

Research statement

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1. OVERVIEW

My research focuses on quantitative descriptions of diffusive processes, obtaining critical thresholds that distinguish qualitatively different phenomena. The tools that I employ include asymptotic analysis, PDE techniques, numerical and continuation methods, and Monte Carlo simulations. One of my main interests is in the modeling of and techniques for analyzing first passage time (FPT) problems. Arising often in ecological and cellular processes, these problems seek either the mean or the full distribution of times at which a random walker will first encounter a target. The vast majority of work in this field has focused on models with *stationary* targets, even though many scenarios lend naturally to models involving *mobile* targets. A 2014 review article [1] noted the modeling and analysis of mobile targets as an important new direction.

In our work, we have recently obtained very interesting results in this new direction. Mobile target problems introduce many new challenges. One must derive the correct boundary value problem with proper geometry, with various tools required to analyze the problem. Results are rich and often counterintuitive. For example, our results are the first to demonstrate that target motion can be *detrimental* for search. FPT problems are relevant and accessible to researchers and students from many backgrounds. As such, this field of research is conducive to student training and interdisciplinary collaboration. To the latter point, we are currently collaborating with biologists to apply some of our methodologies to their model of T cell receptor triggering.

The other main component of my research is analyzing bifurcations and stability of weakly and fully nonlinear patterns in reaction-diffusion systems. We have studied the phenomenon of delayed bifurcations of fully nonlinear patterns [2], noise-driven vegetation patterns [3], 3D pattern formation [4], and mechanisms of vortex formation in a Bose-Einstein condensate model [5].

My works have ranged from a strong focus on applications to more emphasis on analytic methods in PDEs. From an applied analysis perspective, I have worked on problems that require very involved asymptotic methods and analysis of nonlocal eigenvalue problems. Numerically, I have used Fourier spectral methods in tandem with numerical continuation tools to obtain very novel results on snaking bifurcations of time-periodic solutions [6]. From a modeling perspective, I have derived equations for first passage times with mobile targets, and confirmed solutions using Monte Carlo simulations. As such, I am able to work with a variety of researchers and contribute to collaborations in many ways.

2. FIRST PASSAGE TIME PROBLEMS

Calculating first passage times answers the following question: how long will it take a target and a randomly diffusing particle to find each other? While works in the past have treated targets as stationary, there are many examples in which the targets are known to be mobile. For example, the critical threshold at which an action potential is initiated has been shown to vary in time due to external stimuli and memory effects from a previous triggering [7] (see Fig. 1). A 2D example is one in which a nucleus moves before and after mitosis, presenting diffusing proteins with a mobile target [7].

It has been shown on an infinite domain [8] that, compared to stationary targets, mobile targets are found by Brownian particles more quickly. In contrast, our results outlined in §2.1 are the first to quantify effects of target

mobility on a *bounded* domain and to show that target motion can *increase* search times. Recently, we have worked on obtaining the full distribution of possible search times rather than computing only the mean. While this problem requires solution of a time-dependent parabolic PDE, we use the method of Laplace transforms to map it into one of elliptic type, which we are then able to solve using asymptotic techniques. This is outlined in §2.2. Having done this for a 2D domain [9], we plan next to apply similar techniques in the 3D setting.

2.1. MOBILE TARGETS IN 1D AND 2D

In [10], we considered the following two problems: for a Brownian particle starting its walk from location x_0 inside the reflective 1D domain $(0, 1)$, how long on average will it take to reach (1) a target oscillating periodically about $y_0 \in (0, 1)$, and (2) a Brownian target starting a random walk from $y_0 \in (0, 1)$? Crucially, how do the search times compare to those with a *stationary* target located at y_0 ?

The key in both (1) and (2) was to reformulate a problem of time-dependent nature in terms of a stationary problem. In (2), the Brownian dynamics of both particle and target allowed the mapping to a single 2D random walk on a triangular domain. In (1), we extended the problem into 2D and derived an elliptic equation with mixed Neumann-Dirichlet-periodic boundary conditions (see Fig. 2).

In limits amenable to asymptotic analysis, we calculated critical velocity thresholds below which target motion becomes detrimental to search times. Our analysis of multiple oscillating targets in [11] shows that the optimal cooperation strategy between the targets continues to change with increasing oscillation frequency.

In [12], on a reflective unit disk in two dimensions with a small target of size $\varepsilon \ll 1$ rotating at distance $r_0 \in (0, 1)$ from the center with angular frequency ω , we sought to solve the optimization problem: for a given ω , what is the radius of rotation r_0^{opt} that minimizes a certain measure of FPT?

The rotational symmetry of the problem made it amenable to a hybrid asymptotic-numerical analysis of a certain 2D elliptic equation. One of our main findings was that when its rate of rotation is below a certain threshold, the mobile target is no longer optimal. Instead, the optimal strategy is to remain stationary at the center of the disk, again demonstrating explicitly that target motion, when too slow, can hinder search. Another main counterintuitive result was that in a certain high rotation frequency regime, the optimal strategy is to rotate asymptotically near the boundary of the unit disk, far from the central location.

These examples serve to illustrate that even simple problems involving mobile targets can yield very unexpected results. The more we understand these basic test problems, the more insights we gain on more general target motion and optimal search strategies.

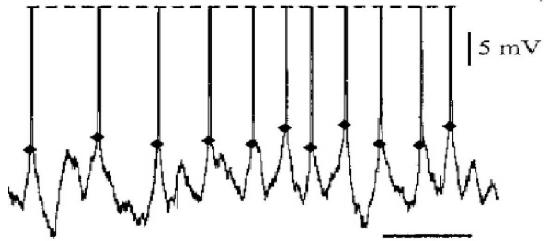


Figure 1. The potential across a membrane undergoes random dynamics until it reaches a critical threshold (black diamonds), at which time an action potential is triggered (spike). The thresholds vary in time (figure from [7]).

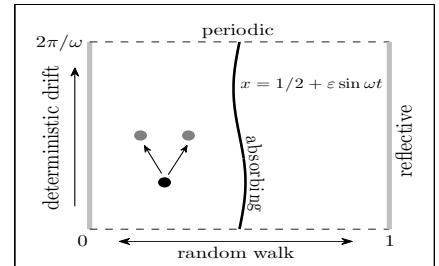


Figure 2. Schematic of the 2D domain on which a 1D moving target problem is posed.

2.2. THE FULL FIRST PASSAGE TIME DISTRIBUTION

In certain applications [13], the mean of the first passage time does not yield sufficient information. In scenarios with small targets, which arise often in the modeling of cellular processes, the mean is reflective only of the fraction of search times that are asymptotically large, yielding little to no information regarding the significant fraction of searches that complete in short time (see Fig. 3).

In [9], we developed a hybrid asymptotic-numerical method for computing the full probability distribution of search times of a random walker in a 2D domain with small targets. We further demonstrated a fully asymptotic method for capturing the critical small time attributes of the distribution, in particular the height and location of the mode. We are currently collaborating with a team of biologists to apply some of these methods to a model of T cell receptor triggering inside a growing domain.

2.3. FUTURE DIRECTION

The area of FTP problems with mobile targets in bounded domains remains new. The main challenge lies in formulating the correct boundary value problem. Asymptotic solutions may be obtained by exploiting certain symmetries and considering special regimes so that regular and singular perturbation methods can be applied in tandem with hybrid asymptotic-numerical methods.

In two dimensions, we plan to work out the effect of a partially absorbing rotating target on FPT. In a recent work [14], we considered a partially absorbing target that remained stationary. The effect of its orientation on FPT was apparent, though small. From numerical solutions, we found that its impact became large under rotation. We can capture this effect analytically by building on the matched asymptotic methods of [14].

Over the longer term, we plan to utilize our method of formulating stationary boundary value problems for mobile targets to quantify effects of more general target motion. In the proper regime, these problems can be treated using a combination of regular and singular perturbation methods, along with hybrid asymptotic-numerical methods like that employed in [9]. For these projects, we are currently planning a collaboration with a computational mathematician who works on closest point methods for solving PDEs on surfaces and domains with moving boundaries [15].

Furthering collaborations with biologists is also a top priority. My contributions would be in helping formulate models and providing numerical and asymptotic tools to solve them. In cases where stochastic simulations of rare events are involved (e.g., escape of ions through a narrow ion channel), FPT results have been shown to be effective in replacing the costly simulations with cheaper Poisson processes [16].

3. PATTERN FORMATION

Analysis of pattern formation usually occurs in two regimes. In the small amplitude regime near a bifurcation point of a homogeneous state, weakly nonlinear theory is used to describe the slow evolution of an unstable perturbation of the homogeneous state. In large amplitude regimes far from the homogeneous state, an assumption of separation of spatial scales allows for application of singular perturbation theory. Analysis can yield stability

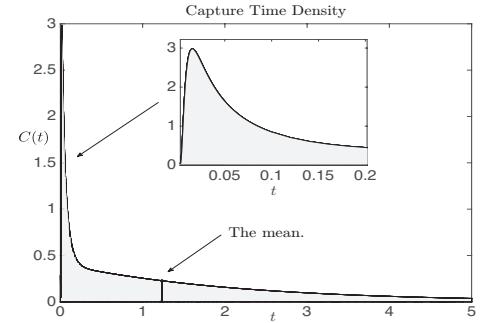


Figure 3. Full probability distribution of search times to a small target. Inset: zoom-in of the small-time behavior, which our asymptotic method is able to capture.

thresholds for slow (drift) and fast (amplitude) instabilities. Below is an outline of work that we have done in both of these regimes.

3.1. DELAYED BIFURCATIONS OF LARGE AMPLITUDE PATTERNS

Slow passage through a bifurcation causes the instability to only manifest when the bifurcation parameter has been tuned well past the threshold predicted by linear stability. It may even not manifest at all. This “delay” in the bifurcation was first quantified in [17]. In a laboratory setting, the slow passage may be an experimenter slowly tuning a feed rate of a chemical reaction to determine when a homogeneous state loses stability to a patterned state. The delay may cause the experimenter to misjudge the stability threshold or to miss it entirely. Analysis of this phenomenon has been restricted to ODE’s, where explicit expressions of eigenvalues may be readily obtained.

In PDE systems exhibiting large amplitude spike patterns, the linear stability problem requires analysis of a nonlocal eigenvalue problem (NLEP) from which the eigenvalues are impossible to compute analytically. However, a special class of *explicitly* solvable NLEP’s recently discovered [18] allows one to analytically extract the eigenvalue. In [2], we exploit this property to describe three examples of delayed bifurcations of spike patterns. An illustration of one of the results is shown in Fig. 4. As the parameter τ is tuned from below to above the Hopf bifurcation threshold τ_H , spike amplitude oscillations cannot be distinguished until τ is tuned well past τ_H . The explicitly solvable NLEP allowed us to analytically predict the delay, which can be seen to agree favorably with that observed numerically. This work was the first to quantify delayed bifurcations in PDEs, and motivated the following study of the interaction between delayed bifurcations and stochastic noise.

3.2. EFFECT OF NOISE ON PATTERNED VEGETATION

One of the mechanisms by which vegetation adapts to stresses of decreasing precipitation is by taking refuge in a patterned state, leaving barren land interspersed by patches of vegetation [19]. This strategy conserves scarce resources while allowing recovery to full vegetation coverage when precipitation returns to normal levels. A rapid decrease in precipitation levels leaves insufficient time for this adjustment to occur, with the result being a collapse to a desert state with little chance of recovery even up on a return to previous precipitation levels [20].

In [3], we quantified this dynamic by studying the competing effects of a delayed Turing bifurcation and spatial noise in a modified Klausmeier model [21]. The main result is illustrated in Fig. 5. When precipitation a decreases past a Turing instability $a = a_p$, spatial noise is required to initiate pattern growth. Only sufficiently slow precipitation decline will allow enough time for this to occur (green path). Otherwise, due to the delay in Turing bifurcation, the system encounters a saddle node and collapses to the desert state (red path). When precipitation returns to normal levels, only the green path shows recovery to a full vegetation state; the red path is irreversibly caught in the desert state. We derived a critical relationship between the level of noise and the rate of precipitation decline that would allow the green scenario to occur. Further, we predicted analytically when the solution would jump to the patterned state (a_d in Fig. 5).

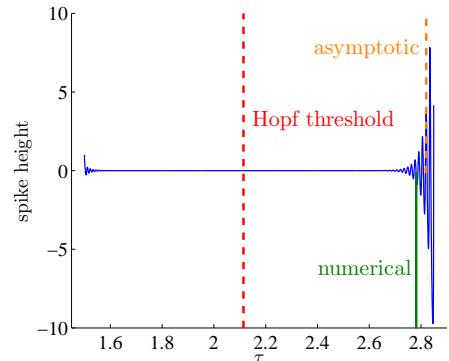


Figure 4. Plot of spike height versus τ . The red line indicates the Hopf bifurcation value τ_H . The green line indicates the value of τ at which the bifurcation is fully triggered as τ is increased. The orange line shows the asymptotically predicted value.

3.3. SPOT PATTERNS IN THREE DIMENSIONS

In [4], we gave the first results for the dynamics and stability of spot patterns in a 3D domain. Using a hybrid asymptotic-numerical approach, we computed quasi-equilibrium spot profiles, derived explicit thresholds for competition (annihilation of spots) and self-replication (splitting of spots) instabilities. We also obtained a coupled ODE system for the slow drift dynamics of multiple spots, finding that the spots evolve as interacting particles according to a gradient flow driven by a discrete energy. The minima of this energy are thus equilibrium configurations of the multi-spot pattern. A distinguishing feature of 3D patterns from their 1D and 2D counterparts is that all spots are identical to leading order regardless of configuration. An intricate analysis is thus required to fully resolve the dependence of stability thresholds on the pattern configuration.

3.4. WEAKLY NONLINEAR ANALYSIS OF VORTEX FORMATION

Vortex formation in a trapped Bose-Einstein condensate under rotation has recently been the subject of analytic, numerical, and experimental studies. One of many analytic approaches was adopted in [22], where a multi-scale stability analysis of a 2D dissipative Gross-Pitaevskii equation (DGPE) showed that an early vortex state emerged as a high-mode instability of a symmetric ground state. It was observed numerically that this instability evolved to select only a few of these vortices (formed at the periphery of the atomic cloud) to grow and move into the bulk.

In [5], we took a significant step toward an analytic description of this early selection mechanism. We used weakly nonlinear theory to obtain a 1D description of the dynamics of the early vortex state near the periphery. Remarkably, we found that although the original dynamics pertain to a 2D self-defocusing DGPE, the reduced 1D azimuthal evolution is of a self-focusing nature. We showed that a subsequent modulational instability was the mechanism that triggered the transition between the early peripheral vortex state and the bulk-vortex steady state. This work sets the stage for a full description of vortex dynamics near the trap periphery.

3.5. FUTURE DIRECTION

In almost all previous studies of localized pattern formation, the assumption has been that the system is insulated from its surroundings, and that any influx of material from the bulk is spatially uniform. An example of a nonuniform bulk feed rate arises in the spatially dependent distribution of ground water due to varying precipitation and slope gradients in a patterned vegetation field [20, 23]. Our current work analyzes the effects of nonuniform feed rates and interaction with environment on the dynamics and equilibrium configurations of spot patterns in a 2D reaction-diffusion system.

In our abovementioned work on vortex formation, we assumed a radially symmetric condensate cloud that led to a uniform formation of vortices around the periphery. The reduction to 1D dynamics near the periphery suggested a topological insulation of the system's boundary. We plan to investigate the effects of an anisotropic

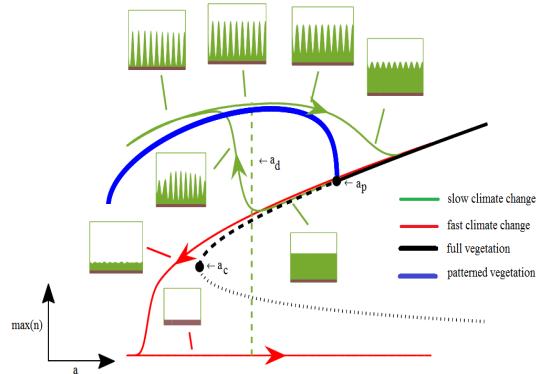


Figure 5. As precipitation decreases, the homogeneous state (black) loses stability to a patterned state (solid blue) at $a = a_p$. If the rate of decrease is too fast (red curve), the delay in Turing bifurcation causes the system to “tip over” the saddle node into the full desert state. If the rate of decrease is sufficiently slow (green curve), there is enough time for spatial noise to perturb the system into the patterned state.

cloud, where we have observed numerically that vortices preferentially nucleate at regions of higher curvature. In particular, we are interested in whether the topological insulation is a consequence of the radial symmetry of the trap or inherent to the DGPE.

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