1. REAL NUMBERS

§1.1. Symbols Representing Numbers

You may not be intending to become a mathematician, but these days practically every profession uses numbers. It’s true that you probably learnt all you need to know about arithmetic in primary school, and in any case you can use calculators, but what you can do with just arithmetic is somewhat limited. If you need to problem solve with numbers you usually have to use algebra.

In high school we learnt that we can represent “unknown” numbers by letters such as \( x, y, z \). These numbers are generally called variables, meaning that either their value is unknown, or it can vary. This process allowed us to express a numerical problem as an equation which we then solve.

Example 1: Find a number which when you double it and add 7 you get 15.

Solution: Let the number be \( x \). Then \( 2x + 7 = 15 \).
Subtracting 7 from both sides of the equation we get \( 2x = 8 \).
Dividing both sides by 2 we get \( x = 4 \).

Generally we use single letters of the alphabet. We don’t usually use combinations of letters because this could lead to confusion. Since we write \( b \) times \( c \) as \( bc \) we should not use \( bc \) as a variable. But since there are only 26 letters of the alphabet we may need more than just \( a - z \). To overcome this problem we use subscripts. If we have a list of numbers we can write them as: \( x_1, x_2, x_3, \ldots \).

Unlike \( x^2 \), where the composite symbol represents a combination of \( x \) and 2, variables with subscripts are to be treated as a single symbol. If you only know the value of \( x \) there is no way you can find the value of \( x_3 \).

The symbols \( x \) and \( x_3 \) represent two, possibly unrelated, numbers. Think of \( x_3 \) as a name with \( x \) as the family name and 3 as the given name.

The subscripts needn’t start with 1. Sometimes we find it convenient to have an \( x_0 \) in the family. It would even be possible to have \( x_{-1} \) as a variable but this would be more uncommon.

Although one subscript is sufficient for infinitely many variables, when we have a lot of variables arranged in a table we usually use two suffixes. We could write the entry in the \( i \)'th row and \( j \)'th column as \( x_{i,j} \). More usually we leave out the comma and write this as \( x_{ij} \).

So \( x_{25} \) would denote the number in the second row and fifth column. You’ve probably spotted a potential problem here. Is \( x_{125} \) in the 12th row and 5th column or is it in the 1st row and 25th column? Clearly when there is danger of such confusion we must include the comma.
In a computer program there’s no provision for subscripts. Here we’d use \( C(12) \) instead of \( C_{12} \). But since a single computer program might contain many hundreds of variables it’s usual to use words, or abbreviations as variables. This helps the programmer to remember what each variable name represents. A consequence, however, is that in a computer program, the multiplication sign must be put in explicitly. While in mathematics we might give the formula for the area of a rectangle as \( A = LB \), in a computer program \( LB \) might be another variable. In any case we would probably write

\[
AREA = LENGTH \times BREADTH.
\]

But we’re doing mathematics here, so we’ll use single letters only, sometimes supplemented by subscripts. But as well as the 26 lower case letters we can use capital letters as well. Sometimes we even borrow symbols from other languages. Mathematicians are quite familiar with the Greek alphabet because these symbols are frequently used in certain contexts. We won’t be using Greek symbols in these notes, but if you do more mathematics you are likely to see symbols such as \( \alpha, \beta, \gamma, \theta, \sigma \) as variables. The only Greeks symbol you need to use is \( \pi \), which represents a constant not a variable. (More of this later in this chapter.)

§1.2. Numbers

The objects that are represented by letters are numbers. These needn’t be whole numbers. They can be fractions or, as we say, rational numbers. Or they can be any number on the number line, numbers we call real numbers. It’s quite difficult to define a real number precisely. For now we can loosely define it to be something that represents a point on the number line.

An integer is a whole number, such as 17 or –8. A rational number is one that can be expressed in the form \( \frac{m}{n} \) where \( m, n \) are integers, such as \( \frac{3}{4} \) or \( \frac{-17}{7} \), which we might write as \( -2\frac{3}{7} \). One way of representing real numbers is by way of decimals.

Example 2:

\[
\begin{align*}
\frac{1}{4} &= 0.25 \\
-\frac{3}{5} &= -0.6 \\
137/10 &= 13.7
\end{align*}
\]

Most numbers require infinitely many decimal places to represent them exactly. In many of these cases the numbers have repeating decimal expansions. They can be written compactly by putting a dot over a repeating digit, or over the first and last digits of a repeating block of digits.

Example 3:

\[
\begin{align*}
\frac{2}{3} &= 0.66666666666666\ldots = 0.\overset{\ldots}{6} \\
\frac{22}{7} &= 3.14285714285714\ldots = 3.142857\overset{\ldots}{7}
\end{align*}
\]

Numbers ending in 9 can be represented in two ways.

Example 4: Show that \( 3.56\overset{\ldots}{9} = 3.57 \).

Solution: Let \( x = 3.56999999\ldots \)

Then \( 10x = 35.699999\ldots \)

Now \( x = 3.569999 \)
Subtracting, $9x = 32.13$.

Hence $x = 3.57$.

It is often argued, wrongly, that $0.9999\ldots$ is a tiny bit less than 1. But, using the above argument we can see that it is simply another way of writing 1. If you’re still not convinced ask yourself what number lies half way between $0.9999\ldots$ and 1?

§1.3. The Number Pi

The ratio of the circumference of a circle to its diameter is approximately 3. This is the approximation given in I Kings 7:23 in the Bible, where a certain vessel is described as being 10 cubits across and 30 cubits round. But some commentators have gone back to the original Hebrew, where the letters have numerical equivalents, and argue that the value of 333/106 can be inferred, which is quite accurate. At school we learnt that $\pi$ is 22/7, but a careful teacher should have explained that this is only a good approximation. A better approximation is 355/113.

The fact that $\pi$ cannot be expressed exactly as a fraction, that is, $\pi$ is irrational means that if we want to write it down exactly we must resort to using the special symbol. So we write the circumference of a circle of radius $r$ as $C = 2\pi r$ and the area as $A = \pi r^2$.

**Example 5:** $\pi = 3.14159265358979323846264338327950288\ldots$ This decimal expansion is non repeating, and has no discernible pattern.

A recent book, which became an excellent film, is *The Life of Pi*. If you’ve seen it you’ll know that isn’t really about mathematics. The main character was named after an uncle who was an excellent swimmer and who trained in a swimming pool that contained the word “piscine”, being the French word for swimming pool. Young Piscine, as he was called, got ragged at school for such a name and so he shortened it to Pi. But he learnt to memorise the value of $\pi$ to hundreds of decimal places, which won him the admiration of his classmates, and confirmed the name Pi.

The number $\pi$ has been computed to over 5 trillion decimal places and no pattern has been detected. Certainly there’s no sign of it repeating. But even 5 trillion decimal places can’t answer the question as to whether the expansion eventually repeats.

This is a good example to show that although computers are extremely useful tools in mathematics, they’ll never replace mathematics. A computer, spewing out the digits of $\pi$, can never, in finite time, answer the question “do the digits of $\pi$ repeat?” Yet a theorem in mathematics proves that the answer is “no”.

§1.4. The Laws of Algebra

There are certain properties that the real numbers possess which underlie basic algebra.

**The Commutative Laws:**
For all real numbers \(a + b = b + a\) and \(ab = ba\).

You've known for a long time that it doesn't matter in which order you add a list of numbers. And you can multiply numbers in any order. But be warned. This is not the usual state of affairs in mathematics, or in life for that matter. The order in which you carry out certain operations is often highly critical. If a watchmaker has to reassemble a watch it does matter in which order he puts in the pieces. Carrying out the operations of putting on your socks and your shoes you get quite a different result if you reverse the usual order!

There are mathematical objects which can be multiplied where \(ab \neq ba\). This leads to all sorts of complications. So remember, despite the heading for this section, we are not really discussing the laws of algebra in general – only the laws of the algebra of real numbers.

**The Associative Laws:**
For all real numbers \((a + b) + c = a + (b + c)\) and \((ab)c = a(bc)\).

This means that when we add three or more terms (expressions or numbers that are added together), or multiply three or more factors (expressions or numbers that are multiplied together), it does not matter how we group them. As a result we usually write the above as \(a + b + c\) and \(abc\) respectively.

Again these aren't laws that operate in all algebraic situations, though they do hold in most. When you come to learn about vector products you’ll discover a system that isn’t associative.

Because of the associative laws for real numbers we write things like \(3x\), meaning \(x + x + x\) and \(x^3 = xxx\). If addition or multiplication were not associative these would be ambiguous. Does \(3x\) mean \((x + x) + x\), or is it \(x + (x + x)\)? Fortunately for us it doesn’t matter for real numbers.

**Identities:**
There are two real numbers that behave specially when it comes to addition and multiplication. These are the numbers 0 and 1.

For all real numbers \(x\), \(0 + x = x + 0 = x\) and \(1x = x1 = x\).

These numbers are called the additive identity (the number 0) and the multiplicative identity (the number 1).

**Inverses:**
For all real numbers \(x\) there is a number, written \(-x\) such that \(x + (-x) = (-x) + x = 0\). The number \(-x\) is called the additive inverse of \(x\). When it comes to multiplication we have to make an exception. For all non-zero real numbers \(x\) there’s a number, written \(1/x\) or \(x^{-1}\), such that \(x x^{-1} = x^{-1}x = 1\). It’s called the multiplicative inverse or reciprocal.
Why the exception? Why can’t 0 have a multiplicative inverse? Why don’t we write $\frac{1}{0} = \infty$? Quite apart from the difficulty of finding a point on the real line to represent it, our whole system would implode if we allowed this. This is because we would then have $0\infty = 1$. But what’s wrong with that? The answer is “nothing, if you want to abandon the distributive laws”. We insist on having the distributive law, so 0 will just have to do without an inverse. Let’s see what the distributive law is.

**Distributive Law:**
The distributive law ties the additive and multiplicative structures of the real numbers together.
For all real numbers $a(b + c) = ab + ac$. Because multiplication is commutative we can also write $(b + c)a = ba + ca$. This is a fundamental property of the real numbers and we use it every time we expand an expression.

**Example 6:** Expand $(3x + 2y)(5x + y)$.
**Solution:**
$(3x + 2y)(5x + y) = (3x + 2y)(5x) + (3x + 2y)(y)$
$= 15x^2 + 10xy + 3xy + 2y^2$
$= 15x^2 + 13xy + 2y^2$.

Here we’re using more than just the distributive law. When we multiplied $3x$ by $5x$ to get $15x^2$ we unconsciously made use of the associative law and the commutative law for multiplication.

And the fact that we wrote the answer as $15x^2 + 13xy + 2y^2$ and not $(15x^2 + 13xy) + 2y^2$ or $15x^2 + (13xy + 2y^2)$ shows that we are mindful of the associative law for addition. Note too that in writing $10xy + 3xy$ as $13xy$ we are unconsciously making use of the distributive law. If we had to justify it we could write $10xy + 3xy = (10 + 3)xy = 13xy$.

When we worked with algebraic expressions in high school we probably didn’t think too much about what we were doing. We just instinctively changed one expression into an equivalent one. But if we want to know *why* things work the way they do we could justify everything from the above fundamental laws.

And what if we wanted a proof that these laws are correct? To do that we’d have to carefully define a real number. Even defining the number 2 in a formal way is quite a sophisticated matter. Now is not the time or place to get bogged down with such fundamentals.

However we can provide a geometric explanation of the distributive law. If you accept that $ab$ is the area of an $a \times b$ rectangle then this picture demonstrates that $a(b + c) = ab + ac$.

```
  |                   |
  |                   |
  |       ab         |
  |  b + c           |
  |                   |
  |                   |
  |       ac         |
  |                   |
  |  b                |
  |                   |
  |  c                |
```

Someone interested in the foundations of mathematics wouldn’t be content with such a proof because it relies on geometric intuition, but it will do for now.

Going back to why we can’t have $0\infty = 1$ if we have the distributive law, consider the following. Suppose $0\infty = 1$. 

Then $1 = 0\infty$
$= (0 + 0)\infty$
$= 0\infty + 0\infty$
$= 1 + 1$
$= 2$, which is nonsense.

Now there are many other laws of algebra that were not given in the above list. For example we all know that “if you multiply by zero you get zero”. Is this yet one more law we have to accept? No, it is a simple consequence of the ones we’ve already given.

**Example 7:** Show that $0x = 0$ for all real numbers $x$.

**Solution:** Let $0x = y$.
Then $y = 0x$
$= (0 + 0)x$
$= 0x + 0x$
$= y + y$.
We would now “cancel” $y$ on both sides to get $y = 0$. But what is exactly going on when we cancel?
From $y = y + y$ we get, by adding $-y$ to both sides:
$y + (-y) = (y + y) + (-y)$
$= y + (y + (-y))$
$= y + 0$
$= y$
But $y + (-y) = 0$, so $y = 0$.
Can you identify the basic laws used here?

Students are often perplexed as to why $-1$ times $-1$ is $+1$. They are often fobbed off by teachers with some vague explanation such as “two negatives make a positive”. Certainly if something is not untidy then it’s tidy, but what has this to do with algebra? A more satisfactory explanation is the following.

**Example 8:** Show that $(-1)(-1) = 1$.

**Solution:** Let $x = -1$.
Then $1 + x = 0$ and so
$0 = 0x$
$= (1 + x)x$
$= x + x^2$.
So $x^2 + x = 1 + x$ and hence $x^2 = 1$.

It follows that the product of two negative numbers is positive, for example $(-3)(-2) = 6$.

**Example 9:** Show that if $xy = 0$ then either $x = 0$ or $y = 0$.

**Solution:** Suppose $xy = 0$. Either $x = 0$ or $x \neq 0$.
If $x \neq 0$ then $x^{-1}$ exists and we can multiply both sides by $x^{-1}$ to get $x^{-1}(xy) = 0$.
By the associative law this becomes $(x^{-1}x)y = 0$. Hence $1y = y = 0$.

**Example 10:** Show that if $xy = xz$ and $x \neq 0$ then $y = z$.

**Solution:** This is proved similarly.
§1.5. Basic Algebraic Identities

There are certain equations that hold for all variables in the algebra of real numbers. We have seen some of them already. Here are some more that are quite useful.

Theorem 1: For all real numbers $x, y$:

1. $(x + y)^2 = x^2 + 2xy + y^2$;
2. $(x - y)^2 = x^2 - 2xy + y^2$;
3. $(x - y)(x + y) = x^2 - y^2$;
4. $(x - y)(x^2 + xy + y^2) = x^3 - y^3$;
5. $(x + y)(x^2 - xy + y^2) = x^3 + y^3$.

Proof:

1. $(x + y)^2 = (x + y)(x + y)$
   
   $= x(x + y) + y(x + y)$
   
   $= x^2 + xy + yx + y^2$
   
   $= x^2 + 2xy + y^2$.

2 to 5 can be proved similarly.

The special cases of (3), (4) and (5), where $y = 1$, should be noted as they arise frequently.

$$x^2 - 1 = (x - 1)(x + 1);$$
$$x^3 - 1 = (x - 1)(x^2 + x + 1);$$
$$x^3 + 1 = (x + 1)(x^2 - x + 1).$$

Also note that although $x^2 - 1$ factorises, $x^2 + 1$ does not.

The first of these identities can be illustrated geometrically as follows.

Example 11: Factorise $x^4 - 1$.

Solution: $x^4 - 1 = (x^2 - 1)(x^2 + 1)$ by replacing $x$ by $x^2$ in the factorisation of $x^2 - 1$

$= (x - 1)(x + 1)(x^2 + 1)$.

§1.6. Solving Equations

An equation is a mathematical sentence where the verb is the equals sign. It differs from an identity in that an identity holds for all values of the variables while an equation does not. If there is just one variable, such as $x$, we can solve the equation, which means find all the values for which the equation is true. Some equations have no solutions, others have a
single, or unique solution. Yet other equations have several solutions and some even have infinitely many solutions.

A linear equation (in one variable) has the form \( ax + b = c \). If \( a \neq 0 \) we can solve for \( x \), and there is just one solution.

Example 12: Solve \( 7x + 5 = 9 \).
Solution: Subtract 5 from both sides, giving \( 7x = 4 \). Divide both sides by 7, giving \( x = 4/7 \).

Often a problem is described in words and involves some real-world situation. In this case we must first express the problem as an equation, and then to solve it. In fact we can identify 6 separate steps.

(1) Define the variable or variables in the problem.
Don’t start throwing \( x \)’s and \( y \)’s around without describing what quantities they represent.

(2) Clearly state the units you are using.
For example you might say “let \( x \) be the length in metres”.

(3) Summarise the facts by a diagram or a table.

(4) Write the facts as an equation, or set of equations.

(5) Solve.

(6) Interpret your solution in the context of the problem, using appropriate units and an appropriate level of approximation.
Don’t just finish with a statement like “\( x = 3.5 \)”. If a person came to you with the problem, and you decided to let \( x \) be the length be length in metres, the person asking the question wouldn’t understand your saying “\( x = 3.5 \)”. You would need to say “the length is 3.5 metres”.

If the answer was \( x = 0.015 \) you should not say “the length is 0.015 metres”, but rather “the length is 1.5 cm”. If the answer, using your calculator, comes out as \( x = 2.87451839201 \), you should not say “the length is 2.87451839201 metres”, but rather “the length is 2.87 metres”, or perhaps “the length is 2.874 metres” or perhaps, rounding up, “the length is 2.875 metres”.

Not every mathematical solution to an equation translates to a solution to the original problem. Suppose that the equation had two solutions \( x = 3 \) and \( x = -2 \). If \( x \) was a length then clearly the solution \( x = -2 \) must be rejected.

Example 13: John is 5 years older than his sister. Fifteen years ago he was exactly twice his sister’s age. How old is John now?
Solution: Let \( x \) be John’s age now (we don’t need to say “in years” – that is understood).

<table>
<thead>
<tr>
<th></th>
<th>John</th>
<th>sister</th>
</tr>
</thead>
<tbody>
<tr>
<td>now</td>
<td>( x )</td>
<td>( x - 5 )</td>
</tr>
<tr>
<td>then</td>
<td>( x - 15 )</td>
<td>( x - 20 )</td>
</tr>
</tbody>
</table>

Since John was exactly twice his sister’s age 15 years ago then 
\( x - 15 = 2(x - 20) = 2x - 40 \).
\[ \therefore x = 25. \]
So John’s age now is 25.
This very simple problem could have been solved just as easily with a little bit of trial and error. But rarely is this the case.

**Example 14:** A 1 metre by 1 metre square is cut off the end of a rectangle. The resulting rectangle, when rotated 90°, has the same proportions as the original rectangle. What is the area of the original rectangle?

**Solution:** The width, or shorter side of the rectangle will be 1 metre. Let \( x \) be the length of the longer side, in metres.

The ratio of the longer side to the shorter is \( \frac{x}{1} \) for the original rectangle, and \( \frac{1}{x - 1} \) for the remaining rectangle. Hence \( \frac{x}{1} = \frac{1}{x - 1} \).

Simplifying, this gives \( x(x - 1) = 1 \), or \( x^2 - x - 1 = 0 \).

This is what is called a quadratic equation, and we will show how to solve these in a later chapter. We will find that there are two solutions: \( x = \frac{1 \pm \sqrt{5}}{2} \) where \( \pm \) means “plus or minus”.

Using a calculator, these are \( x = 1.618033989 \) and \( x = -0.618033989 \). Clearly we must reject the negative solution. Moreover, for most purposes, the degree of accuracy given is far more than is appropriate. We should report that the longer side is about 1.62 metres, or perhaps 162 cm.

But the question doesn’t ask for the length of the longer side, but rather the area. Since the shorter side is 1, the answer is still 1.62, but rather than metres, it is now square metres. When you think that you have finished always check your answer by asking three questions.

1. Is it a sensible solution?
2. Is it intelligible to someone who has read the question but not the solution.
3. Have you answered what was asked for?

§1.7. Fractions

A fraction is an expression of the form \( \frac{m}{n} \). We sometimes write it as \( m/n \). The number on the top is called the **numerator** and the number on the bottom is called the **denominator**. When the numerator and denominator are integers, we call the fraction a **rational number**.

Adding and subtracting fractions, whether algebraic or simply arithmetic, can be difficult, but if the denominators are the same it is very easy.

**Example 15:** \( \frac{3}{17} + \frac{4}{17} = \frac{7}{17} \).

Adding seventeenths is no more difficult than adding apples. Three plus four more is seven of them.
Example 16: (1) $\frac{x}{x^2+1} + \frac{1}{x^2+1} = \frac{x+1}{x^2+1}.$

(2) $\frac{x^2}{x^2+1} + \frac{1}{x^2+1} = \frac{x^2+1}{x^2+1} = 1.$

In this last case we were able to simplify the fraction by dividing the numerator and denominator by the same expression.

When the denominators are different we have to find a common multiple, preferably the least common multiple, though that is not absolutely necessary.

Example 17: Simplify $\frac{2}{15} - \frac{3}{20}.$

Solution: Here the least common multiple of the denominators is 60.

$$\frac{2}{15} - \frac{3}{20} = \frac{8}{60} - \frac{9}{60} = -\frac{1}{60}.$$ However, if we simply used the product of the denominators, 300, we would eventually get the same answer.

$$\frac{2}{15} - \frac{3}{20} = \frac{40}{300} - \frac{45}{300} = -\frac{5}{300} = -\frac{1}{60}.$$  

Example 18: Simplify $\frac{x}{x^2-1} - \frac{1}{x^2+2x+1}.$

Solution: $x^2 - 1 = (x-1)(x+1)$ and $x^2 + 2x + 1 = (x+1)^2.$ Hence both divide $(x^2 - 1)(x+1).$ This will be our common denominator.

$$\frac{x}{x^2-1} - \frac{1}{x^2+2x+1} = \frac{x(x+1)}{(x^2-1)(x+1)} - \frac{x-1}{(x^2-1)(x+1)}$$

$$= \frac{x^2+1}{(x^2-1)(x+1)}.$$  

Multiplying fractions is much easier than adding or subtracting them. You simply multiply the numerators and multiply the denominators. And to divide fractions you “invert and multiply”. The fraction that is being divided by has its numerator and denominator swapped to form its multiplicative inverse.

Example 19: Simplify $\frac{18}{35} \times \frac{20}{27}.$

$$\frac{18}{35} \times \frac{20}{27} = \frac{360}{945} = \frac{8}{21}$$ after “cancelling down” by 45.

But what is much simpler is to do the cancelling before the final multiplications.

$$\frac{18}{35} \times \frac{20}{27} = \frac{35 \times 3}{35 \times 3} = \frac{4}{3} = \frac{8}{21}.$$  

It helps, in algebraic cases, to factorise the numerators and denominators first.

Example 20: Simplify $\frac{x^2 + 1}{x^3 + 1} \times \frac{x + 1}{x^3 + x}.$

Solution: $x^3 + 1 = (x + 1)(x^2 - x + 1)$ and $x^3 + x = x(x^2 + 1).$ So

$$\frac{x^2 + 1}{x^3 + 1} \times \frac{x + 1}{x^3 + x} = \frac{x^2 + 1}{(x + 1)(x^2 - x + 1)} \times \frac{x + 1}{x(x^2 + 1)}.$$
\[
\frac{1}{(x+1)(x^2-x+1)} \times \frac{x+1}{x} = \frac{1}{x(x^2-x+1)}.
\]

**Example 21:** Simplify \( \frac{18}{55} \div \frac{16}{25} \).

**Solution:**
\[
\frac{18}{55} \div \frac{16}{25} = \frac{18}{55} \times \frac{25}{16} = \frac{9}{11} \times \frac{5}{8} = \frac{45}{88}.
\]

**Example 22:** Simplify \( \frac{x^4-1}{x^4+1} + \frac{x^2+1}{x+1} \).

**Solution:**
\[
\frac{x^4-1}{x^4+1} + \frac{x^2+1}{x+1} = \frac{x^4-1}{x^4+1} \times \frac{x+1}{x^2+1} = \frac{(x-1)(x+1)(x^2+1)}{x^4+1} \times \frac{x+1}{x^2+1} = \frac{(x-1)(x+1)^2}{x^4+1}.
\]