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Interaction of Turing and Hopf modes in the superdiffusive Brusselator model ☆

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ABSTRACT

Long-wave stability of spatiotemporal patterns near a codimension-2 Turing–Hopf point of the one-dimensional superdiffusive Brusselator model is analyzed. The superdiffusive Brusselator model differs from its regular counterpart in that the Laplacian operator of the regular model is replaced by $\partial^\alpha/\partial|\xi|^\alpha$, $1 < \alpha < 2$, an integro-differential operator that reflects the nonlocal behavior of superdiffusion. The order of the operator, α, is a measure of the rate of superdiffusion, which, in general, can be different for each of the two components. We first find the basic (spatially homogeneous, time independent) solution and study its linear stability, determining both Turing and Hopf instabilities, as well as a point at which both instabilities occur simultaneously. We then employ a weakly nonlinear stability analysis to derive two coupled amplitude equations describing the slow time evolution of the Turing and Hopf modes. We seek special solutions of the amplitude equations: a pure Turing solution, a pure Hopf solution, and a mixed mode solution, and analyze their stability to long-wave perturbations. We find that the stability criteria of all three solutions depend strongly on the superdiffusion rates. Also, when compared to the regular model and depending on specific values of the orders of the operators, the effect of anomalous diffusion may change the stability characteristics of the special solutions.

1. Introduction

Studies of anomalous diffusion have recently been appearing in the literature as more processes have been observed to exhibit behavior that cannot be described in terms of regular (Fickian) diffusion. These processes can often be described using models with subdiffusion or superdiffusion, where, under a random walk description, the mean square displacement of a particle scales as $\langle x^2(t)\rangle \sim t^\gamma$, with $0 < \gamma < 1$ for subdiffusion, and $1 < \gamma < 2$ for superdiffusion, rather than linearly in time. Subdiffusion has been observed in many applications, including charge carrier transport in amorphous semiconductors, and nuclear magnetic resonance diffusometry in percolative and porous systems, while superdiffusion has been observed in e.g. transport in heterogeneous rocks, quantum optics, and single-molecule spectroscopy [1]. We consider an especially interesting case of superdiffusion, Lévy flights, which is characterized by a jump length distribution having infinite moments. On the macroscopic scale, Lévy flights are described by a diffusion equation where the second-order spatial derivative is replaced by a fractional derivative $\partial^\alpha/\partial|\xi|^\alpha$, $1 < \alpha < 2$, defined as a non-local integro-differential operator [2]. Previous works on reaction–superdiffusion equations have derived and studied amplitude equations near a Hopf [3] or Turing [2] bifurcation point. There have also been similar studies near a codimension-2 Turing–Hopf point (C2THP) of the regular Brusselator model [4]. In this work, we investigate the effects of superdiffusion on the interactions between Hopf and

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Turing instabilities of the Brusselator model by deriving amplitude equations analogous to those given in [4] and studying instabilities of their solutions to long-wave perturbations, thus leading to the identification of parameter values at which new solutions may bifurcate.

2. The model, the basic solution, and its linear stability

We consider the Brusselator model, long a paradigm for nonlinear analysis, given by

$$\frac{\partial f}{\partial \tau} = D_f \frac{\partial^\alpha f}{\partial |\xi|^\alpha} + E - (B+1)f + f^2 g \tag{1}$$

$$\frac{\partial g}{\partial \tau} = D_g \frac{\partial^\beta g}{\partial |\xi|^\beta} + Bf - f^2 g, \quad \tau > 0, \quad \xi \in \mathbb{R}. \tag{2}$$

The diffusion coefficients D_f, D_g, the activator input rate E, and the control parameter B are positive quantities. The action of the operator $\partial^\gamma/\partial|\xi|^\gamma$ in Fourier space is $\mathscr{F}[\partial^\gamma u(\xi)/\partial|\xi|^\gamma](k) = -|k|^\gamma \mathscr{F}[u(\xi)](k)$ for $1 < \gamma < 2$. The equilibrium (basic) state of this system is $(f, g) = (E, B/E)$ for all values of the parameters.

Rescaling (1) and (2) using $f = E + u_* u$, $g = B/E + v_* v$, $\tau = t$, and $\xi = \ell_* x$, where $u_* = (D_g/D_f^{\beta/\alpha})^{1/2}$, $v_* = 1/u_*$, and $\ell_* = D_f^{1/\alpha}$, the Brusselator system becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^\alpha u}{\partial |x|^\alpha} + (B-1)u + Q^2 v + \frac{B}{Q}u^2 + 2Quv + u^2 v \tag{3}$$

$$\eta^2 \frac{\partial v}{\partial t} = \frac{\partial^\beta v}{\partial |x|^\beta} - Bu - Q^2 v - \frac{B}{Q}u^2 - 2Quv - u^2 v, \tag{4}$$

where $\eta = \sqrt{D_f^{\beta/\alpha}/D_g} > 0$, $Q = E\eta > 0$, and x and t represent the rescaled spatial and temporal variables, respectively. The equilibrium state is now at $u = v = 0$.

To determine the stability of the critical point, we consider the normal mode solution, obtaining the dispersion relation between the growth rate σ and the wavenumber $k > 0$, $\eta^2\sigma^2 + M_1\sigma + M_2 = 0$, where $M_1 = Q^2 + k^\beta - \eta^2(B - 1 - k^\alpha)$, and $M_2 = BQ^2 + (k^\beta + Q^2)(1 + k^\alpha - B)$.

Hopf bifurcation occurs if $M_1 = 0$ and $M_2 > 0$, which yields two pure imaginary eigenvalues. $M_1 = 0$ corresponds to $B = k^\beta/\eta^2 + k^\alpha + 1 + Q^2/\eta^2$, which has a minimum, $B_{cr}^{(H)} = 1 + Q^2/\eta^2$ at $k = 0$. The basic state is stable (unstable) for $B < B_{cr}^{(H)}$ ($B > B_{cr}^{(H)}$). In the unstable case, a spatially homogeneous oscillatory mode emerges. For $k = 0$ and $B = B_{cr}^{(H)}$, the eigenvalue $\sigma = \mathrm{i}Q/\eta \equiv \mathrm{i}\omega$, where ω is the frequency of the oscillatory mode, while $\mathbf{c}^\dagger = \big(1, Q\eta^2/(Q + \mathrm{i}\eta)\big)$ and $\mathbf{c} = \big(1, (\mathrm{i}\eta - Q)/(Q\eta^2)\big)^\top$ are the left and right eigenvectors, respectively.

Turing instability occurs when $M_2 = 0$, which yields $B = (Q^2 + k^\beta)(1 + k^\alpha)/k^\beta$. It has a single minimum $(k_{cr}, B_{cr}^{(T)})$, given parametrically by

$$B_{cr}^{(T)} = \frac{(1+z)^2}{1 + (1-s)z}, \qquad Q^2 = \frac{sz^{1+1/s}}{1 + (1-s)z}, \quad k_{cr} = z^{1/\alpha},$$

where $s = \alpha/\beta$. Since Q is real, we find that $0 < z < \infty$ if $1/2 < s \leq 1$, and $0 < z < 1/(s-1)$ if $1 < s < 2$. The corresponding left and right eigenvectors of the zero eigenvalue are, respectively, $\mathbf{a}^\dagger = \big(1, sz\eta_c^2/(1+z)\big)$ and $\mathbf{a} = \big(z^{1/s}, -1-z\big)^\top$. For the Turing instability, a time-independent spatially periodic pattern emerges, with spatial wavenumber $k = k_{cr}$.

Turing and Hopf instability thresholds coincide at the C2THP where $B = B_{cr}^{(T)} = B_{cr}^{(H)} \equiv B_{cr}$, which occurs when $\eta = \eta_c \equiv \sqrt{sz^{1/s}/(z + s + 1)}$. Thus, as the control parameter B is increased beyond B_{cr}, a Turing mode and a Hopf mode simultaneously bifurcate from the basic state, giving rise to terms of the form $A\mathbf{a}e^{\mathrm{i}k_{cr}x}$ and $C\mathbf{c}e^{\mathrm{i}\omega t}$, respectively, in $(u, v)^\top$. We note that Q, η_c, and the activator input rate, $E = Q/\eta_c$, are increasing functions of z for all allowed values of z.

3. Weakly nonlinear analysis

We analyze the system near the C2THP, i.e., let $\eta = \eta_c + \epsilon^2\eta_2$ $(0 < \epsilon \ll 1)$. If $\eta_2 > 0$ (< 0), the Hopf (Turing) mode appears first as the parameter B is increased. We interpret this as changing the parameter E, keeping Q constant. Thus, changing η_2 will only affect the Hopf stability curve, not the Turing curve. Also, let $B = B_{cr} + \epsilon^2\mu$, where $\mu > 0$ is a real $O(1)$ quantity. This leads to the presence of two time scales. The original time scale, t, appears with oscillation frequency ω, while the slow time scale, $T = \epsilon^2 t$, accounts for the slow time evolution of the Turing and Hopf modes. The three relevant spatial scales are x, $X_1 = \epsilon x$, and $X_{2/\alpha} = \epsilon^{2/\alpha}x$, where the scaling for $X_{2/\alpha}$ is chosen under the condition that $\alpha < \beta$. If $\alpha > \beta$, the third spatial scale would instead be $X_{2/\beta}$. While we consider both cases in Section 4, the explicit expressions are for $\alpha < \beta$.

With the relevant scales established, we allow for the possibility that both A and C are functions of the slow time scale as well as the two long spatial scales. Then, since the Turing mode may be a function of all three spatial scales and the Hopf mode a function of the two long spatial scales, we require analogues of the chain rule to obtain expressions for how the operator $\partial^\gamma/\partial|x|^\gamma$ acts on u and v. While nothing in the linear stability analysis of Section 2 prevents the Hopf mode from being a function of the two long spatial scales, solvability conditions discussed below in the weakly nonlinear analysis limit the Hopf mode dependence to $X_{2/\alpha}$ only. Then, since the expression obtained by applying $\mathrm{d}^\gamma/\mathrm{d}|x|^\gamma$ to a function of the form $F(x, X_1, X_{2/\alpha})$ does not reduce to the expression obtained by applying the operator to a function of the form $G(X_{2/\alpha})$ simply by letting $\partial/\partial x = \partial/\partial X_1 = 0$, we decompose the solutions u and v into sums of functions of the form $F(x, X_1, X_{2/\alpha}, t, T)$ and $G(X_{2/\alpha}, t, T)$. Since F accounts for all x-dependent terms, whether or not they depend on X_1 and/or $X_{2/\alpha}$, while G accounts for all x-independent terms, this decomposition captures all possible terms that can arise in u and v. We utilize the product rule [5] for $1 < \gamma < 2$,

$$\frac{\mathrm{d}^\gamma(fg)}{\mathrm{d}|x|^\gamma} = \sum_{j=0}^{\infty} \binom{\gamma}{j} \frac{\mathrm{d}^{\gamma-j}f}{\mathrm{d}|x|^{\gamma-j}} \frac{\mathrm{d}^j g}{\mathrm{d}x^j},$$

to compute (suppressing time dependence for now)

$$\frac{\mathrm{d}^\gamma F(x, X_1, X_{2/\alpha})}{\mathrm{d}|x|^\gamma} = \left(\frac{\partial^\gamma}{\partial|x|^\gamma} + \gamma\frac{\partial^{\gamma-1}}{\partial|x|^{\gamma-1}}\left(\epsilon\frac{\partial}{\partial X_1} + \epsilon^{2/\alpha}\frac{\partial}{\partial X_{2/\alpha}}\right) + \epsilon^2\frac{\gamma(\gamma-1)}{2}\frac{\partial^{\gamma-2}}{\partial|x|^{\gamma-2}}\frac{\partial^2}{\partial X_1^2} + \cdots\right)F(x, X_1, X_{2/\alpha}), \tag{5}$$

where we have discarded terms smaller than $O(\epsilon^2)$. The computation of $\mathrm{d}^\gamma G(X_{2/\alpha})/\mathrm{d}|x|^\gamma$ requires a simpler version of the chain rule, which gives $\mathrm{d}^\gamma G/\mathrm{d}|x|^\gamma = \epsilon^{2\gamma/\alpha}\mathrm{d}^\gamma G/\mathrm{d}|X_{2/\alpha}|^\gamma$, where γ is either α or β.

Due to the fractional powers of ϵ in (5), we include fractional powers in the expansions of u and v:

$$\begin{pmatrix} u \\ v \end{pmatrix} \sim \epsilon\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \epsilon^{2/\alpha}\begin{pmatrix} u_{2/\alpha} \\ v_{2/\alpha} \end{pmatrix} + \epsilon^2\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} + \epsilon^{1+2/\alpha}\begin{pmatrix} u_{1+2/\alpha} \\ v_{1+2/\alpha} \end{pmatrix} + \epsilon^3\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} + \cdots. \tag{6}$$

We decompose u_i and v_i as

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} u_i^{(A)}(x, X_1, X_{2/\alpha}, t, T) \\ v_i^{(A)}(x, X_1, X_{2/\alpha}, t, T) \end{pmatrix} + \begin{pmatrix} u_i^{(C)}(X_{2/\alpha}, t, T) \\ v_i^{(C)}(X_{2/\alpha}, t, T) \end{pmatrix}, \tag{7}$$

where we associate the letter A with the Turing mode (though $u_i^{(A)}$ and $v_i^{(A)}$ also account for products of pure Turing and pure Hopf terms), and the letter C with the Hopf mode. If $\alpha > 4/3$, we must also include an $O(\epsilon^{4/\alpha})$ term in the expansion. Recalling the decomposition in (7), we substitute (6) into (3) and (4), and find that u_1 and v_1 satisfy

$$\left(\frac{\partial}{\partial t} - \mathbf{D}_0\mathscr{D} - \mathbf{M}_0\right)\begin{pmatrix} u_1^{(A)} \\ v_1^{(A)} \end{pmatrix} + \left(\frac{\partial}{\partial t} - \mathbf{M}_0\right)\begin{pmatrix} u_1^{(C)} \\ v_1^{(C)} \end{pmatrix} = 0, \tag{8}$$

where

$$\mathbf{D}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{\eta_c^2} \end{pmatrix}, \qquad \mathscr{D} \equiv \begin{pmatrix} \dfrac{\partial^\alpha}{\partial|x|^\alpha} & 0 \\ 0 & \dfrac{\partial^\beta}{\partial|x|^\beta} \end{pmatrix}, \qquad \mathbf{M}_0 = \begin{pmatrix} B_{cr} - 1 & Q^2 \\ -\dfrac{B_{cr}}{\eta_c^2} & -\dfrac{Q^2}{\eta_c^2} \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = A(X_1, X_{2/\alpha}T)\mathbf{a}\mathrm{e}^{\mathrm{i}k_{cr}x} + C(X_{2/\alpha}, T)\mathbf{c}\mathrm{e}^{\mathrm{i}\omega t} + c.c.,$$

where $c.c.$ denotes complex conjugate. We have allowed only A to depend on both long scales. If we had assumed that C also depended on both long scales, $O(\epsilon^\alpha)$ and $O(\epsilon^\beta)$ terms would need to be included in (6). In this case, solvability conditions at $O(\epsilon^{1+\alpha})$ and $O(\epsilon^{1+\beta})$ would require that C be independent of X_1. These are the solvability conditions mentioned above that dictate that the Hopf mode can only be a function of $X_{2/\alpha}$.

The $O(\epsilon^{2/\alpha})$ equation is the same as the $O(\epsilon)$ equation, with $u_{2/\alpha}$ and $v_{2/\alpha}$ satisfying the same homogeneous equation as u_1 and v_1. Thus we may take $u_{2/\alpha} = v_{2/\alpha} = 0$ without loss of generality (the same applies for $u_{4/\alpha}$ and $v_{4/\alpha}$).

While the left hand side of the $O(\epsilon^2)$ equation is the same as that in (8), its right hand side contains secular-producing terms proportional to $\mathrm{e}^{\mathrm{i}k_{cr}x}$. However, the solvability condition is satisfied (the secular-producing terms are orthogonal to $\mathbf{a}^\dagger$), which leads to the solution

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = A^2\mathbf{p}_{2s}\mathrm{e}^{\mathrm{i}2k_{cr}x} + C^2\mathbf{p}_{2t}\mathrm{e}^{\mathrm{i}2\omega t} + AC\mathbf{p}_L\mathrm{e}^{\mathrm{i}\phi_L} + AC^*\mathbf{p}_R\mathrm{e}^{\mathrm{i}\phi_R} + |A|^2\mathbf{p}_{0s} + |C|^2\mathbf{p}_{0t} + \mathbf{p}_s\mathrm{e}^{\mathrm{i}k_{cr}x} + c.c.,$$

where $\phi_L = k_{cr}x + \omega t$ and $\phi_R = k_{cr}x - \omega t$.

The $O(\epsilon^{1+2/\alpha})$ equation, like the $O(\epsilon^2)$ equation, contains secular-producing terms orthogonal to $\mathbf{a}^\dagger$. However, while $u_{1+2/\alpha}$ and $v_{1+2/\alpha}$ are non-zero, they do not enter the $O(\epsilon^3)$ equation. Upon solving for the vectors $\mathbf{p}_{2s}$, $\mathbf{p}_{2t}$, etc., and applying the solvability condition at $O(\epsilon^3)$, upon rescaling, we obtain the amplitude equations

$$\frac{\partial A}{\partial T} = A + \frac{\partial^2 A}{\partial X_1^2} + \zeta A|A|^2 + \psi_2 A|C|^2, \tag{9}$$

$$\frac{\partial C}{\partial T} = \rho C + (\alpha_1 + \mathrm{i}\alpha_2)\frac{\partial^\alpha C}{\partial |X_{2/\alpha}|^\alpha} + (-|\beta_1| + \mathrm{i}\beta_2)\, C|C|^2 + (\delta_1 + \mathrm{i}\delta_2)\, C|A|^2, \tag{10}$$

where $\zeta = \pm 1$ and $\psi_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1$, and δ_2, are real and functions of α, β, and z. The coefficient ρ, while also real, is a function of α, β, and z, as well as μ and η_2.

We restrict z to the interval I such that $\zeta = -1$, i.e., there exists nonlinear saturation of the Turing mode. I increases (decreases) as β (α) increases, as do the corresponding intervals of Q, η_c, and the activator input rate E. Also, if $\rho < 0$, the system only exhibits the Turing bifurcation while if $\rho > 0$, it also exhibits the Hopf bifurcation. Finally, it can be shown that $\alpha_1 > 0$. We also note that Eqs. (9) and (10) were derived under the necessary condition that C be independent of X_1. Since (10) contains terms involving both A and C, $|A|$ must be spatially homogeneous. Thus, (9) and (10) describe only amplitudes A whose dependence on X_1 takes the form $\mathrm{e}^{\mathrm{i}h(X_1)}$ for a real function h. Since A can depend on both X_1 and $X_{2/\alpha}$, there are no restrictions on the way in which C can depend on $X_{2/\alpha}$.

Lastly, the techniques used here resemble those used for the regular diffusion Brusselator model. There are, however, important differences, one being that two long spatial scales are present instead of the single long scale X_1 in the regular model. Secondly, the expansions of u and v include fractional orders of ϵ, whereas only integer powers were required in the regular model. Thirdly, the rules of differentiation require that the solution be decomposed into separate functions that depend differently on the relevant variables. The resulting forms of the amplitude equations are also different in that the equation for C is now an integro-differential equation.

4. Solutions of the amplitude equations and their long-wave instabilities

In this section, we seek special solutions of Eqs. (9) and (10), namely a pure Turing solution, a pure Hopf solution, and a mixed mode solution. We then study the stability of these solutions to long-wave perturbations. We first consider the pure Turing solution, given by $C = 0$ and $A = \tilde{A}\mathrm{e}^{\mathrm{i}K_A X_1}$ with $\tilde{A} = (1 - K_A^2)^{1/2}$. To study its stability, we linearize around it using $A = (\tilde{A} + a(X_1, T))\mathrm{e}^{\mathrm{i}K_A X_1}$, and $C = c(X_{2/\alpha}, T)$. The resulting linearized equations decouple, so we analyze each separately. A long-wave perturbation $a(X_1, T)$ yields the familiar Eckhaus stability criterion $|K_A| < 1/\sqrt{3}$. If $|K_A| > 1/\sqrt{3}$, the perturbation grows, changing the spatial frequency of the solution. A long-wave perturbation $c(X_{2/\alpha}, T)$ with wavenumber $k \ll 1$ results in the dispersion relation $\sigma = \rho - \alpha_1|k|^\alpha + \tilde{A}^2\delta_1 \pm \mathrm{i}| - \alpha_2|k|^\alpha + \tilde{A}^2\delta_2|$, whose real part must be negative for long-wave stability. If the real part is positive, the perturbation grows, changing the spatial structure of the solution and also introducing a time-oscillatory component. The long-wave stability criterion is $\rho + \tilde{A}^2\delta_1 < 0$. If $\delta_1 < 0$, long-wave perturbations of the Hopf amplitude decay for all $\rho < 0$, or even if $\rho > 0$ as long as ρ remains sufficiently small. If $\delta_1 > 0$, long-wave perturbations of the Hopf amplitude can grow even for $\rho < 0$, as long as ρ is sufficiently close to 0.

For the regular diffusion model, $\delta_1 > 0$ for $z \lesssim 0.26$, or equivalently, $\eta_c \lesssim 0.34$, meaning that the inhibitor (g) diffuses significantly faster than the activator (f). In the anomalous model, we obtain an analogous result for α and β, since these two parameters have a greater impact on the rate of diffusion than do the diffusion coefficients. In contrast to the regular model case, δ_1 can be positive even if $\alpha < \beta$, that is, if the inhibitor diffuses more slowly than the activator. For $\alpha \lesssim 1.67$, $\delta_1 < 0$ for all $z \in I$, meaning that sufficiently fast diffusion of the activator makes it impossible for long-wave perturbations of the Hopf mode to grow if $\rho < 0$. This behavior is not seen in the regular model.

Next, we consider the pure Hopf solution, given by $A = 0$ and $C = \tilde{C}\mathrm{e}^{\mathrm{i}K_C X_{2/\alpha} + \mathrm{i}\Omega T}$ with $\tilde{C} = ((\rho - \alpha_1|K_C|^\alpha)/|\beta_1|)^{1/2}$ and $\Omega = -\alpha_2|K_C|^\alpha + \beta_2\tilde{C}^2$. We note that, since the quantity $\rho - \alpha_1|K_C|^\alpha$ must be positive, ρ must be positive for the pure Hopf solution to exist. Long-wave perturbations of the form $\mathrm{e}^{\sigma T + ikX_{2/\alpha}}$ yield two growth rates, one of which is negative, the other of which has the expansion for small k, $\sigma = a_1 k + a_2 k^2 + O(k^3)$, where

$$a_1 = -\mathrm{i}\frac{\alpha\,(\alpha_1\beta_2 + \alpha_2|\beta_1|)\,|K_C|^{\alpha-1}}{|\beta_1|}$$

and

$$a_2 = \frac{\alpha(\alpha-1)\,(\alpha_2\beta_2 - \alpha_1|\beta_1|)}{2|\beta_1|} + \frac{\alpha^2\alpha_1^2(\beta_1^2 + \beta_2^2)|K_C|^\alpha}{2|\beta_1|^3\tilde{C}^2}.$$

Requiring $a_2 < 0$ for stability leads to the generalized Eckhaus criterion,

$$|K_C|^\alpha < \frac{\rho}{R\alpha_1} \tag{11}$$

where

$$R = 1 + \frac{\alpha\alpha_1\left(\beta_1^2+\beta_2^2\right)}{(\alpha-1)|\beta_1|(\alpha_1|\beta_1|-\alpha_2\beta_2)}.$$

Thus, if (11) is not satisfied, both the spatial and temporal structures of the solution are altered as a long-wave perturbation grows with amplitude oscillating at a frequency different from Ω. As in the regular diffusion case, the magnitude of R is greater than unity for all $z \in I$. However, if $\alpha < \beta$, unlike for the regular diffusion case, R is positive only for z sufficiently small. Beyond this interval, R becomes negative so that (11) is never satisfied, in which case the pure Hopf solution must be long-wave unstable. Restricting z to sufficiently small values for which R is positive implies that Q, η_c, and E must all be sufficiently small. For example, η_c must be less than $\sim$0.62, and thus, while the rate of diffusion is dominated by the diffusion exponents α and β, η_c still impacts whether or not the Hopf mode can be long-wave stable. Note, however, that η_c is not a strict comparison between the diffusion coefficients D_f and D_g, as these parameters do not even have the same units. The qualitative behavior of R in the $\alpha > \beta$ case is the same as for regular diffusion, where $R > 1$ for all $z \in I$, suggesting that faster diffusion of the activator versus the inhibitor may contribute to instability of the pure Hopf solution.

Finally we consider the mixed mode solution, given by $A = \tilde{A}e^{iK_AX_1}$ and $C = \tilde{C}e^{iK_CX_{2/\alpha}+i\Omega T}$ with

$$\tilde{A} = \left(\frac{\psi_2\left(\rho-\alpha_1|K_C|^\alpha\right)+|\beta_1|\left(1-K_A^2\right)}{\Delta}\right)^{1/2}, \qquad \tilde{C} = \left(\frac{\rho-\alpha_1|K_C|^\alpha+\delta_1\left(1-K_A^2\right)}{\Delta}\right)^{1/2},$$

where $\Delta = |\beta_1| - \psi_2\delta_1$, and $\Omega = -\alpha_2|K_C|^\alpha + \beta_2\tilde{C}^2 + \delta_2\tilde{A}^2$. Of course, we must restrict K_A and K_C so that $\tilde{A}$ and $\tilde{C}$ are real. Linearizing (9) and (10) around this mixed mode solution with small perturbations $a(X_{2/\alpha}, T)$ and $c(X_{2/\alpha}, T)$ results in the coupled equations

$$\frac{\partial a}{\partial T} = (1-K_A^2)a - \tilde{A}^2(a^*+2a) + \psi_2\left[\tilde{A}\tilde{C}\left(c^*+c\right)+a\tilde{C}^2\right], \tag{12}$$

and

$$\begin{aligned}\frac{\partial c}{\partial T} &= -i\Omega c + \rho c + (\alpha_1+i\alpha_2)\left(-|K_C|^\alpha c + i\alpha|K_C|^{\alpha-1}\frac{\partial c}{\partial X_{2/\alpha}} + \frac{\alpha(\alpha-1)}{2}|K_C|^{\alpha-2}\frac{\partial^2 c}{\partial X_{2/\alpha}^2}\right)\\ &\quad + (-|\beta_1|+i\beta_2)\tilde{C}^2(c^*+2c) + (\delta_1+i\delta_2)\left[\tilde{A}\tilde{C}\left(a^*+a\right)+c\tilde{A}^2\right].\end{aligned} \tag{13}$$

Eqs. (12) and (13) contain terms involving both a and c, and so, for consistency, we require a to depend only on $X_{2/\alpha}$. We consider perturbations of two types: spatially homogeneous perturbations of a spatially dependent solution ($K_A, K_C \neq 0$), and long-wave perturbations of a spatially homogeneous solution ($K_A = K_C = 0$). For the first case, the resulting dispersion relation yields two zero eigenvalues and two negative eigenvalues as long as $\Delta > 0$, with one of the eigenvalues turning positive if $\Delta < 0$. If $\Delta > 0$, the amplitudes of the Turing and Hopf modes grow in time, with the spatial and temporal frequencies remaining unchanged. Thus, a necessary (and sufficient) condition for stability of the spatially dependent mixed mode solution to homogeneous perturbations is $\Delta > 0$. As in the regular case, there are values of α and β for which stability is possible for both sufficiently large and small values of z. These occur for (α, β) pairs that are near $(2, 2)$. Sufficiently small (large) z refers to an interval of z that ranges from the smallest (largest) $z \in I$ to some larger (smaller) $z \in I$. For (α, β) pairs where α is sufficiently large, it is possible that $\Delta > 0$ only for z sufficiently small. More specifically, for all such (α, β) pairs, stability is possible only if $\eta_c \lesssim 0.42$. Similarly, for all (α, β) for which stability is only possible for sufficiently large z, stability is possible only if $\eta_c \gtrsim 0.65$. For some (α, β) pairs with β sufficiently small, stability is impossible.

For the spatially homogeneous solution, taking $K_A = K_C = 0$, (12) remains the same, while the derivative term in (13) is replaced by $\partial^\alpha c/\partial|X_{2/\alpha}|^\alpha$. Upon inserting long-wave perturbations of the form $e^{\sigma T+ikX_{2/\alpha}}$, the resulting dispersion relation yields two zero eigenvalues and one negative eigenvalue as long as $\Delta > 0$, while the fourth eigenvalue has the expansion for small k, $\sigma \sim a_\alpha|k|^\alpha$, where

$$a_\alpha = \frac{\alpha_2(\beta_2+\psi_2\delta_2)-\alpha_1\Delta}{\Delta}.$$

Thus, long-wave stability of the spatially homogeneous solution requires $\Delta > 0$ and $a_\alpha < 0$. If either one of these conditions is not satisfied, a long-wave spatial pattern appears, breaking the spatial homogeneity. As in the regular diffusion case, there exist (α, β) pairs such that stability is possible only for sufficiently large z. More specifically, for all such (α, β) pairs, long-wave stability of the spatially homogeneous mixed mode solution is possible only if $\eta_c \gtrsim 0.65$. These occur for α sufficiently close to β, but only for $\alpha > \beta$. As in the pure Hopf stability analysis, it appears that the $\alpha > \beta$ case more closely resembles regular diffusion in terms of stability properties. For $\alpha < \beta$ with α and β sufficiently large, stability is possible only for z sufficiently small. For all such (α, β) pairs, $\eta_c \lesssim 0.37$. For both mixed mode solutions, α and β determine whether or not there exist values of parameters, such as η_c, for which stability is possible, as for many (α, β) pairs, stability is impossible.

In summary, the evolution equations (9) and (10) appear similar to their regular diffusion counterparts, but differ both in the behavior of their coefficients, as well as their overall form, as (10) reflects non-local effects. As a result, the stability criteria differ greatly from those of regular diffusion. In the stability analysis of the pure Turing solution, there exist (α, β) such that long-wave perturbations of the Hopf mode cannot grow if $\rho < 0$ for any value of z. This is contrary to the regular diffusion case, for which growth is possible if the inhibitor diffuses sufficiently faster than the activator. Further, we found that there exist (α, β) for which long-wave perturbations of the Hopf mode can grow for $\rho < 0$ even if the inhibitor diffuses more slowly than the activator. We also found that, for $\alpha < \beta$, there exist values of $z \in I$ such that stability of the pure Hopf solution is impossible, while for $\alpha > \beta$, the stability criteria remain qualitatively similar. Finally, for the mixed mode, there exist (α, β) pairs sufficiently close to $(2, 2)$ for which the stability requirements are similar to those of regular diffusion. Away from this regime, these requirements can either change or stability may simply not be possible.

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