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# EQUIDISTRIBUTION MODULO 1 AND SALEM NUMBERS

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À Władysław Narkiewicz à l'occasion
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**Abstract:** Let $\theta$ be a Salem number. It is well-known that the sequence $(\theta^n)$ modulo 1 is dense but not equidistributed. In this article we discuss equidistributed subsequences. Our first approach is computational and consists in estimating the supremum of $\lim_{n\to\infty} n/s(n)$ over all equidistributed subsequences $(\theta^{s(n)})$. As a result, we obtain an explicit upper bound on the density of any equidistributed subsequence. Our second approach is probabilistic. Defining a measure on the family of increasing integer sequences, we show that relatively to that measure, almost no subsequence is equiditributed.
**Keywords:** Salem number, Equidistribution modulo 1, $J_0$ Bessel function.

## 1. Subsequences

Let $u = \big(u(n)\big)$ be an infinite sequence of real numbers. A subsequence $u \circ s = \big(u(s(n))\big)$ is said to have density $d \leqslant 1$ if as $n$ increases $n/s(n) \to d$. Suppose the sequence $u$ is dense (mod 1). Answering a question of one of us in 1973, Y. Dupain and J. Lesca [6] established that the set of densities $d$ of equidistributed (mod 1) subsequences of $u$ is a closed interval $[0, d_0]$ where $d_0 \leqslant 1$ depends on $u$. They also showed how to compute $d_0$. For $0 \leqslant x \leqslant 1$, define the *repartition function*

$$f(x) = \lim_{N\to\infty} \frac{1}{N} \mathrm{card}\big\{n < N \mid \{u(n)\} < x\big\}$$

where $\{u(n)\}$ is the fractional part of $u(n)$. We only consider those $x$ where $f(x)$ and its derivative $f'(x)$ both exist, i.e. almost everywhere. Y. Dupain and J. Lesca proved that $d_0 = \inf_x f'(x)$.

A particularly striking example of such an instance concerns the distribution (mod 1) of the powers of Salem numbers $\theta > 1$. A Salem number [10] (see also [3]) is a real algebraic integer whose algebraic conjugates other than $\theta$ all lie in the unit disc $|z| \leqslant 1$ with one conjugate at least on the boundary $|z| = 1$. It is then known that one and only one of these conjugates $\theta^{-1}$ is inside the disc while the others are on the boundary. The degree $2t$ of $\theta$ is necessarily even and at least equal to 4.

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Denote the different conjugates by $\theta, \theta^{-1}, \exp(\pm 2i\pi\omega_1), \dots, \exp(\pm 2i\pi\omega_{t-1})$. The sum of all conjugates of an algebraic integer is an integer and therefore for all $n \in \mathbb{N}$,

$$\theta^n + \theta^{-n} + 2\sum_{j=1}^{t-1} \cos 2\pi n\omega_j \equiv 0 \pmod 1$$

so that the distribution of $\theta^n \pmod 1$ is essentially that of $-2\sum_{j=1}^{t-1} \cos 2\pi n\omega_j$. Ch. Pisot and R. Salem [9] observed that $1, \omega_1, \dots, \omega_{t-1}$ are $\mathbb{Z}$-linearly independent so that, according to Kronecker, the $(t-1)$ dimensional sequence $(\omega_1 n, \dots, \omega_{t-1} n)$ is equidistributed in $\big(\mathbb{R}/\mathbb{Z}\big)^{t-1}$. As a consequence, the sequence $(\theta^n)$ is therefore clearly dense (mod 1). Furthermore, for all $k \in \mathbb{N} \setminus \{0\}$

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \exp 2i\pi k\theta^n &= \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \prod_{j=1}^{t-1} \exp(-2i\pi k.2\cos 2\pi n\omega_j) \\ &= \Big(\int_0^1 \exp(-4i\pi k \cos 2\pi x)\, dx\Big)^{t-1} \\ &= J_0(4\pi k)^{t-1} \neq 0 \qquad (1.1)\end{aligned}$$

where $J_0(\cdot\,)$ is the Bessel function of the first kind of index 0.

Since $|J_0(\alpha)| < 1$ for all real $\alpha \neq 0$, the above limit tends to 0 as $t \to \infty$. Y. Dupain and J. Lesca conclude that for large degrees $t$, the sequence $\big(\theta^n \pmod 1\big)$ is close to being equidistributed, a fact that S. Akiyama and Y. Tanigawa [1] make very explicit in their article. This is quite remarkable since even though for almost all real $\tau > 1$, $(\tau^n)$ is equidistributed (mod 1), no explicit $\tau$ is known (J. F. Koksma [8]).

We know the existence of $d_0 < 1$ (and quite obviously $d_0 > 0$) such that $s(n) \sim \frac{1}{d_0} n$ and $\big(\theta^{s(n)}\big)$ equidistributed (mod 1). We shall see later on that those sequences are rare. But we can already guess why these sequences $s(n)$ are exceptional. This is a consequence of our first rather trivial theorem.

**Theorem 1.1.** *If $s(n)$ is an increasing sequence of integers such that $\big(\theta^{s(n)}\big)$ is equidistributed* (mod 1)*, then there exists an irrational $x$ such that $xs(n)$ is not equidistributed* (mod 1)*.*

**Proof.** We note that

$$\theta^{s(n)} \equiv -2\sum_{j=1}^{t-1} \cos 2\pi\omega_j s(n) - \theta^{-s(n)} \pmod 1.$$

The $(t-1)$ dimensional sequence $\big(\omega_1 s(n), \dots, \omega_{t-1} s(n)\big)$ is not equidistributed in $\big(\mathbb{R}/\mathbb{Z}\big)^{t-1}$ since if it were, $\big(\theta^{s(n)}\big)$ would not be equidistributed (mod 1). Therefore there exist integers $h_1, \dots, h_{t-1}$ not all 0 such that

$$h_1\omega_1 s(n) + \cdots + h_{t-1}\omega_{t-1} s(n)$$

is not equidistributed (mod 1). The theorem is established with

$$x = \sum_{j=1}^{t-1} h_j \omega_j \ . \qquad \blacksquare$$

Next, we develop a method to approximate $d_0$ for the sequence $\big(\theta^n (\mathrm{mod}\ 1)\big)$, where $\theta$ is a Salem number of degree $2t$. The results indicate that $d_0$ tends to 1 very quickly as $t$ tends to infinity. A key result in this approach is the study of the minimum of a cosine series on $]0,1[$. Under certain conditions, we show that the minimum is always attained at $x = 1/2$, cf. Theorem 2.1.

## 2. Explicit Computations of $d_0$

The repartition function is explicitly determined for a Salem number of degree 4, cf. [5]. Namely,

$$f(x) = \frac{5}{2} - \frac{1}{\pi}\left(\arccos\frac{x-2}{2} + \arccos\frac{x}{2} + \arccos\frac{x-1}{2} + \arccos\frac{x+1}{2}\right) \ .$$

It follows that

$$f'(x) = \frac{1}{2\pi}\left(\frac{1}{\sqrt{1-\left(\frac{x}{2}-1\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{x-1}{2}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{x+1}{2}\right)^2}}\right) \ .$$

A direct study of $f'(x)$ shows that it attains its minimum for $x = \frac{1}{2}$ and gives the exact value of $d_0$, i.e.

$$\frac{1}{\pi}\left(\frac{4}{\sqrt{7}} + \frac{4}{\sqrt{15}}\right) = 0.809988350\ldots \qquad (2.1)$$

For a Salem number of degree $2t$ with $t > 2$, we want to estimate the corresponding $d_0$. First, let us show the following lemma.

**Lemma 2.1.** *Let $\theta$ be a Salem number of degree $2t$, then the repartition function $f(x)$ of the sequence $(\theta^n)$ modulo 1 satisfies*

$$f'(x) = 1 + 2\sum_{k=1}^{\infty} J_0(4k\pi)^{t-1} \cos 2\pi k x$$

*on $]0,1[$, for all $t \geqslant 2$.*

**Proof.** We have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \exp 2i\pi k\theta^n = \int_0^1 \exp 2i\pi kx \, d\nu$$

where $\nu$ is the repartition function $f(x)$. According to Y. Dupain [5] the measure $d\nu = f'(x)\,dx$ is absolutely continuous. It follows from (1.1) that

$$J_0(4\pi k)^{t-1} = \int_0^1 \exp 2i\pi kx\, f'(x)\,dx\ .$$

We can associate with $f'(x)$ its Fourier series

$$\sum_{k\in\mathbb{Z}} J_0(4\pi k)^{t-1}\exp(-2i\pi kx) = 1 + 2\sum_{k=1}^{\infty} J_0(4\pi k)^{t-1}\cos 2\pi kx. \tag{2.2}$$

If this series converges uniformly, then its sum is continuous and equals $f'(x)$. The lemma is clear for $t > 3$, since $J_0(x) = O\big(x^{-\frac{1}{2}}\big)$ and we even have equality on $[0,1]$. For $t = 2$ and 3, we need the following result.

**Lemma 2.2.** *The sequence* $\big(J_0(4\pi k)\big)$ *is positive for all* $k > 0$ *and strictly decreasing.*

**Proof.** In [1, Lemma 2], it is shown that

$$J_0(2\pi k) = \frac{1}{\pi\sqrt{k}}\left(\frac{1}{\sqrt{2}} - \frac{1}{16\sqrt{2}\pi k} + R\right),\ \text{with } |R| \leqslant \frac{9}{512\pi^2 k^2}\cdot$$

It is straightforward to deduce that

$$0 \leqslant \frac{1}{2\pi\sqrt{k}} - J_0(4\pi k) \leqslant \frac{1}{61\pi^2 k^{\frac{3}{2}}}\cdot \tag{2.3}$$

This proves the first part of the lemma. Now

$$\frac{1}{2\pi}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) \geqslant \frac{1}{8\pi k^{\frac{3}{2}}} > \frac{2}{61\pi^2 k^{\frac{3}{2}}}\cdot$$

This shows that

$$\frac{1}{2\pi\sqrt{k}} - \frac{1}{61\pi^2 k^{\frac{3}{2}}} > \frac{1}{2\pi\sqrt{k+1}} - \frac{1}{61\pi^2 (k+1)^{\frac{3}{2}}}$$

which implies that $J_0(4\pi k) > J_0\big(4\pi(k+1)\big)$, for $k > 0$. ■

We deduce that the series (2.2) is uniformly convergent on the compact $[\varepsilon, 1-\varepsilon]$, for any $\varepsilon > 0$ and therefore $f'(x)$ is equal to this series on $]0,1[$. ■

A consequence of Lemma 3.2 is that $d_0$ only depends on $t$ and satisfies

$$d_0 = \inf_{x\in]0,1[}\left(1 + 2\sum_{k=1}^{\infty} J_0(4k\pi)^{t-1}\cos 2\pi kx\right)\ .$$

Next let us recall a definition we shall use later.

**Definition 2.1.** *Let $(b_k)$ be a sequence of real numbers and let $\Delta^0 b_k = b_k$ and $\Delta^n b_k = \Delta^{n-1} b_k - \Delta^{n-1} b_{k+1}$, for all $n > 0$. The sequence $(b_k)$ is said to be totally monotone if $\Delta^n b_k \geqslant 0$ for all $k$, and $n = 0, 1, 2, \ldots$*

By a famous result of Hausdorff [7], the total monotonicity of $(b_k)$ is equivalent to the existence of a nonnegative measure $\mu$ on $[0, 1]$ such that the $b_k$'s are the moments of $\mu$, i.e.

$$b_k = \int_0^1 u^k \, d\mu \, .$$

**Example 2.1.** Let $s$ be a real positive number. The sequence $(b_k)$ defined by

$$b_k = \frac{1}{(k+1)^s}$$

for all $k \geqslant 0$ is totally monotone.

**Theorem 2.1.** *Let $(a_k)$ be a sequence of nonnegative real numbers (except maybe for $a_0$). Assume that $(a_{k+1})$, $k \geqslant 0$ is totally monotone, then the function*

$$g(x) = \sum_{k=0}^{\infty} a_k \cos 2\pi k x$$

*is well-defined and decreasing on the interval $]0, 1/2]$. As a corollary, $g(x)$ attains its minimum for $x = \frac{1}{2}$.*

**Proof.** Let us introduce

$$h(x) = \sum_{k=1}^{\infty} a_k \cos 2\pi k x = \sum_{k=0}^{\infty} b_k \cos 2\pi (k+1) x \, .$$

Since, $g$ and $h$ only differ by $a_0$, it is enough to study $h$ to prove the theorem on $g$. Since $(b_k) = (a_{k+1})$, $\Delta b_k \geqslant 0$, for all $k$. So the sequence $(b_k)$ is decreasing and this shows that the series $h(x)$ is convergent for all $x \in ]\varepsilon, 1-\varepsilon[$, for all $\varepsilon > 0$. Since $h(x) = h(1-x)$, it is enough to study $h$ on $]0, 1/2]$.

Since the $b_k$'s are the moments of a certain nonnegative measure $\mu$, we obtain

$$\begin{aligned} h(x) &= \sum_{k=0}^{\infty} b_k \cos 2\pi (k+1) x \\ &= \sum_{k=0}^{\infty} \int_0^1 u^k \cos 2\pi (k+1) x \, d\mu \\ &= \Re \int_0^1 \frac{e^{2i\pi x}}{1 - e^{2i\pi x} u} \, d\mu. \end{aligned}$$

The last equality being justified by the nonnegativity of $\mu$. It follows that

$$h(x) = \int_0^1 \frac{\cos 2\pi x - u}{1 + u^2 - 2u\cos 2\pi x}\, d\mu\ .$$

To show that $h(x)$ is decreasing on $]0, 1/2]$, evaluate $h(x) - h(y)$ for $0 < x \leqslant y \leqslant 1/2$. Let

$$j_x(u) = \frac{\cos 2\pi x - u}{1 + u^2 - 2u\cos 2\pi x}\cdot$$

Then reducing to the same (positive) denominator, we see that the numerator of $j_x(u) - j_y(u)$ is $(\cos 2\pi x - \cos 2\pi y)(1 - u^2)$ which is nonnegative for all $u \in [0, 1]$.

Since $\mu$ is a nonnegative measure, we deduce that $h(x) \geqslant h(y)$ whenever $x \leqslant y \leqslant 1/2$ and that $h(x) \geqslant h(1/2)$ for all $x \in ]0, 1/2]$. These results apply trivially to the function $g$. ■

**Corollary 2.1.** *Let $s > 0$. Then the series*

$$g(x) = a_0 + \sum_{k=1}^{\infty} \frac{\cos 2\pi k x}{k^s}$$

*is decreasing on* $]0, 1/2]$ *and satisfies*

$$g(x) \geqslant a_0 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s}\cdot$$

**Remark.** It is possible to compute $g(1/2)$ very efficiently following the method explained in [4]. For instance, for the sequence $(a_k)$ defined by a given $a_0$ and $a_k = 1/\sqrt{k}$, for $k \geqslant 1$, we have that

$$g(x) \geqslant g(1/2) = a_0 - 0.6048986434216303702472659142359554997597625451\ldots$$

All the digits in the last equality are correct as can be established knowing the first 60 $a_k$'s.

Unfortunately, we are not able to show that the sequence $\big(J_0(4\pi k)^{t-1}\big), k > 0$ is totally monotone, though the extensive numerical computations of its first $n$-th forward differences seem to indicate that this is the case. Based on the case $t = 2$ and also on direct computations of $f'(x)$ for various $x$, we conjecture that $\inf_x f'(x) = f'(1/2)$ for $t \geqslant 2$. However, to be totally rigorous, we cannot directly apply Theorem 2.1 to obtain the value of $d_0$. Nevertheless, this result will give an approximation of $d_0$, for $t > 2$.

The idea is to apply (2.3) to deduce that

$$\left| J_0(4\pi k)^{t-1} - \frac{1}{(2\pi\sqrt{k})^{t-1}} \right| \leqslant \frac{1}{61\pi^2 k^{\frac{3}{2}}} \frac{(t-1)}{(2\pi\sqrt{k})^{t-2}}\cdot$$

It follows that

$$\left| f'(x) - 1 - 2\sum_{k=1}^{\inf} \frac{\cos 2\pi k x}{(2\pi\sqrt{k})^{t-1}} \right| \leqslant \sum_{k=1}^{\inf} \frac{2}{61\pi^2 k^{\frac{3}{2}}} \frac{(t-1)}{(2\pi\sqrt{k})^{t-2}} ,$$

which, combined with Theorem 2.1, implies that for all $x \in ]0,1[$

$$f'(x) \geqslant \underbrace{1 + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{(2\pi\sqrt{k})^{t-1}}}_{S_1} - \underbrace{\sum_{k=1}^{\infty} \frac{2}{61\pi^2 k^{\frac{3}{2}}} \frac{(t-1)}{(2\pi\sqrt{k})^{t-2}}}_{S_2}.$$

The main contribution, i.e. $S_1$, can be obtained using the acceleration convergence method explained in [4], whereas the second series $S_2$ is simply (up to a constant) an evaluation of the $\zeta$ function at the point $(t+1)/2$. This gives a lower bound for $d_0$. An upper bound is given by $d_0 \leqslant f'(1/2)$, where $f'(1/2)$ is bounded, for any $K$ even, by the truncated alternating series

$$1 + 2\sum_{k=1}^{K} (-1)^k J_0(4\pi k)^{t-1} .$$

The convergence is quite slow for $t = 3$ so that we fixed $K = 2.10^6$ to obtain a relevant upper bound. Much less terms are necessary for larger $t$. A conjectured value $d_0^*$ is also given relying on the assumption that $d_0 = f'(1/2)$ and on the computation of $f'(1/2)$ using [4]. The method seems to converge and at most the first 10 terms are sufficient to give a result with an error less than $10^{-10}$. Also, we checked for $t = 2$ that the value given in (2.1) is, up to several hundred digits, equal to the one computed with this approach.

Note that if the sequence $\big(J_0(4\pi k)^{t-1}\big)$, defined for $k > 0$ is totally monotone, then both assumptions are valid, and therefore $d_0 = d_0^*$. All the figures are given in Table 1.

Table 1: Lower bound, upper bound, and conjectured value of $d_0$

| t | $S_1$ | $S_2$ | $S_1 - S_2$ | $f'(1/2)$ | $d_0^*$ |
|---|---|---|---|---|---|
| 3 | 0.964884753 | 0.000869699 | 0.964015054 | 0.965745539 | 0.965745543 |
| 4 | 0.993830708 | 0.000112882 | 0.993717825 | 0.994046008 | 0.994046007 |
| 5 | 0.998944571 | 0.000016098 | 0.998928472 | 0.998991788 | 0.998991787 |
| 6 | 0.999822887 | 0.000002401 | 0.999820485 | 0.999832498 | 0.999832497 |
| 7 | 0.999970695 | 0.000000367 | 0.999970328 | 0.999972560 | 0.999972559 |
| 8 | 0.999995201 | 0.000000056 | 0.999995144 | 0.999995551 | 0.999995550 |
| 9 | 0.999999220 | 0.000000008 | 0.999999211 | 0.999999285 | 0.999999284 |
| 10 | 0.999999874 | 0.000000001 | 0.999999872 | 0.999999886 | 0.999999885 |

In the next section we shall define the notion of "almost all" increasing sequences of integers $\big(s(n)\big)$. For almost all sequences $\big(s(n)\big)$ and for all irrational numbers $x$, $\big(xs(n)\big)$ is equidistributed. This already shows how exceptional those sequences $\big(s(n)\big)$ are for which $\big(\theta^{s(n)}\big)$ is equidistributed.

Furthermore R. Salem [11] demonstrated that if $\big(s(n)\big)$ is any increasing sequence such that $s(n) = O(n)$, then the Hausdorff dimension of the set of $x$ for which $\big(xs(n)\big)$ is not equidistributed (mod 1), vanishes. The $x$'s in Theorem 1.1 are therefore "rare" if indeed $s(n) \sim \frac{1}{d_0}n$.

## 3. Metrical Results

Let $S$ be the family of finite or infinite strictly increasing sequences of positive integers. To each $s = \big(s(n)\big) \in S$ corresponds a unique sequence $\chi \in D = \{0,1\}^{\mathbb{N}}$ (characteristic sequence) and conversely:

$$\chi(n) = \begin{cases} 1 & \text{if } n \in s, \\ 0 & \text{if not.} \end{cases}$$

Any measure on $D$ lifts to a measure on $S$.

Let $0 < d < 1$. Put $m\{1\} = d$ and $m\{0\} = 1 - d$. Then $\mu = \prod m$ is a probability measure on $D$ to which corresponds a probability measure on $S$ which we still denote by $\mu$ or $\mu_d$ if we wish to emphasize the parameter $d$.

**Theorem 3.1.** *Consider the polynomial $P(X) = \sum_{\ell=0}^{\nu} a_\ell X^\ell$ where at least one of the coefficients $a_\ell$, $1 \leqslant \ell \leqslant \nu$ is irrational. Then for $\mu$-almost all sequences $s \in S$, $P(s) = \big(P(s(n))\big)$ is equidistributed* (mod 1).

**Theorem 3.2.** *If $\theta$ is a Salem number then $\mu$-almost no sequence $\big(\theta^{s(n)}\big)$ is equidistributed* (mod 1). *More generally, if $P$ is any positive integer valued polynomial, $\theta^{P(s)} = \big(\theta^{P(s(n))}\big)$ is $\mu$-almost never equidistributed* (mod 1).

We have seen in Section 1 that there exists a $d_0 \in \,]0,1[$ for which no sequence $s = \big(s(n)\big)$ exists such that $s(n) \sim \frac{1}{d}n$ $(d > d_0)$ and $\big(\theta^{s(n)}\big)$ equidistributed (mod 1). For $d \leqslant d_0$ there do exist $d$-density equidistributed subsequences $\big(\theta^{s(n)}\big)$ but they are $\mu_d$-rare.

**Remark.** For $d \in [0,1]$ let $T(d)$ be the family of increasing sequences $\big(s(n)\big)$ of density $d$ such that $\big(\theta^{s(n)}\big)$ is equidistributed (mod 1). We know that $T(d) = \emptyset$ as long as $d > d_0$. Could it be true that as $d$ decreases to 0 the family $T(d)$ "increases in size"? Could one devise a way to show that this is so, e.g. by defining a fractal dimension adapted to the question?

## 4. Proof of Theorem 3.1

A sequence $\chi \in \{0,1\}^{\mathbb{N}}$ is said to be $d$-normal if all finite words $w = w_1 \dots w_\ell \in \{0,1\}^\ell$ occur in $\chi$ with the frequency $d^k(1-d)^{\ell-k}$ where $k$ is the number of 1's in $w$. It is well known that $\mu_d$-almost all $\chi$ are $d$-normal. For such a sequence

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \big(\chi(n) - d\big) = 0$$

and more generally, for all $k \geqslant 1$ and all integers $h_1 \leqslant \cdots \leqslant h_k$ where at least one couple $h_i < h_{i+1}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \prod_{i=1}^{k} \bigl(\chi(n+h_i) - d\bigr) = 0 \, .$$

A sequence $Y$ is said to be *uncorrelated* if for all $k \geqslant 1$ and all integers $h_1 \leqslant \cdots \leqslant h_k$ where at least one couple $h_i < h_{i+1}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \prod_{i=1}^{k} Y(n+h_i) = 0 \, .$$

If $\chi \in \{0,1\}^{\mathbb{N}}$ is $d$-normal, then as remarked above, $\chi - d$ is uncorrelated.

**Lemma 4.1.** *For all real polynomials $P$ and all uncorrelated sequences $Y$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} Y(n) \exp 2i\pi P(n) = 0 \, .$$

**Proof.** The result is obviously true if $\deg P = 0$. We now argue by induction and assume the truth of the lemma for all $P$ with $\deg P = \nu - 1 \geqslant 0$. Let $Q$ be any polynomial of degree $\nu$ and let $h \geqslant 1$ be an arbitrary integer. Put $f(n) = Y(n) \exp 2i\pi Q(n)$ and consider the correlation

$$\begin{aligned} &= \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \overline{f(n)} f(n+h) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} Y(n) Y(n+h) \exp 2i\pi \bigl(Q(n+h) - Q(n)\bigr). \end{aligned}$$

The product $Z(n) = Y(n)Y(n+h)$ is again uncorrelated and the polynomial $P(n) = Q(n+h) - Q(n)$ is of degree $\nu - 1$. Therefore

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \overline{f(n)} f(n+h) = 0$$

for all $h \geqslant 1$. A classical result (see J. Bass [2]) then implies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} f(n) = 0 \, . \qquad \blacksquare$$

We now prove Theorem 3.1. Suppose $P(X) = \sum_{\ell=0}^{\nu} a_\ell X^\ell$ where at least one of the coefficients $a_1, \ldots, a_\nu$ is irrational. Consider the exponential mean

$$\begin{aligned} &= \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \exp 2i\pi h P\bigl(s(n)\bigr) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{\ell<s(N)} \chi(\ell) \exp 2i\pi h P(\ell) \end{aligned}$$

where $h \geqslant 1$ is an integer, and where $\chi$ is the characteristic function of $s$.

For $\mu = \mu_d$-almost all $s$, $s(N) \sim \frac{1}{d}N = L$. The theorem will be established if for $L \to \infty$

$$\frac{1}{L}\sum_{\ell<L}\chi(\ell)\exp 2i\pi hP(\ell) \to 0\ .$$

The above average can be decomposed into two parts

$$\frac{1}{L}\sum_{\ell<L}\big(\chi(\ell)-d\big)\exp 2i\pi hP(\ell) + \frac{d}{L}\sum_{\ell<L}\exp 2i\pi hP(\ell)\ .$$

For $\mu_d$-almost all $s$, $\chi - d$ is uncorrelated and therefore the first average converges to 0. As for the second average, it converges to 0 because the sequence is well known to be equidistributed (mod 1) [12].

## 5. Proof of Theorem 3.2

Let $P(X) = \sum_{\ell=0}^{\nu} a_\ell X^\ell$, $a_\nu > 0$, be a polynomial which takes integer values when $X$ runs through $\mathbb{N}$. If $s \in S$,

$$\theta^{P\left(s(n)\right)} \equiv -2\sum_{j=1}^{t-1}\cos 2\pi\omega_j P\big(s(n)\big) + o(1)$$

if $P$ is nonconstant (if $P$ is constant the theorem is trivial). The $(t-1)$ polynomials $\omega_1 P, \ldots, \omega_{t-1}P$ all have irrational coefficients. According to Theorem 3.1, the sequences $\big(\omega_j P\big(s(n)\big)\big)$ are $\mu_d$-almost surely equidistributed (mod 1) and more to the point, for all $\underline{h} = (h_1, \ldots, h_{t-1}) \in \mathbb{Z}^{t-1}\setminus\{0\}$ the sequences $\underline{h}\,\underline{\omega}P(s)$ are equidistributed (mod 1). Here $\underline{h}\,\underline{\omega}P(s)$ is the scalar product of $\underline{h}$ and $\underline{\omega} = (\omega_1, \ldots, \omega_{t-1})$. Therefore the $(t-1)$ dimensional sequence $\big(\omega_1 P(s), \ldots, \omega_{t-1}P(S)\big)$ is equidistributed in $\big(\mathbb{R}/\mathbb{Z}\big)^{t-1}$ and as in the first section, we conclude that

$$\frac{1}{N}\sum_{n<N}\exp 2i\pi kP\big(s(n)\big) \underset{N\to\infty}{\longrightarrow} J_0(4k\pi)^{t-1} \neq 0\ .$$

## 6. A Final Remark

All our arguments are based on the fact that $\theta^n$ is essentially a finite sum of $\cos 2\pi\omega_j n$. We could probably extend some of our results to the study of sequences $u = \big(u(n)\big)$ of the type

$$u(n) = \sum_{j=1}^{t} F(n\omega_j)\ .$$

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