Almost everywhere convergence of inverse Fourier transforms

Leonardo Colzani, Christopher Meaney, and Elena Prestini

Abstract. We show that if \( \log(2 - \Delta)f \in L^2(\mathbb{R}^d) \) then the inverse Fourier transform of \( f \) converges almost everywhere. Here the partial integrals in the Fourier inversion formula come from dilates of a closed bounded neighborhood of the origin which is star shaped with respect to 0. Our proof is based on a simple application of the Rademacher-Menshov Theorem. In the special case of spherical partial integrals, the theorem was proved by Carbery and Soria. We obtain some partial results when \( \sqrt{\log(2 - \Delta)}f \in L^2(\mathbb{R}^d) \) and \( \log \log(4 - \Delta)f \in L^2(\mathbb{R}^d) \). We also consider sequential convergence for general elements of \( L^2(\mathbb{R}^d) \).

1. Introduction

We treat the almost everywhere convergence of partial integrals of inverse Fourier transforms on Euclidean space, for functions in \( L^2 \) with logarithmic Sobolev properties. The partial integrals are formed by integrating over dilates of a fixed closed bounded region \( V \) which is star shaped with respect to the origin and has the origin in its interior. Particular choices of \( V \) give rise to the familiar cases of spherical and polyhedral partial integrals. Our results are proved by a very simple application of the Rademacher-Menshov Theorem. In particular, we show that if the Fourier transform satisfies

\[
\int_{\mathbb{R}^d} \left( \log(2 + |y|^2) \right)^2 |\widehat{f}(y)|^2 \, dy < \infty,
\]

then the partial integrals

\[
S_R f(x) = \int_{RV} \widehat{f}(y)e^{2\pi ix \cdot y} \, dy
\]

converge almost everywhere as \( R \to \infty \). If we reduce the power of the logarithmic factor we have a partial result. We show that if

\[
\int_{\mathbb{R}^d} \log(2 + |y|^2) |\widehat{f}(y)|^2 \, dy < \infty,
\]

1991 Mathematics Subject Classification. Primary 42B10, 43A50; Secondary 42C15.

Key words and phrases. Rademacher-Menshov theorem, inverse Fourier transform, series of orthogonal functions.

CM: Partially supported by Progetto cofinanziato MIUR "Analisi Armonica". We are grateful to Fulvio Ricci and the Centro di Ricerca Matematica Ennio De Giorgi for their hospitality.

EP: Partially supported by Progetto cofinanziato MIUR "Analisi Armonica".
then $S_{R_n}f(x)$ converges almost everywhere as $R_n = n^{\log(n)} \to \infty$. When the logarithm is replaced by log log then we find that if
\begin{equation}
\int_{\mathbb{R}^d} \left( \log \log(4 + |y|^2) \right)^2 \left| \hat{f}(y) \right|^2 dy < \infty,
\end{equation}
then $S_{m}f(x)$ converges almost everywhere as $m \to \infty$, for unbounded sequences $(r_m)$ whose terms are in a second order lacunary set, as defined in [2].

When $V$ is a sphere the first case was done by Carbery and Soria [4, Theorem 3]. The introduction to their paper provides a broad description to the background of this area of Fourier Analysis. See also [6] for some weighted norm estimates in the spherical case. The third case is a slight extension of the main result in [2], where they work with integrals over spheres.

Our contribution is the simplicity of the proof and the fact that it is independent of the geometry of $V$. The method seems to depend only on the Plancherel formula, and follows the same idea as used in [8].

2. The Rademacher-Menshov Theorem

**Theorem 1.** Suppose that $(X, \mu)$ is a positive measure space. There is a positive constant $c$ with the following property.

For each orthogonal subset $\{P_n : n \in \mathbb{N}\}$ in $L^2(X, \mu)$ which satisfies
\begin{equation}
\sum_{n=1}^{\infty} (\log(n + 1))^2 \|P_n\|^2 < \infty,
\end{equation}
the maximal function $M(x) = \sup_{N \geq 1} \left| \sum_{n=1}^{N} P_n(x) \right|$ is in $L^2(X, \mu)$ and
\begin{equation}
\|M\|_2 \leq c \left( \sum_{n=1}^{\infty} (\log(n + 1))^2 \|P_n\|^2 \right)^{1/2}.
\end{equation}

In particular, when (4) holds then the series $\sum_{n=1}^{\infty} P_n(x)$ converges almost everywhere on $X$. 

See Theorem XIII.10.21 from [11], Proposition 2.3.1 and Theorem 2.3.2 from [1, Pages 79–80]. Here log means logarithm with base 2. For an application in $L^2(\mathbb{R}^d)$ see part (b) of Lemma 5.1 in [5].

3. Setting up the partial integrals

Suppose that $V$ is a bounded closed subset of $\mathbb{R}^d$ having 0 as an interior point and star shaped with respect to 0. Let $\beta = d(0, \partial V) > 0$. For each $R > 0$ dilate $V$ to get $RV = \{Ry : y \in V\}$, so that the dilated set has measure

$|RV| = R^d |V|$ and $d(0, \partial(RV)) = R \beta$.

Define partial integrals by
\begin{equation}
S_R f(x) = \int_{RV} \hat{f}(y) e^{2\pi i x \cdot y} dy, \quad \forall f \in L^2(\mathbb{R}^d),
\end{equation}
which give Fourier inversion in norm, $\lim_{R \to \infty} \|S_R f - f\|_2 = 0$. If $f \in L^2(\mathbb{R}^d)$ and $S_R f(x)$ converges almost everywhere as $R \to \infty$ then its limit equals $f(x)$ almost everywhere.
Now let \((R_n)_{n=1}^\infty\) be an unbounded increasing sequence of positive real numbers and fix an element \(f \in L^2(\mathbb{R}^d)\). We think of the partial integrals \(S_{R_n} f(x)\) as partial sums of the orthogonal expansion

\[
S_{R_1} f(x) + \sum_{n=2}^{\infty} \left( S_{R_n} f(x) - S_{R_{n-1}} f(x) \right). \tag{7}
\]

Define \(P_n f \in L^2(\mathbb{R}^d)\) by setting

\[
P_n f(x) = \begin{cases} 
S_{R_1} f(x) & \text{if } n = 1, \\
S_{R_n} f(x) - S_{R_{n-1}} f(x) & \text{if } n \geq 2,
\end{cases} \tag{8}
\]

then the partial sums of (7) are

\[
S_{R_n} f(x) = \sum_{k=1}^{n} P_k f(x), \quad \forall n \geq 1, x \in \mathbb{R}^d,
\]

and \(m \neq n\) implies that \(P_m f \perp P_n f\). The Plancherel formula says that

\[
\|P_n f\|_2^2 = \int_{R_n^{-1} \setminus R_{n-1}} \left| \hat{f}(y) \right|^2 \, dy. \tag{9}
\]

4. Convergent subsequences

Suppose that \(f \in L^2(\mathbb{R}^d)\). Since \(S_{R} f\) converges to \(f\) in norm, there exists a sequence \((R_n)_{n=0}^\infty\) with \(\lim_{n \to \infty} S_{R_n} f(x) = f(x)\) almost everywhere. The Rademacher-Menshov Theorem gives a way of describing one such sequence.

**Proposition 2.** Suppose \(f \in L^2(\mathbb{R}^d)\). If an increasing unbounded sequence \(0 = R_0 < R_1 < R_2 < \cdots\) has the property

\[
\sum_{n=1}^{\infty} \left( \log(n + 1) \right)^2 < \infty, \tag{10}
\]

then \(\lim_{n \to \infty} S_{R_n} f(x) = f(x)\) almost everywhere. Furthermore, for each \(f \in L^2(\mathbb{R}^d)\) there is an increasing unbounded sequence \((R_n)_{n=0}^\infty\) with property (10).

**Proof.** The first statement is a direct consequence of Theorem 1. It remains to prove the second statement. If \(\hat{f}\) has bounded support then it is integrable and the statement is immediate. Now suppose that \(\hat{f}\) is not compactly supported. The function \(R \mapsto F(R) = \|S_R f\|_2\) is continuous, its values are non-negative, if \(R' < R''\) then \(F(R') \leq F(R'')\), and \(\lim_{R \to \infty} F(R) = \|f\|_2\). Let \((a_n)_{n=1}^\infty\) be a sequence of positive numbers with

\[
\sum_{n=1}^{\infty} a_n = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} (\log(n + 1))^2 a_n < \infty.
\]

There is an increasing unbounded sequence \((R_n)_{n=1}^\infty\) with the property

\[
F(R_n)^2 = \|f\|_2^2 \sum_{m=1}^{n} a_m, \forall n \geq 1.
\]
In particular, \( \|S_{R_{n+1}}f\|_2^2 - \|S_{R_n}f\|_2^2 = a_{n+1}\|f\|_2^2 \), for all \( n \geq 1 \). Define the projections as in (8). Then we have that
\[
\sum_{n=1}^{\infty} (\log(n+1))^2 \|P_n f\|_2^2 < \infty
\]
and we can apply Theorem 1.

The Cauchy-Schwarz inequality and the Plancherel formula imply that when a sequence of partial integrals converges, then the sequence can be perturbed slightly and still preserve convergence.

**Lemma 3.** Suppose \((R_n)_{n=1}^{\infty}\) is an increasing unbounded sequence. For each \( \rho > 0 \) and \( n \geq 1 \) define the set
\[
E_\rho(n) = \left\{ r > 0 : |r^d - R_n| \leq \rho \right\}.
\]
For these sets and \( f \in L^2(\mathbb{R}^d) \) there is the inequality,
\[
\sup_{n \geq 1} \left( \sup_{r \in E_\rho(n)} |S_r f(x) - S_{R_n} f(x)| \right) \leq \|f\|_2 \sqrt{\rho |V|}, \quad \forall x \in \mathbb{R}^d.
\]

Now fix \( f \in L^2(\mathbb{R}^d) \) and suppose \((R_n)_{n=1}^{\infty}\) is an increasing unbounded sequence for which \( S_{R_n} f(x) \) converges almost everywhere. Furthermore, let \( E_\rho = \bigcup_{n=1}^{\infty} E_\rho(n) \). If \((r_m)_{m=1}^{\infty}\) is an increasing unbounded sequence whose terms belong to a set \( E_\rho \), then \( \lim_{m \to \infty} S_{r_m} f(x) = f(x) \), almost everywhere.

**Proof.** If \( 0 \leq R_n^d - r^d \leq \rho \) then \( rV \subset R_n V \) and \( |R_n V \setminus V| \leq \rho |V| \), so that
\[
|S_{R_n} f(x) - S_r f(x)| \leq \left( \int_{R_n V \setminus rV} \hat{f}(y)^2 \, dy \right)^{1/2} \sqrt{\rho |V|}.
\]

Since \( f \in L^2(\mathbb{R}^d) \), the right hand side tends to zero as \( R_n \to \infty \). A similar argument applies to the case \( 0 \leq r^d - R_n^d \leq \rho \).

We can apply the Rademacher-Menshov Theorem again to give a minor extension of Proposition 2.

**Lemma 4.** Suppose that \( f \in L^2(\mathbb{R}^d) \) and \((R_n)_{n=0}^{\infty}\) satisfy (10) of Proposition 2. If \((r_m)_{m=1}^{\infty}\) is an unbounded increasing sequence with the property that
\[
|\{m : R_n \leq r_m \leq R_{n+1}\}| \leq cn^\gamma, \quad \forall n \geq 1,
\]
for some positive constants \( c \) and \( \gamma \), then
\[
\lim_{m \to \infty} S_{r_m} f(x) = f(x), \quad \text{almost everywhere}.
\]

**Proof.** For each \( n \geq 1 \), suppose that there is a finite set of \( M_n \) real numbers arranged in the interval \((R_n, R_{n+1})\), say
\[
R_n = r_1(n) < \cdots < r_{M_n}(n) = R_{n+1}
\]
and define functions
\[
Q_{k,n}(x) = S_{r_{k+1}(n)} f(x) - S_{r_k(n)} f(x), \quad 1 \leq k < M_n.
\]
These functions form an orthogonal subset of $L^2(\mathbb{R}^d)$ and so the Rademacher-Menshov Theorem says that

$$\max_{1 \leq m < M_n} \left| S_{r_m(n)} f(x) - S_{R_n} f(x) \right| = \max_{1 \leq m < M_n} \left| \sum_{k=1}^m Q_k, n(x) \right|$$

has $L^2$ norm bounded by

$$c (\log M_n) \left\| S_{R_{n+1}} f - S_{R_n} f \right\|_2 = c (\log M_n) \left\| P_{n+1} f \right\|_2.$$

Suppose that

$$\log M_n \leq \gamma \log n = \log (n^\gamma), \quad \forall n \geq 2.$$ 

Because of (10) we see that

$$\sum_{n=1}^\infty \max_{1 \leq m \leq M_n} \left| S_{r_m(n)} f(x) - S_{R_n} f(x) \right|^2$$

is in $L^1(\mathbb{R}^d)$. We then have that as $n \to \infty$,

$$\max_{1 \leq m \leq M_n} \left| S_{r_m(n)} f(x) - S_{R_n} f(x) \right| \to 0, \text{ almost everywhere.}$$

5. The Main Result

**Proposition 5.** Suppose that $f \in L^2(\mathbb{R}^d)$ satisfies the condition (1). Then

$$\lim_{R \to \infty} S_{R_n} f(x) = f(x), \text{ almost everywhere on } \mathbb{R}^d.$$ 

Furthermore, there is a constant $c > 0$ so that for all $w \in \mathbb{R}^d$,

$$\int_{|x-w| \leq 1} \sup_{R > 0} |S_R f(x)|^2 \, dx \leq c \int_{\mathbb{R}^d} (\log (2 + |y|^2))^2 \left| \hat{f}(y) \right|^2 \, dy. \tag{12}$$

**Proof.** Take the sequence $R_n = n^{1/d}$ in setting up (8) and let

$$M f(x) = \sup_{n \geq 1} |S_{R_n} f(x)|, \quad \forall x \in \mathbb{R}^d.$$ 

When $y$ is in the shell $R_n V \setminus R_{n-1} V$ it satisfies $|y| \geq (n-1)^{1/d} \beta$ and for large $n$ there is a constant $c > 0$ for which

$$\log(n+1) \leq c \log (2 + |y|^2), \quad \forall y \in R_n V \setminus R_{n-1} V.$$ 

Combine this with (9) to see that

$$(\log(n+1))^2 \left\| P_n \right\|_2^2 \leq c \int_{R_n V \setminus R_{n-1} V} (\log (2 + |y|^2))^2 \left| \hat{f}(y) \right|^2 \, dy.$$ 

Since $f$ satisfies inequality (1), the sum of the terms on the right hand side is finite. This verifies the hypothesis (4) in Theorem 1 and so $S_{R_n} f(x)$ converges almost everywhere as $n \to \infty$. Furthermore, we see that since (1) holds then inequality (5) says that

$$\left\| M f \right\|_2 \leq c \left( \int_{\mathbb{R}^d} (\log(2 + |y|^2))^2 \left| \hat{f}(y) \right|^2 \, dy \right)^{1/2}. \tag{13}$$
We can dominate the maximal function over $R \geq 1$ by the maximal function over the sequence $(R_n)_{n=1}^{\infty}$ plus a remainder,

$$\sup_{R \geq 1} |S_R f(x)| \leq M f(x) + \sup_{n > 0} \left( \sup_{R_n \leq r < R_{n+1}} |S_r f(x) - S_{R_n} f(x)| \right).$$

We chose the sequence $R_n = n^{1/d}$ so that the increments in the measure of the dilates of $V$ are constant,

$$|R_n V \setminus R_{n-1} V| = n|V| - (n-1)|V| = |V|.$$

If $R_n \leq r < R_{n+1}$ then $n \leq r^d < n+1$ and $|r^d - n| = |r^d - R_n^d| \leq 1$, so that we can apply Lemma 3 with $\rho = 1$. Hence

$$\left\| \sup_{n > 0} \left( \sup_{R_n \leq r < R_{n+1}} |S_r f - S_{R_n} f| \right) \right\|_\infty \leq c \|f\|_2.$$ 

Combine inequalities (13) and (14) to prove (12). \hfill \Box

See [6, Chapter 2] for more sophisticated methods for estimating $S_{R_n} f(x) - S_r f(x)$.

6. The case of one power of logarithm

The first part of the method used above can be applied to other sequences.

**Proposition 6.** Suppose that $f \in L^2(\mathbb{R}^d)$ satisfies the condition (2) and that $R_n = n^{\log n}$, for $n \geq 1$. Then $\lim_{n \to \infty} S_{R_n} f(x) = f(x)$, almost everywhere on $\mathbb{R}^d$.

**Proof.** We have that $\log(R_n) = (\log n)^2$ and for large $n$ there is a constant $c$ for which

$$\log(n+1)^2 \|P_n f\|_2^2 \leq c \int_{R_n V \setminus R_{n-1} V} \log \left(2 + |y|^2 \right) \left|\hat{f}(y)\right|^2 dy.$$

Inequality (2) means that the sum of the terms on the right hand side is finite and so Theorem 1 applies. \hfill \Box

Note that $n^{\log n} = 2^{(\log n)^2}$ grows slower than any unbounded geometric progression but faster than $n^k$, for each $k \in \mathbb{N}$. The measure of the shell $R_n V \setminus R_{n-1} V$ grows too rapidly to use the estimate from Lemma 3. However, Lemma 4 gives convergence for some other sequences.

**Corollary 7.** Suppose that $f \in L^2(\mathbb{R}^d)$ satisfies (2) and $(r_m)_{m=1}^{\infty}$ is an unbounded increasing sequence with the property that

$$\left| \left\{ m : n^{\log n} \leq r_m \leq (n+1)^{\log(n+1)} \right\} \right| \leq cn^\gamma, \quad \forall n \geq 1,$$

for some positive constants $c$ and $\gamma$, then $\lim_{m \to \infty} S_{r_m} f(x) = f(x)$, almost everywhere.
7. Iterated Logarithm

Fix $a > 1$ and define the geometric progression $R_n = a^n$, for all $n \geq 1$. For $y \in R_n \setminus R_{n-1}$ we have $|y| \geq a^{n-1} \beta$ and for large $n$ there is a constant $\kappa > 0$ with

$$\log \log (4 + |y|^2) \geq \kappa \log (n+1).$$

This means that for large $n$ we have

$$\kappa^2 (\log(n+1))^2 \int_{R_n \setminus R_{n-1}} |\hat{f}(y)|^2 \, dy \leq \int_{R_n \setminus R_{n-1}} (\log \log (4 + |y|^2))^2 |\hat{f}(y)|^2 \, dy$$

and we can again apply Theorem 1.

**Corollary 8.** Suppose that $f \in \mathbb{L}^2(\mathbb{R}^d)$ satisfies (3) and that $a > 1$ is fixed. Then $\lim_{n \to \infty} S_{a^n} f(x) = f(x)$, almost everywhere on $\mathbb{R}^d$.

**Remark 7.1.** For lacunary spherical partial integrals there is a much stronger result in [3, Theorem B] and in [7].

Lemma 4 can be applied to the case of $R_n = a^n$.

**Corollary 9.** Fix $a > 1$ and let $(r_m)_{m=1}^\infty$ be an unbounded increasing sequence with the property that

$$|\{m : a^n \leq r_m \leq a^{n+1}\}| \leq cn^\gamma, \quad \forall n \geq 1,$$

for some positive constants $c$ and $\gamma$. If $f \in \mathbb{L}^2(\mathbb{R}^d)$ satisfies (3) then

$$\lim_{m \to \infty} S_{r_m} f(x) = f(x), \quad \text{almost everywhere.}$$

We can combine Corollary 8 with Lemma 3 and Corollary 9 to extend the result of [2] to the case of general $V$.

**Corollary 10.** Fix $a > 1$ and suppose $f \in \mathbb{L}^2(\mathbb{R}^d)$ satisfies (3). Let

$$A = \{a^n(1-a^{-k}) : n, k \in \mathbb{N}\}$$

and suppose that $(r_m)_{m=1}^\infty$ is an increasing unbounded sequence whose terms belong to $A$. Then $\lim_{m \to \infty} S_{r_m} f(x) = f(x)$, almost everywhere.

**Proof.** Let $R_n = a^n$ and consider the set $E_1$, as defined in Lemma 3. We need to count how many elements are in $(A \setminus E_1) \cap [a^{n-1}, a^n]$, for each $n \geq 1$. That is, we count how many $k$ satisfy

$$a^n - a^{n-1} (1 - a^{-k}) > 1.$$  \hspace{1cm} (16)

This is equivalent to the inequality

$$1 - (1 - a^{-k})d > a^{-dn}$$

and the left hand side is equal to $da^{-k}y^{d-1}$ for some $1 - a^{-k} \leq y \leq 1$. Taking logarithms, we see that if $k$ satisfies the inequality (16) then we must have $k \leq cn$, for some constants $c$. This shows that $A \setminus E_1$ satisfies the criterion of Corollary 9. If a sequence has its values in $A$ then it is made up of subsequences in $A \cap E_1$ and $A \setminus E_1$. Apply Lemma 3 for $A \cap E_1$ and Corollary 9 for $A \setminus E_1$. 

\[\Box\]
8. Capacity

We conclude with an extension of Theorem 1.3 of [9] to summation based on the set \( V \). Following Definition 2 in [9], for each \( 0 < \alpha < d \) the \((\alpha, 2)\)-capacity of a subset \( X \subset \mathbb{R}^d \)

\[
C_\alpha(X) = \inf \left\{ \|f\|^2_2 : f \in L^2_+(\mathbb{R}^d), \quad G_\alpha * f(x) \geq 1, \forall x \in X \right\}.
\]

Here \( G_\alpha \) is the Bessel kernel, with \( G_\alpha(y) = \left(1 + |y|^2\right)^{-\alpha/2} \). Its properties are cataloged in [10, Section V.3]. Most importantly, \( G_\alpha(x) \geq 0 \) for all \( x \neq 0 \). Notice that if \( f \in L^2_+(\mathbb{R}^d) \) and

\[
X \subseteq \{x : G_\alpha * f(x) \geq \lambda\} \text{ then } C_\alpha(X) \leq \lambda^{-2}\|f\|^2_2.
\]

Capacity is subadditive and sets of capacity zero have Lebesgue measure zero.

Let \( R_n = n^{1/d} \) for each \( n \geq 1 \), as in the proof of Proposition 5, and define \( Mf(x) = \sup_{n \geq 1} |S_{R_n}f(x)| \). Recall that this satisfies inequality (13).

Lemma 11. Suppose that \( \varphi \in L^2(\mathbb{R}^d) \) satisfies

\[
N(\varphi, \alpha) := \int_{\mathbb{R}^d} |\hat{\varphi}(y)|^2 \left(1 + |y|^2\right)^{\alpha} (\log (2 + |y|))^2 \, dy < \infty,
\]

for some \( 0 < \alpha < d \). There is a positive constant \( c \) so that

\[
C_\alpha \left( \{x : M\varphi(x) \geq \lambda\} \right) \leq c\lambda^{-2}N(\varphi, \alpha), \quad \forall \lambda > 0.
\]

Proof. Since \( \varphi \) satisfies (17), there is a \( \psi \in L^2(\mathbb{R}^d) \) with \( \varphi = G_\alpha * \psi \) and

\[
N(\varphi, \alpha) = N(\psi, 0) = \int_{\mathbb{R}^d} \left|\hat{\psi}(y)\right|^2 (\log (2 + |y|))^2 \, dy < \infty.
\]

Inequality (13) can be applied to both \( \psi \) and to \( \varphi = G_\alpha * \psi \), so that the maximal functions satisfy \( M\psi \in L^2(\mathbb{R}^d) \) and \( M(G_\alpha * \psi) \in L^2(\mathbb{R}^d) \). Since \( G_\alpha \) is positive, the observation on page 1419 of [9] can be adapted to our sequential maximal function so that

\[
M(G_\alpha * \psi)(x) \leq G_\alpha * (M\psi)(x), \quad \forall x \in \mathbb{R}^d.
\]

For each \( \lambda > 0 \) let

\[
X_\lambda = \{x : M(G_\alpha * \psi)(x) \geq \lambda\} \subseteq \{x : G_\alpha * (M\psi)(x) \geq \lambda\}.
\]

From the definition of capacity, \( C_\alpha(X_\lambda) \leq \lambda^{-2}\|M\psi\|^2_2 \leq c\lambda^{-2}N(\psi, 0) \).

Proposition 12. Suppose that \( \varphi \in L^2(\mathbb{R}^d) \) satisfies (17) for some \( 0 < \alpha < d \). The set on which \( S_{R\varphi}(x) \) does not converge to \( \varphi(x) \), as \( R \to \infty \), has \((\alpha, 2)\)-capacity zero.

Proof. The argument based on Lemma 3 shows that it is enough to consider the convergence of \( S_{R_n}\varphi(x) \) as \( n \to \infty \). Let \( \psi \) be the function in the previous proof, so that \( \varphi = G_\alpha * \psi \). For \( \delta > 0 \) let \( H \in C_\alpha^\infty(\mathbb{R}^d) \) satisfy

\[
N(\psi - H, 0) = \int_{\mathbb{R}^d} \left|\hat{\psi}(y) - \hat{H}(y)\right|^2 (\log (2 + |y|))^2 \, dy < \delta.
\]

We know that \( \lim_{R \to \infty} S_R(G_\alpha * H)(x) = G_\alpha * H(x) \), for all \( x \). For each \( \eta > 0 \),

\[
\left\{ x : \limsup_{n \to \infty} |S_{R_n}\varphi(x) - \varphi(x)| > \eta \right\} \subseteq
\]
Lemma 11 shows that

\[
C_\alpha \left( \left\{ x : \sup_{n \geq 1} |S_{R_n} (\psi - G_\alpha \ast H) (x) | > \frac{\eta}{2} \right\} \right) \leq 4c \eta^{-2} \delta.
\]

Observe that \(|G_\alpha \ast \psi - G_\alpha \ast H| \leq G_\alpha \ast |\psi - H|\). The definition of capacity shows that

\[
C_\alpha \left( \left\{ x : |G_\alpha \ast \psi (x) - G_\alpha \ast H (x) | > \frac{\eta}{2} \right\} \right) \leq 4\eta^{-2} \| \psi - H \|_2^2 < 4\eta^{-2} c_2^2 \delta.
\]

Letting \(\delta \to 0\), we find that

\[
C_\alpha \left( \left\{ x : \limsup_{n \to \infty} |S_{R_n} \varphi (x) - \varphi (x) | > \eta \right\} \right) = 0,
\]

for every \(\eta > 0\). The set of divergence is

\[
\bigcup_{k \geq 1} \left\{ x : \limsup_{n \to \infty} |S_{R_n} \varphi (x) - \varphi (x) | > \frac{1}{k} \right\},
\]

which is a countable union of sets of \((\alpha, 2)\)-capacity zero and so it also has \((\alpha, 2)\)-capacity zero.

One consequence of this Proposition is that the partial inverse Fourier integrals of functions in Sobolev classes \(L^2_\alpha (\mathbb{R}^d)\) converge pointwise, with the possible exception of sets with zero \((\alpha - \varepsilon, 2)\)-capacity, for every \(\varepsilon > 0\).

References


Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, Edificio U5, via Cozzi 53, 20125 Milano, Italy
E-mail address: leonardo@matapp.unimib.it

Department of Mathematics, Macquarie University, North Ryde NSW 2109, Australia
E-mail address: chrism@maths.mq.edu.au

Dipartimento di Matematica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail address: prestini@mat.uniroma2.it