'_j then

$$R \cup R' \subset (A_i \times A_j) \cup (A'_i \times A'_j) \subset (A_i \cup A'_i) \times (A_j \cup A'_j)$$

may be viewed as a relation between $A_i \cup A'_i$ and $A_j \cup A'_j$.

Suppose that $(A, \mathsf{E}, \mathsf{B})$ is a pasting scheme. Recall that E stands for a collection of relations E^i_j between A_i and A_j, $j \leq i$. If A' is a subgraded set of A write $\mathsf{E}|_{A'}$ for the collection of relations $\mathsf{E}^i_j|_{A'_i,A'_j}$ and similarly $\mathsf{B}|_{A'}$, but note that $(A', \mathsf{E}|_{A'}, \mathsf{B}|_{A'})$ is not in general a pasting scheme. If $(A', \mathsf{E}', \mathsf{B}')$ is another pasting scheme write $\mathsf{E} \cup \mathsf{E}'$ for the collection of relations $(\mathsf{E} \cup \mathsf{E}')^i_j = \mathsf{E}^i_j \cup \mathsf{E}'^i_j$ between $A_i \cup A'_i$ and $A_j \cup A'_j$ and similarly $\mathsf{B} \cup \mathsf{B}'$, but note that $(A, \mathsf{E}, \mathsf{B}) \cup (A', \mathsf{E}', \mathsf{B}') \stackrel{\text{def}}{=} (A \cup A', \mathsf{E} \cup \mathsf{E}', \mathsf{B} \cup \mathsf{B}')$ need not be a pasting scheme because the two pasting scheme structures might not agree on their intersection.

A set $Q = \{(A, \mathsf{E}, \mathsf{B}), (A', \mathsf{E}', \mathsf{B}')\}$ containing two pasting schemes is called *allowable* when the pasting schemes $(A \cap A', \mathsf{E}|_{A\cap A'}, \mathsf{B}|_{A\cap A'})$ and $(A \cap A', \mathsf{E}'|_{A\cap A'}, \mathsf{B}'|_{A\cap A'})$ are equal whence

$$\bigcup Q = (A \cup A', \mathsf{E} \cup \mathsf{E}', \mathsf{B} \cup \mathsf{B}')$$

is a pasting scheme. In general, a set Q of pasting schemes is called *allowable* when every two element subset of Q is allowable. For example, any collection of subpasting schemes of a fixed pasting scheme is allowable. If Q is allowable then $\bigcup Q$ is a pasting scheme and is called the *total pasting scheme of* Q.

A pasting scheme A of dimension k is called a *singleton pasting scheme* if A_k is a singleton.

A *situation* (P, S, f) is a collection P of well-formed pasting schemes of the same dimension, say k, a collection S of well-formed singleton pasting schemes of dimension $k + 1$, and a realization f satisfying

1. $P \cup S$ is an allowable collection of pasting schemes with total pasting scheme T say
2. If $A \in P$ and $s \in S$ with $\operatorname{dom} s \subset A$ then $\operatorname{cod}(A \cup s) \in P$
3. f is a realization of T in some ω-category C.

A situation is called *coherent* when any two well-formed $k + 1$ dimensional subpasting schemes of the total pasting scheme T, with the same domain and codomain, both in P, have the same realization.

Example 4.1 Suppose that K is a category on which is defined a multiplication functor $\otimes : K \times K \to K$ which is associative up to a specified natural isomorphism

$$a : - \otimes (- \otimes -) \to (- \otimes -) \otimes -.$$

Let C be the 3-category with one object $*$, the natural numbers as 1-cells and addition as 0-composition, functors $K^n \to K^m$ as 2-cells from n to m, and natural transformations between functors as 3-cells. Let P be the collection of 2-dimensional well-formed simplicial sets and let S be the collection of singleton 3-dimensional well-formed simplicial sets. Then $P \cup S$ is certainly allowable with total pasting scheme T the ω-simplex viewed as a loop-free pasting scheme (Section 3.3) and truncated at the third dimension. We realize T in the 3-category C as follows:

$$\begin{array}{rcll} f_0(x) & = & * & \text{for all } x \in T_0 \\ f_1(x) & = & 1 & \text{for all } x \in T_1 \\ f_2(x) & = & \otimes & \text{for all } x \in T_2 \\ f_3(x) & = & a & \text{for all } x \in T_3 \end{array}$$

This defines an appropriate realization (Section 2.6).

Now (P, S, f) so defined forms a situation. Furthermore, if the natural associativity a satisfies Mac Lane's pentagon condition [15]

$$(a \otimes 1) *_2 a *_2 (1 \otimes a) = a *_2 a : - \otimes (- \otimes (- \otimes -)) \to ((- \otimes -) \otimes -) \otimes -$$

(cf. (4.3) below) then the situation is coherent. We will prove coherence in Section 4.3.

4.2 Coherence Lemmas

We establish two lemmas to assist us in proving coherence. The lemmas correspond to two common types of coherence problems: In the second the maps whose coherence is sought are usually isomorphisms and the proof is inspired by Mac Lane's original study of coherence for associativity isomorphisms [15]; the first lemma is inspired be Laplaza's treatment of coherence for associativities which are not isomorphisms [14]. We begin with some definitions.

A situation is said to be *loop free* if its total pasting scheme is a loop-free pasting scheme. Suppose X is a well-formed subpasting scheme of the total pasting scheme of a loop-free situation (P, S, f), and $\operatorname{dom} X = A$, $\operatorname{cod} X = B$ with $A, B \in P$, then X will be referred to as a *path from A to B* (often just as a *path*). Because a well-formed $k+1$ dimensional subpasting scheme X of a loop-free pasting scheme can be decomposed as

$$X = \operatorname{dom} X \cup \mathsf{R}(x_1) \cup \mathsf{R}(x_2) \cup \cdots \cup \mathsf{R}(x_m)$$

with $x_i \in X_{k+1}$ and $\operatorname{dom} \mathsf{R}(x_i) \subset \operatorname{dom} X \cup \mathsf{E}(\{x_j : j < i\}) - \mathsf{B}(\{x_j : j < i\})$, we will sometimes write the path X from A to B as $(x_1, x_2, \ldots x_m)$ and depict it as a string of arrows $A \xrightarrow{x_1} \xrightarrow{x_2} \cdots \xrightarrow{x_m} B$ (but do not be fooled—a path is really $k+1$ dimensional and different strings may correspond to the same path).

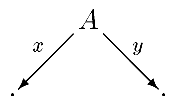
If X is a $k+1$ dimensional pasting scheme denote the cardinality of X_{k+1} by $|X|$. When we think of X as a path we call $|X|$ the *length* of X. If (P, S, f) is a loop-free situation and $A \in P$ then let the *rank of* A, $\operatorname{rank} A$, be the least upper bound of the lengths of paths from A, i.e.

$$\operatorname{rank} A = \sup\{|X| : X \subset T \text{ is well formed and } \operatorname{dom} X = A\}.$$

If every $A \in P$ has finite rank say that (P, S, f) *has rank.* Write $\operatorname{rank} A \,(\operatorname{rel} B)$ for the least upper bound of the lengths of paths from A to B.

Suppose (P, S, f) is a situation. A *fork* in (P, S, f) (or just a fork when the situation is understood) is an element $A \in P$ and two distinct elements $x, y \in S$ with $\operatorname{dom} x, \operatorname{dom} y \subset A$. The fork

(A, x, y) is depicted by



A fork (A, x, y) is said to have a *commuting completion* when there exist X, Y well-formed subpasting schemes of T with

1. $x \subset X$, $y \subset Y$
2. $\operatorname{dom} X = \operatorname{dom} Y = A$
3. $\operatorname{cod} X = \operatorname{cod} Y$
4. $f(X) = f(Y)$

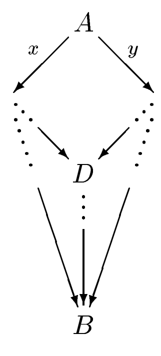
If $\operatorname{cod} X = D$ then the commuting completion is said to have *codomain* D. Note that a commuting completion need not commute in T where formally different subpasting schemes are different, but must commute when realized.

A loop-free situation (P, S, f) is called *coherent into* B, where $B \in P$, when, for any $A \in P$, and any two paths U, V from A to B, U and V have the same realization.

Lemma 4.2 *Suppose (P, S, f) is a loop-free situation which has rank, and suppose $B \in P$. If for any $A \in P$ and any two paths $(x, \ldots), (y, \ldots)$ from A to B, the fork (A, x, y) has a commuting completion with codomain D such that either $D = B$ or there exists a path from D to B, then (P, S, f) is coherent into B. If the hypotheses hold for all $B \in P$, then (P, S, f) is coherent.*

Proof. We prove that for all $A \in P$, all paths from A to B have the same realization by induction over $\operatorname{rank} A\,(\operatorname{rel} B)$. If $\operatorname{rank} A\,(\operatorname{rel} B) = 0$ then there is nothing to prove.

Suppose that for all $A \in P$ with $\operatorname{rank} A\,(\operatorname{rel} B) < j$ all paths from A to B have the same realization. Suppose $A \in P$ has $\operatorname{rank} A\,(\operatorname{rel} B) = j$ and that $U = (x, \ldots)$, $V = (y, \ldots)$ are two paths from A to B. Now (A, x, y) is a fork which can be commutatively completed with the codomain of the completion D say, and $\operatorname{cod}(A \cup x)$ and $\operatorname{cod}(A \cup y)$ have rank $(\operatorname{rel} B)$ less than j. Therefore, in the following diagram, the 'square' and the two 'triangles' commute when realized, hence the two paths U and V are equal when realized.□



Suppose $(A, (s_n, t_n, *_n)_{n \in \omega})$ is an ω-category and x a k-cell of A. We say that x is $k-1$ *monic* (respectively *epic, iso*) when x is monic (respectively epic, iso) in the ordinary category $(A, s_{k-1}, t_{k-1}, *_{k-1})$.

Lemma 4.3 *Suppose given a loop-free situation (P, S, f) which has rank and satisfies*

1. *Every fork in (P, S, f) has a commuting completion*
2. *For all $x \in S$, x is of dimension k and $f(x)$ is $k-1$ monic*

then (P, S, f) *is coherent.*

Proof. We establish first that for any $A \in P$, if there exist paths U from A to B, and V from A to B', with $B, B' \in P$ and $\operatorname{rank} B = \operatorname{rank} B' = 0$, then $B = B'$. The proof is by induction over the rank of A. If $\operatorname{rank} A = 0$ then there is nothing to prove.

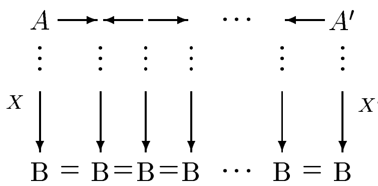
Suppose that the result is established for all A with $\operatorname{rank} A < j$ and suppose $A \in P$, $\operatorname{rank} A = j$, and $U = (x, \ldots)$ and $V = (y, \ldots)$ are two paths from A to B and B' respectively. Then (A, x, y) can be commutatively completed. Let the codomain of the commutative completion be D. Now $\operatorname{cod}(A \cup x)$ and $\operatorname{cod}(A \cup y)$ have rank less than j so if $\operatorname{rank} D = 0$ then applying the inductive hypothesis $B = D = B'$, while if $\operatorname{rank} D > 0$ then there is a path from D to some B'' of rank zero whence, applying the inductive hypothesis to $\operatorname{cod}(A \cup x)$, $B'' = B$, and to $\operatorname{cod}(A \cup y)$, $B'' = B'$.

Furthermore we have shown that for all B of rank zero the hypotheses for Lemma 4.2 are satisfied and so (P, S, f) is coherent into such B.

Now finally suppose $A, A' \in P$ and U, V are paths from A to A'. If $\operatorname{rank} A' = 0$ then U and V have the same realization by Lemma 4.2. Otherwise there exists a path X from A' to some B of rank zero and $X *_{k-1} U$, $X *_{k-1} V$ both have the same realization being paths from A to B. But then, since $f(X)$ can be expressed as a $*_{k-1}$-composite of monics, $f(U) = f(V)$.□

Remark 4.4 If the maps whose coherence we seek are isomorphisms, as in Example 4.1 and in many of our applications, then in order to obtain a situation (P, S, f) which is loop free and has rank, we must include in S only one of each pair of inverse isomorphisms. Then however, Lemma 4.3 only asserts coherence for the maps in S. To extend the result to the isomorphisms in S and their inverses we use an observation of Mac Lane [15].

Suppose (P, S, f) is loop free and has rank and let S^{-1} be the collection of singleton pasting schemes obtained by taking the $dual_k$ of each pasting scheme in S. Suppose that U is a path from A to A' with elements from $S \cup S^{-1}$. We depict such a path as a string of arrows with each arrow pointing in the direction chosen for S and note that, by the proof of Lemma 4.3, there exists a unique B of rank zero with each arrow part of a path in (P, S, f) to B.



Now since all the 'rectangles' commute when realized the realization of the path from A to A' is fully determined by the realizations of the paths X from A to B and X' from A' to B which are independent of the particular path from A to A'. Thus (P, S, f) coherent implies $(P, S \cup S^{-1}, g)$ coherent where g is the realization defined for $x \in S$ by $g(x) = f(x)$ and $g(dual_k\, x) = f(x)^{-1}$, and for $A \in P$ by $g(A) = f(A)$.

Remark 4.5 In both of Lemmas 4.2 and 4.3 it is not necessary to check that all forks can be commutatively completed—for many of them commutative completions are automatic: Suppose (P, S, f) is a situation with the elements of P k-dimensional and (A, x, y) a fork in (P, S, f) with $\mathsf{B}_k(x) \cap \mathsf{B}_k(y) = \emptyset$ then (A, x, y) can be commutatively completed since $A \cup x$ and $(\operatorname{cod}(A \cup x)) \cup y$, and $A \cup y$ and $(\operatorname{cod}(A \cup y)) \cup x$ are both well formed and k-composable with composite $A \cup x \cup y$. If $\mathsf{B}_k(x) \cap \mathsf{B}_k(y) = \emptyset$, say that x and y *have no overlap*. We need only check for commutative completion those forks which have overlap.

4.3 Coherence of Multiplications and Associativities

In this and the next section we apply the coherence lemmas to establish in a unified way several known coherence results.

Example 4.6 Coherence of Multiplication: Suppose that K is a set equipped with a multiplication function $M : K \times K \to K$. Let C be the 2-category with one object $*$, with the natural numbers as 1-cells and addition as 0-composition, and with functions $K^n \to K^m$ as 2-cells from n to m. Let P be the collection of 1-dimensional well-formed simplicial sets and let S be the collection of singleton 2-dimensional well-formed simplicial sets. Let T be the total pasting scheme of $P \cup S$ and realize T in C by

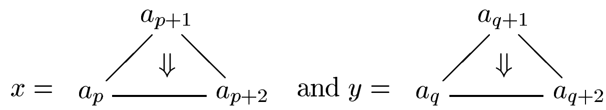
$$\begin{array}{rcll} f_0(x) & = & * & \text{for all } x \in T_0 \\ f_1(x) & = & 1 & \text{for all } x \in T_1 \\ f_2(x) & = & M & \text{for all } x \in T_2 \end{array}$$

Thus defined (P, S, f) forms a situation which is loop free and has rank, with

$$\operatorname{rank} A = |A| - 1.$$

To say that (P, S, f) is coherent is to say that any two composites of iterated M's $K^n \to K^m$ are equal—i.e., M is associative. Lemma 4.2 gives sufficient conditions for the coherence of (P, S, f). We will show that the hypotheses of the lemma are always satisfied by paths $(x, \ldots), (y, \ldots)$ where x and y have no overlap, and that for x and y with overlap, the hypotheses amount precisely to the usual associativity condition $M(1 \times M) = M(M \times 1)$.

Suppose $A \in P$, say A is the simplicial set $a_0 \to a_1 \to \cdots \to a_j$, $a_i \in \omega$, and let $(x, \ldots), (y, \ldots)$ be two paths from A to some $B \in P$. Then

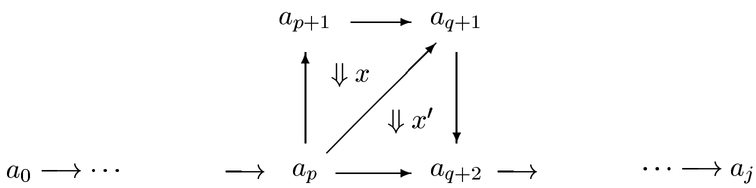


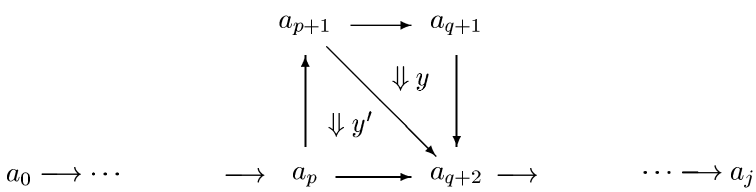
for some $p, q \in [j-2]$. Suppose without loss of generality $q > p$. If x and y have no overlap then $(A, x, , y)$ can be commutatively completed (Remark 4.5) with codomain

$$D = a_0 \to \cdots a_p \to a_{p+2} \to \cdots a_q \to a_{q+2} \to \cdots \to a_j.$$

To see that there is a path from D to B notice that the vertices of B are contained in the vertices of D and that applying elements of S simply eliminates vertices.

If x and y have some overlap it must be that $a_{p+1} = a_q$ whence we can complete the fork (A, x, y) by following x by x' and y by y' as shown

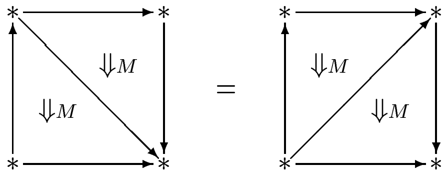




and again there will be a path from

$$D = a_0 \cdots \to a_p \to a_{q+2} \to \cdots a_j$$

to B. It remains only to check that the completion is commutative (and then Lemma 4.2 will apply to give coherence). Commutativity will be obtained if and only if

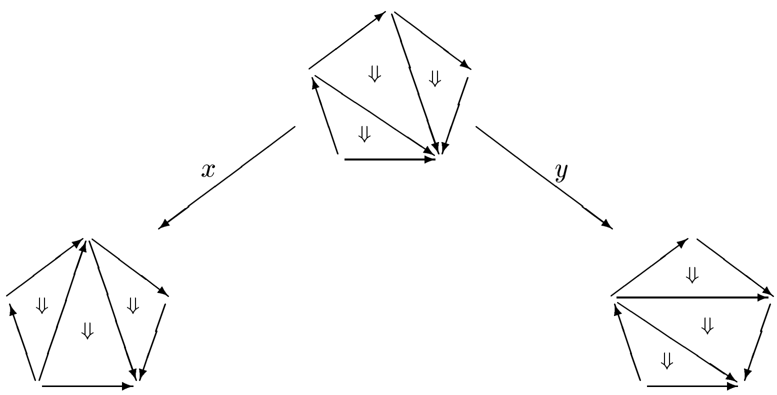


(4.2)

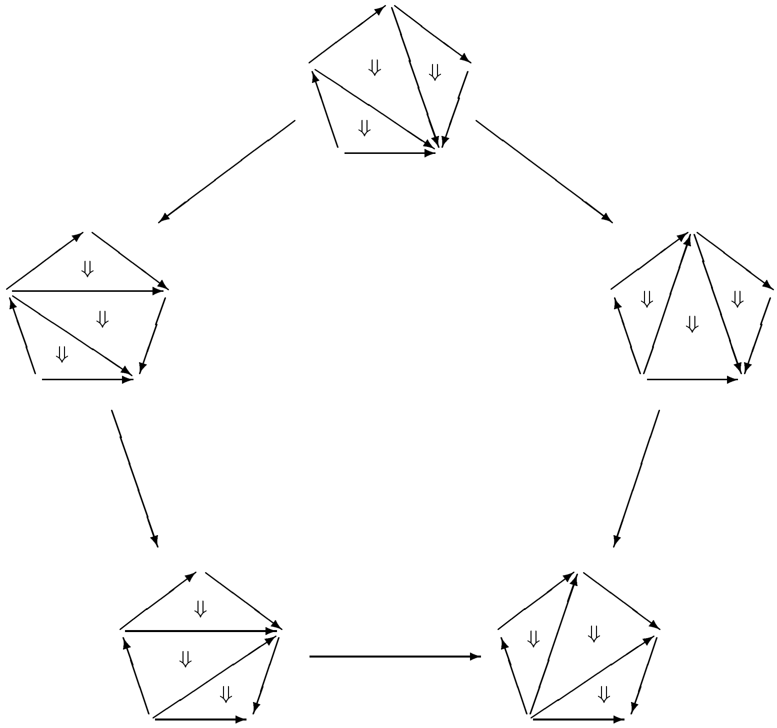
which depicts the usual associativity condition.

Example 4.7 Coherence of Associativity Isomorphisms: Suppose that K is a category with a multiplication functor $\otimes$ as in Example 4.1. We could require $\otimes$ to satisfy the same condition (4.2) as M in which case $\otimes$ is called strictly associative, but when K is a category it is more reasonable to ask for a natural isomorphism a in place of the equality in (4.2). We investigate the coherence of such an isomorphism.

Suppose (P, S, f) is the loop-free situation described in Example 4.1, which has rank because, if we view a well-formed 2-dimensional simplicial set as a bracketing of the 1-dimensional elements of its domain, then applying an element of S moves a pair of brackets towards the *left*, which can only be done finitely many times. Because a, being an isomorphism, is monic we will apply Lemma 4.3 and Remark 4.4, so we need only check that any fork (A, x, y) in (P, S, f) can be commutatively completed. As before if x and y have no overlap then commutatively completing (A, x, y) is straightforward. Suppose x and y have some overlap. Geometrically x and y are tetrahedra, and the requirement that $\mathsf{B}_2(x) \cap \mathsf{B}_2(y) \neq \emptyset$, means that their domains must overlap by at least a triangle so we obtain



Further applications of the elements of S shown yield the pentagonal completion



(4.3)

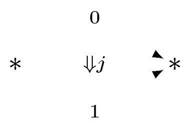
If this completion is always commutative then (P, S, f) is coherent. The commutativity of (4.3) when realized is precisely Mac Lane's pentagonal condition for the coherence of associativity isomorphisms.

Furthermore, Laplaza [14] has shown that the completions (4.3) and the trivial completions satisfy the hypotheses of Lemma 4.2 so that when a is not an isomorphism we again obtain coherence provided that (4.3) commutes when realized.

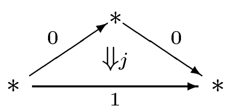
4.4 Coherence of Identities and Commutativities

Example 4.8 Coherence of Identities: Suppose that K is a set equipped with a multiplication $M : K \times K \to K$ (cf. Example 4.6). An identity for the multiplication M is usually given by a function $j : K^0 = I \to K$ which picks out the identity element.

Let C be the 2-category described in Example 4.6: one object $*$, natural numbers as arrows, and functions as 2-cells. Then the function j may be written as a 2-cell



but in order to maintain the geometry we write it as a function $j : K^0 \times K^0 \to K$ and obtain the *triangle*



This departure from the normal description is not necessary to obtain the results below, but it does simplify the treatment because all the pasting schemes we use can be made into labelled well-formed simplicial sets. Let T be the pasting scheme given by

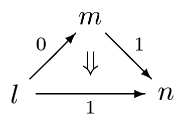
$$T_0 = \omega$$

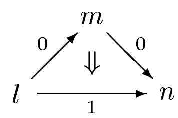
$$T_1 = \{(m,n,a) : m,n \in \omega,\ m < n,\ a \in [1]\}$$
$$T_2 = \{(l,m,n,a,b) : l,m,n \in \omega,\ l < m < n,\ a,b \in [1]\}$$

and with

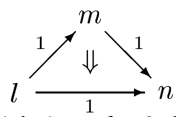
$$\begin{aligned}
\mathsf{B}_1(l,m,n,a,b) &= \{(l,m,a),(m,n,b)\}\\
\mathsf{B}_0(l,m,n,a,b) &= \{m\}\\
\mathsf{E}_1(l,m,n,a,b) &= \{(l,n,1)\}\\
\mathsf{E}_0(l,m,n,a,b) &= \emptyset\\
\mathsf{B}_0(m,n,a) &= \{m\}\\
\mathsf{E}_0(m,n,a) &= \{n\}
\end{aligned}$$

For example:

$\mathsf{R}(l,m,n,0,1) =$ 

, $\mathsf{R}(l,m,n,0,0) =$ 

, and

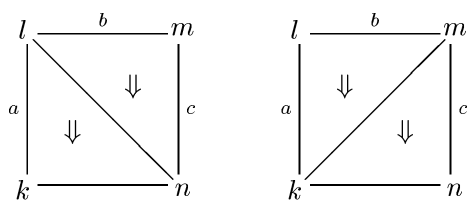
$\mathsf{R}(l,m,n,1,1) =$ 

Then T is a loop-free pasting digram because it's just the 2-dimensional well-formed simplicial sets except that there are two copies of each 1-dimensional element and four copies of each 2-dimensional element. Let P be the well-formed 1-dimensional subpasting schemes of T, and let S be the well-formed singleton 2-dimensional subpasting schemes of T. Realize T in C by

$$\begin{aligned}
f_0(x) &= *, && \text{for all } x \in T_0\\
f_1(m,n,a) &= 0, && \text{if } a = 0\\
&= 1, && \text{if } a = 1\\
f_2(l,m,n,a,b) &= M, && \text{if } a = b = 1\\
&= j, && \text{if } a = b = 0\\
&= \text{Identity} && \text{if } a = 1,\ b = 0 \text{ or } a = 0,\ b = 1
\end{aligned}$$

Thus defined (P,S,f) forms a situation which is loop free and has rank.

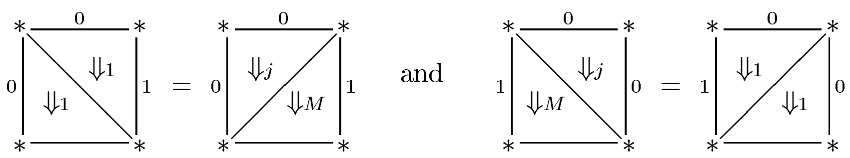
As in Example 4.6, Lemma 4.2 will apply provided that all the completions of the form



(4.4)

commute when realized. We investigate what this means for the operations j and M.

When $a = b = c = 1$ we obtain (4.2) and so discover, as there, that M is associative. When two of a, b, and c are 1 then the two sides of (4.4) are both M and so commutativity is automatic. When $a = b = c = 0$ the two sides of (4.4) are both j and so commutativity is automatic. When $a = c = 0$ and $b = 1$ the two sides of (4.4) are both the same identity 2-cell and so commutativity is automatic. Finally, when $a = b = 0$ and $c = 1$ or $b = c = 0$ and $a = 1$ commutativity is respectively



(4.5)

which are precisely the two usual identity laws.

Thus (P, S, f) is coherent if and only if K is a *monoid* (a set with an associative multiplication which has a two-sided identity).

Example 4.9 Coherence of Identity Isomorphisms: Suppose that K is a category with a multiplication functor $\otimes : K \times K \to K$ as in Example 4.7. Suppose further that $j : K^0 \to K$ is a functor. If j and $\otimes$ satisfy (4.5) and (4.2) (with $\otimes$ in place of M) then K is a *strict monoidal category.* Suppose that (4.5) and (4.2) are only satisfied up to specified isomorphisms, say l, r, and a respectively. We investigate the coherence of such isomorphisms.

Let C be the 3-category described in Example 4.1. Let T be the pasting schemes described in Example 4.8, but with an extra dimension

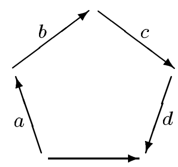
$$T_3 = \{(k, l, m, n, a, b, c) : k, l, m, n \in \omega,\ k < l < m < n,\ a, b, c, \in [1]\}$$

and extend the realization of Example 4.8 by

$$\begin{aligned} f_3(k, l, m, n, 1, 1, 1) &= a, \\ f_3(k, l, m, n, 0, 0, 1) &= l, \\ f_3(k, l, m, n, 1, 0, 0) &= r, \quad \text{and} \\ f_3(k, l, m, n, a, b, c) &= \text{Identity}, \quad \text{for all other } a, b, c. \end{aligned}$$

Let P be the collection of well-formed subpasting schemes of T and let S be the collection of well-formed singleton 3-dimensional subpasting schemes of T. Then (P, S, f) is loop free and has rank and, since all the cells in the image of f_3 are isomorphisms we may prove coherence by applying Lemma 4.3 provided that all forks can be commutatively completed. Of course, forks which have no overlap may be commutatively completed. Those which have overlap come in a number of types.

Since two overlapping tetrahedra must have a common triangle in their domains, we obtain diagrams of the form (4.3). In such a diagram, label the common external 1-cells as shown:



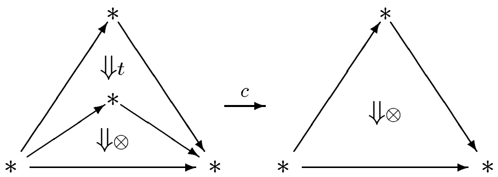
In all but the following cases the two 'legs' of the pentagon (4.3) are automatically equal when realized. If $a = b = c = d = 1$ then asking for commutativity is precisely asking for Mac Lane's pentagonal coherence of associativity condition to hold. If $a = d = 1$ and $b = c = 0$ then equality of the legs of (4.3) when realized is, up to choice of orientation, Kelly's refinement [7] of Mac Lane's conditions [15] for the coherence of an identity isomorphism.

We have proved the usual coherence theorem for monoidal categories.

Example 4.10 Coherence for Commutativity Isomorphisms: The theory of coherent situations presented so far is quite sufficient for our main purpose—the establishment of the higher coherence conditions. However, in its present form, it is not sufficient for the analysis of all coherence questions. This example is included to show how the simple minded analyses above will not always lead directly to the required conditions, and is presented less rigorously—the aim is only to exhibit the difficulties which arise.

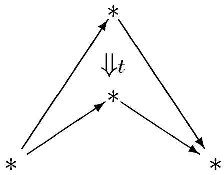
Suppose that K is a monoidal category i.e., a category K_0 together with a multiplication functor $\otimes : K_0 \times K_0 \to K_0$ which is associative and has identities up to specified coherent isomorphisms a, l, r as in Example 4.9. Recall that the twist functor $t : K_0 \times K_0 \to K_0 \times K_0$ is defined by $(f, g) \mapsto (g, f)$.

We will view a commutativity isomorphism as a natural transformation $\otimes t \to \otimes$ i.e.,



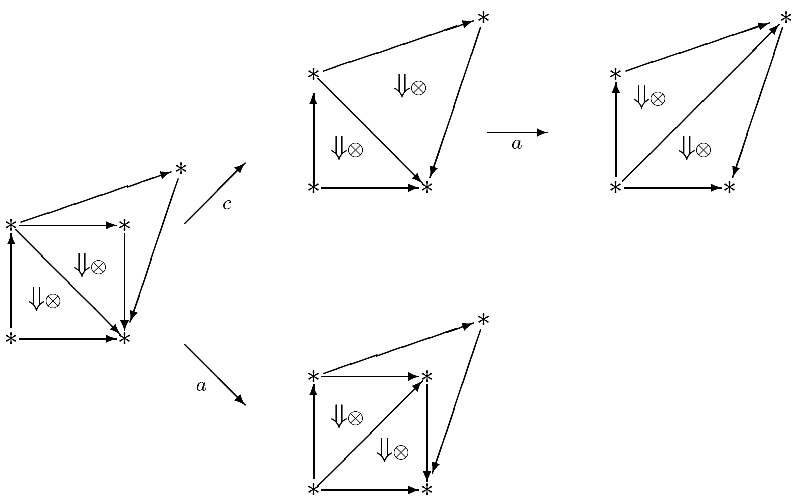
(4.6)

Notice that we have not managed to force the geometry into the well-formed simplicial set mould because at the 2-dimensional level we need a ‘quadrilateral’



Therefore, to be precise we must define a new pasting scheme Q generated by squares and triangles, and verify for it properties like loop-freeness (as we did for well-formed simplicial sets in Chapter 3). We will not attempt to formally define the pasting scheme here, but suppose that we had done so. Then let P be the collection of well-formed 2-dimensional subschemes of Q. Let S be the collection of singleton pasting schemes of the form c (4.6) or a (Example 4.1). Suppose $P \cup S$ is allowable and let T be the total pasting scheme of $P \cup S$. Realize T in the 3-category C of Example 4.1 in the obvious way.

Now to investigate the coherence of the situation just sketched we aim to apply Lemma 4.3 and so we look for forks (A, x, y) which overlap. We will find forks where x and y are two instances of a, which may be commutatively completed if the usual pentagonal condition holds. Forks involving two instances of c either fail to overlap or overlap completely and so commutative completions are automatic. Finally, the interesting case is a fork where one of x and y is an instance of a, say x, and the other, y, is an instance of c (in the following diagrams the quadrilaterals which contain a twist functor have been left blank):

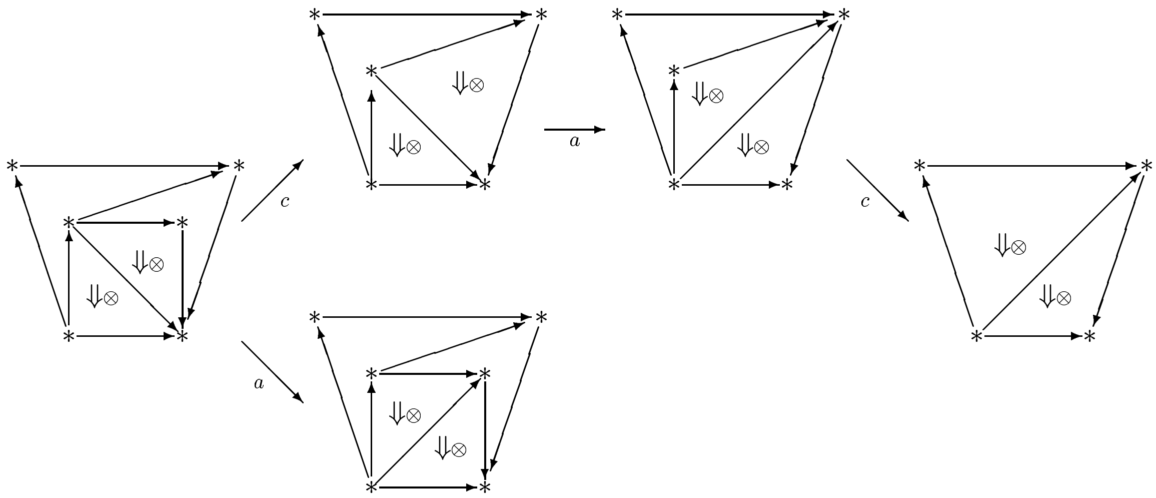


(4.7)

Unfortunately, this fork cannot be commutatively completed as it stands—the application of an instance of a shown in the upper right of (4.7) is the ony path which can be appended to the fork.

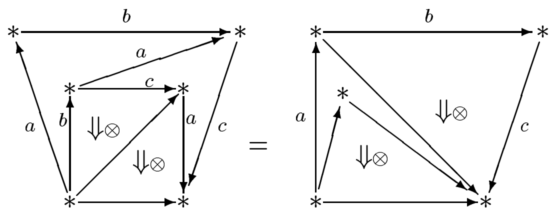
Nevertheless, a little ingenuity will produce the usual condition for coherence of a commutativity isomorphism!

We begin by introducing a further twist functor:



(4.8)

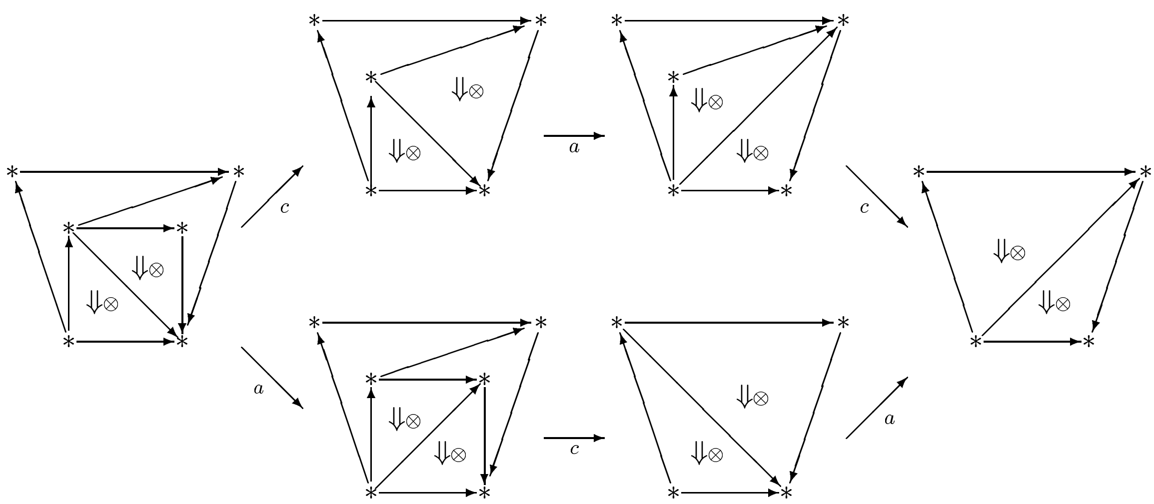
Whether such an addition can be justified in the current framework will not be argued because a more severe difficutly remains. Once again the fork cannot be completed—(4.8) shows the most that can be done without the realization that



(4.9)

(moving a past b and c and then multiplying b and c is the same as multiplying b and c and moving

a past the product). Using this we obtain



$$(4.10)$$

which depicts the usual hexagonal condition for coherence of a commutativity.

But what does the Equation 4.9 really say? Certainly it cannot be an equation of pasting schemes because formally different schemes are always distinct, and if the pasting schemes are not equal then the lower "path" in (4.10) is not a path in (P, S, f) at all.

It appears that the analysis of coherence described so far is a little too strict to treat coherence for a commutativity although the idea of searching for commutative completions of forks with overlap does seem to be right. Some untested modifications which may extend the applicability of the theory are recorded in the prospectus.

4.5 The Higher Coherence Conditions

Notice that in Examples 4.6 and 4.7 above, a single instance of a multiplication, M or $\otimes$, is a 2-simplex viewed as a pasting scheme. To require that M be coherent is to require commutativity (upon realization) of a 3-simplex—the associative law. If the associative law is weakened to an isomorphism a, then the coherence of a follows from the commutativity (when realized) of a 4-simplex. In this section we prove that this process continues. The *higher coherence conditions* for associativity are simply commutativities of n-simplexes for appropriate n, and such commutativities are easy to understand in the light of our knowledge of the well-formed simplicial sets.

The first higher coherence question arises from a multiplication $\otimes$ with specified reassociating maps a (Example 4.1) for which (4.3) need not necessarily commute when realized, but is equipped with a specified 4-cell isomorphism p between its left and right legs.

The situation which expresses this question is obtained by letting P be the collection of well-formed simplicial sets of dimension 3, letting S be the collection of singleton well-formed simplicial sets of dimension 4, and letting f be the evident extension of the realization of Example 4.1. By Lemma 4.3, (P, S, f) is coherent if every fork can be commutatively completed. By Remark 4.5 it suffices to check forks which have overlap so suppose (A, x, y) is such a fork. Naively, x and y may overlap in several ways—since each has a domain consisting of three tetrahedra they might overlap by one, two or three tetrahedra. However, counting vertices we see that there is only one possible form of overlap: x and y are each 4-simplexes and so have five vertices, a tetrahedron has four vertices and, to overlap at all, x and y must have a tetrahedron, i.e. four of their five vertices, in common. Any greater overlap would make them equal, any lesser overlap would be no overlap at all ($\mathsf{B}_3(x) \cap \mathsf{B}_3(y) = \emptyset$). Thus, by Lemma 4.3 we have:

Proposition 4.11 *The situation (P, S, f) just described is coherent if the realization of every 5-simplex commutes.*$\square$

We will show that this continues.

Let W be the set of well-formed simplicial sets and let SW be the set of singleton well-formed simplicial sets. Write $(W)_k$ for the set of elements of W of dimension k and $(SW)_k$ for the set of elements of SW of dimension k. Recall that $(S', \mathsf{E}', \mathsf{B}')$ is the pasting scheme defined on the ω-simplex on page 25 and that $|S'|_n$ is the subpasting scheme of S' with elements of dimension n or less. Thus for $k \leq n$, elements of $(SW)_k$ and $(W)_k$ are well-formed subpasting schemes of $|S'|_n$. In fact $|S'|_n$ is the total pasting scheme of the collection $(W)_{n-1} \cup (SW)_n$.

Proposition 4.12 *Suppose that f is a realization of $|S'|_n$ with $f(u)$ $(n-1)$-monic for each $u \in (S')_n$ then $((W)_{n-1}, (SW)_n, f)$ is a loop free situation which is coherent if and only if for every $a \in (S')_{n+1}$, $f(\operatorname{dom} \mathsf{R}(a)) = f(\operatorname{cod} \mathsf{R}(a))$ (i.e. if and only if the realization of every $n+1$ simplex commutes).*

Proof. The fact that $((W)_{n-1}, (SW)_n, f)$ forms a loop-free situation follows from the remarks immediately preceeding the proposition.

Suppose the situation is coherent and $a \in (S')_{n+1}$ then $\operatorname{dom}\operatorname{dom} \mathsf{R}(a) = \operatorname{dom}\operatorname{cod} \mathsf{R}(a) = A$ say and $\operatorname{cod}\operatorname{dom} \mathsf{R}(a) = \operatorname{cod}\operatorname{cod} \mathsf{R}(a) = B$ say are both in $(W)_{n-1}$ by Theorem 3.6. But $\operatorname{dom} \mathsf{R}(a)$, $\operatorname{cod} \mathsf{R}(a)$ are two paths from A to B in $|S'|_n$ (Theorem 3.6 again) so they must have the same realization by the definition of coherence.

Conversely, suppose the realization of every $n+1$ simplex commutes. Suppose (A, x, y) is a fork in $((W)_{n-1}, (SW)_n, f)$, say $x = \mathsf{R}(\{x_1, x_2, \ldots x_{n+1}\})$ and $y = \mathsf{R}(\{y_1, y_2, \ldots y_{n+1}\})$. If x and y have overlap then $\mathsf{B}_{n-1}(x) \cap \mathsf{B}_{n-1}(y) \neq \emptyset$ so exactly n of the x_i occur as y_i's. Thus $a = \{x_i\} \cup \{y_i\} \in (S')_{n+1}$. Furthermore, x and y are not both contained in $\operatorname{dom} a$ nor in $\operatorname{cod} a$ by the compatibility of those schemes. Thus $\operatorname{dom} a$ and $\operatorname{cod} a$ form a completion of (A, x, y) which must be commutative by assumption and so Lemma 4.3 applies giving $((W)_{n-1}, (SW)_n, f)$ coherent.$\square$

Remark 4.13 The argument of Proposition 4.12 remains unchanged if we replace well formed simplicial sets with labelled well formed simplicial sets. Thus the result gives the higher coherence conditions (commutativity of appropriate simplexes) for associativities and identities jointly as well as for associativies.

Chapter 5

Parametrized Theories

The Moore construction of a category of paths in a topological space has a number of forms. One variant is as follows.

Let I_n denote the real interval $[0, n]$ of length n with the usual Euclidean topology. For each n let $\lceil 0 \rceil$ be the continuous map $I_0 \to I_n$ given by $\lceil 0 \rceil(0) = 0$ and let $\lceil n \rceil$ be the continuous map $I_0 \to I_n$ given by $\lceil n \rceil(0) = n$. Let X be any topological space and let $M_X = \sum_{n \in \omega} \mathbf{Top}(I_n, X)$ denote the set of continuous maps into X each of which has some I_n as its domain. Then M_X has a category structure given by: Suppose $f \in M_X$, say $f : I_j \to X$ then let $s(f)$ be the map $f\lceil 0 \rceil : I_0 \to X$ and $t(f)$ the map $f\lceil n \rceil : I_0 \to X$, and if $f : I_j \to X$, $g : I_k \to X$ are such that $s(g) = t(f)$ then we define $g * f : I_{j+k} \to X$ by

$$\begin{aligned} g * f(a) &= f(a) \qquad \text{if } 0 \leq a \leq j \\ &= g(a - j) \qquad \text{if } j \leq a \leq j + k. \end{aligned}$$

The verification that this definition satisfies the associative and identity laws (with identities given by maps $I_0 \to X$) is routine.

The Moore construction is functorial: Given any other topological space X' and a continuous map $h : X \to X'$ then composition with h defines a functor $M_h : M_X \to M_{X'}$. Arguments due to Eckmann-Hilton (see e.g. [6]) show that if such an algebraic structure is borne by a single hom set $\mathcal{C}(D, X)$, then the parametrizing object D is a coalgebra in $\mathcal{C}$ (Section 1.3). However, the 'multi-sorted' case, where D is replaced by many objects (I_n), and $\mathcal{C}(D, X)$ by a coproduct of many hom sets $(\sum_{n \in \omega} \mathbf{Top}(I_n, X))$ has not been treated.

This chapter develops the multi-sorted theory in order to understand, in the manner of Eckmann-Hilton, the basis of the Moore construction. Our chief goal is to generalize the Moore construction to include information about 'higher dimensional paths' (paths between paths) in a 'higher dimensional category' (an ω-category).

In section 1 we define parametrized (multi-sorted) algebras in a category $\mathcal{C}$ by shifting our attention to a category of families of objects of $\mathcal{C}$. Section 2 returns the problem to the category $\mathcal{C}$ and gives an alternative description of parametrized algebras as algebras in $\mathcal{C}$ for certain theories—the parametrized theories. Section 3 explores some of the relationships between a theory (e.g. the theory of categories) and its parametrized forms (the parametrized theories of categories) and their respective algebras. Finally Section 4 includes examples of algebras for parametrized theories demonstrating that several well known constructions are examples of 'multi-sorted homming out constructions' although they are not representable in the ordinary sense; and that the well-formed simplicial sets form a co-parametrized-ω-category providing a solution to the problem of generalizing the Moore construction to higher dimensions.

5.1 Parametrized Algebras

The Moore construction and several others (see below) depend upon homming out of a collection of objects $(D_i)_{i \in I}$ of a category $\mathcal{C}$ and obtaining a structure on the collection of morphisms

$\sum_{i \in I} \mathcal{C}(D_i, X)$. In this section we interpret constructions of this sort in the category $\mathrm{Fam}\,\mathcal{C}^{\mathrm{op}}$.

The section begins by showing that the collection of morphisms $\sum_{i \in I} \mathcal{C}(D_i, X)$ is a single hom set in $\mathrm{Fam}\,\mathcal{C}^{\mathrm{op}}$. We then define parametrized algebras, show that Moore's domains $(I_n)_{n \in \omega}$ are a co-parametrized category in **Top**, and note that this yields a 'classical' explanation of the Moore construction.

Notice first that $\mathcal{C}$ is a full subcategory of $(\mathrm{Fam}\,\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ with the inclusion I given by IC is the singleton family (C) and for $f : C \to C'$, $f^{\mathrm{op}} : C' \to C$ in $\mathcal{C}^{\mathrm{op}}$ is the single component of a map $(C') \to (C)$ in $\mathrm{Fam}\,\mathcal{C}^{\mathrm{op}}$ whose opposite is by definition If.

Lemma 5.1 *Suppose $\mathcal{C}$ is a category and $(D_i)_{i \in I}$ a collection of objects of $\mathcal{C}$ then for any $X \in \mathcal{C}$*

$$\sum_{i \in I} \mathcal{C}(D_i, X) \cong \mathrm{Fam}\,\mathcal{C}^{\mathrm{op}}(I^{\mathrm{op}}X, (D_i)_{i \in I})$$

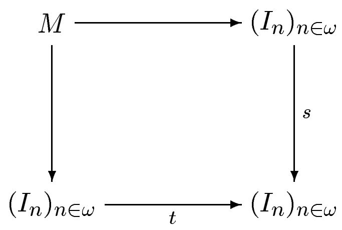
Proof. The bijection is clear if we note that a morphism in the right hand side is a function ϕ from the indexing set of the family $I^{\mathrm{op}}X$, i.e. from a one point set $\{*\}$, into the set I, together with a $\mathcal{C}^{\mathrm{op}}$ morphism $X \to D_{\phi(*)}$. In other words, an element of the right hand side is given by an element $i \in I$, and a $\mathcal{C}$ morphism $D_i \to X$.□

Now suppose T is a theory and $\mathcal{C}$ a category. A *parametrized algebra of* T *in* $\mathcal{C}$ is an (ordinary) algebra of T in $\mathrm{Fam}\,\mathcal{C}$ and a *co-parametrized algebra of* T *in* $\mathcal{C}$ is a T-algebra in $\mathrm{Fam}\,\mathcal{C}^{\mathrm{op}}$.

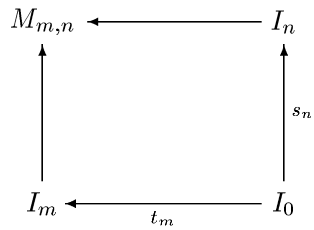
Example 5.2 The collection $(I_n)_{n \in \omega}$ is a co-parametrized category in **Top**. To see this we verify that $(I_n)_{n \in \omega}$ has a category structure in $\mathrm{Fam}\,\mathbf{Top}^{\mathrm{op}}$.

Firstly define $s : (I_n)_{n \in \omega} \to (I_n)_{n \in \omega}$ by $s = (\phi, (s_n)_{n \in \omega})$ with $\phi : \omega \to \omega$ given by $\phi(n) = 0$, and $s_n : I_n \to I_{\phi(n)}$ in $\mathbf{Top}^{\mathrm{op}}$ given by the map $\lceil 0 \rceil : I_0 \to I_n$ in **Top**. Similarly $t = (\phi, (t_n)_{n \in \omega})$ with $t_n : I_n \to I_{\phi(n)}$ determined by the map $\lceil n \rceil : I_0 \to I_n$. It is easy to see that s and t satisfy the equations required for source and target maps in a category.

The 'composable maps' in the category (I_n) are given by the pullback



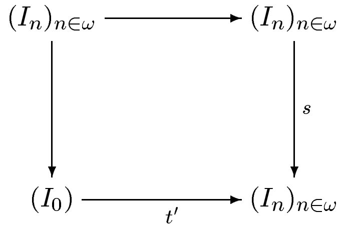
which is easily calculated in $\mathrm{Fam}\,\mathbf{Top}^{\mathrm{op}}$: M is the family of *pushouts* (calculated in **Top**)



indexed by $\omega \times \omega$. Notice that $M_{m,n}$ may be chosen to be I_{m+n} with the injections given by $a(x) = x$, $x \in [0, m]$ and $b(y) = m + y$, $y \in [0, n]$ (and of course any other choice of pushout is canonically isomorphic to this).

The 'composition' in the category is a $\mathrm{Fam}\,\mathbf{Top}^{\mathrm{op}}$ morphism $c : (M_{m,n})_{m,n \in \omega \times \omega} \to (I_n)_{n \in \omega}$. To give such a morphism is to give, for each $m, n \in \omega \times \omega$ a natural number $\phi(m, n)$ and a continuous map $c_{m,n} : I_{\phi(m,n)} \to M_{m,n}$. In this case ϕ is given by $\phi(m, n) = m + n$ and $c_{m,n}$ is the identity $I_{m+n} \to M_{m,n}$ (or the canonical isomorphism if some other choice of pushout is made above).

That c is associative is easy to see. Identities for c come from the object of objects (the equalizer of s and the identity $(I_n) \to (I_n)$ or equivalently of t and the identity) which is the singleton family (I_0). 'Composable maps', the first of which is an identity, are given by the pullback



(where t' is the composition of t with the equalizer) and then c is equal to the projection of the pullback onto the second factor. Similarly for left identities.□

Thus, by the classical argument of Eckmann-Hilton, homming into (I_n) in Fam $\mathbf{Top}^{\mathrm{op}}$, which by Lemma 5.1 is the same as homming out of each of the I_n in $\mathbf{Top}$, yields an ordinary category—the Moore category.

5.2 Parametrized Theories

The fact that $\mathcal{C}(X, D)$ inherits an algebra structure if D is an algebra in $\mathcal{C}$ has long been understood. The work of Eckmann-Hilton demonstrated that $\mathcal{C}(D, X)$ inherits an algebra structure if D is an algebra, *not* in $\mathcal{C}$, but in $\mathcal{C}^{\mathrm{op}}$. The preceding section demonstrates that $\sum_{i\in I} \mathcal{C}(D_i, X)$ inherits an algebra structure if (D_i) is an algebra in Fam $\mathcal{C}^{\mathrm{op}}$ and that this is the basis of the Moore construction.

Although a coalgebra D in $\mathcal{C}$ is defined as an algebra in $\mathcal{C}^{\mathrm{op}}$, the structure that D bears in $\mathcal{C}$ is easy to describe. In this section we describe the structure borne in $\mathcal{C}$ by the collection of objects D_i which corresponds to an algebra structure on the family (D_i) in Fam $\mathcal{C}^{\mathrm{op}}$.

Let T be a theory, $\mathcal{C}$ a category and F a co-parametrized T-algebra in $\mathcal{C}$ (i.e. $F : \mathsf{T} \to \mathrm{Fam}\,\mathcal{C}^{\mathrm{op}}$ preserves finite limits). Recall that $p : \mathrm{Fam}\,\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ given by $(C_i)_{i\in I} \mapsto I$ and $(\phi, (f_i)_{i\in I}) \mapsto \phi$ preserves finite limits. The composite $G = pF : \mathsf{T} \to \mathbf{Set}$ is a T-algebra and will be called *the algebra of sorts* of F.

Example 5.3 Section 5.1 showed that the family $(I_n)_{n\in\omega}$ is a co-parametrized category. Its algebra of sorts is the category ω which has a single object and arrows corresponding to the natural numbers with composition given by addition. Explicitly: If we write the elements of the set ω as n (as usual) then source and target maps are given by $s(n) = 0 = t(n)$ for all $n \in \omega$; the composable pairs are given by the pullback of s along t which is $\omega \times \omega$ (the set indexing the composable maps of the category (I_n)); and composition is given by the image of the indexing map ϕ for the composition c of (I_n) which sends $(n, m) \mapsto n + m$.

Let $*$ be a one point set. If $F : \mathsf{T} \to \mathbf{Set}$ recall that $e\ell\, F$, the category of elements of F, is the comma category $*/F$. The elements construction is functorial—considering the category $\mathbf{set}$ of small sets as an object of $\mathbf{Cat}$, we have a functor $e\ell : \mathbf{Cat}/\mathbf{set} \to \mathbf{Cat}$. The Fam construction used above is also functorial with Fam $: \mathbf{Cat} \to \mathbf{Cat}/\mathbf{set}$.

Proposition 5.4 *The functor $e\ell$ is left adjoint to the functor* Fam *.*

Proof. The isomorphism

$$\mathbf{Cat}(e\ell\,(G : \mathcal{A} \to \mathbf{set}), \mathcal{C}) \cong \mathbf{Cat}/\mathbf{set}(G : \mathcal{A} \to \mathbf{set}, p : \mathrm{Fam}\,\mathcal{C} \to \mathbf{set})$$

is easily seen: A functor in the left hand side gives for each $A \in \mathcal{A}$ a collection of objects of $\mathcal{C}$, one for each element of GA, while functors in the right hand side give a family of objects of $\mathcal{C}$ indexed by the set GA.□

Remark 5.5 Indeed the bijection is an isomorphism of categories, so we have a 2-adjoint, and it carries left exact functors to left exact functors.

Proposition 5.6 *Suppose* T *is small and finitely complete, and* $G : \mathsf{T} \to \mathbf{Set}$ *preserves finite limits then* $e\ell\, G$ *is small and finitely complete.*

Proof. The projection $Q : e\ell\, G \to \mathsf{T}$ creates limits (see [16, page 117]).□

Thus for any theory T and any T-algebra G the comma category $e\ell\, G$ is a theory. The remark says that the category of parametrized algebras of T in $\mathcal{C}$ with algebra of sorts G is isomorphic to the category of $e\ell\, G$-algebras in $\mathcal{C}$. In other words to give a T-algebra structure on (D_i) in $\mathrm{Fam}\,\mathcal{C}$ is to give an $e\ell\, G$-algebra structure on the objects D_i in $\mathcal{C}$ and dually (replacing $\mathrm{Fam}\,\mathcal{C}$ by $\mathrm{Fam}\,\mathcal{C}^{\mathrm{op}}$ and $e\ell\, G$-algebra by $e\ell\, G$-coalgebra). Thus $e\ell\, G$ is *the theory of parametrized* T*-algebras with algebra of sorts* G.

In what follows we will move freely between descriptions of parametrized T-algebras in $\mathcal{C}$ as T-algebras in $\mathrm{Fam}\,\mathcal{C}$ and as $e\ell\, G$-algebras in $\mathcal{C}$ for an appropriate choice of G.

An explicit description of the structure of a parametrized monoid in **Set** appears in Section 5.4.

5.3 Algebras and Parametrized Algebras

This section records a few elementary results about the relationships between ordinary and parametrized algebras and their theories. Specifically, it demonstrates that every algebra is a degenerate parametrized algebra, and every parametrized algebra can be made into an ordinary algebra by 'joining up the sorts'.

Let T be a theory and let G be a T-algebra. Let $\mathsf{T}' = e\ell\, G$ be the theory of parametrized T-algebras with algebra of sorts G. Since $\mathsf{T}' = e\ell\, G = */G$ (where $*$ is a one point set and $G : \mathsf{T} \to \mathbf{Set}$), there is a projection Q of the comma category T' onto the theory T.

Lemma 5.7 *The projection* $Q : \mathsf{T}' \to \mathsf{T}$ *preserves finite limits.*

Proof. Immediate from [16, page 113, Theorem 2] since Q creates limits.□

Corollary 5.8 *Every algebra in* $\mathcal{C}$ *is a parametrized algebra in* $\mathcal{C}$.

Proof. Suppose $F : \mathsf{T} \to \mathcal{C}$ is an algebra in $\mathcal{C}$ and $G : \mathsf{T} \to \mathbf{Set}$ an algebra in **Set**. Let T' be the theory of parametrized T-algebras with algebra of sorts G, and let Q be the projection $\mathsf{T}' \to \mathsf{T}$. Since Q preserves finite limits the composite FQ is a T'-algebra. The corollary as stated is the usual abuse of notation: If $D \in \mathcal{C}$ is the image under F of a generic object of T then D is often referred to as a T-algebra and D is also the image under FQ of the many generic objects of T' and so is called a T'-algebra.□

A parametrized T-algebra $F' : \mathsf{T}' \to \mathcal{C}$ which factors as FQ where $F : \mathsf{T} \to \mathcal{C}$ is a T-algebra and $Q : \mathsf{T}' \to \mathsf{T}$ is the projection of the comma category T', is called a *degenerate* parametrized T-algebra. In the $\mathrm{Fam}\,\mathcal{C}$ description of a parametrized T-algebra the degeneracy manifests itself through the image containing only families of the form $(C)_{i \in I}$ (that is, a copy of the same object C of $\mathcal{C}$ for each $i \in I$). Notice however that the algebra of sorts of a parametrized T-algebra retains its structure even when the parametrized T-algebra is degenerate.

Conversely every parametrized T-algebra (in **Set**) yields, by Kan extension, an ordinary T-algebra.

Proposition 5.9 *Suppose* $F' : \mathsf{T}' \to \mathbf{Set}$ *is a parametrized* T*-algebra and let* $Q : \mathsf{T}' \to \mathsf{T}$ *be the projection of the comma category* T' *then the left Kan extension of* F' *along* Q *preserves finite limits.*

Proof. Since the comma category Q/B is a filtered category for each $B \in \mathsf{T}$, the colimits involved in computing the Kan extension [16, page 234] are filtered colimits. The proposition follows since filtered colimits commute with finite limits in **Set**.□

Furthermore, the left Kan extension is easy to compute.

Proposition 5.10 *Suppose $F' : \mathsf{T}' \to \mathbf{Set}$ is a parametrized T-algebra with algebra of sorts G and $Q : \mathsf{T}' \to \mathsf{T}$ is the projection of the comma category $e\ell\, G = \mathsf{T}'$ onto T. If $B \in \mathsf{T}$ then*

$$(\mathrm{Lan}_Q F')B = \sum_{B' \in GB} F'B'$$

Proof. In Q/B the full subcategory determined by the objects $QB' \xrightarrow{\mathrm{Id}} B$ is final and discrete so the colimit may be computed over this subcategory and becomes a coproduct.□

The preceding two results provide another explanation of the Moore construction of a category of paths in a topological space. In **Top** the collection of objects I_n, $n \in \omega$, forms a co-parametrized category (an $e\ell\, G$-coalgebra where G is the category ω described in Example 5.3). Thus, classically, homming out of the I_n yields a parametrized category (an $e\ell\, G$-algebra in **Set**). The parametrized category in turn yields an ordinary category—the Moore category—via Proposition 5.10. Indeed the coproduct which is computed there amounts to 'forgetting' the distinction between the different sorts which in this case means collecting all the paths together irrespective of the length of their domains $[0, n]$.

5.4 Further Examples

Example 5.11 The first example is the description of a parametrized monoid (in **Set**) showing its structure both in Fam **Set** and as an algebra for a theory of the form $e\ell\, G$. This example plays no direct part in the analysis of the Moore and related constructions, but because monoids are well understood it may illuminate some of the preceding definitions.

A parametrized monoid is an object in Fam **Set**, say $(M_i)_{i\in I}$, together with a multiplication $m : (M_i)_{i\in I} \times (M_i)_{i\in I} \to (M_i)_{i\in I}$ which is associative and has a left and right identity given by $e : (*) \to (M_i)_{i\in I}$ (where $(*)$ is a terminal object in Fam **Set**, say the singleton family containing a one point set $*$). Of course this is strictly only a *presentation* of a monoid in Fam **Set**—the whole structure is given by a finite limit preserving functor from the theory T of monoids into Fam **Set**, but the relationships between such a presentation and a T-algebra are well understood and will not be entered into here.

In terms of sets a parametrized monoid is a collection of sets M_i, one for each $i \in I$, together with a number of multiplications. The multiplication given in Fam **Set**

$$m : (M_i)_{i\in I} \times (M_i)_{i\in I} \cong (M_i \times M_j)_{(i,j)\in I\times I} \to (M_i)_{i\in I}$$

gives for each $(i,j) \in I \times I$ an index $\phi(i,j)$ and a function $m_{i,j} : M_i \times M_j \to M_{\phi(i,j)}$. Associativity amounts to asking that

$$\phi(\phi(i,j),k) = \phi(i,\phi(j,k)) \tag{5.1}$$

and that

$$m_{\phi(i,j),k}(m_{i,j} \times 1) = m_{i,\phi(j,k)}(1 \times m_{j,k}) : M_i \times M_j \times M_k \to M_{\phi(\phi(i,j),k)}.$$

The identity amounts to choosing a $\psi(*) \in I$ and an element of $M_{\psi(*)}$ which acts as a left and right identity in any multiplication $m_{\psi(*),j}$ or $m_{j,\psi(*)}$, $j \in I$, in which it is involved. (We are allowing $*$ to ambiguously denote either the one point set or its single element.)

The sets M_i, $i \in I$, are called *sorts* and the parametrized monoid is sometimes called a *multi-sorted monoid*—it is a monoid in the sense that any two elements can be multiplied together (associatively) and there exists left and right identities, but the sorts are a partitioning of the elements into sets which behave similarly in that the product of any two elements from the sets M_i and M_j is an element of the set $M_{\phi(i,j)}$.

Notice that the sorts themselves have a monoid structure (the *monoid of sorts* of the parametrized monoid). Any two sorts may be multiplied with the product of M_i and M_j given by $M_{\phi(i,j)}$. This product is associative by (5.1) and $M_{\psi(*)}$ acts as a left and right identity. The reader may wish to convince himself that if this monoid of sorts is thought of as a monoid $G : \mathsf{T} \to \mathbf{Set}$ then the category $e\ell\, G$ has all the structure required of a multi-sorted monoid and that the collection M_i is an $e\ell\, G$-algebra in **Set**.

Finally Proposition 5.10 says that the left Kan extension of this $e\ell\, G$-algebra along the projection $e\ell\, G \to \mathsf{T}$ (which by Proposition 5.9 gives a monoid) is the monoid obtained by forming the coproduct of the M_i—forgetting the partitioning and viewing the whole collection of elements as a single ordinary monoid.

Dually a co-parametrized monoid is a monoid in Fam $\mathbf{Set}^{\mathrm{op}}$ which amounts to a multi-sorted comonoid—a collection of sets M_i together with, for each $(i,j) \in I \times I$, a *co*multiplication $m_{i,j} : M_{\phi(i,j)} \to M_i + M_j$ which is coassociative etc.

Example 5.12 Write $[k] = \{0, 1, \ldots k-1\}$ and hence $[0] = \emptyset$ (note that this is a different definition of $[k]$ to that used in earlier chapters). In **Set** the collection $[k]$ for $k = 0, 1, \ldots$ is a co-parametrized monoid. To verify this we show that $([k])_{k\in\omega}$ is a monoid in Fam $\mathbf{Set}^{\mathrm{op}}$.

The multiplication $m : ([k])_{k\in\omega} \times ([k])_{k\in\omega} \cong ([j]+[k])_{(j,k)\in\omega\times\omega} \to ([k])_{k\in\omega}$ in Fam $\mathbf{Set}^{\mathrm{op}}$ is given by a function $\phi : \omega \times \omega \to \omega$ which sends $(j,k) \mapsto j+k$ and functions $m_{j,k} : [\phi(j,k)] \to [j] + [k]$ in **Set**. Notice that $[j+k]$ is canonically isomorphic to $[j] + [k]$ and choose $m_{j,k}$ to be this isomorphism. Finally let $e : (*) \to ([k])_{k\in\omega}$ be given by (ψ, e_*) where ψ picks out the index 0 and e_* is the opposite of the unique map $[0] = \emptyset \to *$. The verification that $(\phi, (m_{j,k}))$ and (ψ, e_*) satisfy the associative and identity laws is routine.

Since the collection $[k]$ is a co-parametrized monoid we know that homming out of it into some set X say yields a parametrized monoid. Furthermore Proposition 5.10 says that forgetting the distinctions between the sorts gives an ordinary monoid. The ordinary monoid has as elements functions from $\{0, 1, \ldots k-1\}$ into X which may be thought of as words of length k in elements of X. The identity for this ordinary monoid is the empty word, and the multiplication is given by concatenation of words.

Thus homming out of this co-parametrized monoid into a set X is a construction of *the free monoid on* X. In a sense the collection $[k]$ is a *representing collection of objects* for the free monoid construction.

Example 5.13 Let L be the category with two distinct objects and two parallel non-identity arrows, then $\mathbf{Set}^L$ is the category **Graph** of graphs and graph morphisms. In **Graph** the graphs

$$[k] = 0 \to 1 \to \cdots \to k$$

for $k = 0, 1, \ldots$ form a co-parametrized category. Homming out of the $[k]$ into some graph G gives a parametrized category which becomes a category if we forget the distinctions between the sorts. The category so obtained is the free category on the graph G.

Example 5.14 In the category of simplicial sets the well-formed simplicial sets form a co-parametrized-ω-category. To see this let (W_i) be the family of well-formed simplicial sets which is an object in Fam $[\Delta^{\mathrm{op}}, \mathbf{Set}]^{\mathrm{op}}$. Define $s_n, t_n : (W_i) \to (W_i)$ by $s_n = (\phi_n, s_{n,i})$ and $t_n = (\phi'_n, t_{n,i})$ where $\phi_n(i)$ is the index of the n-dimensional source of W_i (Chapter 3) and $s_{n,i}$ is the opposite of the inclusion of the n-dimensional source of W_i into W_i in $[\Delta^{\mathrm{op}}, \mathbf{Set}]$ (and similarly for ϕ'_n and $t_{n,i}$).

The $*_n$-composable arrows are given by the pullback of s_n along t_n which may be chosen to be a differently indexed family of well formed simplicial sets since the pushout of the inclusion of an n-dimensional source of a well formed simplicial set along the inclusion of an n-dimensional target of another well formed simplicial set is again a well formed simplicial set (by Theorem 2.12). As before we may define the composition $*_n$ to be the identity on each of its components. The verification that s_n, t_n and $*_n$ satisfy the axioms of an ω-category is routine.

Thus homming out of (W_i) in the category of simplicial sets, or equivalently homming out of the realizations of the (W_i) in **Top**, yields an ω-category. This provides a solution to the problem of generalizing the Moore construction to an ω-category of 'higher dimensional paths' in a topological space.

Prospectus

This thesis explores a number of closely related problems—the pasting theorem, Street's orientals, the higher coherence conditions, and the Moore n-category of a space. As such, its scope is rather narrow—questions and research possibilities which became apparent during the work have not been addressed because they are not an integral part of "the story". This Prospectus records some of the related work which is being carried out as well as some suggestions for future research.

Theory of Pasting Schemes

The notion of pasting schemes presented here was motivated by Street's consideration in [20] of what amounts to the particular pasting scheme $(S, \mathsf{E}, \mathsf{B})$ of Chapter 3, and by the need to specify composable diagrams for the treatment of the pasting theorem. Street went on to work with Aitchison on the cubical analogue of $(S, \mathsf{E}, \mathsf{B})$. They obtained a clear description of the cubical complex and provided notation which, together with Theorem 3.7 led the author to a proof that a certain pasting scheme structure on the ω-cube is loop free, and hence to a construction of the "free n-category on the n-cube".

The development of the general theory has continued. Street and the author have obtained a construction of the product of two loop-free pasting schemes and they hope to continue the development in joint work. Currently, Street is working on improving the axioms for a loop-free pasting scheme.

Concurrently Eilenberg-Street have been working on explicit constructions of free n-categories. It is the author's hope that better developed theories of pasting schemes and parametrized n-categories will have an influence on this work.

Coherence

As noted in Chapter 4 the framework for the analysis of coherence questions presented there is not sufficiently general. The problem seems to be that pasting schemes are "too free"—equations like (4.9) may be clear enough in particular realizations but they can never hold in pasting schemes. This suggests that the theory should be developed in 'quotients' of pasting schemes where the independence of a particular realization remains but extra equations which hold in all intended realizations can be introduced. In addition the use of Eilenberg-Kelly generalized natural transformations may be more appropriate than the ordinary natural transformations used in Chapter 4.

One of the intended applications of the results of Chapter 4 is the definition of a pseudo-n-category. A bicategory may be thought of as a pseudo-2-category because some of the axioms of a 2-category are only satisfied up to coherent isomorphisms. The idea of extending this to 3-categories and higher is quite old but the requisite coherence conditions have not been available. The higher coherence conditions proved in Chapter 4 solve this problem for associativities and identities and so the definition may now be within reach.

Categories of Paths

The Moore ω-category of paths which was defined in Chapter 5 was constructed for use in algebraic topology. Walters has conjectured that the homotopy groups of a space may be obtained from its path ω-category and that generalized homotopy theory should be carried out in a topos containing a co-parametrized-ω-category. These ideas are as yet little developed but Walters and the author hope to continue the work.

The general theory of parametrized theories is only beginning. As noted in the introduction, algebras for parametrized theories include parametrized notions such as indexed categories and fibrations but whether there is anything to be gained from this observation has not yet been explored.

Finally, current joint work of Carboni and the author has shed light on the role played by $(\mathrm{Fam}\,\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ in Chapter 5. Our present goal is to obtain explicit criteria for the existence of a co-parametrized-T-algebra which represents (in $(\mathrm{Fam}\,\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$) the construction of free T-algebras in $\mathcal{C}$.

Bibliography

[1] Aitchison, I., The geometry of oriented cubes, unpublished manuscript.

[2] Barr, M. and Wells, C. *Toposes Triples and Theories,* Springer, 1985.

[3] Benabou, J., Introduction to bicategories, *Lecture Notes in Math.* 47 (1967) 1–77.

[4] Eilenberg, S. and Street, R.H., manuscript in preparation.

[5] Gabriel, P. and Zisman, M. *Calculus of fractions and homotopy theory,* Springer, 1967.

[6] Hilton, P.J., *Homotopy Theory and Duality,* Gordon & Breach, 1965.

[7] Kelly, G.M., On Mac Lane's conditions for coherence of natural associativities, commutativities, etc., *J. Algebra* 1 (1964) 397–402.

[8] Kelly, G.M., Many variable functorial calculus I, *Lecture Notes in Math.* 281 (1972) 66–105.

[9] Kelly, G.M., An abstract approach to coherence, *Lecture Notes in Math.* 281 (1972) 106–147.

[10] Kelly, G.M., Coherence theorems for lax algebras and for distributive laws, *Lecture Notes in Math.* 420 (1974) 281–375.

[11] Kelly, G.M., *Basic concepts of enriched category theory,* London Math. Soc. Lecture Notes 64 (1982).

[12] Kelly, G.M. and Mac Lane, S., Coherence in closed categories, *J. Pure and Applied Algebra* 1 (1971) 97–140.

[13] Kelly, G.M. and Street, R.H., Review of the elements of 2-categories, *Lecture Notes in Math.* 420 (1974) 75–103.

[14] Laplaza, M.L., Coherence for associativity not an isomorphism, *J. Pure and Applied Algebra* 2 (1972) 107–120.

[15] Mac Lane, S., Natural associativity and commutativity, *Rice University Studies* 49 (1963) 28–46.

[16] Mac Lane, S., *Categories for the working mathematician,* Springer, 1972.

[17] May, J.P., *Simplicial objects in algebraic topology,* Van Nostrand, 1968.

[18] Roberts, J.E., Mathematical aspects of local cohomology, *Proc. of the Colloquium on Operator Algebras and their Application to Mathematical Physics,* Marseille, 1977.

[19] Street, R.H., Limits indexed by category-valued 2-functors, *J. Pure and Applied Algebra* 8 (1976) 149–181.

[20] Street, R.H., The algebra of oriented simplexes, *J. Pure and Applied Algebra,* to appear (page and theorem numbers are references to a preprint).

[21] Street, R.H. and Walters, R.F.C., Yoneda structures on 2-categories *J. Algebra* 50 (1978) 350–379.

[22] Walters, R.F.C., A categorical approach to universal algebra, Doctoral Thesis, Australian National University, 1970.

[23] Walters, R.F.C., Lecture to the Sydney Category Seminar, December, 1971.