EQUIDISTRIBUTION MODULO 1 AND SALEM NUMBERS

Christophe Doche, Michel Mendès France, Jean-Jacques Ruch

Abstract: Let \( \theta \) be a Salem number. It is well-known that the sequence \((\theta^n)\) modulo 1 is dense but not equidistributed. In this article we discuss equidistributed subsequences. Our first approach is computational and consists in estimating the supremum of \(\lim_{n \to \infty} n/s(n)\) over all equidistributed subsequences \((\theta^{s(n)})\). As a result, we obtain an explicit upper bound on the density of any equidistributed subsequence. Our second approach is probabilistic. Defining a measure on the family of increasing integer sequences, we show that relatively to that measure, almost no subsequence is equidistributed.

Keywords: Salem number, Equidistribution modulo 1, \(J_0\) Bessel function.

1. Subsequences

Let \( u = (u(n)) \) be an infinite sequence of real numbers. A subsequence \( u \circ s = (u(s(n))) \) is said to have density \( d \leq 1 \) if as \( n \) increases \( n/s(n) \to d \). Suppose the sequence \( u \) is dense (mod 1). Answering a question of one of us in 1973, Y. Dupain and J. Lesca \cite{6} established that the set of densities \( d \) of equidistributed (mod 1) subsequences of \( u \) is a closed interval \([0, d_0]\) where \( d_0 \leq 1 \) depends on \( u \). They also showed how to compute \( d_0 \). For \( 0 \leq x \leq 1 \), define the repartition function

\[
f(x) = \lim_{N \to \infty} \frac{1}{N} \text{card}\{n < N \mid \{u(n)\} < x\}
\]

where \( \{u(n)\} \) is the fractional part of \( u(n) \). We only consider those \( x \) where \( f(x) \) and its derivative \( f'(x) \) both exist, i.e. almost everywhere. Y. Dupain and J. Lesca proved that \( d_0 = \inf_x f'(x) \).

A particularly striking example of such an instance concerns the distribution (mod 1) of the powers of Salem numbers \( \theta > 1 \). A Salem number \cite{10} (see also \cite{3}) is a real algebraic integer whose algebraic conjugates other than \( \theta \) all lie in the unit disc \(|z| \leq 1\) with one conjugate at least on the boundary \(|z| = 1\). It is then known that one and only one of these conjugates \( \theta^{-1} \) is inside the disc while the others are on the boundary. The degree \( 2t \) of \( \theta \) is necessarily even and at least equal to 4.

Mathematics Subject Classification: 11K06, 11J71.
Denote the different conjugates by $\theta, \theta^{-1}, \exp(\pm 2i\pi \omega_1), \ldots, \exp(\pm 2i\pi \omega_{t-1})$. The sum of all conjugates of an algebraic integer is an integer and therefore for all $n \in \mathbb{N}$,

$$\theta^n + \theta^{-n} + 2 \sum_{j=1}^{t-1} \cos 2\pi n \omega_j \equiv 0 \pmod{1}$$

so that the distribution of $\theta^n \pmod{1}$ is essentially that of $-2 \sum_{j=1}^{t-1} \cos 2\pi n \omega_j$. Ch. Pisot and R. Salem [9] observed that $1, \omega_1, \ldots, \omega_{t-1}$ are $\mathbb{Z}$-linearly independent so that, according to Kronecker, the $(t-1)$ dimensional sequence $(\omega_1 n, \ldots, \omega_{t-1} n)$ is equidistributed in $(\mathbb{R}/\mathbb{Z})^{t-1}$. As a consequence, the sequence $(\theta^n)$ is therefore clearly dense $(\pmod{1})$. Furthermore, for all $k \in \mathbb{N} \setminus \{0\}$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n<N} \exp 2i\pi k \theta^n = \lim_{N \to \infty} \frac{1}{N} \sum_{n<N} \prod_{j=1}^{t-1} \exp(-2i\pi k \cdot 2 \cos 2\pi n \omega_j)$$

$$= \left( \int_0^1 \exp(-4i\pi k \cos 2\pi x) \, dx \right)^{t-1}$$

$$= J_0(4\pi k)^{t-1} \neq 0 \quad (1.1)$$

where $J_0(\cdot)$ is the Bessel function of the first kind of index 0.

Since $|J_0(\alpha)| < 1$ for all real $\alpha \neq 0$, the above limit tends to 0 as $t \to \infty$. Y. Dupain and J. Lesca conclude that for large degrees $t$, the sequence $(\theta^n \pmod{1})$ is close to being equidistributed, a fact that S. Akiyama and Y. Tanigawa [1] make very explicit in their article. This is quite remarkable since even though for almost all real $\tau > 1$, $(\tau^n)$ is equidistributed $(\pmod{1})$, no explicit $\tau$ is known (J. F. Koksma [8]).

We know the existence of $d_0 < 1$ (and quite obviously $d_0 > 0$) such that $s(n) \sim \frac{1}{d_0 n}$ and $(\theta^{s(n)})$ equidistributed $(\pmod{1})$. We shall see later on that those sequences are rare. But we can already guess why these sequences $s(n)$ are exceptional. This is a consequence of our first rather trivial theorem.

**Theorem 1.1.** If $s(n)$ is an increasing sequence of integers such that $(\theta^{s(n)})$ is equidistributed $(\pmod{1})$, then there exists an irrational $x$ such that $xs(n)$ is not equidistributed $(\pmod{1})$.

**Proof.** We note that

$$\theta^{s(n)} \equiv -2 \sum_{j=1}^{t-1} \cos 2\pi \omega_j s(n) - \theta^{-s(n)} \pmod{1}.$$ 

The $(t-1)$ dimensional sequence $(\omega_1 s(n), \ldots, \omega_{t-1} s(n))$ is not equidistributed in $(\mathbb{R}/\mathbb{Z})^{t-1}$ since if it were, $(\theta^{s(n)})$ would not be equidistributed $(\pmod{1})$. Therefore there exist integers $h_1, \ldots, h_{t-1}$ not all 0 such that

$$h_1 \omega_1 s(n) + \cdots + h_{t-1} \omega_{t-1} s(n)$$
is not equidistributed \((\text{mod } 1)\). The theorem is established with

\[
x = \sum_{j=1}^{t-1} h_j \omega_j .
\]

Next, we develop a method to approximate \(d_0\) for the sequence \((\theta^n \text{ (mod } 1))\), where \(\theta\) is a Salem number of degree \(2t\). The results indicate that \(d_0\) tends to 1 very quickly as \(t\) tends to infinity. A key result in this approach is the study of the minimum of a cosine series on \([0, 1]\). Under certain conditions, we show that the minimum is always attained at \(x = 1/2\), cf. Theorem 2.1.

2. Explicit Computations of \(d_0\)

The repartition function is explicitly determined for a Salem number of degree 4, cf. [5]. Namely,

\[
f(x) = \frac{5}{2} - \frac{1}{\pi} \left( \arccos \frac{x-2}{2} + \arccos \frac{x}{2} + \arccos \frac{x-1}{2} + \arccos \frac{x+1}{2} \right) .
\]

It follows that

\[
f'(x) = \frac{1}{2\pi} \left( \frac{1}{\sqrt{1 - \left(\frac{x}{2} - 1\right)^2}} + \frac{1}{\sqrt{1 - \left(\frac{x-1}{2}\right)^2}} + \frac{1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}} + \frac{1}{\sqrt{1 - \left(\frac{x+1}{2}\right)^2}} \right) .
\]

A direct study of \(f'(x)\) shows that it attains its minimum for \(x = \frac{1}{2}\) and gives the exact value of \(d_0\), i.e.

\[
\frac{1}{\pi} \left( \frac{4}{\sqrt{7}} + \frac{4}{\sqrt{15}} \right) = 0.809988350\ldots
\]

(2.1)

For a Salem number of degree \(2t\) with \(t > 2\), we want to estimate the corresponding \(d_0\). First, let us show the following lemma.

**Lemma 2.1.** Let \(\theta\) be a Salem number of degree \(2t\), then the repartition function \(f(x)\) of the sequence \((\theta^n)\) modulo 1 satisfies

\[
f'(x) = 1 + 2 \sum_{k=1}^{\infty} J_0(4k\pi)^{t-1} \cos 2\pi k x
\]

on \([0, 1]\), for all \(t \geq 2\).

**Proof.** We have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n < N} \exp 2i\pi k \theta^n = \int_0^1 \exp 2i\pi k x \, d\nu
\]
where $\nu$ is the repartition function $f(x)$. According to Y. Dupain [5] the measure $d\nu = f'(x) \, dx$ is absolutely continuous. It follows from (1.1) that

$$J_0(4\pi k)^{t-1} = \int_0^1 \exp 2i\pi kx f'(x) \, dx.$$  

We can associate with $f'(x)$ its Fourier series

$$\sum_{k \in \mathbb{Z}} J_0(4\pi k)^{t-1} \exp(-2i\pi kx) = 1 + 2 \sum_{k=1}^{\infty} J_0(4\pi k)^{t-1} \cos 2\pi kx. \quad (2.2)$$

If this series converges uniformly, then its sum is continuous and equals $f'(x)$. The lemma is clear for $t > 3$, since $J_0(x) = O(x^{-\frac{1}{2}})$ and we even have equality on $[0, 1]$. For $t = 2$ and 3, we need the following result.

**Lemma 2.2.** The sequence $(J_0(4\pi k))$ is positive for all $k > 0$ and strictly decreasing.

**Proof.** In [1, Lemma 2], it is shown that

$$J_0(2\pi k) = \frac{1}{\pi \sqrt{k}} \left( \frac{1}{\sqrt{2}} - \frac{1}{16\sqrt{2}\pi k} + R \right), \text{ with } |R| \leq \frac{9}{512\pi^2 k^2}.$$  

It is straightforward to deduce that

$$0 \leq \frac{1}{2\pi \sqrt{k}} - J_0(4\pi k) \leq \frac{1}{61\pi^2 k^\frac{3}{2}}. \quad (2.3)$$

This proves the first part of the lemma. Now

$$\frac{1}{2\pi} \left( \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) \geq \frac{1}{8\pi k^\frac{3}{2}} > \frac{2}{61\pi^2 k^\frac{3}{2}}.$$  

This shows that

$$\frac{1}{2\pi \sqrt{k}} - \frac{1}{61\pi^2 k^\frac{3}{2}} > \frac{1}{2\pi \sqrt{k+1}} - \frac{1}{61\pi^2 (k+1)^\frac{3}{2}}$$

which implies that $J_0(4\pi k) > J_0(4\pi (k+1))$, for $k > 0$. □

We deduce that the series (2.2) is uniformly convergent on the compact $[\epsilon, 1-\epsilon]$, for any $\epsilon > 0$ and therefore $f'(x)$ is equal to this series on $]0, 1[$. □

A consequence of Lemma 3.2 is that $d_0$ only depends on $t$ and satisfies

$$d_0 = \inf_{x \in [0, 1]} \left( 1 + 2 \sum_{k=1}^{\infty} J_0(4k\pi)^{t-1} \cos 2\pi kx \right).$$

Next let us recall a definition we shall use later.
Definition 2.1. Let \((b_k)\) be a sequence of real numbers and let \(\Delta^0 b_k = b_k\) and \(\Delta^n b_k = \Delta^{n-1} b_k - \Delta^{n-1} b_{k+1},\) for all \(n > 0.\) The sequence \((b_k)\) is said to be totally monotone if \(\Delta^n b_k \geq 0\) for all \(k,\) and \(n = 0, 1, 2, \ldots.\)

By a famous result of Hausdorff [7], the total monotonicity of \((b_k)\) is equivalent to the existence of a nonnegative measure \(\mu\) on \([0, 1]\) such that the \(b_k\)'s are the moments of \(\mu,\) i.e.

\[
b_k = \int_0^1 u^k \, d\mu.
\]

Example 2.1. Let \(s\) be a real positive number. The sequence \((b_k)\) defined by

\[
b_k = \frac{1}{(k + 1)^s}
\]

for all \(k \geq 0\) is totally monotone.

Theorem 2.1. Let \((a_k)\) be a sequence of nonnegative real numbers (except maybe for \(a_0\)). Assume that \((a_{k+1}), k \geq 0\) is totally monotone, then the function

\[
g(x) = \sum_{k=0}^{\infty} a_k \cos 2\pi k x
\]

is well-defined and decreasing on the interval \([0, 1/2].\) As a corollary, \(g(x)\) attains its minimum for \(x = \frac{1}{2}.\)

Proof. Let us introduce

\[
h(x) = \sum_{k=1}^{\infty} a_k \cos 2\pi k x = \sum_{k=0}^{\infty} b_k \cos 2\pi (k + 1) x.
\]

Since, \(g\) and \(h\) only differ by \(a_0,\) it is enough to study \(h\) to prove the theorem on \(g.\) Since \((b_k) = (a_{k+1}), \Delta b_k \geq 0,\) for all \(k.\) So the sequence \((b_k)\) is decreasing and this shows that the series \(h(x)\) is convergent for all \(x \in ]\varepsilon, 1 - \varepsilon[,\) for all \(\varepsilon > 0.\) Since \(h(x) = h(1 - x),\) it is enough to study \(h\) on \([0, 1/2].\)

Since the \(b_k\)'s are the moments of a certain nonnegative measure \(\mu,\) we obtain

\[
h(x) = \sum_{k=0}^{\infty} b_k \cos 2\pi (k + 1) x
\]

\[
= \sum_{k=0}^{\infty} \int_0^1 u^k \cos 2\pi (k + 1) x \, d\mu
\]

\[
= \Re \int_0^1 \frac{e^{2i\pi x}}{1 - e^{2i\pi x} u} \, d\mu.
\]
The last equality being justified by the nonnegativity of $\mu$. It follows that
\[
h(x) = \int_0^1 \frac{\cos 2\pi x - u}{1 + u^2 - 2u \cos 2\pi x} \, d\mu.
\]

To show that $h(x)$ is decreasing on $[0, 1/2]$, evaluate $h(x) - h(y)$ for $0 < x \leq y \leq 1/2$. Let
\[
j_x(u) = \frac{\cos 2\pi x - u}{1 + u^2 - 2u \cos 2\pi x}.
\]

Then reducing to the same (positive) denominator, we see that the numerator of $j_x(u) - j_y(u)$ is $(\cos 2\pi x - \cos 2\pi y)(1 - u^2)$ which is nonnegative for all $u \in [0, 1]$.

Since $\mu$ is a nonnegative measure, we deduce that $h(x) \geq h(y)$ whenever $x \leq y \leq 1/2$ and that $h(x) \geq h(1/2)$ for all $x \in [0, 1/2]$. These results apply trivially to the function $g$.

**Corollary 2.1.** Let $s > 0$. Then the series
\[
g(x) = a_0 + \sum_{k=1}^{\infty} \frac{\cos 2\pi k x}{k^s}
\]
is decreasing on $[0, 1/2]$ and satisfies
\[
g(x) \geq a_0 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s}.
\]

**Remark.** It is possible to compute $g(1/2)$ very efficiently following the method explained in [4]. For instance, for the sequence $(a_k)$ defined by a given $a_0$ and $a_k = 1/\sqrt{k}$, for $k \geq 1$, we have that
\[
g(x) \geq g(1/2) = a_0 - 0.604898643421630370247265914235954997597625451 \ldots
\]

All the digits in the last equality are correct as can be established knowing the first 60 $a_k$’s.

Unfortunately, we are not able to show that the sequence $(J_0(4\pi k)^{t-1})$, $k > 0$ is totally monotone, though the extensive numerical computations of its first $n$th forward differences seem to indicate that this is the case. Based on the case $t = 2$ and also on direct computations of $f'(x)$ for various $x$, we conjecture that $\inf_x f'(x) = f'(1/2)$ for $t \geq 2$. However, to be totally rigorous, we cannot directly apply Theorem 2.1 to obtain the value of $d_0$. Nevertheless, this result will give an approximation of $d_0$, for $t > 2$.

The idea is to apply (2.3) to deduce that
\[
\left| J_0(4\pi k)^{t-1} - \frac{1}{(2\pi \sqrt{k})^{t-1}} \right| \leq \frac{1}{61\pi^2 k^2} \frac{(t-1)}{(2\pi \sqrt{k})^{t-2}}.
\]
It follows that

\[
\left| f'(x) - 1 - 2 \sum_{k=1}^{\infty} \frac{\cos 2\pi k x}{(2\pi \sqrt{k})^{t-1}} \right| \leq \sum_{k=1}^{\infty} \frac{2}{61\pi^2 k^2 (2\pi \sqrt{k})^{t-2}},
\]

which, combined with Theorem 2.1, implies that for all \( x \in [0, 1] \)

\[
f'(x) \geq 1 + 2 \left( \sum_{k=1}^{\infty} \frac{(-1)^k}{(2\pi \sqrt{k})^{t-1}} - \sum_{k=1}^{\infty} \frac{2}{61\pi^2 k^2 (2\pi \sqrt{k})^{t-2}} \right).
\]

The main contribution, i.e. \( S_1 \), can be obtained using the acceleration convergence method explained in [4], whereas the second series \( S_2 \) is simply (up to a constant) an evaluation of the \( \zeta \) function at the point \( (t + 1)/2 \). This gives a lower bound for \( d_0 \). An upper bound is given by \( d_0 \leq f'(1/2) \), where \( f'(1/2) \) is bounded, for any \( K \) even, by the truncated alternating series

\[
1 + 2 \sum_{k=1}^{K} (-1)^k J_0(4\pi k)^{t-1}.
\]

The convergence is quite slow for \( t = 3 \) so that we fixed \( K = 2.10^6 \) to obtain a relevant upper bound. Much less terms are necessary for larger \( t \). A conjectured value \( d_0^* \) is also given relying on the assumption that \( d_0 = f'(1/2) \) and on the computation of \( f'(1/2) \) using [4]. The method seems to converge and at most the first 10 terms are sufficient to give a result with an error less than \( 10^{-10} \). Also, we checked for \( t = 2 \) that the value given in (2.1) is, up to several hundred digits, equal to the one computed with this approach.

Note that if the sequence \( (J_0(4\pi k)^{t-1}) \), defined for \( k > 0 \) is totally monotone, then both assumptions are valid, and therefore \( d_0 = d_0^* \). All the figures are given in Table 1.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_1 - S_2 )</th>
<th>( f'(1/2) )</th>
<th>( d_0^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.964884753</td>
<td>0.000869699</td>
<td>0.964015054</td>
<td>0.965745539</td>
<td>0.965745543</td>
</tr>
<tr>
<td>4</td>
<td>0.993830708</td>
<td>0.000112882</td>
<td>0.993717825</td>
<td>0.994046008</td>
<td>0.994046007</td>
</tr>
<tr>
<td>5</td>
<td>0.998944571</td>
<td>0.000016098</td>
<td>0.998928472</td>
<td>0.998991788</td>
<td>0.998991787</td>
</tr>
<tr>
<td>6</td>
<td>0.999822887</td>
<td>0.000000241</td>
<td>0.999820485</td>
<td>0.999832498</td>
<td>0.999832497</td>
</tr>
<tr>
<td>7</td>
<td>0.999970695</td>
<td>0.000000367</td>
<td>0.999970328</td>
<td>0.999972560</td>
<td>0.999972559</td>
</tr>
<tr>
<td>8</td>
<td>0.999995201</td>
<td>0.000000056</td>
<td>0.999995144</td>
<td>0.999995551</td>
<td>0.999995550</td>
</tr>
<tr>
<td>9</td>
<td>0.999999220</td>
<td>0.000000008</td>
<td>0.999999211</td>
<td>0.999999285</td>
<td>0.999999284</td>
</tr>
<tr>
<td>10</td>
<td>0.999999874</td>
<td>0.000000001</td>
<td>0.999999872</td>
<td>0.999999886</td>
<td>0.999999885</td>
</tr>
</tbody>
</table>

In the next section we shall define the notion of "almost all" increasing sequences of integers \( (s(n)) \). For almost all sequences \( (s(n)) \) and for all irrational
numbers \( x \), \( (xs(n)) \) is equidistributed. This already shows how exceptional those sequences \( (s(n)) \) are for which \( (\theta^{s(n)}) \) is equidistributed.

Furthermore R. Salem [11] demonstrated that if \( (s(n)) \) is any increasing sequence such that \( s(n) = O(n) \), then the Hausdorff dimension of the set of \( x \) for which \( (xs(n)) \) is not equidistributed \( (\text{mod } 1) \), vanishes. The \( x \)'s in Theorem 1.1 are therefore "rare" if indeed \( s(n) \sim \frac{1}{d_0} n \).

3. Metrical Results

Let \( S \) be the family of finite or infinite strictly increasing sequences of positive integers. To each \( s = (s(n)) \in S \) corresponds a unique sequence \( \chi \in D = \{0, 1\}^N \) (characteristic sequence) and conversely:

\[
\chi(n) = \begin{cases} 
1 & \text{if } n \in s, \\
0 & \text{if not.}
\end{cases}
\]

Any measure on \( D \) lifts to a measure on \( S \).

Let \( 0 < d < 1 \). Put \( m\{1\} = d \) and \( m\{0\} = 1 - d \). Then \( \mu = \prod m \) is a probability measure on \( D \) to which corresponds a probability measure on \( S \) which we still denote by \( \mu \) or \( \mu_d \) if we wish to emphasize the parameter \( d \).

**Theorem 3.1.** Consider the polynomial \( P(X) = \sum_{\ell=0}^{\nu} a_\ell X^\ell \) where at least one of the coefficients \( a_\ell \), \( 1 \leq \ell \leq \nu \) is irrational. Then for \( \mu \)-almost all sequences \( s \in S \), \( P(s) = (P(s(n))) \) is equidistributed \( (\text{mod } 1) \).

**Theorem 3.2.** If \( \theta \) is a Salem number then \( \mu \)-almost no sequence \( (\theta^{s(n)}) \) is equidistributed \( (\text{mod } 1) \). More generally, if \( P \) is any positive integer valued polynomial, \( \theta^P(s) = (\theta^P(s(n))) \) is \( \mu \)-almost never equidistributed \( (\text{mod } 1) \).

We have seen in Section 1 that there exists a \( d_0 \in ]0, 1[ \) for which no sequence \( s = (s(n)) \) exists such that \( s(n) \sim \frac{1}{d} n \) \( (d > d_0) \) and \( (\theta^{s(n)}) \) equidistributed \( (\text{mod } 1) \). For \( d \leq d_0 \) there do exist \( d \)-density equidistributed subsequences \( (\theta^{s(n)}) \) but they are \( \mu_d \)-rare.

**Remark.** For \( d \in [0, 1] \) let \( T(d) \) be the family of increasing sequences \( (s(n)) \) of density \( d \) such that \( (\theta^{s(n)}) \) is equidistributed \( (\text{mod } 1) \). We know that \( T(d) = \emptyset \) as long as \( d > d_0 \). Could it be true that as \( d \) decreases to 0 the family \( T(d) \) "increases in size"? Could one devise a way to show that this is so, e.g. by defining a fractal dimension adapted to the question?

4. Proof of Theorem 3.1

A sequence \( \chi \in \{0, 1\}^N \) is said to be \( d \)-normal if all finite words \( w = w_1 \ldots w_\ell \in \{0, 1\}^\ell \) occur in \( \chi \) with the frequency \( d^\ell (1 - d)^{\ell - k} \) where \( k \) is the number of 1’s in \( w \). It is well known that \( \mu_d \)-almost all \( \chi \) are \( d \)-normal. For such a sequence

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n < N} (\chi(n) - d) = 0
\]
and more generally, for all \( k \geq 1 \) and all integers \( h_1 \leq \cdots \leq h_k \) where at least one couple \( h_i < h_{i+1} \)

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n<N} \prod_{i=1}^{k} (\chi(n+h_i) - d) = 0 .
\]

A sequence \( Y \) is said to be uncorrelated if for all \( k \geq 1 \) and all integers \( h_1 \leq \cdots \leq h_k \) where at least one couple \( h_i < h_{i+1} \)

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n<N} \prod_{i=1}^{k} Y(n+h_i) = 0 .
\]

If \( \chi \in \{0,1\} \) is \( d \)-normal, then as remarked above, \( \chi - d \) is uncorrelated.

**Lemma 4.1.** For all real polynomials \( P \) and all uncorrelated sequences \( Y \)

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n<N} Y(n) \exp 2i\pi P(n) = 0 .
\]

**Proof.** The result is obviously true if \( \deg P = 0 \). We now argue by induction and assume the truth of the lemma for all \( P \) with \( \deg P = \nu - 1 \geq 0 \). Let \( Q \) be any polynomial of degree \( \nu \) and let \( h \geq 1 \) be an arbitrary integer. Put \( f(n) = Y(n) \exp 2i\pi Q(n) \) and consider the correlation

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n<N} f(n) f(n+h)
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{n<N} Y(n)Y(n+h) \exp 2i\pi (Q(n+h) - Q(n)).
\]

The product \( Z(n) = Y(n)Y(n+h) \) is again uncorrelated and the polynomial \( P(n) = Q(n+h) - Q(n) \) is of degree \( \nu - 1 \). Therefore

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n<N} f(n) f(n+h) = 0
\]

for all \( h \geq 1 \). A classical result (see J. Bass [2]) then implies

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n<N} f(n) = 0 .
\]

We now prove Theorem 3.1. Suppose \( P(X) = \sum_{\ell=0}^{\nu} a_{\ell} X^\ell \) where at least one of the coefficients \( a_1, \ldots, a_\nu \) is irrational. Consider the exponential mean

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n<N} \exp 2i\pi h P(s(n))
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{\ell<s(N)} \chi(\ell) \exp 2i\pi h P(\ell)
\]

where \( h \geq 1 \) is an integer, and where \( \chi \) is the characteristic function of \( s \).
For $\mu = \mu_d$-almost all $s$, $s(N) \sim \frac{1}{d}N = L$. The theorem will be established if for $L \to \infty$

$$\frac{1}{L} \sum_{\ell < L} \chi(\ell) \exp 2i\pi hP(\ell) \to 0.$$ 

The above average can be decomposed into two parts

$$\frac{1}{L} \sum_{\ell < L} (\chi(\ell) - d) \exp 2i\pi hP(\ell) + \frac{d}{L} \sum_{\ell < L} \exp 2i\pi hP(\ell).$$

For $\mu_d$-almost all $s$, $\chi - d$ is uncorrelated and therefore the first average converges to 0. As for the second average, it converges to 0 because the sequence is well known to be equidistributed $\pmod{1}$ [12].

5. Proof of Theorem 3.2

Let $P(X) = \sum_{\ell=0}^{\nu} a_\ell X^\ell$, $a_\nu > 0$, be a polynomial which takes integer values when $X$ runs through $\mathbb{N}$. If $s \in S$,

$$\theta^P(s(n)) \equiv -2 \sum_{j=1}^{t-1} \cos 2\pi \omega_j P(s(n)) + o(1)$$

if $P$ is nonconstant (if $P$ is constant the theorem is trivial). The $(t-1)$ polynomials $\omega_1P, \ldots, \omega_{t-1}P$ all have irrational coefficients. According to Theorem 3.1, the sequences $(\omega_jP(s(n)))$ are $\mu_d$-almost surely equidistributed $\pmod{1}$ and more to the point, for all $h = (h_1, \ldots, h_{t-1}) \in \mathbb{Z}^{t-1} \setminus \{0\}$ the sequences $h\omega P(s)$ are equidistributed $\pmod{1}$. Here $h\omega P(s)$ is the scalar product of $h$ and $\omega = (\omega_1, \ldots, \omega_{t-1})$. Therefore the $(t-1)$ dimensional sequence $(\omega_1P(s), \ldots, \omega_{t-1}P(s))$ is equidistributed in $(\mathbb{R}/\mathbb{Z})^{t-1}$ and as in the first section, we conclude that

$$\frac{1}{N} \sum_{n<N} \exp 2i\pi kP(s(n)) \xrightarrow{N \to \infty} J_0(4k\pi)^{t-1} \neq 0.$$ 

6. A Final Remark

All our arguments are based on the fact that $\theta^n$ is essentially a finite sum of $\cos 2\pi \omega_j n$. We could probably extend some of our results to the study of sequences $u = (u(n))$ of the type

$$u(n) = \sum_{j=1}^{t} F(n\omega_j).$$

Acknowledgement. The authors had long discussions with Vitaly Bergelson and with Bahman Saffari whom we would very much like to thank warmheartedly.
References


**Addresses:** Christophe Doche: Division of ICS, Building E6A Room 360, Macquarie University, NSW 2109 Australia, Michel Mendès France, Jean-Jacques Ruch: Laboratoire A2X, Université Bordeaux I, 351, cours de la Libération, F-33405 Talence Cedex France

**E-mail:** doche@ics.mq.edu.au, michel.mendes-france@math.u-bordeaux1.fr, jean-jacques.ruch@math.u-bordeaux1.fr, mmf@math.u-bordeaux.fr, ruch@math.u-bordeaux.fr

**Received:** 18 December 2007; **revised:** 16 July 2008