6. OPTIMISATION PROBLEMS

§6.1 Applied Mathematics and Modelling

While it may be interesting to develop mathematics for its own sake, for most people the whole point of studying the subject is to be able to apply it to the real world. The Babylonians took the first steps in geometry to help them with surveying problems. Calculus was invented for astronomers to be able to work out what was going on with planetary motion.

The phrase “applied mathematics”, with its counterpart “pure mathematics”, can be misleading in that it implies that only certain parts of mathematics are useful. History has shown that what seemed to be merely an esoteric curiosity in one era can be extremely useful in a later one. For example the theory of prime numbers and divisibility was long considered to be the purest of pure mathematics. But in the last few decades it has been extremely important in connection with cryptography, especially in relation to encoding confidential information electronically.

A phrase that has come to replace “applied mathematics” is “mathematical modelling”. It describes the process of setting up a mathematical model to describe a part of reality. A mathematical model is an equation, or more usually a whole system of interlocking equations, that “model” some system of interest in the real world.

Mathematical models have been set up to describe systems in physics and astronomy, in chemistry and biology, and in economics and meteorology. A scale model of an aircraft can be tested in a wind tunnel to learn about the strengths and weaknesses of some new design. The small model is expected to behave in a similar way to the full-scale version but is cheaper and more convenient to test. A mathematical model is not so much a scaled down version as an abstract version of actual reality. A well-constructed mathematical model can be expected to behave in an analogous way to the real-world system, yet because it can be manipulated on paper and on a computer, it is very much cheaper and convenient to use the model than the real system.

A simple mathematical model applies to an object being fired into the air. As a result of experiments we’re led to make the assumption that the object is pulled downwards by gravity with a constant acceleration. Then, using calculus, we’re able to deduce that the height of the object above the ground, after some time \( t \) has elapsed, is given by the formula: 

\[ h = -at^2 + bt + c \]

where \( a, b \) and \( c \) are constants. These constants relate to the strength of gravitational pull and the speed and height of the object at the moment when it’s fired.

Don’t worry, for now, how one derives this formula. The important thing is that it’s a mathematical model for a physical reality. Not only can we use it to predict the position of the object at any moment (assuming that we know the values of the three constants), but we can carry out mathematical analysis to answer such related questions as “how high does the object reach before falling back down” and “how long does it take to hit the ground”.

The maximum height question is simply a question about the maximum value of \( h \). We differentiate \( h \) to get 

\[ \frac{dh}{dt} = -2at + b. \]

This is zero when \( t = \frac{b}{2a} \). This gives the time when the object reaches its maximum height and, by substituting this back into the formula for \( h \) we get the maximum height. And the time taken to reach the ground is answered by putting 

\[ -at^2 + bt + c = 0 \]

and solving for \( t \).

Now a simple mathematical model such as this makes certain simplifying assumptions. It ignores air resistance. It ignores the fact that the higher the object goes, the further it is from the centre of the earth and the weaker the force of gravity. For a stone being thrown up into the air, or even a shell from a canon, these effects are tiny and can be safely ignored.

The important thing to realise with a mathematical model is that there will always be simplifying assumptions that are, strictly speaking, not quite correct. This is unavoidable because it’s impossible to take every factor into account. There’s no point in having a very comprehensive mathematical model that includes all sorts of minor effects if the resulting equations are too
complicated and unwieldy to work with. The trick is to make as many simplifying assumptions as you can get away with.

In the case of a stone being thrown into the air, we’d probably ignore the effect of slowing down due to air resistance and the minute changes to gravitational force as the stone went higher. But if we were modelling the motion of a feather we couldn’t ignore air resistance. If we wanted to model a rocket going into the upper atmosphere we’d need to build into our model the fact that the gravitational force decreases as the rocket moves away from the earth. And because of the high speeds of a rocket, air resistance couldn’t be ignored. Moreover we’d have to take into account the fact that air resistance would diminish as the air became thinner at higher altitudes. And, of course, we’d have to incorporate the thrust of the rockets. The resulting mathematical model would be considerably more complicated than the simple one above.

§ 6.2 Max-Min Problems

A true real-world application requires not only a knowledge of the relevant mathematics; it also needs a good understanding of the basic principles of the area of the application as well as modelling skills for linking the application to the mathematics.

But when it comes to teaching modelling skills we have to resort to problems that require very little expert knowledge in the relevant application area. We can’t expect everyone to have a good grounding in physics, chemistry or economics. For this reason the problems we discuss may often seem to be somewhat artificial or are an over-simplification with too many factors ignored. This is so that we can focus on the skill of uncovering the mathematics from the problem and so that the resulting mathematics is within our somewhat limited scope.

The type of problem that we’ll focus on here is an optimisation problem, more loosely described as a “Max-Min Problem”. We have something that we want to maximise or minimise. (The word “optimise” covers these two extremes.) And we have one or more quantities we can vary to achieve this optimum. In general there’ll be several variables that can be altered independently and the problem will be to find the right combination to achieve the maximum, or minimum, value of our target variable.

At this stage we’ve only developed calculus in one variable so we’ll have to limit ourselves to one independent variable. If we call the independent variable \( x \) (that’s the one we are able to vary directly) and the dependent variable \( y \) (that’s the one we want to optimise) we have to work out an equation which expresses \( y \) in terms of \( x \).

Now there may be certain limitations to the values of \( x \) that we may use. For example if we had to vary the length of a rectangle whose perimeter is 4 units then the length would have to lie within the interval \([0, 2]\). This is because the length cannot be negative, and if bigger than 2 the width would be negative. So any optimisation problem involving the length of such a rectangle would become the problem of finding the global maximum or minimum over this interval. This could occur at a stationary point, or perhaps at an endpoint.

Students find these so-called “word problems” intimidating. “Give me an equation and I’ll solve it or differentiate it, or do whatever you like with it”, they say, “but what am I suppose to do if I’m confronted by a mass of words and no equations?”

Here are some useful guidelines for tackling word problems of a max-min type. They all involve reading the words carefully a couple of times to familiarise yourself with the details and then to extract certain information.

(1) What you are trying to optimise? In other words, what is your dependent variable?

(2) What are you varying in order to achieve this optimised value? In other words, what is your independent variable?

(3) What values of this independent variable are possible? In other words, over what range of values will you be working?
(4) What information is irrelevant?
Some data may seem obviously irrelevant, such as the fact that a field belongs to Farmer Brown. But other information that’s provided may involve numbers that look as though they might enter into the solution but which, when you think about it more closely, are simply “red herrings”. It may seem unfair for a problem to include information that’s not needed, but in the real world this happens all the time. When a mathematician talks to an economist about some problem, he or she may get inundated with a whole lot of redundant information. The job of the mathematician is to decide what information is needed and what to ignore. And very often, after sifting through a whole lot of irrelevant facts, the mathematician discovers that there is some pertinent information that hasn’t been provided. At least you can be confident that, when you are given a word problem as an exercise here, all the relevant facts will be included. But beware – we may throw some irrelevant information into the mix.

(5) What simplifying assumptions can be made?
In order to come up with a manageable model you may have to ignore certain factors that, while they technically affect the outcome, they have only a small effect. This requires some knowledge of the application area as well as common sense. Also you would need to know what level of accuracy is required. The more accurate the result needs to be the more factors you may have to take into account. In the max-min problems in this chapter you have to cut back to just one independent variable and ignore all the rest.

(6) Does common sense expect a maximum or a minimum?
You may see the $y$ value increasing as you move away from the lower endpoint and later decrease as you approach the upper end-point. Then clearly (assuming the function is continuous in between) you expect to find a maximum somewhere between.

(7) Summarise the relevant information.
This may involve drawing a rough diagram or putting numbers into a table.

(8) Define your variables.
Your solution should communicate your ideas to someone else. If you start throwing $x$’s and $y$’s around in your solution your client needs to be told what they stand for. It is a good idea to use $A$ for area and $t$ for time etc. This would help the reader to guess what they mean, but you should still state explicitly what they are.

(9) Specify your units.
The equations will involve numbers, not quantities. The units should be specified at the time you define your variables. So you might say “let $x$ be the length of the field (in metres)”. So if you later say $x = 4$ this would be interpreted as “the length is 4 metres”. Never say “$x = 4$ metres”. Symbols in algebra always represent numbers, not measurements. Also be aware that different units may be used in different places. You may be given the length of a floor tile in centimetres but its thickness in millimetres. You should choose the same unit for all lengths and make appropriate adjustments.

(10) Find equations relating the variables.
Sometimes these will be given, in words, in the text of the problem. Sometimes you have to know a formula from the application area. Because we don’t all have the same background in physics, chemistry, biology or economics we very often use geometry as our application area. So you may need to know the formula for the area of a rectangle or the volume of a cylinder. Sometimes you have to develop your own formula, using the given information given and common sense. “If he sells them at $20 he’ll be able to sell 100 units, but for each extra $1 he charges he’ll sell 2 fewer
units.” If $P$ is the price charged in dollars and $S$ is the number of units sold, then the above information gives the equation $S = 100 - 2(P - 20)$, which can be simplified to $S = 140 - 2P$.

(11) **Eliminate variables.**

You may have several quantities in a problem, all related to one another. To be able to solve the problem using one variable optimisation techniques you’d have to be able to eliminate variables, expressing them in terms of others, until you’re left with just one equation connecting two variables. (If it’s impossible to do this then you need calculus of several variables to solve the problem.)

(12) **Find the global maximum or minimum.**

Put the derivative equal to zero and solve. Also look at endpoints.

(13) **Interpret you answer.**

Having solved the mathematical problem you need to interpret this answer in terms of the original problem. If a businessman gave you information about his business and asked how he could maximise his profits he doesn’t want to be told “$x = 32$”. Even worse would be if the answer came out to be “$x = \frac{\pi}{\sqrt{20}}$”. You should write your answer as a complete English sentence and this should specify units. So your final answer might read “You will maximise your profits if you charge $35 per unit. The profit, at that price, will be $2450.”

(14) **Use appropriate units and an appropriate degree of accuracy.**

“The cost of making the drums will be minimised by making the diameter $\frac{\pi}{\sqrt{20}}$ metres”. Nobody wants to be told the answer to a manufacturing problem in such a highly mathematical way.

“The cost of making the drums will be minimised by making the diameter 0.7024814731 metres” is a little better but it’s not appropriate to use metres as the unit of measurement here.

“The cost of making the drums will be minimised by making the diameter 70.24814731 centimetres” is much better. But such a high degree of accuracy is inappropriate. Apart from the fact that the manufacturer wouldn’t be able to achieve such precision, there’ll be certain factors that might have been ignored which, if taken into account, could perhaps have changed the answer by one percent or more.

“The cost of making the drums will be minimised by making the diameter 70.25 centimetres” would be a reasonable conclusion, or even “the cost of making the drums will be minimised by making the diameter 70 centimetres”.

**Example 1:**

Find the maximum area, in hectares, of a rectangular field that is enclosed by 3km of fencing if one side of the field needs no fence because it runs along a 20 metre wide river.

**Solution:**

The width of the river is irrelevant and we’ll assume that the field is flat.

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![Diagram of a rectangular field with river]

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x

A

y

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Let \( x = \) width of field (metres)

\( y = \) length of field (metres)

\( A = \) area of field (metres\(^2\))

\( L = \) length of fence (metres)

Endpoints: \( 0 \leq x \leq 1500 \). If \( x = 0 \) the field is a thin sliver running 3km along the river. Clearly the area at this extreme would be zero. If \( x = 1500 \) the fence would run 1½ km away from the river and 1½ km back. The length, and hence the area, would be zero at this extreme. Clearly there’ll be a maximum area for some \( x \) between 0 and 1500.

\( A = xy \).

\( L = 2x + y = 3000 \). (Note that we converted the 3km to metres here.)

\[ \therefore A = x(3000 - 2x) = 3000x - 2x^2. \]

\[ \therefore \frac{dA}{dx} = 3000 - 4x = 0 \text{ when } x = 750. \text{ The corresponding value of } A = 1125000. \]

The values of \( A \) at the endpoints are zero. So the global maximum occurs at \( x = 750 \). Therefore to enclose the maximum area you should make the width (perpendicular to the river) 750 metres and the length (parallel to the river) 1500 metres. The maximum area thus enclosed will be 11.25 hectares. (One hectare = 10000 square metres.)

**Example 2:** A power line must be laid to connect the two towns \( A \) and \( B \) on opposite sides of a broad river, 1 kilometre wide as shown on the following map.

![Diagram of river and towns A and B with power line laid from D to C to B](image)

Every kilometre of cable laid on land costs $1000 and the work takes 5 working days. Under water each kilometre costs $2000 and takes 15 working days. What route should be followed in order to minimize the cost?

**Solution:** We must run the cable across the river to some point \( C \) and then along the opposite shore until we reach \( B \). Clearly each of these should be straight line segments.

\[ \frac{dC}{dx} = 2000 \frac{2x}{2\sqrt{x^2 + 1}} - 1000. \text{ This is zero when } \frac{2x}{\sqrt{x^2 + 1}} = 1, \text{ ie when } 2x = \sqrt{x^2 + 1}. \]
This is the equation we must now solve, to find the stationary points. To solve an equation with square roots we square both sides. This gives $4x^2 = x^2 + 1$ and so $3x^2 = 1$ or $x = \pm \frac{1}{\sqrt{3}}$. But the negative value doesn’t satisfy the original equation $2x = \sqrt{x^2 + 1}$. The local minimum is at $x = \frac{1}{\sqrt{3}} \approx 0.577$. So we should cross to the point 577 metres from $D$ (towards the other town).
EXERCISES FOR CHAPTER 6
(There are hints, starting on page 92, as well as full solutions, starting on page 93)

Exercise 1: The cross-section of a waterfall is in the shape of part of the curve \( y = x^3 - 3x + 3 \) (in metres). A pool forms in the curve formation of the rock. What is the greatest depth of water that can accumulate in the rock pool?

Exercise 2: A box with a volume of 2 cubic metres is to be cut out of a rectangular sheet of metal as follows:

\[
\begin{array}{ccc}
  & h & x \\
 2x & & h \\
  & h & \\
\end{array}
\]

The black regions are to be cut out and the left part has the four flaps folded up to form the bottom and four sides of the box and the right portion is to form the lid, to be soldered on. The base of the box is to be twice as long as it is wide.

(a) If \( A \) is the area of the rectangular sheet of metal, in square metres, show that
\[
A = 4x^2 + \frac{8}{x} + \frac{4}{x^2}.
\]

(b) Show that \( \frac{dA}{dx} = \frac{8x^6 - 8x^3 - 16}{x^3} \).

(c) Find the dimensions of the box (to the nearest centimetre) which minimize the area of metal required.

Exercise 3: A large cylindrical steel tank 10 cm thick is required to have a volume of 32 cubic metres. What should its dimensions be in order to minimize the total surface area (including both ends)?

Exercise 4: A tool rental shop finds that it can rent 30 floor sanders per day at a daily rental of $12 per sander (including GST at 10%). The owner believes that for each additional dollar per day that he charges one fewer sander would be rented. If he wants to maximize his gross revenue, what should he charge?

Exercise 5: A road is to be built between two towns, \( A \) and \( B \), on opposite sides of a river of uniform width 1km. A bridge must be built, running perpendicular to the banks. \( A \) is 1km from the river and \( B \) is 2km from the river. The distance between the two towns, as the crow flies, is 5km. Where should the bridge be built so as to minimize the road distance between the towns?

Exercise 6: An oil field, \( O \), is 10km from shore. The nearest point on land is at \( S \). This would have been a good place to build a refinery except for the fact that there is already a refinery at \( R \), 10km further down the coast. It is planned to build a pipeline running in a straight line under water to some point \( P \) on the coast, between \( S \) and \( R \) and then run it along the shore to \( R \).

It costs $1 million per km to build pipeline on land and $2 million per km under water. How far from \( S \) should the point \( P \) be located in order to minimize the cost of laying the pipe?
Exercise 7: You are walking north along a straight road beside some fields. Your destination lies two kilometres north and one kilometre west of your current position. Your speed along the road is 6 kph and across the fields it is 5 kph. At what point should you leave the road to cut across the fields?

HINTS FOR CHAPTER 6

Exercise 1: The depth of the pool is the difference between the local maximum and the local minimum.

Exercise 2: Show that \( \frac{dA}{dx} = \frac{8x^6 - 8x^3 - 16}{x^5} \). In solving \( \frac{dA}{dx} = 0 \), multiply by \( x^5 \) to get rid of fractions and substitute \( X = x^3 \). This will give a simple quadratic. Having found \( X \), use \( x = X^{1/3} \) to find the corresponding values of \( x \). You will get two solutions for \( x \) but only one of them can possibly represent a length.

Exercise 3: Let \( V \) be the volume and \( A \) the surface area of the tank. The volume of a cylinder is \( \pi r^2 h \) where \( r \) is the radius and \( h \) is the height. The surface area of the side is \( 2\pi rh \). On top of this there are the two ends, each of which is a circle with area \( \pi r^2 \). We have two variables here, \( r \) and \( h \) but the fact that the volume has to be 32 cubic metres means that we can express \( h \) in terms of \( r \). Once you’ve done this you should get \( A = \frac{64}{r} + \pi r^2 \). You should know what to do now.

Exercise 4: Let \( SC \) be the amount charged per day, per sander and let \( n \) be the number that can be rented at this price. The number of additional dollars charged, over the base figure of \$12 is \( C - 12 \). From the information given, for each of these dollars the number of sanders that can be rented per day will reduce by 1 so that if the cost is set at \$C \) the number rented per day will be \( n = 30 - (C - 12) \). You can now work out the total daily revenue, \( SR \), that would be received by renting out this many sanders at \$C \) each, per day. You’ll now have an expression that gives \( R \) in terms of \( C \). You now want to find what value of \( C \) maximises \( R \).

Exercise 5:

The vertical distance between \( A \) and \( B \) is 4 and the diagonal distance is 5 so the horizontal distance is 3. Using Pythagoras’ Theorem the length of the road from \( A \) to the river is \( \sqrt{x^2 + 1} \) and the length of the road from the river to \( B \) is \( \sqrt{(3-x)^2 + 4} = \sqrt{x^2 - 6x + 13} \). If \( L \) is the total length of the road from \( A \) to \( B \), including the bridge, we can write \( L \) in terms of \( x \).

Do this, and then put \( \frac{dL}{dx} = 0 \).
Exercise 6:

Suppose $P$ is $x$ kilometres from $S$, towards $R$. Then the land section is $10 - x$ kilometres long. Use Pythagoras to work out the length of the under water section. Now work out the total cost, as a function of $x$. Differentiate to find the value of $x$ that minimises the total cost.

Exercise 7:

Suppose you leave the road $x$ kilometres from the point $C$ that’s due east from your destination. You can use Pythagoras’ Theorem to work out the length of the diagonal section in terms of $x$. The remaining section along the first road will be $1 - x$ kilometres.

Now speed = distance / time so time = distance / speed. With the information provided you can work out the time taken along the two portions of your route, in terms of $x$. Calculus will now enable you to work out the value of $x$ that minimises the total time.

SOLUTIONS FOR CHAPTER 6

Exercise 1: $\frac{dy}{dx} = 3x^2 - 3$. So the stationary points are at $x = \pm 1$. Now $\frac{d^2y}{dx^2} = 6x$.

When $x = 1$, $\frac{d^2y}{dx^2} > 0$ so there is a local minimum at $(1, 1)$.

When $x = -1$, $\frac{d^2y}{dx^2} < 0$ so there is a local minimum at $(-1, 5)$.

The depth of the pool is the difference between the local maximum and the local minimum and this is $5 - 1 = 4$. So the depth of the pool is 4 metres.

Exercise 2:

(a) $A = (2x + 2h)^2 = 4x^2 + 8xh + 4h^2$.

The volume, when folded into a box, is $2x \cdot x \cdot h = 2x^2 h$. Since this is required t equal 2 we have $x^2 h = 1$, ie $h = \frac{1}{x^2}$. Substituting into the expression for $A$ gives:

$$A = 4x^2 + 8x \frac{1}{x^2} + 4 \frac{1}{x^2}.$$  
(b) $\frac{dA}{dx} = 8x - \frac{8}{x^3} - 16x^{-5} = \frac{8x^6 - 8x^3 - 16}{x^5}.$$

(c) Let $\frac{dA}{dx} = 0$. Then $8x^6 - 8x^3 - 16 = 0$, so $x^6 - x^3 - 2 = 0$.

This is what’s known as an equation reducible to a quadratic. Let $X = x^3$. 

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Then $X^2 - X - 2 = (X - 2)(X + 1) = 0$. Hence $X = 2$ or $-1$.

Now $x = X^{1/3}$ so this gives two solutions for $x$: $x = -1$ or $x = 2^{1/3}$.

But $x$ represents a length and so can’t be negative. So the only feasible solution for $x$ is $x = 2^{1/3}$, which is approximately 1.26. So $h = \frac{1}{x^2} \approx 0.63$.

So to minimise the surface area we need to make the box about $126 \text{cm} \times 252 \text{cm} \times 63 \text{cm}$.

**Exercise 3:** Let the radius of the cylinder be $r$ and the height $h$ (both in metres). Let $V$ be the volume in cubic metres and $A$ the surface area in square metres.

Then $V = \pi r^2 h = 32$ so $h = \frac{32}{\pi r^2}$.

$A = 2\pi r h + 2\pi r^2 = 2\pi \frac{32}{\pi r^2} + 2\pi r^2 = \frac{64}{r} + 2\pi r^2$.

\[
\frac{dA}{dr} = -\frac{64}{r^2} + 4\pi r. \text{ This is zero when } 4\pi r^3 = 64, \text{ ie } r^3 = \frac{16}{\pi}. \text{ This gives } r = \left(\frac{16}{\pi}\right)^{1/3} \approx 1.72 \text{ and } h = \frac{32}{\pi r^2} \approx 3.44.
\]

So to minimise the total surface area the tank should have a radius of about 172cm and a height of about 344cm.

**Exercise 4:** Let $SC$ be the amount charged per day, per sander and let $n$ be the number that can be rented at this price. The number of additional dollars charged, over the base figure of $12$ is $C - 12$.

So $n = 30 - (C - 12) = 42 - C$.

Let the total daily revenue be $R$. So $R = nC = (42 - C)C = 42C - C^2$.

\[
\frac{dR}{dC} = 42 - 2C \text{ so } \frac{dR}{dC} = 0 \text{ when } C = 21. \text{ So to maximise his gross revenue he should charge } $21 per sander per day. If he does his daily revenue will be $441.
\]

**Exercise 5:**

Let $R$ be the point on the river bank closest to $A$. The vertical distance between $A$ and $B$ is 4 and the diagonal distance is 5 so the horizontal distance is 3 (Pythagoras’ Theorem). Using Pythagoras’ Theorem again, the length of the road from $A$ to the river is $\sqrt{x^2 + 1}$ and the length of the road from the river to $B$ is $\sqrt{(3-x)^2 + 4} = \sqrt{x^2 - 6x + 13}$.

If $L$ is the total length of the road from $A$ to $B$, including the bridge, then $L = \sqrt{x^2 + 1} + \sqrt{x^2 - 6x + 13} + 1$.

\[
\frac{dL}{dx} = \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x + \frac{1}{2\sqrt{x^2 - 6x + 13}} \cdot (2x - 6).
\]

Let $\frac{dL}{dx} = 0$. Then $\frac{x}{\sqrt{x^2 + 1}} + \frac{x - 3}{\sqrt{x^2 - 6x + 13}} = 0$. 

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\[
\frac{x}{\sqrt{x^2 + 1}} = \frac{3 - x}{\sqrt{x^2 - 6x + 13}}.
\]
This looks quite horrible, but just hang in there! To remove the square roots we square both sides.
\[
\frac{x^2}{x^2 + 1} = \frac{(3 - x)^2}{x^2 - 6x + 13}.
\]
\[
x^2(x^2 - 6x + 13) = (x^2 + 1)(x^2 - 6x + 9)
\]
\[
x^4 - 6x^3 + 13x^2 = x^4 - 6x^3 + 9x^2 + x^2 - 6x + 9
\]
\[
x^4 - 6x^3 + 10x^2 - 6x + 9
\]
\[
x^2 + 6x - 9 = 0
\]
\[
x^2 + 2x - 3 = 0
\]
\[
(x + 3)(x - 1) = 0
\]
\[
x = -3 \text{ or } x = 1.
\]
Although \(x\) represents a distance it could be negative. The solution \(x = -3\) would mean that the road should from A should hit the river 3 km along from \(R\) \textit{in the opposite direction to B}.

![Diagram of the problem](image)

You’d probably want to reject this on common sense grounds. You can’t possibly minimise the length of the road by going in the opposite direction.

But there’s a mathematical reason for rejecting \(x = -3\). Remember that whenever you square both sides of an equation to eliminate square roots you can introduce “false solutions”. These are solutions to the equation obtained when one side is multiplied by \(-1\). The difference between these two equations disappears on squaring.

So in such cases it’s important that you substitute your so-called solutions into the original equation and reject any that don’t work.

If \(x = -3\) then
\[
\frac{x}{\sqrt{x^2 + 1}} = -\frac{3}{\sqrt{10}} \quad \text{while} \quad \frac{3 - x}{\sqrt{x^2 - 6x + 13}} = \frac{6}{\sqrt{40}} = \frac{3}{\sqrt{10}}.
\]
One side of the equation is negative while the other is positive. It doesn’t work.. In fact \(x = -3\) is the solution to another equation: \(\frac{x}{\sqrt{x^2 + 1}} = \frac{x - 3}{\sqrt{x^2 - 6x + 13}}\). Both our original equation, and this new one, give the same equation when we square each side.

So our original equation has only one solution: \(x = 1\). Therefore the bridge should be built 1 km from \(R\) (towards \(B\)).

\textbf{Exercise 6:}

\begin{align*}
S & \quad x & \quad P & \quad 10 - x & \quad R \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \latex
Suppose \( \frac{dC}{dx} = 0 \). Then \( \frac{2x}{\sqrt{x^2 + 1}} = 1 \), so \( 2x = \sqrt{x^2 + 1} \).

Square both sides (remember we will have to test our solutions because they may not all work in the equation before we squared): \( 4x^2 = x^2 + 1 \).

\[
3x^2 = 1, \text{ so } x^2 = \frac{1}{3} \text{ and hence } x = \pm \sqrt{\frac{1}{3}}.
\]

But \( x = -\sqrt{\frac{1}{3}} \) doesn’t satisfy the equation \( 2x = \sqrt{x^2 + 1} \) so we have just one solution: \( x = \frac{1}{\sqrt{3}} \approx 0.577 \). So \( P \) should be about 577 metres from \( S \) in order to minimise the cost.

Exercise 7:

![Diagram](image)

Suppose you leave the road \( x \) kilometres from the point \( C \) which is exactly due east from your destination. By Pythagoras’ Theorem the length of the diagonal section is \( \sqrt{x^2 + 1} \). The remaining section along the first road will be \( 2 - x \) kilometres. Let the time taken be \( t \) hours.

Then \( t = \frac{\sqrt{x^2 + 1}}{5} + \frac{2-x}{6} \).

\[
\frac{dt}{dx} = \frac{1}{5} \cdot \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x - \frac{1}{6} = \frac{x}{5\sqrt{x^2 + 1}} - \frac{1}{6} = \frac{6x - 5\sqrt{x^2 + 1}}{30\sqrt{x^2 + 1}}.
\]

So \( \frac{dt}{dx} = 0 \) when \( 6x = 5\sqrt{x^2 + 1} \).

Again we have to square both sides (watch out for “false solutions”).

\( 36x^2 = 25x^2 + 25 \), so \( 11x^2 = 25 \). This gives \( x = \pm \frac{5}{\sqrt{11}} \). But \( x = -\frac{5}{\sqrt{11}} \) doesn’t satisfy the original equation. So \( x = \frac{5}{\sqrt{11}} \approx 1.507556723 \). So you should leave the road, and cut across the fields about 1500 metres before \( C \) or about 500 north of your current position.