The Eckmann-Hilton argument, higher operads and $E_n$-spaces.

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Abstract

The classical Eckmann-Hilton argument shows that two monoid structures on a set, such that one is a homomorphism for the other, coincide and, moreover, the resulting monoid is commutative. This argument immediately gives a proof of the commutativity of the higher homotopy groups. A reformulation of this argument in the language of higher categories is: suppose we have a one object, one arrow 2-category, then its $\text{Hom}$-set is a commutative monoid. A similar argument due to A. Joyal and R. Street shows that a one object, one arrow tricategory is ‘the same’ as a braided monoidal category.

In this paper we extend this argument to arbitrary dimension. We demonstrate that for an $n$-operad $A$ in the author’s sense there exists a symmetric operad $S^n(A)$ called the $n$-fold suspension of $A$ such that the category of one object, one arrow, . . . , one $(n - 1)$-arrow algebras of $A$ is equivalent to the category of algebras of $S^n(A)$. Moreover, under some mild conditions, we present an explicit formula for $S^n(A)$ which involves taking the colimit over a remarkable categorical $E_n$-operad. In the case, where $A$ is contractible in an appropriate sense, this formula provides us with an action of the $E_n$-operad on algebras of $S^n(A)$.

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1 How can symmetry emerge from nonsymmetry?

Hopf and Alexandrov pointed out to Čech that his higher homotopy groups were commutative. The proof follows from the following statement which is known since $\cite{13}$ as the Eckmann-Hilton argument: two monoid structures on a set such

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that one is a homomorphism for the other coincide and, moreover, the resulting monoid is commutative. A reformulation of this argument in the language of higher categories is: suppose we have a one object, one arrow 2-category, then its \( \text{Hom} \)-set is a commutative monoid. A higher dimensional generalization of this argument was provided by Joyal and Street in [19]. Essentially they proved that a 1-object, 1-arrow tricategory is a braided monoidal category and a one object, one arrow, one 2-arrow tetracategory is a symmetric monoidal category.

Obviously we have here a pattern of some general higher categorical principle. Almost nothing, however, is known precisely except for the above low dimensional examples and some higher dimensional cases which can be reduced to the classical Eckmann-Hilton argument [3]. Yet, there are plenty of important conjectures which can be seen as different manifestations of this principle. First of all there is a bunch of hypotheses from Baez and Dolan [2] about the so called ‘\( k \)-tuply monoidal’ \( n \)-categories, which are \((n + k)\)-categories with one object, one arrow etc. up to \((k - 1)\). Basically these hypotheses state that these ‘\( k \)-tuply monoidal’ \( n \)-categories are \( n \)-categorical analogues of \( k \)-fold loop spaces i.e. \( n \)-categories equipped with an additional monoidal structure together with some sort of higher symmetry structures similar to the structure of a \( k \)-fold loop space. In particular, ‘\( k \)-tuply monoidal’ weak \( \omega \)-groupoids should model \( k \)-fold loop spaces. Many other hypothesis from [2] are based on this analogy.

Another problem, which involves the passage to the \( k \)-tuply monoidal \( n \)-categories, is the definition of higher centers [2, 11, 31]. Closely related to this problem is the Deligne’s conjecture from deformation theory [22, 23] which tells us that there is an action of an \( E_2 \)-operad on the Hochschild complex of an associative algebra. This conjecture is now proved by several people. In higher dimensions the generalised Deligne conjecture has been understood by Kontsevich [22] as a problem of existence of some sort of homotopy center of any \( d \)-algebra. This homotopy center must have a structure of \((d + 1)\)-algebra. Here a \( d \)-algebra is an algebra of the little \( d \)-cubes operad [35]. To the best of our knowledge this hypothesis is not proved yet in this generality, but there is progress on it [32].

In this paper we consider a generalisation of the Eckman-Hilton argument in higher dimensions using the apparatus of higher-dimensional nonsymmetric operads [5]. They were introduced in [5] for the purpose of defining weak \( n \)-categories for higher \( n \). A weak \( n \)-category in our sense is an algebra of a contractible \( n \)-operad.

Now consider the algebras of an \( n \)-operad \( A \) which have only one object, one arrow, ... , one \((k - 1)\)-arrow. The underlying \( n \)-globular object of such an algebra can be identified with an \((n - k)\)-globular object and we can ask ourself what sort of algebraic structure the action of \( A \) induces on this \((n - k)\)-globular object. Here we restricts ourself by considering only \( k = n \). This provides a great simplification of the theory, yet clearly shows how higher symmetries can appear. We must say that we do not know the answer for arbitrary \( k \). For this, perhaps, we need to develop the theory of symmetric higher operads, and some steps in this directions have already been taken [35].

Returning to the case \( k = n \) we show that for an \( n \)-operad \( A \) one can con-
struct a symmetric operad \( S^n(A) \) (which in this case is just a classical symmetric operad in the sense of May \[26\] in a symmetric monoidal category), called \( n \)-fold suspension of \( A \), such that the category of one object, one arrow, \( \ldots \), one \((n-1)\)-arrow algebras of \( A \) is equivalent to the category of algebras of \( S^n(A) \). Moreover, under mild conditions we present an explicit formula for \( S^n(A) \) involving the colimit over a remarkable categorical \( E_n \)-operad. In the case where \( A \) is contractible in an appropriate sense, this formula provides us with an action of an \( E_n \)-operad on algebras of \( S^n(A) \).

Fortunately, the restriction \( n = k \) not only simplifies our techniques, but also makes almost unnecessary the use of variable category theory from \[28, 30\] which our paper \[5\] used. We can reformulate our theory of higher operads in a way that makes it very similar to the theory of classical symmetric operads. So the reader who does not need to understand the full structure of a higher operad may read the present paper without looking at \[5, 29, 30, 8\]. In several places we do refer to some constructions from \[5, 8\] but these references are not essential for understanding the main results.

We now provide a brief description of each section.

In section 2 we introduce the notion of the \( n \)-fold suspension of an \( n \)-operad. This is the only section where we seriously refer to the notion of monoidal globular category from \[5\]. Nevertheless, we hope that the main notion of suspension will be clear even without understanding all the details of the definition of \( n \)-operad in a general monoidal globular category because the proposition \[2.1\] shows that the problem of finding a \( k \)-fold suspension of an \( n \)-operad \( A \) can be reduced to the case when \( A \) is of a special form, which we call \((k-1)\)-terminal. The latter is roughly speaking an operad which has strict \((k-1)\)-categories as algebras of its \((k-1)\)-skeleton. The reader, therefore, can start to read our paper from section 3.

Section 3 and 4 contain most of the combinatorics we need in this paper. We recall the definition of the \( \omega \)-category of trees and of the category \( \Omega_n \) \[3, 5, 20\] which is an \( n \)-dimensional analogue of simplicial \( \Delta \) and plays an important role here. More generally, we believe that the categories \( \Omega_n \) must be one of the central objects of studies in higher dimensional category theory, at least on the combinatorial side of the theory. It appears that \( \Omega_n \) contains all the information on the coherence laws available in weak \( n \)-categories.

In section 4 we also relate \( \Omega_n \) with the combinatorics of symmetric groups. The lemma \[4.1\] will be one of the principal tools in our study of \( n \)-operads.

In section 5 we give a definition of \( n \)-operad in a symmetric monoidal category \( V \), which is just an \( n \)-operad in the monoidal globular category \( \Sigma^n V \). This definition is much simpler than the definition of general \( n \)-operad and is reminiscent of the classical definition of nonsymmetric operad.

Section 6 is devoted to the construction of a functor \( Des_n \) from symmetric operads to \( n \)-operads which incorporates the action of symmetric groups. We also show here that each endomorphism operad is invariant with respect to this functor. Here we again refer to our paper \[5\] for a construction of the endomorphism \( n \)-operad. However, the reader, can accept our construction here as a definition of endomorphism \( n \)-operad, so again does not need to understand
the technical construction from \([5]\). Our main activity for the rest of the paper will be an actual construction of the left adjoint to \(\text{Des}_n\).

In section 7 we consider \(n\)-operads in \(\text{Cat}\). These operads have one extra dimension which allows us to speak about internal \(n\)-operads in a categorical \(n\)-operad and in a symmetric categorical operad, in analogy, for example, with monoids in a monoidal category. So we have just another manifestation of the *microcosm principle* so named in \([1]\).

In section 8 we consider \(n\)-operads in \(\text{Cat}\). These operads have one extra dimension which allows us to speak about internal \(n\)-operads in a categorical \(n\)-operad and in a symmetric categorical operad, in analogy, for example, with monoids in a monoidal category. So we have just another manifestation of the *microcosm principle* so named in \([1]\).

In section 8 we construct a categorical operad \(h^n\) as the operad which represents the functor which assigns to a categorical symmetric operad the category of internal \(n\)-operads in it. The operads \(h^n\) generalise to arbitrary dimensions the operad \(h^1\) introduced earlier in \([6]\). By definition, \(h^n\) is the symmetric \(\text{Cat}\)-operad freely generated by an internal \(n\)-operad.

In section 9 we relate the operad \(h^n\) with the categorical operads \(M^n\) constructed in \([3]\). The operad \(M^n\) represents the theory of categories with \(n\) strict monoidal structures which are related by possibly noninvertible interchange laws. It was proved in \([3]\) that \(M^n\) has the homotopy type of the little \(n\)-cube operad and, therefore, its algebras model \(n\)-fold loop spaces (up to group completion, as usual). We prove here, that \(M^n\) contains an internal \(n\)-operad and, hence, we have a canonical operad morphism

\[ \kappa : h^n \to M^n. \]

The main result of this section is theorem 9.2 which states that \(\kappa\) is a weak equivalence.

Finally, in section 10 we provide our Eckmann-Hilton formula for the suspension of an \((n-1)\)-terminal \(n\)-operad \(A\) in a symmetric closed monoidal category \(V\). The formula is

\[ S^n(A)_k \simeq \colim_{\tilde{h}^n_k} \tilde{A}_k \]

where \(\tilde{A}\) is an operadic functor on \(h^n\) which appears from the universal property of \(h^n\).

We also consider several cases when we can actually calculate this suspension or at least its homotopy type. For example, if \(A\) is contractible, then \(S^nA\) is often an \(E_n\)-operad.

We do not examine here what general conditions ensure that \(S^nA\) is an \(E_n\)-operad. This interesting question will be the topic of a further paper.

Perhaps, a truly useful result would be a version of the Eckmann-Hilton formula up to all higher homotopies. It is not so hard now to work it out using the methods developed in the present paper. This also will be a topic for another paper.

Deeper applications of our results will include:
- a construction of a basis of higher dimensional homological algebra, and in particular as a first test for it, finding a conceptual and short proof of Deligne’s hypothesis in all dimensions;
- coherence theorems for weak \(n\)-categories;
- the problem of modelling of \(n\)-homotopy types;
- establishing of relations to the other definitions of weak $n$-categories, for example, to the definition of P.May which also makes use of operadic methods.

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2 $n$-fold suspension of higher operads

We introduce here a general notion of $n$-fold suspension of an $n$-operad in an augmented monoidal $n$-globular category.

Let $M$ be an augmented monoidal $n$-globular category. Let $I$ be the unit object of $M_0$. Fix an integer $k > 0$. Then we can construct the following augmented monoidal $n$-globular category $M^{(k)}$. The category $M^{(k)}_l$ is the terminal category when $l < k$. If $l \geq k$ then $M^{(k)}_l$ is the full subcategory of $M_l$ consisting of objects $x$ with $s_{k-1}x = t_{k-1}x = z^{k-1}I$.

There is an obvious inclusion $j : M^{(k)} \to M$.

We also can form an augmented monoidal $(n-k)$-globular category $\Sigma^{-k}M^{(k)}$ with $(\Sigma^{-k}M^{(k)})_l = M^{(k)}_{l+k}$ and obvious augmented monoidal structure.

Recall that a globular object of $M$ is a globular functor from the terminal $n$-globular category $1$ to $M$. We will call a globular object $x : 1 \to M$ $(k-1)$-terminal if $x$ can be factorised through $j$. Analogously, a morphism between two $(k-1)$-terminal globular objects is a natural transformation which can be factorised through $j$.

Let us denote by $gl_n(M)$ and $gl_n^{(k)}(M)$ the categories of globular objects in $M$ and $(k-1)$-terminal globular objects in $M$ correspondingly. Then we have isomorphisms of categories $gl_n^{(k)}(M) \simeq gl_n(M^{(k)}) \simeq gl_{n-k}(\Sigma^{-k}M^{(k)})$. 

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In the same way we can define \((k - 1)\)-terminal collections in \(M\) and \((k - 1)\)-terminal \(n\)-operads in \(M\). Again the category of \((k - 1)\)-terminal \(n\)-operads in \(M\) is isomorphic to the category of \(n\)-operads in \(M^{(k)}\) but is different from the category of \((n - k)\)-operads in \(\Sigma^{-k}M^{(k)}\).

Suppose now \(A\) is an \(n\)-operad in \(M\) and colimits in \(M\) commute with the augmented monoidal structure \([5]\). Then \(A\) generates a monad \(A\) on the category of \((k - 1)\)-terminal \(n\)-operads in \(M\). The algebras of \(A\) are, by definition, the algebras of the monad \(A\).

More generally, let \(A\) be an arbitrary monad on \(gl_n(M)\). An algebra \(x\) of \(A\) is called \((k - 1)\)-terminal provided its underlying globular object is \((k - 1)\)-terminal. A morphism of \((k - 1)\)-terminal algebras is a morphism of underlying \((k - 1)\)-terminal objects which is also a morphism of \(A\)-algebras.

Now let \(Alg^{(k)}(\mathcal{A})\) be the category of \((k - 1)\)-terminal algebras of \(A\). We have a forgetful functor

\[
U^{(k)} : Alg^{(k)}(\mathcal{A}) \to gl_n(M^{(k)}) \cong gl_{n-k}(\Sigma^{-k}M^{(k)}).
\]

**Definition 2.1** If \(U^{(k)}\) is monadic then we call the corresponding monad on \(gl_{n-k}(\Sigma^{-k}M^{(k)})\) a \(k\)-fold suspension of \(A\).

In the special case \(X = Span(Set)\) this definition was given by M.Weber in his PhD thesis [33]. He also proved that in this case the suspension exists for a large class of monads on globular sets. Observe that \(gl_{\infty}(Span(Set)^{(k)})\) is equivalent to the category of globular sets again.

Suppose now that \(A\) is obtained from an \(n\)-operad \(A\) in \(M\). Even if a \(k\)-fold suspension of \(A\) exists it is often not true that the suspension comes from an operad in \(\Sigma^{-k}M^{(k)}\). To handle this situation we need a more general notion of operad which is not available now. M.Weber has a notion of symmetric globular operad in a special case \(M = Span(Set)\) which seems to be a good candidate in this situation [33].

However, there is one case when such a notion already exists. Indeed, if \(k = n\) the globular category \(M^{(n)}\) has only one nontrivial category \(M^{(n)}_n = \Sigma^{-n}M^{(n)} = V\). This category has to be braided monoidal if \(n = 1\) and symmetric monoidal if \(n > 1\). An \(n\)-fold suspension of a monad \(A\) on \(gl_n(M)\) generates, therefore, a monad on \(V\). It is now natural to ask if this monad comes from a symmetric (nonsymmetric if \(n = 1\)) operad in \(V\).

Recall that a symmetric operad in \(V\) is a collection \(A_n, n \ge 0\) of objects of \(V\) equipped with an action of the symmetric group \(\Sigma_n\) on \(A_n\), a map

\[
m : A_k \otimes A_{n_1} \otimes \ldots \otimes A_{n_k} \to A_n
\]

for each subdivision \(n = n_1 + \ldots + n_k\), and also a map

\[
e : I \to A_1
\]

satisfying associativity, unit and equivariancy conditions [26]. If we forget about the symmetric groups we get a definition of nonsymmetric operad.
Definition 2.2 Let $A$ be an $n$-operad in $M$ such that the $n$-fold suspension of $A$ exists and comes from a symmetric (nonsymmetric if $n = 1$) operad $S^n(A)$ on $V$. Then we call $S^n(A)$ the $n$-fold suspension of $A$.

Remark 2.1 It may happen that $n = 1$ but $V$ is still a symmetric monoidal category. In this case it makes sense to speak about the symmetric 1-fold suspension of $A$. We will see that this symmetric 1-fold suspension is a symmetrisation of the nonsymmetric operad $S^1(A)$. So these two notions agree and we will use the notation $S^1(A)$ for symmetric suspension as well if it does not lead to confusion.

Now we will show that the problem of finding an $n$-fold suspension of an operad in $M^n$ can often be reduced to the problem of finding a suspension of the $(n - 1)$-terminal operad.

Let $t : gl_n(M^{(n)}) \to gl_n(M)$ be the natural inclusion functor. Let $\tau$ be the other obvious inclusion

$$\tau : O_n(M^{(n)}) \to O_n(M^n)$$

where $O_n(C)$ means the category of $n$-operads in $C$. Let $x$ be a globular object of $M^{(n)}$. And suppose there exist an endooperads $End(x)$ and $End(t(x))$ in $M^{(n)}$ and $M$ correspondingly [5]. Then it is not hard to check that $\tau(End(x)) \simeq End(t(x))$.

If now $x$ is an algebra of some $n$-operad $A$ in $M$ then we have an operadic morphism

$$k : A \to End(t(x)) \simeq \tau(End(x))$$

Suppose in addition that $\tau$ has a left adjoint $\lambda$. This is true in the most interesting cases. Then we have that $k$ is uniquely determined and determines an operadic map

$$k' : \lambda(A) \to End(x).$$

Thus we have

**Proposition 2.1** The category of $(n - 1)$-terminal algebras of an $n$-operad $A$ in $M$ is isomorphic to the category of algebras of the $n$-operad $\lambda(A)$ in $M^{(n)}$.

In general, if $V$ is a symmetric monoidal category, we can form the augmented monoidal $n$-globular category $L = \Sigma^n V$ where $L$ has $V$ in dimension $n$ and terminal categories in other dimensions. The monoidal structure is given by $\otimes_i = \otimes$ where $\otimes$ is tensor product in $V$. For example, $M^{(n)} = \Sigma^n (M_n^{(n)})$. So in the rest of the paper we will study the case $M = \Sigma^n V$. We will show that the most interesting phenomena appear already in this situation. The passage from $A$ to $\lambda(A)$ will be studied elsewhere by a method similar to the method developed in this paper.
3 Trees and their morphisms.

Let us introduce some notation. For a natural number \( n \) we will denote by \([n]\) the ordinal
\[
1 < 2 < \ldots < n.
\]
In particular \([0]\) will denote the empty ordinal. A morphism from \([n] \to [k]\) is any map between underlying sets. It can be order preserving or not. It is clear, that we then have a category. We denote this category by \( \Omega^n \). Of course, \( \Omega^n \) is equivalent to the category of finite sets. In particular, the symmetric group \( \Sigma_n \) is the group of automorphisms of \([n]\).

**Definition 3.1** A tree of height \( n \) (or simply \( n \)-tree) is a chain of order preserving maps of ordinals
\[
T = [k_n] \xrightarrow{\rho_{n-1}} [k_{n-1}] \xrightarrow{\rho_{n-2}} \ldots \xrightarrow{\rho_0} [1]
\]
If \( i \in [k_m] \) and there is no \( j \in [k_{m+1}] \) such that \( \rho_m(j) = i \) then we call \( i \) a leaf of \( T \) of height \( i \). We will call the leaves of \( T \) of height \( n \) the tips of \( T \). If for an \( n \)-tree \( T \) all its leaves are tips we call such a tree pruned.

We illustrate the definition in a picture

The tree on the right side of the picture has the empty ordinal at the highest level. We will call such trees degenerate. There is actually an operation on trees which we denote by \( z(\cdot) \) which assigns to the \( n \)-tree \([k_n] \to [k_{n-1}] \to \ldots \to [1]\) the \((n+1)\)-tree
\[
[0] \to [k_n] \to [k_{n-1}] \to \ldots \to [1].
\]
Two other operations on trees are truncating
\[
\partial([k_n] \to [k_{n-1}] \to \ldots \to [1]) = [k_{n-1}] \to \ldots \to [1]
\]
and suspension
\[
S([k_n] \to [k_{n-1}] \to \ldots \to [1]) = [k_n] \to [k_{n-1}] \to \ldots \to [1] \to [1].
\]

**Definition 3.2** A tree \( T \) is called a \( k \)-fold suspension if it can be obtained from another tree by applying the operation of suspension \( k \)-times. The suspension index \( \text{sus}p(T) \) is the maximum integer \( k \) such that \( T \) is a \( k \)-fold suspension.
The only $n$-tree with suspension index equal to $n$ is the linear tree

$$U_n = [1] \rightarrow \ldots \rightarrow [1].$$

We now define source and target of a tree $T$ to be equal $\partial(T)$. So we have a globular structure on the set of all trees. We actually have more. The trees form an $\omega$-category $Tr$ with the set of $n$-cells being equal to the set of the trees of height $n$. If two $n$-trees $S$ and $T$ have the same $k$-sources and $k$-targets (i.e. $\partial^{n-k}T = \partial^{n-k}S$) then they can be composed, and the composite will be denoted by $S \otimes_k T$. Then $z(T)$ is the operation of taking identity of the $n$-cell $T$. Here is an example of the $2$-categorical operations on trees

\[\begin{array}{ccc}
\begin{tikzpicture}[scale=0.5]
\node (A) at (0,0) {$*$};
\node (B) at (1,1) {$*$};
\node (C) at (2,2) {$*$};
\node (D) at (3,3) {$*$};
\node (E) at (4,4) {$*$};
\draw (A) -- (B) -- (C) -- (D) -- (E);
\end{tikzpicture}
\end{array} = \begin{array}{ccc}
\begin{tikzpicture}[scale=0.5]
\node (A) at (0,0) {$*$};
\node (B) at (1,1) {$*$};
\node (C) at (2,2) {$*$};
\node (D) at (3,3) {$*$};
\node (E) at (4,4) {$*$};
\draw (A) -- (B) -- (C) -- (D) -- (E);
\end{tikzpicture}
\end{array}\]

horizontal composition \hspace{1cm} vertical composition

\[\begin{array}{ccc}
\begin{tikzpicture}[scale=0.5]
\node (A) at (0,0) {$*$};
\node (B) at (1,1) {$*$};
\node (C) at (2,2) {$*$};
\node (D) at (3,3) {$*$};
\node (E) at (4,4) {$*$};
\draw (A) -- (B) -- (C) -- (D) -- (E);
\end{tikzpicture}
\end{array} = \begin{array}{ccc}
\begin{tikzpicture}[scale=0.5]
\node (A) at (0,0) {$*$};
\node (B) at (1,1) {$*$};
\node (C) at (2,2) {$*$};
\node (D) at (3,3) {$*$};
\node (E) at (4,4) {$*$};
\draw (A) -- (B) -- (C) -- (D) -- (E);
\end{tikzpicture}
\end{array}\]

identity morphism

The $\omega$-category $Tr$ is actually the free $\omega$-category generated by the terminal globular set. Every $n$-tree can be considered as a special sort of $n$-pasting diagram called globular. This construction was called the $\ast$-construction in \textsuperscript{[5]}. Here are a couple of examples.

For a globular set $X$ one can form then the set $D(X)$ of all globular pasting diagrams labelled in $X$. This is a free $\omega$-category generated by $X$. In this way we have a monad $(D, \mu, \epsilon)$ on the category of globular sets, which plays a central role in our work \textsuperscript{[5]}. In particular, $D(1) = Tr$. We also can consider $D(Tr) = D^2(1)$. It was observed in \textsuperscript{[5]} that the $n$-cells of $D(Tr)$ can be identified with the morphisms of
another category introduced by A. Joyal in [20]. This category was called $\Omega_n$ and some properties of it were studied later in [8, 10]. More precisely, it was found that the collection of categories $\Omega_n$ forms an $\omega$-category in $\text{Cat}$ and, moreover, it is freely generated by an internal $\omega$-category. So it is a higher dimensional analogue of the simplicial category $\Delta$ (which is of coarse free monoidal category generated by a monoid [24]). The $\Omega_n$ will be of primary importance for us.

**Definition 3.3** The category $\Omega_n$ has as objects the trees of height $n$. The morphisms of $\Omega_n$ are commutative diagrams

\[
\begin{array}{c}
[k_n] \xrightarrow{\rho_{n-1}} [k_{n-1}] \xrightarrow{\rho_{n-2}} \cdots \xrightarrow{\rho_0} [1] \\
\downarrow \sigma_n \downarrow \sigma_{n-1} \downarrow \cdots \downarrow \sigma_0 \\
[s_n] \xrightarrow{\xi_{n-1}} [s_{n-1}] \xrightarrow{\xi_{n-2}} \cdots \xrightarrow{\xi_0} [1]
\end{array}
\]

such that for all $i$ and all $j \in [k_{i-1}]$ the restriction of $\sigma$ on $\rho_{i-1}^{-1}(j)$ preserves the natural order on it.

Let $T$ be an $n$-tree and let $i$ be a leaf of height $m$ of $T$. Then $i$ determines a unique morphism $\xi_i : z^{n-m}U_m \to T$ in $\Omega_n$ such that $\xi_m(1) = i$. We will often identify the leaf with this morphism.

Let $\sigma : T \to S$ be a morphism in $\Omega_n$ and let $i$ be a leaf of $T$. Then the fiber of $\sigma$ over $i$ is the following pullback in $\Omega_n$

\[
\begin{array}{ccc}
\sigma^{-1}(i) & \xrightarrow{} & z^{n-m}U_m \\
\downarrow & & \downarrow \xi_i \\
T & \xrightarrow{\sigma} & S
\end{array}
\]

which can be calculated as a levelwise pullback in $\text{Set}$.

Now, for a $\sigma : T \to S$ one can construct a labelling of the pasting scheme $S^*$ in $\omega$-category $Tr$ by associating to a vertex $i$ from $S$ the fiber of $\sigma$ over $i$. The result of the pasting operation will be exactly $T$. We will use extensively this correspondence between morphisms in $\Omega_n$ and pasting schemes in $Tr$.

Some trees will play a special role in our paper. We will denote by $M^j_i$ the tree

\[
\underbrace{U_n \otimes_1 \cdots \otimes_1 U_n}_{j\text{-times}}
\]

A picture for $M^j_i$ is the following
Let now $T$ be a tree with $\text{susp}(T) = l$. Then it is easy to see that we have a unique representation

$$T = T_1 \otimes_l \ldots \otimes_l T_j$$

where $\text{susp}(T_i) > l$. In [3] we called this representation canonical decomposition of $T$. We also will refer to the canonical decomposition when talking about the morphism

$$T \rightarrow M^j_l$$

it generates.

4 The combinatorics of the symmetric groups operad.

It is easy to see that there are exactly $k!$ morphisms from $M^k_0$ to $M^k_{n-1}$ in $\Omega_n$ which induce bijection in dimension $n$. A little bit more work required to prove that the morphisms from $M^k_0$ to $M^l_{n-1}$ in $\Omega_n$ which are surjections in the dimension $n$ are in one to one correspondence with the set of the $(k - l)$-dimensional faces of the permutohedron $P_k$. This example suggests that the morphisms of trees are closely related to permutation groups.

Now we describe how one can associate a permutation to each morphism in $\Omega_n$. It is not a functorial correspondence, but still satisfies an interesting relation with respect to composition. This relation will play a central role in the development of our theory.

Recall that the symmetric groups form a symmetric operad in $\text{Set}$. Let $\Gamma$ be its multiplication. Thinking of elements of $\Sigma_n$ as bijections from $[n]$ to $[n]$ we can demonstrate this multiplication using the picture:

\[ \Gamma((132),(12,1,1)) \]
Now, every morphism $\sigma : [n] \to [k]$ in $\Omega^s$ has a unique factorisation

$$\sigma = \pi(\sigma) \cdot \nu(\sigma)$$

where $\nu(\sigma)$ is order preserving, while $\pi(\sigma)$ is bijective and preserves order on the fibers of $\sigma_n$.

The following lemma describes the behaviour of this factorisation with respect to composition.

**Lemma 4.1** Let

$$[n] \xrightarrow{\sigma} [l] \xrightarrow{\omega} [k]$$

be a composite of morphisms of ordinals. Then

$$\Gamma(1_k; \pi(\sigma_1), ..., \pi(\sigma_k))\pi(\sigma \cdot \omega) = \Gamma(\pi(\omega); 1|_{\sigma^{-1}(1)}|_{1}, ..., 1|_{\sigma^{-1}(l)}|_{1})\pi(\sigma)$$

where $\sigma_i$ is the $i$-th fiber of $\sigma$ i.e the pullback

\[
\begin{array}{cccc}
[\sigma^{-1}(\omega^{-1}(i))] & \xrightarrow{\sigma_i} & [\omega^{-1}(i)] & \xrightarrow{\xi_i} [1] \\
\downarrow & & \downarrow & \downarrow \\
[n] & \xrightarrow{\sigma} & [l] & \xrightarrow{\omega} [k]
\end{array}
\]

The idea of the lemma is presented in the diagram.
\[\pi(\sigma) \quad \nu(\sigma) \quad \pi(\omega) \quad \nu(\omega)\]

\[\pi(\nu(\sigma) \cdot \pi(\omega)) = \Gamma(\pi(\omega); 1, 1, 1)\]

\[\nu(\nu(\sigma) \cdot \pi(\omega)) \quad \nu(\omega)\]

where the bottom diagram is just an appropriate deformation of the top one. The formal proof follows.

**Proof.** Consider the diagram

\[
\begin{array}{c}
[n] \\
\pi(\sigma \cdot \omega) \quad \nu(\sigma \cdot \omega) \\
[\sigma] \quad [\omega] \\
[\nu(\sigma)] \quad [\nu(\omega)] \\
\pi(\nu(\sigma) \cdot \pi(\omega)) \quad \nu(\nu(\sigma) \cdot \pi(\omega))
\end{array}
\]
By definition of $\Gamma$

$$\pi(\nu(\sigma) \cdot \pi(\omega)) = \Gamma(\pi(\omega); 1_{[\sigma^{-1}(1)]}, \ldots, 1_{[\sigma^{-1}(l)]})$$

Now we have a decomposition

$$\sigma \cdot \omega = \pi(\sigma) \cdot \Gamma(\pi(\omega); 1_{[\sigma^{-1}(1)]}, \ldots, 1_{[\sigma^{-1}(l)]}) \cdot \nu(\nu(\sigma) \cdot \pi(\omega)) \cdot \nu(\omega).$$

But $\nu(\nu(\sigma) \cdot \pi(\omega)) \cdot \nu(\omega)$ is order preserving and $\pi(\sigma) \cdot \Gamma(\pi(\omega); 1_{[\sigma^{-1}(1)]}, \ldots, 1_{[\sigma^{-1}(l)]})$ is bijective. So we have

$$\nu(\nu(\sigma) \cdot \pi(\omega)) \cdot \nu(\omega) = \nu(\omega),$$

but

$$\pi(\sigma) \cdot \Gamma(\pi(\omega); 1_{[\sigma^{-1}(1)]}, \ldots, 1_{[\sigma^{-1}(l)]})$$

is not necessary order preserving on fibers.

Consider the restriction of $\pi(\sigma) \cdot \Gamma(\pi(\omega); 1_{[\sigma^{-1}(1)]}, \ldots, 1_{[\sigma^{-1}(l)]})$ to $\sigma^{-1}(1)$. Let us denote this restriction $r_i$. Then obviously we have $r_i \cdot \nu(\nu(\sigma) \cdot \pi(\omega)) = \pi(\omega) \cdot \sigma_i$. But $\pi(\omega)$ was order preserving on fibers of $\omega$ so $\pi(\omega) \cdot \sigma_i = \sigma_i$. So we have a decomposition of $\sigma_i$ into a bijection and order preserving map. In addition, $r_i$ preserves order on fibers of $\sigma_i$ as $\nu(\sigma)$ and $\Gamma(\pi(\omega); 1_{[\sigma^{-1}(1)]}, \ldots, 1_{[\sigma^{-1}(l)]})$ do. Hence, $r_i = \pi(\sigma_i)$ by uniqueness of decomposition.

Since $r_i$ acts on fibers of $\nu(\nu(\sigma) \cdot \pi(\omega)) \cdot \nu(\omega) = \nu(\sigma \cdot \omega)$ we have another decomposition

$$\sigma \cdot \omega = \Gamma(1; r_1^{-1}, \ldots, r_k^{-1}) \cdot \pi(\sigma) \cdot \Gamma(\pi(\omega); 1_{[\sigma^{-1}(1)]}, \ldots, 1_{[\sigma^{-1}(l)]}) \cdot \nu(\sigma \cdot \omega).$$

Finally, $\Gamma(1; r_1^{-1}, \ldots, r_k^{-1}) \cdot \pi(\sigma) \cdot \Gamma(\pi(\omega); 1_{[\sigma^{-1}(1)]}, \ldots, 1_{[\sigma^{-1}(l)]})$ is order preserving on fibers by construction. Therefore, it is equal to $\pi(\sigma \cdot \omega)$ and our formula follows.

There is an obvious functor $\Omega_n \to \Omega^*$ which assigns to a tree $T = [k_n] \xrightarrow{\rho_{n-1}} [k_{n-1}] \xrightarrow{\rho_{n-2}} \ldots \xrightarrow{\rho_0} [1]$ the ordinal $|T| = |k_n|$. We also introduce the notation $|T|$ for the cardinality of $|T|$ i.e. for the number of tips of $T$.

For a morphism of trees $\sigma : T \to S$ we will denote by $\pi(\sigma)$ the permutation $\pi(\sigma_n)$.

5 $n$-operads in symmetric monoidal categories.

Here and in all subsequent sections the fiber of a morphism $\sigma : T \to S$ in $\Omega_n$ will mean only the fiber over a tip of $S$. So every $\sigma : T \to S$ with $|S| = k$ determines a list of trees $T_1, \ldots, T_k$ being fibers over tips of $S$ ordered according to the order in $|S|$. From now on we will always relate to $\sigma$ this list of trees in this order.

The definition below is a specialisation of a general definition of $n$-operad in an augmented monoidal category $M$ given in [3]. Let $(V, \otimes, I)$ be a (strict)
symmetric monoidal category. Put \( M = \Sigma^n V \), which means that \( M \) has terminal categories up to dimension \( n - 1 \) and \( V \) in dimension \( n \). The augmented monoidal structure is given by \( \otimes_1 = \otimes \) for all \( i \). Then an \( n \)-operad in \( V \) will mean an \( n \)-operad in \( \Sigma^n V \). Explicitly it means the following:

**Definition 5.1** An \( n \)-operad in \( V \) is a collection \( A_T, T \in \Omega_n \) of objects of \( V \) equipped with the following structure:

- a morphism \( e : I \to A_{U_n} \) (the unit);
- for every morphism \( \sigma : T \to S \) in \( \Omega_n \) a morphism
  \[ m_\sigma : A_S \otimes A_{T_1} \otimes \ldots \otimes A_{T_k} \to A_T \] (the multiplication).

They must satisfy the following identities:

- for any composite \( T \Rightarrow S \Rightarrow R \) the associativity diagram

\[
\begin{array}{c}
A_R \otimes A_{S_1} \otimes A_{T_1} \otimes \ldots \otimes A_{T_k} \\
\downarrow \\
A_R \otimes A_{S_1} \otimes A_{T_1} \otimes \ldots \otimes A_{T_k}
\end{array}
\begin{array}{c}
A_S \otimes A_{T_1} \otimes \ldots \otimes A_{T_k} \\
\downarrow \\
A_S \otimes A_{T_1} \otimes \ldots \otimes A_{T_k}
\end{array}
\begin{array}{c}
A_T \\
\downarrow \\
A_T
\end{array}
\begin{array}{c}
A_R \otimes A_{T_1} \otimes \ldots \otimes A_{T_k} \\
\downarrow \\
A_R \otimes A_{T_1} \otimes \ldots \otimes A_{T_k}
\end{array}
\begin{array}{c}
A_S \otimes A_{T_1} \otimes \ldots \otimes A_{T_k} \\
\downarrow \\
A_S \otimes A_{T_1} \otimes \ldots \otimes A_{T_k}
\end{array}
\begin{array}{c}
A_R \\
\downarrow \\
A_R
\end{array}
\]

commutes, where

\[ A_{S_1} = A_{S_1} \otimes \ldots \otimes A_{S_k} \]
\[ A_{T_1} = A_{T_1} \otimes \ldots \otimes A_{T_{m_i}} \]

and

\[ A_{T_i} = A_{T_1} \otimes \ldots \otimes A_{T_k} ; \]

- for an identity \( \sigma = \text{id} : T \to T \) the diagram

\[
\begin{array}{c}
A_T \otimes A_{U_n} \otimes \ldots \otimes A_{U_n} \\
\downarrow \\
A_T \otimes A_T \otimes \ldots \otimes A_T
\end{array}
\begin{array}{c}
A_T \\
\downarrow \\
A_T
\end{array}
\begin{array}{c}
A_T \otimes A_{U_n} \otimes \ldots \otimes A_{U_n} \\
\downarrow \\
A_T \otimes A_T \otimes \ldots \otimes A_T
\end{array}
\begin{array}{c}
A_T \\
\downarrow \\
A_T
\end{array}
\]

commutes;

- for a unique morphism \( T \to U_n \) the diagram

\[
\begin{array}{c}
A_{U_n} \otimes A_T \\
\downarrow \\
A_T
\end{array}
\begin{array}{c}
I \\
\downarrow \\
I
\end{array}
\begin{array}{c}
A_{U_n} \otimes A_T \\
\downarrow \\
A_T
\end{array}
\begin{array}{c}
I \otimes A_T \\
\downarrow \\
I
\end{array}
\begin{array}{c}
A_T \\
\downarrow \\
A_T
\end{array}
\]

commutes.
The definition of morphisms of $n$-operads is obvious, so we have a category of $n$-operads in $V$ which we will denote by $O_n(V)$.

We give an example of a 2-operad to provide the reader with a feeling of how these operads may look like. Other examples will appear later in course of the paper.

**Example 5.1** One can construct a 2-operad $G$ in $Cat$ such that the algebras of $G$ in $Cat$ are braided strict monoidal categories. If we apply the functor $\tau$ to $G$ (i.e. consider it as a 1-terminal operad in $Span(Cat)$) then the algebras of $\tau(G)$ are Gray-categories [18]. The categories $G_T$ are chaotic groupoids with objects corresponding to so called $T$-shuffles. A nice geometrical picture shows these groupoids in some low dimensions.

In general, the groupoid $G_T$ is generated by the 1-faces of the so called $T$-shuffle convex polytopes $P_T$ which themself form a topological 2-operad. The polytope $P_T$ is a point if $T$ has only one tip. The polytope $P_{T_1 \otimes T_2}$ is the permutohedron $P_j$, and the polytope $P_{M_T \otimes G_{T_2}}$ is the resultohedron $N_{pq}$ [21] [16].

We finish this example by presenting a picture for multiplication in $P$ (or $G$ if you like). The reader might find this picture somewhat familiar.
There are similar pictures for higher dimensions. In general the multiplication in $P$ produces some subdivisions of $P_T$ into products of shuffle polytopes of low dimensions. Some special cases of these subdivisions were discovered in [21]. Two examples are presented below.

6 Symmetric operads as $n$-operads.

Let $A$ be a symmetric operad in $V$ with multiplication $m$ and unit $e$. Then an $n$-operad $Des_n(A)$ is defined by

$$Des_n(A)_T = A_{|T|},$$

with unit morphism

$$e : I \to A_{U_n} = A_1$$
and multiplication

\[ m: A_{|S|} \otimes A_{|T_1|} \otimes \cdots \otimes A_{|T_k|} \rightarrow A_{|T|} \]

for \( \sigma: T \to S \).

**Lemma 6.1** \( \text{Des}_n A \) is an \( n \)-operad in \( V \).

**Proof.** The only nonobvious condition is associativity. The following diagram supplies the proof. The internal squares are commutative either by associativity and equivariance of \( m \) or by lemma 4.1.

Let us denote by \( SO(V) \) the category of symmetric operads in \( V \). So we have a functor

\[ \text{Des}_n: SO(V) \rightarrow O_n(V). \]

Let us recall a construction from [5]. Let \( M \) be a monoidal globular category then a corollary from the coherence theorem for monoidal globular categories [6] says that \( M \) is a strong algebra of the monad \( D \) on the 2-category of globular categories. So we have an action \( k: D(M) \to M \) and an isomorphism in the square

\[ \pi(\omega)^{-1} \otimes 1 \]

\[ \Gamma(\pi(\omega)^{-1};1) \]
If \( x : 1 \to M \) is a globular object from \( M \) then the composite

\[
t : D(1) \xrightarrow{D(x)} D(M) \xrightarrow{k} M
\]

can be considered as a globular version of the tensor power functor. The value of this functor on a tree \( T \) is denoted by \( x^T \). Moreover, the square above gives us an isomorphism \( \chi \):

\[
\begin{array}{ccc}
D^2(M) & \xrightarrow{D(t)} & D(M) \\
\mu & \searrow & \swarrow k \\
D(M) & \xrightarrow{k} & M
\end{array}
\]

This isomorphism \( \chi \) gives a canonical isomorphism

\[
\chi : x^{T \otimes S} \to x^T \otimes x^S,
\]

for example.

In the special case of \( M = \Sigma^n(V) \) we identify globular objects of \( \Sigma^n(V) \) with objects of \( V \) and follow the constructions from [5] to get the following inductive description of the tensor power functor and isomorphism \( \chi \).

For the \( k \)-tree \( T \), \( k \leq n \), and an object \( x \) from \( V \), let us define the object \( x^T \) in the following inductive way:

- if \( k < n \), then \( x^T = I \);
- if \( k = n \) and \( T = zT' \) then \( x^T = I \);
- if \( k = n \) and \( T = U_n \), then \( x^T = x \);
- now we use the induction on \( \text{susp}(T) \); suppose we already have defined \( x^S \) for \( S \) such that \( \text{susp}(S) \geq k + 1 \), and let \( T = T_1 \otimes_k \ldots \otimes_k T_j \) be a canonical decomposition of \( T \). Then we define

\[
x^T = x^{T_1} \otimes \ldots \otimes x^{T_j}.
\]

Clearly, \( x^T \) is isomorphic to \( x \otimes \ldots \otimes x \), Now we want to provide an explicit description of \( \chi \).
Lemma 6.2 For $\sigma : T \to S$ the isomorphism

$$\chi_\sigma : x^{T_1} \otimes \ldots \otimes x^{T_k} \to x^T$$

is induced by the permutation inverse of the permutation $\pi(\sigma)$.

Proof. We will prove the lemma by induction.

If $S = U_n$ then $\chi_\sigma$ is the identity morphism. Suppose we already have proved our lemma for all $\sigma$'s with codomain being an $(l + 1)$-fold suspension. As a first step we study $\chi_\sigma$ in a special case $T = \mathcal{M}_k$.

Now we start another induction on $\text{susp}(T)$. If $\text{susp}(T) > l$ then $\sigma$ factorises through one of the tips, so the fibers are either $T$ or degenerate trees and we get

$$\chi_\sigma = \text{id} : I \otimes \ldots \otimes x^T \otimes \ldots I \to x^T.$$  

If $\text{susp}(T) = l$ then we get

$$\chi_\sigma = \text{id} : x^{T_1} \otimes \ldots \otimes x^{T_k} \to x^T.$$  

Suppose we already have proved our lemma for all $T$ with $\text{susp}(T) > m$. And suppose we have a $\sigma$ with $\text{susp}(T) = m < l$. In this case we have the canonical decomposition

$$T_i = T_1^i \otimes_m \ldots \otimes_m T_l^j,$$

where $j$ is the same for all $1 \leq i \leq k$. Then $\chi_\sigma$ is equal to the composite

$$x^{T_1} \otimes \ldots \otimes x^{T_k} = \left( x^{T_1^1} \otimes \ldots \otimes x^{T_1^l} \right) \otimes \ldots \otimes \left( x^{T_k^1} \otimes \ldots \otimes x^{T_k^l} \right) \xrightarrow{\Sigma^{-1}}$$

$$\xrightarrow{\pi} \left( x^{T_1^1} \otimes \ldots \otimes x^{T_1^l} \right) \otimes \ldots \otimes \left( x^{T_k^1} \otimes \ldots \otimes x^{T_k^l} \right) \xrightarrow{\chi_1 \otimes \ldots \otimes \chi_j}$$

$$\xrightarrow{\chi_1 \otimes \ldots \otimes \chi_k} x^{T_1^1 \otimes \ldots \otimes T_1^l} \otimes \ldots \otimes x^{T_k^1 \otimes \ldots \otimes T_k^l} = x^T$$

where $\pi$ is the corresponding permutation and $\chi_i$ is already constructed by the inductive hypothesis as $\text{susp}(T_1^i \otimes \ldots \otimes T_l^i) > m$. Again by induction $\chi_i$ is induced by the permutation $\pi(\phi_i)^{-1}$, where

$$\phi_i : T_1^i \otimes \ldots \otimes T_l^i \to M_l^k.$$  

So $\chi_\sigma$ is induced by $\Gamma(\pi(\omega); \pi(\phi_1), \ldots, \pi(\phi_j))^{-1}$.

From the point of view of morphisms in $\Omega_n$ what we have used here is a decomposition of $\sigma$ into

$$T \xrightarrow{\xi} M_0^l \otimes_l \ldots \otimes_l M_0^l \xrightarrow{\omega} M_l^k.$$  

Then we have $\pi = \Gamma(\pi(\omega); 1, \ldots, 1)$. By construction we have $\pi(\xi_j) = 1$ and by the inductive hypothesis, $\pi(\xi) = \Gamma(1; \pi(\phi_1), \ldots, \pi(\phi_j))$. By the lemma [1,2]

$$\pi(\sigma) = \Gamma(\pi(\omega); 1, \ldots, 1)\pi(\xi) = \Gamma(\pi(\omega); 1, \ldots, 1)\Gamma(1; \pi(\phi_1), \ldots, \pi(\phi_j)) =$$

$$= \Gamma(\pi(\omega); \pi(\phi_1), \ldots, \pi(\phi_j)).$$
So we have completed our first induction.

To complete the proof it remains to show the lemma when $S$ is an arbitrary tree with $\text{susp}(S) = l$. Then we have a canonical decomposition $\omega : S \to M_l^f$ and we can form the composite

$$T \xrightarrow{\sigma} S \xrightarrow{\omega} M_l^f.$$ 

Since $\omega$ is a monotone map, by our lemma 4.1 again,

$$\pi(\omega) = \Gamma(1; \pi(\sigma_1), \ldots, \pi(\sigma_j))\pi(\sigma \cdot \omega).$$

By the inductive hypothesis again we can assume that we already have proved our lemma for the $\sigma_i$’s and, by the previous argument, for $\sigma \cdot \omega$ as well. So the result follows.

We now recall the construction of endomorphism operad from [5] in the special case of an augmented monoidal globular category equal to $\Sigma^n V$, where $V$ is a closed symmetric monoidal category.

Let $x$ be an object of $V$. The endomorphism $n$-operad of $a$ is the following $n$-operad $\text{End}_n(a)$ in $V$. For a tree $T$,

$$\text{End}_n(x)_T = V(x^T, x);$$

the unit of this operad is given by the identity

$$I \to V(x^n, x) = V(x, x).$$

of $x$. For a morphism $\sigma : T \to S$ the multiplication is given by

$$V(x^{T_1}, x) \otimes \ldots \otimes V(x^{T_k}, x) \otimes V(x^S, x) \to V(x^{T_1} \otimes \ldots \otimes x^{T_k}, x) \otimes V(x^S, x) \to$$

$$\xrightarrow{V(x^{\sigma^{-1}}_\sigma, x^\sigma)} V(x^T, x^\sigma) \otimes V(x^S, x) \to V(x^T, x).$$

We also can form the usual symmetric endoperad of $x$ in the symmetric closed monoidal category $V$. Let us denote this operad by $\text{End}(x)$. Now we want to compare $\text{End}_n(x)$ with the $n$-operad $\text{Des}_n(\text{End}(x))$. In $\text{End}(x)$ the action of a bijection $\pi : [n] \to [n]$ is defined to be

$$V(x^n, x) \xrightarrow{V(\pi, x)^{-1}} V(x^n, x).$$

So for $\sigma : T \to S$ we have in $\text{Des}_n(\text{End}(x))$ the multiplication

$$V(x^{[T_1]}, x) \otimes \ldots \otimes V(x^{[T_k]}, x) \otimes V(x^{[S]}, x) \to V(x^{[T_1]} \otimes \ldots \otimes x^{[T_k]}, x^k) \otimes V(x^{[S]}, x) \to$$

$$\xrightarrow{V(\pi(\sigma), x)} V(x^{[T]}, x) \xrightarrow{V(\pi(\sigma), x)} V(x^{[T]}, x).$$

But $V(\pi(\sigma), x) = V(\chi^{-1}_\sigma, x)$ and we proved the following

**Proposition 6.1** For any object $x \in V$, there is a natural isomorphism of $n$-operads

$$\text{End}_n(x) \simeq \text{Des}_n(\text{End}(x)).$$
7 Internal operads.

Let now $A$ and $B$ be two $n$-operads in $\text{Cat}$. A lax-morphism

$$f : B \to A$$

consists of a collection of functors

$$f_T : B_T \to A_T$$

together with natural transformations

$$\begin{array}{c}
B_S \times B_{T_1} \times \ldots B_{T_k} \\
\downarrow \\
A_S \times A_{T_1} \times \ldots A_{T_k}
\end{array} \xrightarrow{\mu} \begin{array}{c}
B_T \\
\downarrow \\
A_T
\end{array}$$

for every $\sigma : T \to S$ and a morphism $\epsilon : e_A \to e_B$, where $e_A, e_B$ are unit objects of $A$ and $B$ respectively. They must satisfy the usual coherence conditions.

In the particular case $B = 1$ the terminal $\text{Cat}$-operad, a lax-morphism $1 \to A$ will be called an internal operad in $A$. Because this notion is of primary importance for us, we give a separate definition.

Definition 7.1 Let $A$ be an $n$-operad in the symmetric monoidal category $\text{Cat}$ with multiplication $m$ and unit object $e \in A_{U_n}$. An internal $n$-operad in $A$ consists of a collection of objects $a_T \in A_T$, $T \in T_{\mathfrak{r}_n}$, together with a morphism

$$\mu_\sigma : m(a_S; a_{T_1}, \ldots, a_{T_k}) \to a_T$$

for every morphism of trees $\sigma : T \to S$ and a morphism

$$\epsilon : e \to a_{U_n}$$

which satisfy obvious conditions analogous to the conditions from the definition of $n$-operad.

Definition 7.2 Let $A$ be a symmetric operad in $\text{Cat}$. Then an internal $n$-operad in $A$ is an internal $n$-operad in $\text{Des}_n(A)$.

The last definition deserves to be unpacked. So an internal $n$-operad in a symmetric $\text{Cat}$-operad is given by a collection of objects $a_T \in A_{|T|}$, $T \in T_{\mathfrak{r}_n}$ together with a morphism

$$\mu_\sigma : m(a_S; a_{T_1}, \ldots, a_{T_k}) \to \pi(\sigma)a_T$$

for every $\sigma : T \to S$ and

$$\epsilon : e \to a_{U_n}$$

which satisfies associativity and unitary conditions.

In analogy with the last definition we give the following.
Definition 7.3 Let $A$ be a symmetric operad in $\text{Cat}$. An internal symmetric operad in $A$ consists of a collection of objects $a_{[n]} \in A_n$, $n \geq 0$ together with a morphism

$$\mu_\sigma : m(a_{[k]}, a_{[n_1]}, \ldots, a_{[n_k]}) \longrightarrow \pi(\sigma)a_{[n]}$$

for every $\sigma : [n] \rightarrow [k]$ in $\Omega^\ast$, and

$$\epsilon : e \longrightarrow a_{[1]}$$

which satisfy associativity and unitary conditions.

It is obvious also how to define morphisms of internal operads. We will denote by $\text{Oper}_n(A)$ the category of internal $n$-operads in $A$ (internal symmetric operads for $n = 1$).

Example 7.1. Let $V$ be a symmetric strict monoidal category. Consider the following symmetric $\text{Cat}$-operad $V$:

$$V_n = V,$$

the multiplication is given by iterated tensor product, the unit of $V$ is the unit object of $V$ and the action of symmetric groups is trivial. Then an internal $n$-operad in $V$ is the same as an $n$-operad in $V$. More precisely, we have an isomorphism of categories of internal $n$-operads in $V$ and $n$-operads in $V$. A little bit more delicate is the following.

Lemma 7.1 The category of internal symmetric operads in $V$ is isomorphic to the category of symmetric operads in $V$.

Proof. An internal symmetric operad in $V$ amounts to the following data. For every ordinal $[n]$ we have an object $a_{[n]}$ of $V$. We have a morphism $I \rightarrow a_1$. Finally, for every map of ordinals $\sigma : [n] \rightarrow [k]$ we have a morphism

$$\mu_\sigma : a_{[k]} \otimes a_{[n_1]} \otimes \ldots \otimes a_{[n_k]} \longrightarrow a_{[n]}$$

which satisfies unit and associativity axioms. Let us define a functor

$$\text{Des}_\infty : \text{SO}(V) \longrightarrow \text{Oper}_\infty(V).$$

For every symmetric operad $A$, the operad $\text{Des}_\infty(A) = a$ will have the same underlying collection but multiplication is given by

$$\mu_\sigma : a_k \otimes a_{n_1} \otimes \ldots \otimes a_{n_k} \longrightarrow a_n$$

The argument from lemma 6.1 shows that $\text{Des}_\infty(A)$ is indeed an internal symmetric operad.

Let us define an inverse functor $\text{Sym}$. For an internal symmetric operad $a$ a symmetric operad $\text{Sym}(a)$ will have $A_n = a_{[n]}$. Let $n = n_1 + \ldots + n_k$ then there is a unique order preserving morphism $\sigma : [n] \rightarrow [k]$ with fibers isomorphic to $[n_1], \ldots, [n_k]$ and we put

$$m = \mu(\sigma) : A_k \otimes A_{n_1} \otimes \ldots \otimes A_{n_k} \longrightarrow A_n.$$
Now we have to define the action of the symmetric groups on $A$. Let $\sigma : [n] \to [n]$ be a permutation. Then the composite

$$A_n \to A_n \otimes I \otimes \ldots \otimes I \otimes A_n \otimes A_1 \otimes \ldots \otimes A_1 \xrightarrow{\mu_\sigma} A_n$$

determines an endomorphism $q_\sigma$. The reader may check as an exercise that $q_\sigma$ is functorial and then determines a contravariant action of the symmetric group. It is also not hard to check that this action is equivariant. Now we make this action covariant by reversing the action. The rest of the proof is obvious.

\*\*\*

**Example 7.2** Let $C$ be a category. We can consider the endomorphism operad $\text{End}(C)$ of $C$ in $\text{Cat}$. An internal 1-operad $a$ in $C$ is what we call a multitensor in $C$ [7]. This is a sequence of functors $a_k : C^k \to C$ satisfying usual associativity and unitarity conditions. If $a_k$, $k \geq 1$, are isomorphisms then $a$ is just a tensor product on $C$. Conversely, every tensor product on $C$ determines in an obvious manner a multitensor on $C$.

It makes sense to consider categories enriched in a multitensor. In [8] we show that the category of algebras of an arbitrary higher operad $A$ in $\text{Span}(C)$ is equivalent to the category of categories enriched over an appropriate multitensor on the category of algebras of another operad $B(A)$ which is a some sort of delooping of $A$.

**Example 7.3** The internal symmetric operads in $\text{End}(C)$ were considering by J.McClure and J.Smith [25] under the name of functor operads. These operads generalise symmetric monoidal structures on $C$ in the same way as multitensors generalise monoidal structures.

Other examples of internal operads will be given in the next two sections.

8 Representing $n$-operads.

Consider the following 2-category $\text{CO}_n$. The objects of $\text{CO}_n$ are categorical $n$-operads, the morphisms are their operadic morphisms and the 2-morphisms are operadic natural transformations. We have a 2-functor

$$O_n : \text{CO}_n \to \text{Cat}$$

which assigns to an operad $A$ the category of internal $n$-operads in $A$.

Analogously, consider $\text{SCO}$ the 2-category of symmetric $\text{Cat}$-operads, their operadic functors and operadic natural transformations. There is a 2-functor

$$\text{Oper}_n : \text{SCO} \to \text{Cat}$$
which assigns to an operad \( A \) the category of internal \( n \)-operads in \( A \). For \( n = \infty \) the functor \( \text{Oper}_\infty \) assigns the category of internal symmetric operads in \( A \). It turns out that \( O_n \) as well as \( \text{Oper}_n \) are representable 2-functors.

**Theorem 8.1**

1. For every \( 1 \leq n < \infty \), there exists a categorical \( n \)-operad \( H^n \) such that the category \( O_n(A) \) is natural isomorphic to the category \( \text{CO}_n(H^n, A) \);

2. For every \( 1 \leq n \leq \infty \), there exists a symmetric \( \text{Cat} \)-operad \( h^n \) such that the category \( \text{Oper}_n(A) \) is natural isomorphic to the category \( \text{SCO}(h^n, A) \).

**Proof.** We concentrate first on the construction of \( h^n \).

The free symmetric operad construction on a nonsymmetric collection \( A \) has been discussed a lot in the literature. In this paper we choose a universal algebraic approach, so the objects are constructed inductively. If we have already the objects \( x \in F(A)_k \) and \( y_i \in F(A)_{n_i}, 1 \leq i \leq k \), then we can form another object \( \mu(x; y_1, \ldots, y_k) \in F(A)_{n_1 + \ldots + n_k} \). Now we want to apply this construction to the following situation.

Consider a grading on the set of objects of \( \Omega_n \) according to the number of tips. We use the notation \( Tr_n \) for this collection. Now we can form a free symmetric operad \( F(Tr_n) \) on this collection. The elements of \( F(Tr_n) \) are the objects of \( h^n \).

Now we want to define morphisms. We will do this by constructing generators and relations.

Suppose we have a morphism \( \sigma : T \rightarrow S \) in \( \Omega_n \) and \( T_1, \ldots, T_k \) is its list of fibers. Then we will have a generator

\[
\gamma(\sigma) : \mu(S; T_1, \ldots, T_k) \rightarrow \pi(\sigma)T
\]

where \( \mu \) is the multiplication in \( F(Tr_n) \). By the equivariancy requirement, we also have morphisms

\[
\mu(\pi S; \xi_1 T_1, \ldots, \xi_k T_k) = \Gamma(\pi; \xi_1, \ldots, \xi_k) \mu(S; T_1, \ldots, T_k) \rightarrow \Gamma(\pi; \xi_1, \ldots, \xi_k) \pi(\sigma)T.
\]

For every composite

\[
T \overset{\sigma}{\rightarrow} S \overset{\omega}{\rightarrow} R
\]

we will have a relation given by the commutative diagram
\[
\mu(\mu(R; S_r; T^*_1, ..., T^*_i, ..., T^*_k)) = \mu(R; \mu(S_1; T^*_1), ..., \mu(S_i; T^*_i), ..., \mu(S_k; T^*_k))
\]

\[
\mu(\pi(\omega); S^*_1, ..., T^*_i, ..., T^*_k) = \mu(R; \pi(\sigma_1) T^*_1, ..., \pi(\sigma_k) T^*_k)
\]

\[
\Gamma(\pi(\omega); 1, ..., 1) \pi(\sigma) T = \Gamma(1_k; \pi(\sigma_1), ..., \pi(\sigma_k)) \pi(\sigma \cdot \omega) T
\]

We also have a generator
\[
\epsilon : e \to U_n
\]

and two commutative diagrams:

\[
\begin{array}{ccc}
\mu(T; U_n, ..., U_n) & \rightarrow & \mu(T; e, ..., e) \\
\downarrow & & \downarrow id \\
T & \leftarrow & id
\end{array}
\]

and

\[
\begin{array}{ccc}
\mu(U_n; T) & \rightarrow & \mu(e; T) \\
\downarrow & & \downarrow id \\
T & \leftarrow & id
\end{array}
\]

as relations.

This operad contains an internal \(n\)-operad given by \(a_T = T\). Obviously, \(h^n\) is freely generated by this internal operad which proves the theorem for finite \(n\).

For \(n = \infty\) we should start from the free operad on the collection of ordinals

\[
Tr_{\infty} = ([0], [1], ..., [n], ...)
\]

which is isomorphic to the terminal collection and then introduce morphisms by generators one for each map of ordinals in \(\Omega^n\). Everything else is obvious.

To construct the operad \(H^n\) we should follow the same steps as above with the difference that we use the free \(n\)-operad construction. So the objects will correspond not for an arbitrary expressions \(\mu(x; y_1, ..., y_k)\), but for each morphism \(\sigma : T \to S\) in \(\Omega_n\), we can form \(\mu(x; y_{T_{\ell_1}}, ..., y_{T_{\ell_k}}) \in F(A)_T\).

The main object of study in this paper will be the symmetric operad \(h^n\). The \(n\)-operad \(H^n\) will play a role in a subsequent paper.
To better understand the structure of $h^n$ we provide the following lemma, the proof of which is an easy use of axioms.

**Lemma 8.1** Every object $a$ from $h^n$ has a unique representation in the form

$$\mu(T; a_1, \ldots, a_k).$$

This lemma show that one can inductively construct an object of $h^n$ as a labeled planar tree with vertices decorated by trees from $T_{r_n}$ in the following sense: to every vertex $v$ of valency $k$ we associate an $n$-tree with $k$-tips. The following picture illustrates the concept for $n = 2$.

So, the objects of $h^n$ are labeled planar trees with some extra internal structure. The morphisms are contractions or growing of internal edges, yet not all contractions are possible. It depends on the extra internal structure. We can simultaneously contract the input edges of a vertex $v$ only if the corresponding $n$-trees in the vertices above $v$ can be past together in the $n$-category $T_{r_n}$, according to the globular pasting scheme determined by the tree at the vertex $v$. In the above example we have that the trees on the highest level are fibers over a map of trees:

So in $h^2$ we have a morphism corresponding to the $\sigma$ from the object above to the object
The two extreme cases \( n = 1 \) and \( n = \infty \) are well studied. Indeed, with \( n = 1 \) all the decorations are meaningless. Yet, the morphisms in \( \mathfrak{h}^1 \) correspond only to order-preserving maps between ordinals.

Therefore the operad \( \mathfrak{h}^1 \) coincides with the symmetrisation of a nonsymmetric operad \( h \) described in [3]. For a discussion on it the reader may also look at [13]. The objects of \( \mathfrak{h}^1 \) are bracketings of the strings consisting of several \( 0 \)'s and symbols \( 1, \ldots, k \) in fixed order without repetition. Multiple bracketing like \(((\ldots))\)) and also empty bracketing ( ) are allowable. The morphisms are throwing off \( 0 \)'s, removing and introducing a pair of brackets and also a morphism ( ) \( \rightarrow \) (1). Symmetric groups act by permuting symbols \( 1, \ldots, k \). The operad multiplication is given by replacing one of the symbols by a corresponding expression. It is clear that \( \pi_0(\mathfrak{h}^1_k) = \Sigma_k \) and all higher homotopy groups vanish.

In other words \( \mathfrak{h}^1 \) is an \( A_\infty \)-operad. The algebras of \( \mathfrak{h}^1 \) in \( \text{Cat} \) are categories equipped with \( n \)-fold tensor product satisfying some obvious associativity and unitarity conditions. For example, instead of associativity isomorphism we will have two, perhaps noninvertible, morphisms from two different combinations of binary products to the triple tensor product

\[
(a \otimes b) \otimes c \longrightarrow a \otimes b \otimes c \longleftarrow a \otimes (b \otimes c).
\]

Instead of the pentagon, we will have a baricentric subdivision of it and so on. Such categories were called lax-monoidal in [13].

In another extreme case \( n = \infty \) all the decorations again collapse to a point. But morphisms are more complicated and correspond to all maps of ordinals. So we can give the following description of the operad \( \mathfrak{h}^\infty \). A typical object of \( \mathfrak{h}^\infty_n \) is a planar tree equipped with an injective function (labeling) from \([n]\) to the set of vertices of this tree. The symmetric group acts by permuting the labels. The morphisms are generated by contraction of an internal edge, growing of an internal edge, and dropping unlabeled leaves with usual relations of associativity and unitarity. We also will have an isomorphism \( T \rightarrow \pi T \) for every permutation \( \pi \in S_n \). This isomorphism commutes with other generators. Again the \( \mathfrak{h}^\infty \)-algebras in \( \text{Cat} \) are symmetric lax-monoidal categories in the terminology of [14].

Observe also that a tree
which corresponds to the generated object $[l]$ is the terminal object in $h_l^\infty$. Hence, the nerve of $h^\infty$ is an $E_\infty$-operad.

**Remark 8.1** The trees formalism from [17] section 1.2 is actually a special case of our theorem 8.1 with $n = \infty$, $A = \mathcal{V}$ for a symmetric monoidal category $\mathcal{V}$.

To clarify the structure of $h^n$ for $2 \leq n < \infty$ we provide the homotopy essential part of the picture of $h_2^2$:

The homotopy type of the nerve of this category is $S^1$. The reader can find an analogy with a diagram from the construction of the braiding in the proposition 5.3 from [19]. The reader may also look at a similar picture for the category $M_2^2$ in the next section, where $M^2$ is a $\text{Cat}$-operad constructed in [3]. We also recommend to the reader to look at the picture of $M_3^2$ in [3], which looks like a two dimensional sphere, and to try to construct a similar picture in $h_3^2$. Of course, these are not accidental coincidences as we will show in next two sections.
9 Homotopy type oh $h^n$.

We study the homotopy type of $h^n$ by comparing it with the operad $\mathcal{M}^n$ constructed in [3]. First of all we briefly review the construction of the operad $\mathcal{M}^n$.

The objects of $\mathcal{M}^n_k$ are all finite expressions generated by the symbols $1, \ldots, k$ and $n$ associative operations $\circ_n, \ldots, \circ_{n-1}$ in which each generating symbol occurs exactly once. We bring to the attention of the reader that in [3] the operations are numbered from 1 to $n$. Yet, it is more convenient for us to use a different numeration. There is yet another difference in the order of the operations which is reflected in our definition of morphisms in $\mathcal{M}^n$. Observe also, that we have an action of $\Sigma_k$ on $\mathcal{M}^n_k$ and an obvious operation of substitution which provides an operadic structure on the objects of $\mathcal{M}^n$.

Now we can describe the morphisms in $\mathcal{M}^n$. They are generated by the middle interchange laws

$$\eta^{ij} : (1 \circ_i 2) \circ_j (3 \circ_4) \to (1 \circ_i 3) \circ_j (2 \circ_4), \ i < j,$$

substitutions and permutations and must satisfy the coherence conditions specified in the first section of [3]. The difference between our version of $\mathcal{M}^n$ and that of [3] is the opposite direction of $\eta^{ij}$.

Here is the above promised picture of $\mathcal{M}_2^2$ (our version).

![Diagram](image)

It was shown in [3] that the the operad $\mathcal{M}^n$ is a poset operad. The algebras of $\mathcal{M}^n$ in $\text{Cat}$ are iterated $n$-monoidal categories, i.e. categories with $n$ strict monoidal structures which are related by interchange morphisms (not necessary isomorphisms) satisfying some natural coherence conditions.

**Theorem 9.1** The operad $\mathcal{M}^n$ contains an internal $n$-operad.

**Proof.** We have to assign an object $a_T \in \mathcal{M}^n_k$ to every $n$-tree $T$ with $|T| = k$. We will do it by induction. We put $a_T = 1$ for all trees with $|T| = 0, 1$. In particular, $a_{U_n} = 1$. Now, suppose we have already constructed $a_T$ for all trees which are $(n - k)$-fold suspensions. Suppose a tree $T$ is an $(n - k - 1)$-fold suspension. Take a canonical decomposition

$$T = T_1 \otimes_{n-k-1} T_2 \otimes_{n-k-1} \cdots \otimes_{n-k-1} T_r$$

Then we put

$$a_T = m(1 \circ_{n-k-1} 2 \circ_{n-k-1} \cdots \circ_{n-k-1} r; a_{T_1}, a_{T_2}, \ldots, a_{T_r}).$$
where $m$ is the multiplication in $\mathcal{M}^n$.

**Remark 9.1** In [3] the full subcategory of $\mathcal{M}^n$ containing the objects of the form $\pi a_T, \pi \in \Sigma_{|T|}$, is called the Milgram subspace. The Milgram subspace does not form an operad but determines a preoperad in the sense of Berger [9] which is closely related to the Milgram model of the free $n$-fold loop space [3].

To give an idea how the operad multiplication in $\sigma$ looks like we present the following 2-dimensional example.

In this picture the map of trees is given by

\[
\sigma(1) = 1, \sigma(2) = 2, \sigma(3) = 1, \sigma(4) = 2, \quad \pi(\sigma) = (1324).
\]

Then

\[
a_T = (1 \circ 2) \circ (3 \circ 1 4).
\]

\[
m(a_S; a_T, a_T) = m(1 \circ 2; 1 \circ 2, 1 \circ 2) = (1 \circ 2) \circ (3 \circ 4),
\]

and operadic multiplication $\mu_\sigma$ is given by the middle interchange morphism

\[
\eta_{1,2,3,4} : (1 \circ 2) \circ (3 \circ 4) \rightarrow (1 \circ 3) \circ (2 \circ 4).
\]

Before we construct the multiplication in general we have to formulate the following lemma whose proof is obtained by an obvious induction.

**Lemma 9.1** Let the $n$-tree $T$ be

\[
T = [k_n] \xrightarrow{\rho_{n-1}} [k_{n-1}] \xrightarrow{\rho_{n-2}} \cdots \xrightarrow{\rho_0} [1]
\]

then an element $u \circ v$ in $a_T$ in the sense of [3] if and only if $u < v$ and

\[
\rho_{n-1} \cdots \rho_i(u) = \rho_{n-1} \cdots \rho_i(v)
\]

but

\[
\rho_{n-1} \cdots \rho_{i+1}(u) \neq \rho_{n-1} \cdots \rho_{i+1}(v).
\]

Now we want to construct the multiplication $m_\sigma$ in the special case where

\[
\sigma : T \rightarrow M^k_\sigma.
\]
So we have to construct a morphism

\[ m(1 \circ 2; a_{T_1}, a_{T_2}) \longrightarrow \pi(\sigma)a_T. \]

According to [3] we have to check that \( u \circ v \) in \( m(1 \circ 2; a_{T_1}, a_{T_2}) \) implies either \( u \circ v \) in \( \pi(\sigma)a_T \) for \( j \leq i \) or \( v \circ u \) in \( \pi(\sigma)a_T \) for \( j < i \).

Recall that \( m(1 \circ 2; a_{T_1}, a_{T_2}) = a_{T_1} \circ a_{T_2} \) where \( a_{T_2} \) is the same expression as \( a_T \) but all numbers are shifted on \( |T_1| \). Let \( \xi_i : T_i \to T, i = 1, 2 \) be inclusions of \( T_i \) as \( i \)-th fiber.

Now, suppose \( u \circ v \) is in \( a_{T_1} \). By our lemma it means that \( u < v \) and

\[ \rho_{n-1} \cdot \ldots \cdot \rho_i(u) = \rho_{n-1} \cdot \ldots \cdot \rho_i(v) \]

but

\[ \rho_{n-1} \cdot \ldots \cdot \rho_{i+1}(u) \neq \rho_{n-1} \cdot \ldots \cdot \rho_{i+1}(v) \]

in \( T_1 \). Hence, we have \( \xi_1(u) < \xi_1(v) \) and

\[ \rho_{n-1} \cdot \ldots \cdot \rho_i(\xi_1(u)) = \rho_{n-1} \cdot \ldots \cdot \rho_i(\xi_1(v)) \]

but

\[ \rho_{n-1} \cdot \ldots \cdot \rho_{i+1}(\xi_1(u)) \neq \rho_{n-1} \cdot \ldots \cdot \rho_{i+1}(\xi_1(v)) \]

in \( T \). But \( \pi(\sigma)(\xi_1(u)) = w \) by definition of \( \pi(\sigma) \). Therefore, \( \pi(\sigma)^{-1}u < \pi(\sigma)^{-1}v \) and

\[ \rho_{n-1} \cdot \ldots \cdot \rho_i(\pi(\sigma)^{-1}u) = \rho_{n-1} \cdot \ldots \cdot \rho_i(\pi(\sigma)^{-1}v) \]

but

\[ \rho_{n-1} \cdot \ldots \cdot \rho_{i+1}(\pi(\sigma)^{-1}u) \neq \rho_{n-1} \cdot \ldots \cdot \rho_{i+1}(\pi(\sigma)^{-1}v) \]

in \( T \). By our lemma it means that \( u \circ v \) is in \( a_T \).

The same argument applies if \( u \circ v \) is in \( a_{T_2} \) but all numbers must be shifted on \( |T_1| \).

Now suppose, \( u \) is in \( a_{T_1} \), but \( v \) is in \( a_{T_2} \). This means that \( u \circ_v v \) is in \( a_{T_1} \circ_v a_{T_2} \).

We have two possibilities. The first is

\[ \rho_{n-1} \cdot \ldots \cdot \rho_k(u) = \rho_{n-1} \cdot \ldots \cdot \rho_k(v) \]

where the first composite is in \( T_1 \) and the second is in \( T_2 \). This means that

\[ \rho_{n-1} \cdot \ldots \cdot \rho_k(\xi_1(u)) = \rho_{n-1} \cdot \ldots \cdot \rho_k(\xi_2(v)) \]

already in \( T \). But \( \sigma \) is a morphism of trees, hence, preserves order on fibers of \( \rho_k \) and we have \( \xi_1(u) < \xi_2(v) \), hence, again \( \pi(\sigma)^{-1}u < \pi(\sigma)^{-1}v \) and

\[ \rho_{n-1} \cdot \ldots \cdot \rho_k(\pi(\sigma)^{-1}u) = \rho_{n-1} \cdot \ldots \cdot \rho_k(\pi(\sigma)^{-1}v) \]

and therefore \( u \circ_v v \) is in \( \pi(\sigma)a_T \).
The last possibility is

\[ \rho_{n-1} \cdots \rho_l(u) = \rho_{n-1} \cdots \rho_l(v) \]

for some \( l < k \) but

\[ \rho_{n-1} \cdots \rho_{l+1}(u) \neq \rho_{n-1} \cdots \rho_{l+1}(v) \]

again in \( T_1 \) and \( T_2 \) respectively. Then

\[ \rho_{n-1} \cdots \rho_l(\xi_1(u)) = \rho_{n-1} \cdots \rho_l(\xi_2(v)) \]

for some \( l < k \) but

\[ \rho_{n-1} \cdots \rho_{l+1}(\xi_1(u)) \neq \rho_{n-1} \cdots \rho_{l+1}(\xi_2(v)) \]

already in \( T \). By the usual argument it means that either \( u \circ v \) or \( v \circ u \) is in \( \pi(\sigma) a_T \), and that finishes the proof of the special case.

Now, suppose we have constructed \( m_\sigma \) for all \( \sigma \) such that the codomain of it is \( S = M_T^j \) and where \( j \leq m \). Then for \( S = M_T^{m+1} \)

\[ a_S = m(1 \circ_k 2; a_{S_1}, a_{S_2}) \]

and an easy inductive argument can be applied.

In general, let \( \sigma : T \to S \) be a morphism of trees. If \( S = U_n \) then we put \( \mu_\sigma = id \). Now suppose we already have constructed \( \mu_\sigma \) for all \( \sigma \) with codomain being an \((n-k)\)-fold suspension. Let \( S \) be an \((n-k-1)\)-fold suspension. Then the canonical decomposition of \( S \) gives us

\[ \omega : S \to M_{n-k-1}^j \]

with \((n-k)\)-fold suspensions \( S_i, 1 \leq i \leq r \), as fibers. We have

\[ m(a_S; a_{T_1}, \ldots, a_{T_k}) = m(m(1 \circ_{n-k-1} \ldots \circ_{n-k-1} r; a_{S_1}, \ldots, a_{S_r}, a_{T_1}, \ldots, a_{T_k}) = \]

\[ = m(1 \circ_{n-k-1} \ldots \circ_{n-k-1} r; m(a_{S_1}; a_{T_1}, \ldots, a_{T_{m_1}}), \ldots, m(a_{S_r}; a_{T_2}, \ldots, a_{T_{m_r}})). \]

By the inductive hypothesis we already have \( m_\sigma \) for the fibers of \( \sigma \). So we have a morphism

\[ m(1, m_{\sigma_1}, \ldots, m_{\sigma_r}) : m(1 \circ_{n-k-1} \ldots \circ_{n-k-1} r; m(a_{S_1}; a_{T_1}, \ldots, a_{T_{m_1}}), \ldots, m(a_{S_r}; a_{T_2}, \ldots, a_{T_{m_r}})) \to m(1 \circ_{n-k-1} \ldots \circ_{n-k-1} r; \pi(\sigma_1) a_{T_1}, \ldots, \pi(\sigma_r) a_{T_r}) \]

where \( T_1', \ldots, T_r' \) are fibers of \( \sigma \cdot \omega \). But

\[ m(1 \circ_{n-k-1} \ldots \circ_{n-k-1} r; \pi(\sigma_1) a_{T_1}, \ldots, \pi(\sigma_r) a_{T_r}) = \]

\[ = \Gamma(1, \pi_{\sigma_1}, \ldots, \pi_{\sigma_r}) m(1 \circ_{n-k-1} \ldots \circ_{n-k-1} r; a_{T_1'}, \ldots, a_{T_r'}). \]
Now we already have the morphism
\[ m_{\sigma, \omega} : m(1 \circ_{n-1} \ldots \circ_{n-1} r; a_{T'}, \ldots, a_{T}) \to \pi(\sigma \cdot \omega)a_T \]
So we have
\[ \Gamma(1, \pi\sigma_1, \ldots, \pi\sigma_r)m_{\sigma, \omega} : \Gamma(1, \pi\sigma_1, \ldots, \pi\sigma_r)m(1 \circ_{n-1} \ldots \circ_{n-1} r; a_{T'}, \ldots, a_{T}) \to \Gamma(1, \pi\sigma_1, \ldots, \pi\sigma_r)\pi(\sigma \cdot \omega)a_T. \]
By lemma 4.1,
\[ \Gamma(1, \pi\sigma_1, \ldots, \pi\sigma_r)\pi(\sigma \cdot \omega) = \Gamma(\pi(\omega), 1, \ldots, 1)\pi(\sigma). \]
But \( \omega \) is order preserving, hence, the last permutation is \( \pi(\sigma) \). So the composite
\[ m(1, m_{\sigma_1}, \ldots, m_{\sigma_r}) \cdot \Gamma(1, \pi\sigma_1, \ldots, \pi\sigma_r)m_{\sigma, \omega} \]
gives us the required morphism
\[ \mu_{\sigma} : m(a_{S}; a_{T_1}, \ldots, a_{T_k}) \to \pi(\sigma)a_T. \]
Associativity and unitarity of this multiplication are trivial because \( \mathcal{M}^n \) is a poset operad.

In [3] a \( \text{Cat} \)-operadic morphism
\[ \mathcal{M}^n \to \mathcal{K}^{(n)} \]
is constructed. Here \( \mathcal{K}^{(n)} \) is the \( n \)-th filtration of Berger’s complete graph operad [9], which plays a central role in his theory of cellular operads. So we have

**Corollary 9.1.1** \( \mathcal{K}^{(n)} \) contains an internal \( n \)-operad.

Let us denote the canonical operadic morphism corresponding to the internal operad in \( \mathcal{M}^n \) by
\[ \kappa : h^n \to \mathcal{M}^n. \]

**Theorem 9.2** The morphism \( \kappa \) induces a weak homotopy equivalence of nerves of the corresponding categories. Hence, \( N(h^n) \) is a simplicial \( E_n \)-operad.

**Proof.** The idea of the proof is simple. First of all we are going to show that a typical fiber of \( \kappa \) has a terminal object. Secondly, we show that the map between objects of \( \mathcal{M}^n \) and \( h^n \) which assigns to an object \( a \in \mathcal{M}^n \) the terminal object in \( \kappa^{-1}(a) \) can be extended to a functor (not operadic). Hence, \( N(\kappa) \) is a deformation retraction in every dimension.

In the cases \( h^1 \) or \( h^\infty \) this is easy to do because we always can make a planar tree ‘collapse’ by contracting its internal edges. But for finite \( n \geq 2 \) it is not
always possible to contract an edge. Nevertheless, we show below that in the
fibers of $\kappa$ some sort of analogue of the contraction exists.

We need now the following inductive technical definition. An object of $h^n$
of the form $T$ is *irreducible* if $T$ is a pruned tree or a tree $z^nU_0$. Otherwise it is *reducible*. We also define the object $e \in h^n_0$ to be irreducible.

An object of $h^n$ in the form $\mu(S; T_1, \ldots, T_k)$ is *reducible (at i-th place)* if there is $0 \leq i \leq k$ such that $T_i \neq e$ and there is a tree $T$ and a morphism $\sigma : T \to S$ such that $\sigma$ has all fibers except for $i$-th equal to $U_n$ and the $i$-th fiber equal to $T_i$. The object $\mu(S; T_1, \ldots, T_k)$ is *irreducible* if $S$ and $T_1, \ldots, T_k$ are irreducible and the object itself is not reducible at any place.

For a tree

$$T = [k_n] \rho_{n-1} [k_{n-1}] \rho_{n-2} \ldots \rho_0 [1]$$

and the $i$-th tip of $T$ let $v_i(T)$ be the maximal integer $k$ such that there exists a tip $j$ such that

$$\rho_{n-1} \cdot \ldots \cdot \rho_k(i) = \rho_{n-1} \cdot \ldots \cdot \rho_k(j).$$

The following lemma can be easily proved.

**Lemma 9.2** The object $\mu(S; T_1, \ldots, T_k)$ is reducible at the $i$-th place if and only if

$$susp(T_i) \geq v_i(S).$$

Let now an object from $h^n$ have the form

$$x = \mu(S; x_1, \ldots, x_k)$$

where

$$x_i = \mu(T_i; y^1_i, \ldots, y^k_i),$$

then it is irreducible provided $x_1, \ldots, x_k$ and $\mu(S; T_1, \ldots, T_k)$ are all irreducible.

An elementary reduction morphism is a morphism of the form:

$$\mu(S; T_1, \ldots, e, \ldots, T_i, \ldots, e, \ldots, T_k) \to \mu(S; T_1, \ldots, U_n, \ldots, T_i, \ldots, U_n, \ldots, T_k) \to T$$

where the last morphism corresponds to some $\sigma : T \to S$. A reduction morphism is a morphism constructed from elementary reduction morphisms using operations of operadic multiplication and composition. A reduction morphism is an analogue in $h^n$ of a morphism of (nonsimultaneous) contraction of some edges of a tree in the extreme cases $h_1$, $h^\infty$. The picture below represents an example of a reduction morphism.
For every tree $T$ there is a maximal pruned subtree $T^{(p)}$ of $T$. The corresponding morphism $i^{(p)}: T^{(p)} \to T$ in $\Omega_n$ has all fibers equal to $U_n$. Suppose now a morphism $\sigma: T \to S$ in $\Omega_n$ is such that its restriction on $T^{(p)}$ gives an identity morphism $T^{(p)} \to S^{(p)}$. Then $\sigma$ also has all fibers equal to $U_n$. Then we can form a composite in $h^n$:

$$S = \mu(S; e, \ldots, e) \longrightarrow \mu(S; U_n, \ldots, U_n) \longrightarrow T.$$  

We will call such a morphism an elementary pruning morphism. The corresponding picture is

Definition 9.1 An object of $h^n_k$, $k \neq 0$, of the form $T$ is pruned if $T$ is a pruned tree. An object of the form $\mu(S; y_1, \ldots, y_k)$ is pruned if $S$ is a pruned tree and objects $y_1, \ldots, y_k$ are pruned.

Now let $P(a)$ be the subcategory of $\kappa^{-1}(a)$ generated by the elementary pruning morphisms.

The following lemma is obvious.
Lemma 9.3. For every object \( x \) from \( \kappa^{-1}(a) \), there exists a unique morphism \( x \to y \) into a pruned object in \( P(a) \).

Lemma 9.4. Let \( a \) be an object of \( \mathcal{M}^n \). For every object \( x \) in \( \kappa^{-1}(a) \), there exists an irreducible object \( y \) in \( \kappa^{-1}(a) \) together with a reduction morphism \( x \to y \) in \( \kappa^{-1}(a) \).

Proof. We will do it inductively. It is obvious that \( \kappa^{-1}(a) \) is a nonempty category. Let \( x \) be an object from \( \kappa^{-1}(a) \).

If \( x \) is represented by a tree \( T \) and \( T \) is not degenerate then we have a morphism \( \mu(T; T_1, \ldots, T_k) = \mu(T; T_1, T_3, \ldots, T_k) \)

If \( T \) is degenerate, then we have a morphism \( \mu(T; U_1, \ldots, U_n) = \mu(T; U_1, T_3, \ldots, T_k) \)

Let \( x \) be an object from \( \kappa^{-1}(a) \). Without loss of generality we can assume that \( S, T_1, \ldots, T_k \) are irreducible. Suppose \( x \) is reducible at the \( i \)-th place and let \( \mu(S; T_1, \ldots, T_k) \).

Without loss of generality we can assume that \( S, T_1, \ldots, T_k \) are irreducible. Suppose \( x \) is reducible at the \( i \)-th place and let \( \mu(S; T_1, \ldots, T_k) \).

Then if the last object was not irreducible it can be reducible only at the places which do not correspond to \( e \). So we can continue this process and end up with an irreducible object. Again observe, that we not only constructed an irreducible object in \( \kappa^{-1}(a) \) but also a morphism in \( \kappa^{-1}(a) \) from \( x \) to this object.

It is clear now how to construct an irreducible object together with a morphism from \( x \) to it in \( \kappa^{-1}(a) \) using elementary pruning morphisms together with the morphisms of the type \((*)\).

We will call a morphism of the type \((*)\) an elementary pure reduction morphism if \( T_i \) is not degenerate. Observe, that if in an elementary pure reduction morphism all \( S, T_1, \ldots, T_k \) are pruned then so is \( T \).

Lemma 9.5. There is only one irreducible object in \( \kappa^{-1}(a) \)

Proof. Suppose \( y = \mu(S; y_1, \ldots, y_k) \) is an irreducible object in \( \kappa^{-1}(a) \). Let \( \sigma : S \to M_i^j \) be a canonical decomposition of \( T \) with fibers \( T_1, \ldots, T_j \). Then the object \( \mu(M_i^j; T_1, \ldots, T_j; y_1, \ldots, y_k) \) belongs to \( \kappa^{-1}(a) \) and moreover the morphism \( \mu(M_i^j; T_1, \ldots, T_j; y_1, \ldots, y_k) \to \mu(S; y_1, \ldots, y_k) \) belongs to \( \kappa^{-1}(a) \) as well.

But

\[
\mu(M_i^j; T_1, \ldots, T_j; y_1, \ldots, y_k) =
\]

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= \mu(M^1_1; \mu(T_1; y_1, \ldots, y_{k_1}), \mu(T_2; y_{k_1+1}, \ldots, y_{k_2}), \ldots, \mu(T_j; y_{k_{j-1}+1}, \ldots, y_k))

and all the objects \( \mu(T_1; y_{k_1+1}, \ldots, y_{k_2}), \ldots, \mu(T_j; y_{k_{j-1}+1}, \ldots, y_k) \) are irreducible by lemma 9.2 (as \( \text{susp}(S) < \text{susp}(T_i) \)) and an easy inductive argument.

Moreover, the object \( a \) must have the form \( a_1 \circ \ldots \circ a_j \) and \( \mu(T_1; y_{k_1+1}, \ldots, y_{k_2}, \ldots, y_n) \) belongs to \( \kappa^{-1}(a) \).

Observe that none of the \( a_i \) can be in the form \( c \circ d \) because \( \sigma \) is the canonical decomposition.

If we have another irreducible object \( y' = \mu(S'; y'_1, \ldots, y'_{\ell}) \), a similar argument shows that it can be covered in \( \kappa^{-1}(a) \) by an object

\[
\mu(M^1_1\mu(T_1; y'_1, \ldots, y'_{k_1})\mu(T_2; y'_{k_1+1}, \ldots, y'_{k_2})\ldots\mu(T_j; y'_{k_{j-1}+1}, \ldots, y'_{\ell})).
\]

So \( a \) has to be in the form \( a_1' \circ \ldots \circ a'_{\ell} \). Again none of the \( a'_i \) can be in the form \( c \circ d \). It means that \( i = i', j = j' \) and \( a_i = a'_i \).

Now, using an obvious induction on \( \text{susp}(T_i) \) we can assume that there is only one irreducible object in \( \kappa^{-1}(a_i) \). So

\[
\mu(T_1; y_{k_{i-1}+1}, \ldots, y_{k_i}) = \mu(T'_i; y'_{k_{i-1}+1}, \ldots, y'_{k_i})
\]

and, hence, \( y = y' \).

\[\blacksquare\]

**Definition 9.2** We will call a morphism \( \sigma : T \to S \) a collapse morphism if \( \sigma_n(v) = \sigma_n(w) \) for any tips \( v, w \) from \( T \). It is a generalised associativity morphism if for any injection \( \eta : M^2_1 \to T \) the composite \( \eta \cdot \sigma \) is an injection or a collapse morphism.

**Lemma 9.6** Let \( \gamma : \mu(S; T_1, \ldots, T_k) \to \pi(\sigma)T \) be a generator corresponding to \( \sigma : T \to S \). Then \( \kappa(\gamma) = \text{id} \) if and only if \( \sigma \) is a generalised associativity morphism. For such \( \sigma \) the permutation \( \pi(\sigma) = 1 \).

**Proof.** The proof is an application of lemma 9.1. \[\blacksquare\]

**Lemma 9.7** Let \( \sigma : T \to S \) be a generalised associativity morphism such that \( T, S \) are pruned trees, \( \sigma_n \) is a surjection and \( S = M^2_1 \). Then \( \text{susp}(T) \geq 1 \).

**Proof.** Suppose \( k = \text{susp}(T) < 1 \). Then \( T = T_1 \varotimes_k T_2 \). Take a tip \( v \) of \( T \) which belongs to \( T_1 \). If \( u \) is a tip of \( T_2 \) then there exists a unique map \( \xi : M^2_k \to T \) which covers \( u \) and \( v \). The composite \( \xi \cdot \sigma : M^2_k \to M^2_1 \) can not be injection, so it must be a collapse morphism. This means, that \( \sigma_n(v) = \sigma_n(u) \). As \( \sigma_n \) is surjective and preserves order we have that \( \sigma_n(v) = 2 \). Because \( v \) is arbitrary we have a contradiction. \[\blacksquare\]

**Lemma 9.8** Let \( \gamma_\sigma : \mu(S; T_1, \ldots, T_k) \to T \) corresponds to a generalised associativity morphism \( \sigma : T \to S \) of trees. Then \( \gamma_\sigma \) is a reduction morphism.
Proof. Suppose first that $T$ is pruned and $\sigma_n$ is surjective. Consider the tip $i$ of $S$. Let $v_i(S) = k$. The last equality means that we can construct an injection $M_k^2 \to S$ which covers the tip $i$. Pulling back $\sigma$ along this injection gives us another generalised associativity morphism $\omega : T' \to M_k^2$ with one of the fibers equal to $T_i$ and satisfying the condition of lemma 9.7. Then we have

$$\text{susp}(T_i) \geq \text{susp}(T') \geq k = v_i(S).$$

By lemma 9.2 it means that $\mu(S; T_1, \ldots, T_k)$ is reducible at the $i$-th place. Doing the reduction again for every tip of $S$ we end up with a reduction morphism from $\mu(S; T_1, \ldots, T_k)$ to $T$. An obvious induction proves that this morphism is equal to $\gamma_\sigma$.

Let now $\sigma : T \to R$ be an arbitrary generalised associativity morphism. Then $\sigma$ can be factorised as $T \xrightarrow{\sigma'} S \xrightarrow{\omega} R$ where $\sigma'$ is surjective in dimension $n$ and $\omega$ is an injection. To do this it is enough to take $S$ equal to a minimal pruned subtree containing the image of $\sigma$. Suppose $S$ has $l$ tips and $\omega_n(m) = i_m$. Then we have a commutative diagram

\[
\begin{array}{ccc}
\mu(R; T_1, \ldots, U_n, \ldots, U_n, \ldots, T_k) & \xrightarrow{\gamma_\sigma} & \mu(R; T_1, \ldots, T_k) \\
\mu(R; T_1, \ldots, T_k) & \xrightarrow{\gamma_{\sigma'}} & \mu(R; T_1, \ldots, U_n, \ldots, U_n, \ldots, T_k) \\
\mu(S; T_1, \ldots, T_l) & \xrightarrow{id} & \mu(R; T_1, \ldots, T_k)
\end{array}
\]

where in $\mu(R; T_1, \ldots, U_n, \ldots, U_n, \ldots, T_k)$ the trees $U_n$ are in places $i_1, \ldots, i_l$. The left vertical composite is a reduction morphism and $\gamma_\sigma'$ is a reduction morphism by previous argument. So is $\gamma_\sigma$.

Let now $G(a)$ be a subcategory of $\kappa^{-1}(a)$ generated by the morphism $\epsilon : e \to U_n$. So the morphisms in $G(a)$ look like

\[
\begin{array}{ccc}
\mu(R; T_1, \ldots, U_n, \ldots, U_n, \ldots, T_k) & \xrightarrow{\gamma_\sigma} & \mu(R; T_1, \ldots, T_k) \\
\mu(S; T_1, \ldots, T_l) & \xrightarrow{id} & \mu(R; T_1, \ldots, T_k)
\end{array}
\]
Let also $R(a)$ be the subcategory of $\kappa^{-1}(a)$ generated by the pruned trees, object $e$ and elementary pure reduction morphisms. Observe, that the objects of $R(a)$ are pruned objects from $\kappa^{-1}(a)$.

**Lemma 9.9** The unique irreducible object $y$ in $\kappa^{-1}(a)$ is the terminal object in $R(a)$.

**Proof.** We already proved that for any object $x$ in $\kappa^{-1}(a)$ there exists a reduction morphism to $y$. If $x$ is from $P(a)$ we can think that this reduction morphism is from $P(a)$ as well by observation after lemma 9.4. Now call the volume $V(T)$ of a tree $T$ the number of its edges. We put also $V(e) = 0$. By induction,

$$V(\mu(T; y_1, \ldots, y_k)) = V(T) + V(y_1) + \ldots + V(y_k).$$

It is not hard to prove that this volume function on trees satisfies the following equation

$$V(S) = V(S_1) + \ldots + V(S_j) - l(k - 1)$$

for the canonical decomposition $S = S_1 \otimes_l \ldots \otimes_l S_j$.

Let us prove that any elementary pure reduction morphism strictly decreases the volume of the objects. For this it is sufficient to prove, that for a morphism of pruned trees $\sigma : T \to S$ with fibers $U_n, \ldots, T_i, \ldots, U_n$ we have

$$V(S) + V(T_i) > V(T).$$

We will prove it by induction on the suspension index of $S$.

The statement is obvious for $\text{sup}(S) = n$, i.e. $S = U_n$. Suppose we have proved it for all trees $S$ with $\text{sup}(S) \geq l + 1$. Let $\text{sup}(S) = l$. This means that $\text{sup}(T) \geq l$ because $\sigma$ has at least one fiber equal to $U_n$. The suspension index of $T$ can not be greater then $l$ either, as $\sigma_n$ is surjective and the fibers are pruned trees. So we have a canonical decomposition

$$S = S_1 \otimes_l \ldots \otimes_l S_j$$
and moreover,
\[ \sigma = \sigma_1 \otimes \cdots \otimes \sigma_j \]
where \( \sigma_p \to T'_p \).

Without loss of generality we can assume that the fibers of \( \sigma_1 \) are \( U_n, \ldots, T_i, \ldots, U_n \) and the fibers of the others \( \sigma_p \) are \( U_n \). So
\[ T'_p = S_p, \ p > 1. \]

Then we have by induction
\[ V(S_1) + V(T_i) > V(T'_1). \]

Next we have
\[ V(T) = V(T'_1) + \ldots + V(T'_j) - l(k - 1) \]
and
\[ V(S) = V(S_1) + \ldots + V(S_j) - l(k - 1) \]

Hence,
\[ V(S) + V(T_i) > V(S) - V(S_1) + V(T'_1) = V(S_1) + \ldots + V(S_j) - l(k - 1) - V(S_1) + \]
\[ + V(T'_1) = V(T). \]

So we can use induction on volume to prove the uniqueness of the reduction morphism to \( y \). It is now easy to do following MacLane’s line of proof of the coherence theorem [24].

In the following lemma the letters \( A \) and \( B \) mean one of the categories \( G(a), P(a) \) or \( R(a) \) and \( A \neq B \) or \( A = B = G(a) \).

**Lemma 9.10** i) If in the composite
\[ x \xrightarrow{f} y \xrightarrow{g} z \]
the morphism \( f \) is from category \( A \) and the morphism \( g \) is from \( B \) then there exists \( g' \) and morphisms \( g' \) from \( B \) and \( f' \) from \( A \) such that the square
\[
\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  g' & \downarrow & g \\
  y' & \xrightarrow{f'} & z
\end{array}
\]
commutes.

ii) If in the composite
\[ x \xrightarrow{f} y \xrightarrow{g} z \]
the morphism $f$ is from category $A$ and the morphism $g$ is from $B$ then there exists $g'$ and morphisms $g'$ from $B$ and $f'$ from $A$ such that the square

\[
\begin{array}{ccc}
y & \xrightarrow{g} & z \\
f \downarrow & & \downarrow f' \\
x & \xrightarrow{g'} & y'
\end{array}
\]

commutes.

This lemma can be proved by an obvious induction and in its turn immediately implies the following lemma.

**Lemma 9.11** Every morphism in $\kappa^{-1}(a)$ can be factorised as $g \cdot p \cdot r$ where $g \in G(a)$, $p \in P(a)$, and $r \in R(a)$.

**Lemma 9.12** The irreducible object $y$ is terminal in $\kappa^{-1}(a)$.

**Proof.** We have to prove the uniqueness of the morphism $x \to y$ in $\kappa^{-1}(a)$. Suppose we have two morphisms from an object $x$ to $y$. Then factorising them by lemma 9.11 we get the following diagram

\[
\begin{array}{ccc}
x & \xrightarrow{g_1} & x_1 \\
\downarrow & & \downarrow p_1 \\
x_2 & \xrightarrow{g_2} & x_2 \\
\downarrow & & \downarrow p_2 \\
& & y_2 \\
\end{array}
\]

By lemma 9.10 this diagram can be completed to

\[
\begin{array}{ccc}
x & \xrightarrow{g_1} & x_1 \\
\downarrow & & \downarrow p_1 \\
x_2 & \xrightarrow{g_2} & x_2 \\
\downarrow & & \downarrow g_2' \\
& & y_2 \\
\end{array}
\]

with commutative squares marked by (*)

Observe, that $y_1'$ and $y_2'$ are both pruned objects, hence $y_1' = y_2'$ and therefore $p_1' = p_2'$ and $r_1' = r_2'$.

It remains to show that the two triangles in the diagram are commutative.

By lemma 9.11 the first triangle can be completed to a commutative square

\[
\begin{array}{ccc}
x & \xrightarrow{g_1} & x_1 \\
\downarrow & & \downarrow p_1 \\
x_2 & \xrightarrow{g_2} & x_2 \\
\downarrow & & \downarrow g_2' \\
& & y_2 \\
\end{array}
\]
Let $r_1'' : z \to y$ be a unique reduction morphism. Then we have $r_1' = r_1'' \cdot r_1'''$.

So

$$g_1'' \cdot r_1' = g_1'' \cdot r_1'' \cdot r_1''' = r_1 \cdot g_1'' \cdot r_1''' .$$

Again by lemma 9.10 we can form a commutative square

\[
\begin{array}{ccc}
y & g_1''' & z \\
\downarrow r_3 & \downarrow & \downarrow r_1'''
\end{array}
\]

Since $y$ is irreducible, we have $r_3 = g_3 = id$. So $g_1''' \cdot r_1''' = id$ and hence $g_1'' \cdot r_1' = g_1'' \cdot r_1'' \cdot r_1''' = r_1$. Analogously, the second triangle commutes. So we have proved our lemma.

**Lemma 9.13** The map from $\text{Ob}(M^n_k)$ to $\text{Ob}(h^n_k)$ which assigns to an object $a$ the terminal object in $\kappa^{-1}(a)$ can be extended to a functor

$$K : M^n_k \to h^n_k .$$

**Proof.** It is enough to construct this functor on generators of $M^n$. Indeed, let

$$\eta^{ij} : (1 \circ 2) \circ (3 \circ 4) \to (1 \circ 3) \circ (2 \circ 4), \ i < j$$

be a generator in $M^n$. Then the terminal object in $\kappa^{-1}((a \circ b) \circ (c \circ d))$ is $\mu(M^2_2; M^2_1, M^2_2)$ and we have a morphism in $h^n$

$$\mu(M^2_2; M^2_1, M^2_2) \to (1342)M^2_2 \otimes_i M^2_2 .$$

which we take as the value of $K$ on $\eta^{ij}$. We leave as an exercise for the reader to check that $K$ respects all the relations between $\eta^{ij}$ given in 8 so we actually get a functor.

Therefore, we proved the lemma and also our theorem 9.2.

**Theorem 9.3** The categorical $n$-operad $H^n$ is contractible, in the sense that the nerve of the category $H^n_{\geq 2}$ is contractible.
Proof. The proof is an easy analogue of the above proof. We can actually prove that the object of $H^n_T$ which corresponds to the tree $T$ is the terminal object of $H^n_T$. This is possible because in the case of $H^n$ the objects correspond only to composable pasting schemes. We skip the details as they are very similar to the proof of theorem 9.2.

10 Eckmann-Hilton formula.

We have a unique map of collections $Tr_n \rightarrow Tr_\infty$. This map generates a $Cut$-operadic map

$$\zeta : h^n \rightarrow h^\infty$$

Restriction along $\zeta$ gives us a functor

$$\zeta^* : Oper_\infty(A) \rightarrow Oper_n(A).$$

Observe, that in the special case $A = V$ we have $\zeta^* = Des_n$.

Lemma 10.1 For $n \geq 2$ the functor $\zeta$ is final (in the sense of [23]).

Proof. The functor $\zeta$ is surjective on objects by construction. Hence, it will be sufficient to prove that for any morphism $f : a \rightarrow b$ in $h^\infty$ and any objects $a', b' \in h^n$ such that $\zeta(a') = a$ and $\zeta(b') = b$ there exist a chain of morphisms in $h^n$

$$b' \leftarrow x_1 \leftarrow \ldots \leftarrow x_{i+1} \leftarrow f' \leftarrow x_i \leftarrow \ldots \leftarrow x_m \rightarrow a'$$

with the following properties:

- there exists an $0 \leq i \leq m$ such that $\zeta(f') = f$;
- the image under $\zeta$ of any other arrow is either a retraction or its right inverse;
- the image under $\zeta$ of a composite of the appropriate morphisms or their inverses gives an identity $\zeta x_i \rightarrow a$;
- the image under $\zeta$ of a composite of the appropriate morphisms or their inverses gives an identity $\zeta x_{i+1} \rightarrow b$.

If it is the case then the following commutative diagram provides a path between any two objects in comma-category of $\zeta$ under an object $a$ from $h^\infty$.
It is clear that it will be enough to show that the above property holds for a generator morphism

\[ f : \mu([k]; [n_1], \ldots, [n_k]) \rightarrow \pi(\sigma)[m] \]

which corresponds to a morphism of ordinals \( \sigma : [m] \rightarrow [k] \).

Let the trees \( T, S \) and \( T_1, \ldots, T_k \) be such that

\[ \zeta(T) = [m], \]
\[ \zeta(S) = [k], \]
\[ \zeta(T_i) = [m_i], 1 \leq i \leq k. \]

Then

\[ \zeta(\mu(S, T_1, \ldots, T_k)) = \mu([k], [m_1], \ldots, [m_k]). \]

Let

\[ T' = M^n_0, \quad S' = M^k_{n-1} \quad \text{and} \quad T'_i = M^m_{0}. \]

Then \( \sigma \) determines a unique morphism \( \sigma' : T' \rightarrow S' \) in \( \Omega_n \) with \( \sigma'_n = \sigma \).

This morphism gives the following morphism in \( h_n \)

\[ f' : \mu(S'; T'_1, \ldots, T'_k) \rightarrow \pi(\sigma)T' \]

with \( \zeta(f') = f \).

There is also a unique morphisms \( S \rightarrow S' \) with \( \zeta_n = id \), which gives a morphism

\[ \xi : \mu(S', S'_1, \ldots, S'_k) \rightarrow S \]

in \( h^n \). Every \( S'_i \) has a unique tip. Hence, we have a morphism in \( h_n \)

\[ \psi : \mu(S', S'_1, \ldots, S'_k) \rightarrow \mu(S'; U_n, \ldots, U_n) \rightarrow S'. \]

Now

\[ \zeta(\xi) = \zeta(\psi) : \mu([k]; [1], \ldots, [1]) \rightarrow [k] \]

is a retraction in \( h^n \). So we get a chain of morphisms in \( h^n \)

\[ \mu(S'; T'_1, \ldots, T'_k) \leftarrow \mu(\mu(S', S'_1, \ldots, S'_k); T'_1, \ldots, T'_k) \rightarrow \mu(S; T'_1, \ldots, T'_k). \]

We continue by choosing a unique morphism \( \phi : T' \rightarrow T \) with \( \phi_n = id \) and construct the other side of the chain analogously. Finally observe, that we have morphisms \( \sigma'_i : T'_i \rightarrow T_i \) with \( (\sigma'_i)_n = id \) which allow us to complete the construction.

**Theorem 10.1** Suppose for a \( \text{Cat-operad} \) \( A \) that the categories \( A_n \) are cocomplete and multiplication in \( A \) preserves colimits in each variables. Then \( \zeta^* \) has a left adjoint \( EH_n \). Moreover, if \( a \) is an internal \( n \)-operad in \( A \) then

\[ (EH_n(a))_k \simeq \operatorname{colim} \tilde{a}_k \]

where \( \tilde{a}_k : h^n_k \rightarrow A_k \) is the operadic functor corresponding to the operad \( a \).
Proof. The case $n = 1$ is well known. For an internal symmetric operad $x$ the internal 1-operad $\zeta^*(x)$ has the same underlying collection as $x$ and the same multiplication for the order preserving maps of ordinals. So the left adjoint to $\zeta^*$ on object $a$ is given by
\[(EH_1(a))_n = \prod_{\Sigma_n} a_n\]
which is the same as the colimit of $\tilde{a}$ over $h_1$ (see description of $h_1$ in section 3).

Let $x : h^n \to A$ be an operadic functor, $n \geq 2$. If we forget about operadic structures on $h^n, h_\infty$ and $A$ we can take a left Kan extension
$$
\begin{array}{ccc}
h^n & \xrightarrow{x} & A \\
\downarrow \zeta & & \downarrow L = Lan_\zeta(x) \\
h_\infty & & \\
\end{array}
$$
of $x$ along $\zeta$ in the 2-category of symmetric $Cat$-collections. Since multiplication $m$ in $A$ preserves colimits in each variables, the following diagram is a left Kan extension
\[
\begin{array}{ccccccc}
h^k_n \times h^k_{n_1} \times \ldots \times h^k_{n_k} & \longrightarrow & A_k \times A_{n_1} \times \ldots \times A_{n_k} & \xrightarrow{m} & A_{n_1+\ldots+n_k} \\
\downarrow \phi_k \times \phi_{n_1} \times \ldots \times \phi_{n_k} & & \downarrow L_k \times L_{n_1} \times \ldots \times L_{n_k} & & \\
h^\infty_k \times h^\infty_{n_1} \times \ldots \times h^\infty_{n_k} & & \\
\end{array}
\]

On the other hand, since $\zeta$ and $x$ are strict operadic functors we have a natural transformation
\[
\begin{array}{ccccccc}
h^k_n \times h^k_{n_1} \times \ldots \times h^k_{n_k} & \longrightarrow & A_k \times A_{n_1} \times \ldots \times A_{n_k} & \xrightarrow{m} & A_{n_1+\ldots+n_k} \\
\downarrow \mu & & \downarrow \phi & & \\
h^\infty_{n_1+\ldots+n_k} & & L & & \\
\end{array}
\]

and by the universal property of Kan extension we have a natural transformation
\[
\rho : m(L_k; L_{n_1}, \ldots, L_{n_k}) \to L_n(\mu)
\]
which determines a structure of lax-operadic functor on \( L \). Moreover, \( \phi \) becomes an operadic natural transformation.

Now the sequence of objects \( L(n) = L([n]) \) has a structure of an internal symmetric operad in \( A \). For a map of ordinals \( \sigma : [n] \to [k] \) let us define an internal multiplication \( \lambda_\sigma \) by the composite

\[
m(L(k); L(n_1), \ldots, L(n_k)) \to L(\mu([k], [n_1], \ldots, [n_k])) \to L(\pi(\sigma)[n]) = \pi(\sigma)L(n).
\]

Let us denote this operad \( L(x) \).

The calculation of \( L(k) \) can be performed by the classical formula for pointwise left Kan extension \([24]\). It is therefore \( \text{colim}_{f \in \zeta/[k]} \delta \), where \( \delta(f) = x(S) \) for an object \( f : \zeta(S) \to [k] \) of the comma category \( \zeta/[k] \). But according to the remark after theorem 8.1, \([k]\) is a terminal object of \( h^\infty_k \) and therefore

\[
\text{colim}_{f \in \zeta/[k]} \delta \simeq \text{colim} \ x_k.
\]

It remains to prove that the internal operad \( L(\tilde{a}) \) is \( EH_n(a) \). Indeed, for a given operadic morphism \( L(\tilde{a}) \to b \) the composite

\[
\tilde{a} \overset{\phi}{\to} \zeta^* L(\tilde{a}) \to \zeta^* \tilde{b}
\]

is operadic since \( \phi \) is operadic. But \( \zeta \) is final and, therefore, the counit of the adjunction \( \zeta^* \dashv \text{Lan}_\zeta \) is an isomorphism. So for a given operadic morphism \( \tilde{a} \to \zeta^* \tilde{b} \) of internal \( n \)-operads the morphism

\[
L(\tilde{a}) \to L(\zeta^* \tilde{b}) \simeq \tilde{b}
\]

is operadic, as well. So the proof of the theorem is completed.

As another immediate application of this formula we provide the following

\[
\text{Theorem 10.2 For an } n\text{-operad } A \text{ in a closed symmetric monoidal category } V \text{ its } n\text{-suspension is given by } EH_n(A).
\]

\textbf{Proof.} Indeed, for an object \( x \in V \), an \( \sigma \)-algebra structure is given by a morphism of operads

\[
k : A \to \text{End}_n(x).
\]

By proposition \([4.1]\)

\[
\text{End}_n(x) \simeq \text{Des}_n(\text{End}(x))
\]

and so \( k \) determines and is uniquely determined by a map of symmetric operads

\[
EH_n(A) \to \text{End}(x).
\]
Theorem 10.3 Let \( A \) be a contractible \( n \)-operad in the category of compactly generated Hausdorff spaces and let \( X \) be an algebra of \( A \). Then \( x \) has a structure of an \( E^n \)-space, so up to group completion \( x \) is an \( n \)-fold loop space. Contractibility means here that every \( A_k \) is a contractible topological space.

Proof. By the previous theorem \( x \) is an algebra of \( EH_n(A) \) and by the Eckmann-Hilton formula \( EH_n(A) \simeq \colim h_k \). It is not hard to check that the sequence \( \hocolim \hat{A}_k \) has a structure of an operad and, moreover, the canonical map

\[
\hocolim \hat{A}_k \to \colim \hat{A}_k
\]

is operadic. But \( \hocolim \hat{A}_k \) has the same homotopy type as \( h_k^n \) because of contractibility of \( A_k \). So \( x \) is an algebra of a \( E^n \)-operad \( \hocolim h_k \).

Example 10.1 It is still possible for \( x \) from the previous theorem to be an \( E^m \)-space for \( m > n \). For example, if \( B \) is any \( E_\infty \)-operad, then \( A = Des_n(B) \) is contractible but \( EH_n(A) \simeq B \). So the question arises when the \( n \)-fold suspension of a contractible topological \( n \)-operad is a genuine \( E_n \)-operad?

We can answer positively in some cases.

Example 10.2 The first case is our categorical \( n \)-operad \( H^n \). It is almost a tautology that

\[
EH_n(H^n) \simeq h^n.
\]

Indeed, if in the condition of the theorem \( A = Cat \) (see example 4.1) then

\[
SCO(EH_n(H^n), X) \simeq \text{Oper}_\infty(EH_n(H^n), X) \simeq \text{Oper}_n(H^n, Des_n(X)) \simeq CO_n(H^n, Des_n(X)) \simeq O_n(Des_n(X)) \simeq \text{Oper}_n(X) \simeq SCO(h^n, X)
\]

for any symmetric Cat-operad \( X \).

So the geometric realization of \( H^n \) provides us with an example of a contractible topological \( n \)-operad with a \( E_n \)-operad as its \( n \)-fold suspension.

Actually, the operad \( M^n \) is also an \( n \)-fold suspension over some categorical operad. One can show (and it is actually the essence of the proof of theorem 9.2) that \( M^n \) is the representing object for a functor from \( SCO \) to \( Cat \) which picks up some special semistrict internal \( n \)-operads in the sense that multiplication in such an operad must be the identity in the case when \( \sigma : T \to S \) is a generalised associativity morphism. In addition, these operads must have strict units (we do not specify here the exact meaning of this statement). It is also not hard to construct a strictified analogue of \( H^n \) by generators and relations. It is also a contractible categorical operad with finite poset for every tree. If \( n = 2 \) one can
show that this poset is closely related to the poset of faces of the polytope \( P_T \)
and it is just the poset of faces of permutohedron \( P_k \) for \( T = M^k_0 \).
For semistrict \( n \)-operads the Eckman-Hilton formula has the form

\[
EH_n(a) = \colim_{\mathcal{M}_k^n} \tilde{a}_k
\]

which is much better since \( \mathcal{M}_k^n \) is a finite poset.

Examples of semistrict 2-operads are the Gray-categories operad \( G \) and the
shuffle polytopes operad \( P \). One can see geometrically how the Eckman-Hilton
formula works in the case of \( P \), for example. The second space of 2-fold suspen-
sion of \( P \) is the following colimit

\[
\begin{array}{ccc}
1 \circ 2 & \to & 1 \\
1 \circ 2 & \to & 2 \circ 1 \\
2 \circ 1 & \to & 1 \\
\end{array}
\]

which is topologically a circle.

**Example 10.3** For the Gray-categories 2-operad, its 2-fold suspension is a
symmetric \( \text{Cat} \)-operad for braided monoidal categories, by the Eckmann-Hilton
argument from [18]. It is, of course, an \( E_2 \)-operad.

**Example 10.4** In order to prove that the operad \( \mathcal{M}_n \) is homotopy equiva-
 lent to the operad of little \( n \)-cubes Balteanu, Fiedorovicz, Schwänzl and Vogt
constructed a functor [3, Definition 6.4]

\[ F : \mathcal{M}_n \to \text{Top} \]

based on some special configurations of little \( n \)-cubes called decomposable con-
figurations. It is proved that the values of \( F \) are contractible spaces. It is not
hard to check using [3, Lemma 6.11], that the collection \( \mathcal{F}_T = \{ F(a_T) \}, T \in Tr_n \)
has a structure of a topological \( n \)-operad. Moreover, one can prove that

\[
\colim_{h^k} \tilde{F}_k \simeq \colim_{\mathcal{M}_k^n} F_k
\]

and, hence, is isomorphic to \( D_k^n \), where \( D_k^n \) is the \( k \)-th space of the operad of
decomposable little \( n \)-cubes [3] which in its turn is homotopy equivalent to the
little \( n \)-cube operad.

In general, we want to have the conditions which ensure that the canonical
map \((\ast)\) is a weak homotopy equivalence. To be able to find such conditions we
are going to develop a homotopy theory of \( n \)-operads in a symmetric monoidal
Quillen model category. This will be done in a further paper.
Example 10.5 Using corollary [9.1.1] it is not hard to check that a contractible topological $n$-operad can be constructed out of any cellular operad in the sense of [9] using a method described in the previous remark. It is not, however, clear how an $n$-fold suspension of this $n$-operad is related to the original operad in general. It would be highly desirable to clarify this point.

References


