8. FROBENIUS GROUPS

§8.1. Permutation Representations

A **permutation representation** is one where the corresponding matrices are permutation matrices.

**Theorem 1:** If $\theta$ is the character of a permutation representation then $g\theta$ is the number of symbols fixed by $g$.

**Proof:** $g\theta$ is the trace of the corresponding matrix. This matrix has entries 0 and 1 only, and there is a “1” on the diagonal in those positions that correspond to a symbol fixed by $g$.

A permutation group $G$ is **transitive** permutation group on $X$ if for all $x, y \in X$ there exists $\alpha \in G$ such that $x\alpha = y$. Let $G_r = \{g \mid rg = s\}$ and let $Gr = G_{rr}$. Then $Gr \leq G$ for all $r$.

**Theorem 2:** If $G \leq S_n$ is transitive then $|G:Gr| = n$ for all $r$.

**Proof:** If $g \in G_r$ then $GrG = G_r$ so $Gr_1, \ldots, G_rn$ are the left cosets of $Gr$ in $G$.

**Theorem 3:** If $G$ is a transitive permutation group on $n$ symbols, and $\theta$ is the permutation representation, then $\sum_{g \in G} g\theta = |G|$.

**Proof:** $\sum_{g \in G} g\theta$ is the size of the set $\{(x, g) \mid xg = x\}$, which is $\sum_{r=1}^{n} |Gr| = n \frac{|G|}{n} = |G|$.

§8.2. Multiply Transitive Permutation Groups

A permutation group $G$, on a set $X$, is $t$-**transitive** if for all $t$-tuples $(x_1, \ldots, x_t) \in X^t$, with the $x_i$’s distinct and the $y_i$’s distinct, there exists $\alpha \in G$ such that for each $i$, $x_i\alpha = y_i$.

**Examples 1:** $S_n$ is $n$-transitive, $A_n$ is $(n - 2)$-transitive. $S_3 \times S_3$ is not transitive.

**NOTE:** Apart from $S_n$ and $A_n$ there are only four 4-transitive groups, called the Mathieu groups.

**Theorem 4:** If $G$ is a 2-transitive permutation group and $\theta$ is the corresponding character then $\sum_{g \in G} (g\theta)^2 = 2|G|$.

**Proof:** $G$ is transitive on $D = \{(x, y) \mid x \neq y\}$, where $(x, y)g$ is defined to be $(xg, yg)$. Now the number of elements of $D$ fixed by $g$ is $(g\theta)(g\theta) - 1$.

So $\sum_{g \in G} [(g\theta)^2 - (g\theta)] = |G|$ and so $\sum_{g \in G} (g\theta)^2 = \sum_{g \in G} (g\theta) + |G| = 2|G|$.

**Theorem 5:** Let $G$ be a permutation group with character $\theta$.

If $G$ is transitive, $\chi_1$ occurs in $\theta$ with multiplicity 1.

If $G$ is doubly transitive, $\theta = \chi_1 + \chi_j$ for some non-trivial irreducible character $\chi_j$.

**Proof:** If $G$ is transitive, $\langle \chi_1 \mid \theta \rangle = \Sigma (g\theta)\chi_1/|G| = 1$.

If $G$ is doubly transitive, $\langle \theta \mid \theta \rangle = \Sigma (g\theta)^2/|G| = 2$. 

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§8.3. Frobenius Permutation Groups

**Theorem 6:** Suppose $G$ is a transitive permutation group on $n$ symbols, and only the identity fixes more than one symbol. Let $H = G_1$ and let $\chi_1$ be the trivial character of $G$, $\theta$ the permutation character of $G$ and let $\psi$ be any irreducible character of $H$, of degree $m$. Then $\psi^* = \psi^G - m(\theta - \chi_1)$ is an irreducible character of $G$.

**Proof:** Let $X = \{g \mid g$ fixes exactly one symbol $\}$ and $Y = \{g \mid g$ fixes no symbols $\}$. Then $G = \{1\} + X + Y$ and each of $X, Y$ is the disjoint union of conjugacy classes.

\[
\begin{array}{c|ccc}
\theta & 1 & X & Y \\
\hline
\chi_1 & n & 1 & 1 & 0 & 0 \\
\alpha & 1 & 0 & 1 & 1 & 0 \\
\end{array}
\]

Since $\langle \theta \mid \chi_1 \rangle > 0$, $\alpha = 0 - \chi_1$ is a character:

\[
\begin{array}{c|ccc}
\theta & 1 & X & Y \\
\hline
\chi_1 & n & 1 & 1 & 0 & 0 \\
\alpha & 1 & 0 & 1 & 1 & 0 \\
\end{array}
\]

The elements of $X$ consist of the non-trivial elements of the $G_r$, for $r = 1, 2, \ldots, n$. Hence $|X| = n(h - 1)$ where $h = |H| = |G|/n$ and so $|Y| = n - 1$. Since the $G_r$ are conjugate in $G$ each conjugacy class within $X$ contains an element of $H$.

Let $1 \neq x \in H$ and let $y \in C_G(x)$. Then $1yx = 1xy = 1y$. Since $x$ fixes both $1$ and $1y$ it must be that $1y = y$ and so $y \in H$. Hence $C_G(x) = C_H(x)$ and so $|x^G| = n|x^H|$. If $g^{-1}xg = y \in H$, $1gy = 1xg = 1g$. Clearly $y \neq 1$ and, since it fixes both $1$ and $1g$, $1g = 1$ and so $g \in H$. It follows that $x^G \cap H = x^H$. Hence the proportion of each conjugacy class within $X$ that lies in $H$ is $1/n$.

Let the conjugacy classes of $H$ be $\Gamma_1, \ldots, \Gamma_k$ with sizes $h_1, \ldots, h_k$. For each $r = 1, 2, \ldots, n$ choose $z_r \in G_{\Gamma_r}$. Then the conjugacy classes of $G$ that lie within $X$ are of the form $\Omega_j = \bigcup_{r=1}^n z_r^{-1}\Gamma_j z_r$, for $j = 2, \ldots, n$, where $\Sigma$ here denotes a disjoint union. Moreover $\Omega_j \cap H = \Gamma_j$. So within $X$ there are $k - 1$ conjugacy classes of sizes $nh_2, \ldots, nh_k$.

\[
\begin{array}{c|ccc}
\alpha & n-1 & 0 & \ldots \\
\psi^G & mn & \psi(\Gamma_2) & \ldots \\
\psi^* & m & \psi(\Gamma_2) & \ldots \\
\psi & m & \psi(\Gamma_2) & \ldots \\
\end{array}
\]

At this stage $\psi^*$ is just a linear combination of irreducible characters, with integer coefficients. It may not be a character, let alone an irreducible one.

\[
\langle \psi^*, \psi^* \rangle = \frac{m^2 + \sum_{j=2}^n n h_j |\psi(\Gamma_j)|^2 + (n-1)m^2}{nh} = \frac{m^2 n - m^2 + n \sum_{j=1}^n h_j |\psi(\Gamma_j)|^2}{nh} = \frac{nh}{nh} = 1.
\]

Thus $\psi^G - m\alpha$ is an irreducible character or minus an irreducible character.
But \((\psi^G - m\alpha)(1) = mn - m(n - 1) = m \geq 0\) so \(\psi^G - m\alpha\) is an irreducible character.  

**Corollary:** The set of permutations in \(G\) which fix no symbols, together with the identity, is a normal subgroup of \(G\).  

**Proof:** For all \(x \in X \cap H\) there exists an irreducible character \(\psi\) of \(H\) such that \(\psi(x) \neq \deg \psi\).  Hence \(x\) lies outside of the kernel of the representation corresponding to \(\psi^G - (\deg \psi)\alpha\).  The intersection of these kernels must therefore be \(\{1\} + Y\).  

**Example 2:** \(G = A_4, H \cong C_3, X = (\times \times), Y = (\times \times)(\times \times), h = 3\).  

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>((\times \times))</th>
<th>((\times \times))</th>
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<tr>
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<td>(\omega^2)</td>
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<tr>
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<td>(\omega^2)</td>
<td>(\omega)</td>
</tr>
<tr>
<td>(\psi_3^G - m\alpha)</td>
<td>1</td>
<td>(\omega^2)</td>
<td>(\omega)</td>
</tr>
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</table>

§8.4. Abstract Frobenius Groups  

A group \(G\) is a Frobenius group if there exists a proper non-trivial subgroup \(H\) of \(G\) such that \(N_G(H) = H\) and \(C_G(h) \leq H\) for all non-trivial \(h \in H\).  

**Example 3:** \(S_3\) is a Frobenius group with \(H\) being any of the three subgroups of order 2.  

**Theorem 7:** \(G\) is Frobenius if and only if there exists a proper non-trivial subgroup \(H\) of \(G\) such that \(H \cap H^g = 1\) for all \(g \in G - H\).  

**Proof:** Suppose \(H \cap H^g = 1\) for all \(g \notin H\) and let \(g \in C_G(h)\) for \(1 \neq h \in H\).  Then \(h \in H \cap H^g\) and so \(g \in H\).  

Suppose now \(G\) is Frobenius and let \(|G:H| = N\).  Suppose \(1 \neq h \in H \cap H^g\) for some \(g \notin H\).  Let \(1 = x_1, g = x_2, x_3, \ldots, x_N\) be a set of left coset representatives of \(H\) in \(G\).  Then \(H, H^g, H^{x_2}, \ldots, H^{x_N}\) are the conjugates of \(H\) in \(G\) and, since \(N_G(H) = H\), they are distinct.  

Clearly \(h^G = \bigcup_{i=1}^{N} h^{x_i}\).  Now \(|h^{x_i}| = |h^H|\) for each \(i\).  Since \(h \in h^{x_i} \cap h^{x_j}\), \(|h^G| < N|h^H|\).  But \(C_G(h) = C_H(h)\) so \(|h^G| = N|h^H|\), a contradiction.  

**Theorem 8:** Suppose \(G\) is Frobenius.  Then \(G\) is a transitive group of permutations on the left cosets of \(H\) and only the identity fixes more than one coset.  

**Proof:** \((Ha)g = Hag\) so \(G\) permutes the right cosets.  Moreover it acts transitively.  

If \((Ha)g = Ha\) and \((Hb)g = Hb\) for \(g \neq 1\) then \(g \in H^a \cap H^b = (H \cap H^{x_2})^a\) so \(Ha = Hb\).  

**Theorem 9:** A Frobenius group has a normal subgroup \(K\) and a subgroup \(H\) such that \(G = KH\) and \(H \cap K = 1\).  \((G\) is called a split extension of \(K\) by \(H\)\).  

The subgroup \(K\) is called the kernel of the Frobenius group.
**Theorem 10:** If \( A \) is any abelian group of odd order and \( G \) is \( A \) extended by a cyclic subgroup \( H = \langle g \rangle \) of order 2 where \( g \) induces the automorphism \( a \rightarrow a^{-1} \) on \( A \), then \( G \) is a Frobenius group with kernel \( A \).

**Proof:** Clearly \( A \) is a normal subgroup of \( G \).
A typical element of \( G \) has the form \( a \) or \( ag \) where \( a \in A \).
If \( a \) commutes with \( g \) then \( ag = ga = a^{-1}g \) in which case \( a^2 = 1 \). Since \( A \) has odd order we must have \( a = 1 \).
If \( ag \) commutes with \( g \) then \((ag)g = g(ag) = a^{-1}g^2\) in which case \( a^2 = 1 \) and so again \( a = 1 \).
Hence \( H \cap H^a = 1 \) and \( H \cap H^{ag} = 1 \) for all \( a \in A \).

**Example 4:** The dihedral group \( D_{2n} = \langle A, B \mid A^n, B^2, BA = A^{-1}B \rangle \) is a Frobenius group, with kernel \( K = \langle A \rangle \). Any of the subgroups of order 2, for example \( \langle B \rangle \) can play the role of \( H \) in the definition.

**Theorem 11:** If \( G \) is Frobenius with kernel \( K \) and complement \( H \) and \( 1 \neq k \in K \) then \( C_G(k) = C_K(k) \).

**Proof:** Let \( g \in C_G(k) \). If \( Hag = Ha \) then \( aga^{-1} \in H \). But \( aka^{-1} \in C_G(aga^{-1}) \) whence \( g = 1 \).
If \( g \) fixes no coset then, by the Frobenius theorem, \( g \in K \).

**Theorem 12:** Suppose \( G \) is Frobenius with kernel \( K \) and complement \( H \).
Then the class equations for \( H \) and \( K \) have the forms:
\[
|H| = N = 1 + h_2 + \ldots + h_s \quad \text{and} \\
|K| = M = 1 + k_2 \ast N + \ldots + k_t \ast N
\]
and the class equation for \( G \) is:
\[
|G| = 1 + k_2N + \ldots + k_tN + Mh_2 + \ldots + Mh_s
\]

**Proof:** Let \( 1 \neq x \in K \) and \( 1 \neq y \in H \). Then \( |x^G| = M|x^K| \) and \( |y^G| = N|y^H| \).
\( |x^G| \) is a union of \( N \) conjugacy classes in \( K \).
Every element \( g \in G - K \) is conjugate to exactly one \( h \in H \) so \( |g^G| = M|h^G| \).

[Note that in the class equation given for \( G \) the terms are in non-descending order.]

**Theorem 13 (THOMPSON):** The kernel of a Frobenius group is nilpotent.

**Proof:** We omit the proof of this very deep theorem.

**Corollary:** \( k_2 = 1 \) in theorem 11.

So far in all our examples of Frobenius groups the kernel has been abelian. However it can be non-abelian.

**Example 5:** Let \( K \) be the set of all \( 3 \times 3 \) uni-upper-triangular matrices over \( \mathbb{Z}_7 \). That is, the elements of \( K \) have the form \[
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix}
\]. This is a non-abelian group of order 343 = \( 7^3 \).

Extend \( K \) by \( H = \langle g \rangle \) of order 3 such that \( g^{-1}\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix} g = \begin{pmatrix}
1 & 2a & 4b \\
0 & 1 & 2c \\
0 & 0 & 1
\end{pmatrix} \). The resulting group \( G \) is a Frobenius group of order 1029 = \( 3 \cdot 7^3 \) with Frobenius kernel \( K \) of order 343. Although \( K \) is not abelian, it is nilpotent, in accordance with Thompson’s Theorem.
Theorem 14: Suppose the Frobenius group $G$ has the class equation:

$$|G| = 1 + a_2 + \ldots + a_n$$

where $a_2 \leq a_3 \leq \ldots \leq a_n$.

Let $N = a_2, M = |G|/N$ and let $a_r$ be the last term divisible by $N$.

1. $M = 1 + a_2 + \ldots + a_r$;
2. $N = (a_{r+1} + \ldots + a_n)/M + 1$;
3. $N | M - 1$;
4. $N | a_i$ if $i \leq r$;
5. $M | a_i$ if $i > r$;
6. if $N$ is even, $a_i = N$ for $i \leq r$.

Proof: Since $N$ doesn’t divide any $Mh_i$, $r = t$ and $a_i = \begin{cases} k_iN & \text{if } i \leq r \\ Mh_{i-r+1} & \text{if } i > r \end{cases}$.

Parts (1) – (5) now follow.

(6) Let $h \in H$ have order 2. If $k \in K$ then $k^{-1}kk^h = k^h k = h^{-1}kk^h h$ so $kh^{-1} \in C_G(kk^h) \leq K$ or $kk^h = 1$. The first is a contradiction so $k^h = k^{-1}$ for all $k \in K$ whence $K$ is abelian.

Theorem 15 (FROBENIUS TEST): Let $p$ be prime. Suppose $|G| = pN$ and $G$ has $p - 1$ conjugacy classes of size $N$. Then $G$ is a Frobenius group with kernel $G'$ of order $N$.

Proof: Let $h$ belong to a conjugacy classes of size $N$ and let $H = C_G(h)$. By the proof of the $pN$ Test, each conjugacy class of size $N$ contains exactly one non-trivial element of $H$.

Suppose $1 \neq h \in H \cap x^{-1}Hx$. Then $H \leq C_G(h)$ and since they both have order $r$, $H = C_G(h)$.

If $x \notin H$ then $h \neq xhx^{-1} \in H$. So the conjugacy class containing $h$ contains two elements of $H$, contradicting the $pN$ Test. Hence if $x \in G - H, H \cap x^{-1}Hx = 1$.

Thus $G$ is a Frobenius group. If the Frobenius kernel is $K$ then $|K| = N$. Now $G/K \cong H$ and so is abelian. Thus $G' \leq K$.

But, if $x, y$ belong to the same conjugacy class of size $N$, then $x^{-1}y \in G'$ and so $|G'| \geq N$. Hence the Frobenius kernel is $G'$. 
